**Ph. D. THESIS** **MATHEMATICS**

# INDUCED AND EDGE INDUCED $V_4$-MAGIC LABELING OF GRAPHS

Thesis submitted to the

**University of Calicut**

for the award of the degree of

**DOCTOR OF PHILOSOPHY**

in Mathematics
under the Faculty of Science

by

**LIBEESHKUMAR K. B.**

**Department of Mathematics, University of Calicut**
**Kerala, India 673 635.**

**OCTOBER 2020**

# Certificate

It is certified that the reports of adjudicators for the thesis **" Induced and Edge Induced $V_4$-Magic Labeling of Graphs"** of Mr. Libeeshkumar K.B., have not been suggested any modifications or corrections of the work .

12.04.2021

Calicut University

Dr. Anilkumar V.

Research Supervisor

# DECLARATION

I hereby declare that the thesis, entitled "**INDUCED AND EDGE INDUCED $V_4$-MAGIC LABELING OF GRAPHS**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

University of Calicut,

27 October 2020. Libeeshkumar K. B.

# ACKNOWLEDGEMENT

I would like to express my immense and heartfelt gratitude to many people in the fulfillment of this work. First and foremost, I thank Dr. Anil Kumar V., Professor, Department of Mathematics, University of Calicut, who enlightened and constantly inspired me throughout the period of my research. I feel fortunate and very much privileged that my academic journey and scholastic achievement have been without much stress and strain because of his presence. His constant motivation, skillful guidance and easy access prompted me to keep on my work even in the difficult situation.

I extend my sincere gratitude to Dr. Preethi Kuttipulackal, Associate Professor and Head, Department of Mathematics, who has been very much lenient on me to provide all facilities of the department for the completion of my thesis. I also thank Dr. Raji Pilakkat, Associate Professor, Department of Mathematics and Dr. P. T. Ramachandran (Former Head, Department of Mathematics), Dr. Sini P., Assistant Professor, Department of Mathematics for their inspiration, assistance and guidance throughout my work.

I am enormously happy to thank my friends and well-wishers including Dr. Vineesh K.P., Assistant Professor, S.N.G College, Chelannur, Kozhikode, Premod, Assistant Professor, Govt.Polytechnic College, Perinthalmanna, Ummer K C, Assistant Professor, Govt. Polytechnic College, Kannur and Dr. Noufal K.M., C.K.G.M. Govt. College, Perambra, Dr. Latheesh Kumar A. R. and Dr. Shikhi M., for their incessant encouragement and assistance for the appropriate modification of my thesis.

Further, I thank all the Research fellows, M. Phil and M.Sc. students of the department for the vibrant presence in making my life a memorable one in the department during these years.

I also appreciate and acknowledge the non teaching staff and Librarian of the department for the cordial support given to me during the entire tenure of my study here.

Words are insufficient to acknowledge the role of my family members including my wife Neelima C.C. and my daughter Vedha Kalyani in my life, whose humble, simple and innocent presence prompted me to materialize my dream.

I also express my gratitude to the Principals and Staff members of C.K.G.M. Govt. College, Perambra, Govt. Polytechnic College, Kannur and S.N.G. College, Chelannur, Kozhikode for their support and co-operation.

University of Calicut,
27 October 2020. Libeeshkumar K. B.

# Contents

# List of Symbols

| | |
|---|---|
| $G$ | Simple, connected and undirected graph |
| $V(G)$ | Vertex set of $G$ |
| $E(G)$ | Edge set of $G$ |
| $I_G$ | The incidence relation |
| $N(u)$ | Neighbourhood set of the vertex $u$ |
| $deg\ (v)$ | Degree of the vertex $v$ |
| $P_n$ | Path on $n$ vertices |
| $C_n$ | Cycle on $n$ vertices |
| $K_n$ | Complete graph |
| $K_{m,n}$ | Complete bipartite graph |
| $K_{1,n}$ | Star graph |
| $G_1 \cup G_2$ | Union of $G_1$ and $G_2$ |
| $G_1 + G_2$ | Join of $G_1$ and $G_2$ |
| $G_1 \times G_2$ or $G_1 \Box G_2$ | Product of $G_1$ and $G_2$ |
| $G_1 \odot G_2$ or $G_1 \circ G_2$ | Corona of $G_1$ and $G_2$ |
| $O(k)$ | Order of $k$ in a group |
| $\Gamma(A)$ | Set of all induced $A$-magic graphs |
| $\Gamma_k(A)$ | Set of all induced $A$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k\}$, for some $k \in A \setminus \{0\}$ |

| | |
|---|---|
| $\Gamma(V_4)$ | Set of all induced $V_4$-magic graphs |
| $\Gamma_k(V_4)$ | Set of all induced $V_4$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k\}$, for some $k \in V_4 \setminus \{0\}$ |
| $\Gamma_{k,0}(V_4)$ | Set of all induced $V_4$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k, 0\}$, for some $k \in V_4 \setminus \{0\}$ |
| $W_n$ | Wheel graph |
| $H_n$ | Helm graph |
| $W(2, n)$ | Web graph |
| $CH_n$ | Closed helm |
| $Fl^n$ | Flower graph |
| $G_n$ | Gear graph |
| $F_n$ | Fan graph |
| $Fl_n$ | Flag graph |
| $SF_n$ | Sunflower graph |
| $J(m, n)$ | Jelly fish graph |
| $Sun_n$ | Sun graph |
| $BS(p, q)$ | Broken sun graph |
| $CBSun_{p,q}$ | Consecutive broken sun graph |
| $C_n^{(t)}$ | One point union of $t$ cycles of length $n$ |
| $C_3^{(t)}$ | Friendship graph or Dutch 3 windmill graph |
| $B(n, k)$ | $n$- gon Book graph |
| $BP(n)$ | Bipyramid graph |
| $CB_n$ | Comb graph |
| $TS_n$ | Triangular snake graph |
| $DTS_n$ | Double triangular snake graph |
| $O(L_n)$ | Open ladder graph |
| $\Theta(a_1, a_2, a_3, \ldots, a_k)$ | Generalized theta graph |
| $B_n$ | Book graph |

| | |
|---|---|
| $B_{m,n}$ | Bistar graph |
| $Bt(n,k)$ | $(n,k)-$ Banana tree |
| $K_m^{(n)}$ | Windmill graph |
| $S(G)$ | Subdivision graph of $G$ |
| $Sh(G)$ | Shadow graph of $G$ |
| $M(G)$ | Middle graph of $G$ |
| $L(G)$ | Line graph of $G$ |
| $\sigma_a(V_4)$ | Set of all edge induced $V_4$-magic graphs with edge induced magic labeling $f$ satisfying $f^{++}(u) = a$ for all $u \in V$ |
| $\sigma_0(V_4)$ | Set of all edge induced $V_4$-magic graphs with edge induced magic labeling $f$ satisfying $f^{++}(u) = 0$ for all $u \in V$ |
| $\sigma(V_4)$ | $\sigma_a(V_4) \bigcap \sigma_0(V_4)$ |

# List of Figures

# Introduction

Recently, Graph Theory has been lauded as an area of research with many applications in our everyday lives. These days, there is an unprecedented increase in research related to graph theory. Graph theory is generally applicable to various branches of mathematics, such as Algebra, Algebraic topology, Number Theory, Algebraic Geometry, Numerical Analysis, Matrix Theory, Operations Science, etc., even though graph theory was originally known by recreational math issues. It also nurtured the growth of other branches of sciences like Physical Sciences, Chemical Sciences, Computer Science, Life Science, Sociology, Economics, Social Sciences, Geography, Architecture, Electrical Engineering, Genetics and so on.

In graph theory, numerous research studies are ongoing, especially in the field of graph labeling. The origin of most Graph labeling methods can be traced back to the theory introduced by A. Rosa [3] in 1967 or given by R. L. Graham and N. J. A. Sloane [15] in 1980. Various Graph Labeling problems are $\beta-$ valuation (or graceful labeling), $\gamma-$ labeling, $\rho-$ labeling, cordial labeling, total magic cordial labeling, elegant labeling, mean labeling, magic labeling, anti magic labeling, prime labeling and so on. Among these, Magic labeling of a graph is a well known one. For any abelian group $A$ with identity element 0, a graph $G = (V(G), E(G))$ is said to be $A$-magic [19] if there exists a labeling $l : E(G) \to A \setminus \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to A$ defined by

$$l^+(v) = \sum\{l(uv) : uv \in E(G)\}$$

is a constant map.

Among various mathematical models, labeled graphs with a wide range of applications serve as a useful model. For example, research fields like conflict resolution in social psychology, electrical circuit theory and energy crisis theory and others are enriched by qualitative labelings of graph. In the same way, quantitative labelings of graphs are applied in coding theory problems, such as radar location codes, missile guidance codes, synch-set codes and convolution codes. There are other applications for labeled graphs as fixing complexities of $X-$ray crystallographic analysis or design communication network addressing systems. Labeled graphs are also used in optimal circuit layout and radio astronomy.

## An Overview of the Thesis

In this thesis, we define some new types of labelings, namely induced $A$-magic labeling of graphs, induced $V_4$-magic labeling of graphs and edge induced $V_4$-magic labeling of graphs. Through out this work, we consider graphs that are connected, finite, simple and undirected. The Klein 4-group, denoted by $(V_4, +) = (\{0, a, b, c\}, +)$ is an abelian group of order 4 with identity element 0, where the operation $+$ is defined as $a+a = b+b = c+c = 0$ and $a+b = c,\ b+c = a,\ c+a = b$.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$ and $(A, +)$ be an abelian group. Suppose $f : V(G) \to A$ be a vertex labeling and $f^* : E(G) \to A$ denote the induced edge labeling of $f$ defined by $f^*(uv) = f(u) + f(v)$ for all $uv \in E(G)$. Then $f^*$ again induces a vertex labeling $f^{**} : V(G) \to A$

defined by $f^{**}(u) = \Sigma f^*(uv)$, where the summation is taken over all the vertices $v$ which are adjacent to $u$. Then a graph $G$ is said to be an induced $A$-magic graph and denoted by IM$A$G or simply IMG if there exists a non zero vertex labeling function $f : V(G) \to A$ such that $f \equiv f^{**}$. The function $f$, so obtained is called an induced $A$-magic labeling of $G$ and denoted by IM$A$L or simply IML.

In a similar way, by taking the Klein 4-group $V_4$ instead of an arbitrary abelian group $A$ in the above definition, we can define induced $V_4$-magic graphs (IM$V_4$G) and induced $V_4$-magic labeling of a graph (IM$V_4$L).

Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : (V(G) \to V_4$ denote the induced vertex labeling of $f$ defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in (V(G)$. Then $f^+$ again induces an edge labeling $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$. Then a graph $G = (V(G), E(G))$ is said to be an edge induced $V_4$-Magic graph if $f^{++}(e)$ is a constant for all $e \in E(G)$. If this constant is $x$ then $x$ is said to be the induced edge sum of the graph $G$, or sometimes induced edge sum of an edge $uv$. The function $f$, so obtained is called a edge induced $V_4$-Magic labeling of $G$ or simply edge induced Magic labeling of $G$ and it is denoted by EIM$V_4$L or simply EIML.
Through out this thesis we will use the following notations:

(i) $\Gamma(A) :=$ set of all induced $A$-magic graphs.

(ii) $\Gamma_k(A) :=$ set of all induced $A$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k\}$, for some $k \in A \setminus \{0\}$.

(iii) $\Gamma(V_4) :=$ set of all induced $V_4$-magic graphs.

(iv) $\Gamma_k(V_4) :=$ set of all induced $V_4$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k\}$, for some $k \in V_4 \setminus \{0\}$.

(v) $\Gamma_{k,0}(V_4) :=$ set of all induced $V_4$-magic graphs induced magic label $f$ satisfying $f(V(G)) = \{k, 0\}$, for some $k \in V_4 \setminus \{0\}$.

(vi) $\sigma_a(V_4) :=$ Set of all edge induced $V_4$-magic graphs with edge induced magic labeling $f$ satisfying $f^{++}(u) = a$ for all $u \in V$.

(vii) $\sigma_0(V_4) :=$ Set of all edge induced $V_4$-magic graphs with edge induced magic labeling $f$ satisfying $f^{++}(u) = 0$ for all $u \in V$.

(viii) $\sigma(V_4) := \sigma_a(V_4) \bigcap \sigma_0(V_4)$.

The thesis contains an introductory chapter and eight other chapters as well. In the introductory chapter, we discuss the motivation of the study of induced $A$-magic labeling, induced $V_4$-magic labeling and edge induced $V_4$-magic labeling of graphs and a literature survey on it.

In **Chapter One**, we list out preliminary definitions from the areas of graph theory and group theory which will be useful for the upcoming chapters in the thesis.

**Chapter Two** introduces the concept of induced $A$-magic graphs, where $A$ is an abelian group and the concept of induced $V_4$-magic graphs. The first section of the chapter gives the definition of induced $A$-magic labeling and some definitions of cycle related graphs. In the second section, we prove the "Induced degree sum theorem" which establishes the necessary and sufficient condition for a vertex label function to be an induced $A$ magic label for a graph. This theorem states that: Let $f$ be a vertex labeling of a graph $G$. Then $f$ is an IAML of $G$, if and only if $[deg\ (u) - 1]f(u) + \sum f(v) = 0$, for any vertex $u \in V(G)$, where the summation is taken over all the vertices $v$ which are adjacent to $u$. We then we prove a necessary and sufficient conditions for the path $P_n$, cycle $C_n$, the complete graph $K_n$ and the complete bipartite graph $K_{m,n}$ belongs to the above said sets (i) and (ii). In the third section we define induced $V_4$-magic labeling of graphs and prove a theorem analogues to "Induced degree sum theorem". Furthermore, we prove that $G \notin \Gamma_k(V_4)$ for any graph $G$. In the last section of this chapter we discuss whether the graphs cycle $C_n$, wheel graph $W_n$, helm $H_n$, web graph $W(2,n)$, closed helm $CH_n$, flower graph $Fl^n$, gear graph $G_n$, fan graph $F_n$, flag graph $Fl_n$, sunflower graph $SF_n$, jelly fish $J(m,n)$, sun graph $Sun_n$, consecutive broken sun graph $CBSun_{p,q}$, one point union of $t$ cycles of length $n$ denoted by $C_n^{(t)}$, $n$- gon book $B(n,k)$, bipyramid graph $BP(n)$ are induced $V_4$ magic or not.

**Chapter Three** deals with induced $V_4$-magic labeling of path and star related graphs. In the first section, we include definitions of some path and star related graphs. Second section discusses induced $V_4$ magic labeling of path related graphs namely path $P_n$, comb graph $CB_n$, triangular Snake graph $TS_n$, double triangular snake graph $DTS_n$, open ladder $O(L_n)$, the book graph $B_n$ and so on. The third section discusses star related graphs namely complete graph $K_n$, complete bipartite graph $K_{m,n}$, star graph $K_{1,n}$, bistar $B_{m,n}$, $< K_{1,n} : m >$, $(n,k)$-banana tree $Bt(n,k)$ and windmill graph $K_m^{(n)}$ admit induced $V_4$-magic labeling or not.

The first section of **Chapter Four,** we include the definition of subdivision graph and prove some theorems regarding induced $V_4$ magic labeling of subdivision of graphs. Moreover, we check whether the subdivision graph of $C_n$, $P_n$, $K_n$, $B_{m,n}$, $K_{m,n}$, $K_{1,n}$, $W_n$, $H_n$, $CB_n$, $J(m,n)$, $SF_n$, $G_n$ and $Fl^n$ admits the induced $V_4$ magic labeling or not. In the second section of this chapter, we prove that shadow graph of any graph is not an induced $V_4$ magic graph.

In the first section of **Chapter Five**, we provide the definitions of Middle graph of a graph and prove the theorem: "Let G be a graph with every vertex is of odd degree, then $M(G) \in \Gamma(V_4)$." In continuation of this section we discuss induced $V_4$-magic labeling of middle graphs like $M(P_n)$, $M(K_{1,n})$, $M(B_{m,n})$, $M(K_{m,n})$, $M(K_n)$, $M(CB_n)$, $M(C_n)$, $M(W_n)$, $M(H_n)$, $M(Fl_n)$ and $M(Sun_n)$. In the second section, we define Line graph of a graph and discuss induced $V_4$-

magic labeling of line graphs $L(C_n)$, $L(P_n)$, $L(K_{1,n})$, $L(B_{m,n})$, $L(Sun_n)$, $L(CB_n)$, $L(W_n)$, $L(TS_n)$, $L(G_n)$ and $L(Fl_n)$.

**Chapter Six** introduces the concept of edge induced $V_4$-magic labeling of graphs and we prove some theorems regarding the concept of edge sum equation of an edge in a graph. The second section of this chapter gives some main results regarding the edge induced $V_4$-magic labeling of graphs. In the third section, we discuss edge induced $V_4$ magic labeling of some graphs like $P_n$, $C_n$, $K_{1,n}$, $K_{m,n}$, $B_{m,n}$, and $K_n$ and in the last section we discuss edge induced $V_4$ magic labeling of some special graphs like $Sun_n$, $CB_n$, $W_n$, $J(m,n)$, $TS_n$ and $O(L_n)$.

**Chapter Seven** contains two sections, in which we discuss, edge induced $V_4$ magic labeling of subdivision graph, line graph of some general graphs and some special graphs.

**Chapter Eight** briefly sums up the overall aspects of the work and the scope for further research also.

Chapter 1

# Preliminaries

*In this chapter, we include a brief overview of the preliminary concepts in graph theory and group theory that we used in the coming chapters. Readers can refer to* [4] and [12] *for notations and terminologies not explicitly specified in this thesis.*

## 1.1 Basic Definitions from Graph Theory

**Definition 1.1.1.** [12] *A graph is an ordered triple $G = (V(G), E(G), I(G))$, where $V(G)$ is a non empty set, $E(G)$ is a set disjoint from $V(G)$ and $I(G)$ is an "incidence" relation that associates with each element of $E(G)$ an unordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the vertices (or nodes or points) of $G$ and elements of $E(G)$ are called the edges (or lines) of $G$. $V(G)$ and $E(G)$ are the vertex set and the edge set of $G$ respectively. If for the edge $e$ of $G$, $I(G)(e) = \{u, v\}$, we write $I_G(e) = uv$.*

**Definition 1.1.2.** [12] *If $I(G)(e) = \{u, v\}$ then the vertices $u$ and $v$ are called the end vertices or ends of the edge $e$. Each edge is said to join its ends; in this case, we say that $e$ is incident with each one of its ends. Also, the vertices $u$ and*

*$v$ are then incident with $e$.*

**Definition 1.1.3.** [12] *A set of two or more edges of a graph $G$ is called a set of multiple or parallel edges if they have the same pair of distinct ends.*

**Definition 1.1.4.** [12] *An edge for which the two ends are the same is called a loop at the common vertex.*

**Definition 1.1.5.** [12] *A vertex $u$ is a neighbour of $v$ in $G$; if $uv$ is an edge of $G$; and $u \neq v$. The set of all neighbours of $v$ is the open neighbourhood of $v$ or the neighbour set of $v$; and is denoted by $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of $v$ in $G$.*

**Definition 1.1.6.** [12] *Vertices $u$ and $v$ are adjacent to each other in $G$ if and only if there is an edge of $G$ with $u$ and $v$ as its ends. Two distinct edges $e$ and $f$ are said to be adjacent if and only if they have a common end vertex.*

**Definition 1.1.7.** [12] *A graph is simple if it has no loops and no multiple edges. Thus, for a simple graph G; the incidence function $I(G)$ is one-to-one. Hence, an edge of a simple graph is identified with the pair of its ends. A simple graph, therefore, may be considered as an ordered pair $(V(G), E(G))$, where $V(G)$ is a non empty set and $E(G)$ is a set of un ordered pairs of elements of $V(G)$.*

**Definition 1.1.8.** [12] *A graph is called finite if both $V(G)$ and $E(G)$ are finite. A graph that is not finite is called an infinite graph.*

**Definition 1.1.9.** [12] *A simple graph $G$ is said to be complete if every pair of distinct vertices of $G$ are adjacent in $G$. A complete graph, which has $n$ vertices is denoted by $K_n$.*

**Definition 1.1.10.** [12] *A graph is bipartite if it's vertex set can be partitioned into two nonempty subsets $X$ and $Y$ such that each edge of $G$ has one end in $X$ and the other in $Y$. The pair $(X, Y)$ is called a bipartition of the bipartite graph.*

*The bipartite graph $G$ with bipartition $(X, Y)$ is denoted by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is complete if each vertex of $X$ is adjacent to all the vertices of $Y$. If $G(X, Y)$ is complete with $|X| = m$ and $|Y| = n$ then $G(X, Y)$ is denoted by $K_{m,n}$. A complete bipartite graph of the form $K_{1,n}$ is called a star.*

**Definition 1.1.11.** [12] *Let $G$ be a graph and $v \in V$. Then the number of edges incident at $v$ in $G$ is called the degree (or valency) of the vertex $v$ in $G$ and is denoted by $d_G(v)$ or $deg(v)$ or $d(v)$.*

**Definition 1.1.12.** [12] *A graph $G$ is called $k-$regular if every vertex of $G$ has degree $k$. A graph is said to be regular if it is $k-$regular for some non negative integer $k$.*

**Definition 1.1.13.** [12] *A vertex of degree 0 is an isolated vertex of $G$. A vertex of degree 1 is called a pendant vertex of $G$; and the unique edge of $G$ incident to such a vertex of $G$ is a pendant edge of $G$.*

## 1.2 Paths and Connectedness

**Definition 1.2.1.** [12] *A walk in a graph $G$ is an alternating sequence $W : v_0e_1v_1e_2v_2 \dots e_pv_p$ of vertices and edges beginning and ending with vertices in which $v_{i-1}$ and $v_i$ are the ends of $e_i$, $v_0$ is the origin and $v_p$ is the terminus of $W$. The walk $W$ is said to join $v_0$ and $v_p$, it is also referred to as a $v_0 - v_p$ walk. If the graph is simple, a walk is determined by the sequence of its vertices. The walk is closed if $v_0 = v_p$ and is open otherwise. The length of a walk is the number of edges in it.*

**Definition 1.2.2.** [12] *A walk is called a trail if all the edges appearing in the walk are distinct.*

**Definition 1.2.3.** [12] *A walk is called a path if all the vertices are distinct.*

*Thus, a path in $G$ is automatically a trail in $G$: When writing a path, we usually omit the edges. A path on $n$ vertices is usually denoted by $P_n$.*

**Definition 1.2.4.** [12] *A cycle is a closed trail in which the vertices are all distinct. A cycle of length $n$ is usually denoted by $C_n$.*

**Definition 1.2.5.** [12] *Let $G$ be a graph. Two vertices $u$ and $v$ of $G$ are said to be connected if there is a $u - v$ path in $G$. A graph $G$ is said to be connected if every pair of vertices in $G$ are connected.*

## 1.3 Operations on Graphs

**Definition 1.3.1.** [4] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$. Then their union $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.*

**Definition 1.3.2.** [4] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$. Then their join is denoted by $G_1 + G_2$ and it is a graph consists of $G_1 \cup G_2$ and all edges joining $V_1$ with $V_2$.*

**Definition 1.3.3.** [4] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$. Then the product of $G_1$ and $G_2$ is denoted by $G_1 \times G_2$ or $G_1 \square G_2$ and it has vertex set $V = V_1 \times V_2$ and two distinct vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $G_1 \times G_2$ are adjacent whenever $[u_1 = v_1$ and $u_2$ adjacent to $v_2]$ or $[u_2 = v_2$ and $u_1$ adjacent to $v_1]$.*

**Definition 1.3.4.** [4] *The Corona $G_1 \odot G_2$ or $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$ is the graph obtained by taking one copy of $G_1$, (which has $p_1$ vertices) and $p_1$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ by an edge to every vertex in the $i^{th}$ copy of $G_2$.*

## 1.4 Basic Definitions from Group Theory

**Definition 1.4.1.** [6] *A binary operation on a set $S$ is a function mapping from $S \times S$ into $S$.*

**Definition 1.4.2.** [6] *A binary operation $*$ on a set $S$ is associative if $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$.*

**Definition 1.4.3.** [6] *A binary operation $*$ on a set $S$ is commutative if $a*b = b*a$ for all $a, b \in S$.*

**Definition 1.4.4.** [6] *A set $S$ together with a binary operation $*$ is called a binary algebraic structure or simply binary structure, denoted by $< S, * >$ .*

**Definition 1.4.5.** [6] *A group $< G, * >$ is a set $G$, closed under a binary operation $*$, such that the following axioms are satisfied:*

*(i) The operation $*$ is associative.*

*(ii) There exists an element $e \in G$ such that $e * x = x = x * e$ for all $x \in G$. (The element $e$ is called the identity element of the binary operation on $*$ $G$.)*

*(iii) For each $a \in G$, there is an element $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$ for all $a \in G$. (The element $a^{-1}$ is called the inverse of the element $a$.)*

**Definition 1.4.6.** [6] *A group $< G, * >$ is said to be an an abelian group or simply abelian or a commutative group if the binary operation $*$ is commutative.*

**Definition 1.4.7.** *Let $V_4 = \{0, a, b, c\}$. Then $V_4$ is an abelian group with identity element $0$, under the binary operation $+$ defined by $a + a = b + b = c + c = 0$ and $a + b = c,\ b + c = a,\ c + a = b$. This abelian group is called Klein-4-group or $V$ group.*

**Definition 1.4.8.** [6] *Let $G$ be a group with identity element $e$. Then the order of an element $a$ in $G$ is the smallest positive integer $m$ such that $a^m = e$ and it is denoted by $O(a)$.*

Chapter 2

# Induced Magic Labeling of Graphs

*The first section of this chapter introduces the concept of induced $A$-magic graph, where $A$ is an Abelian group. Some well known cycle-related graphs are also included. The second section of the chapter gives a necessary and sufficient condition for some general graphs, that admits induced $A$ magic labeling. The third section of the chapter introduces the concept of induced $V_4$-magic graphs and the last section deals with induced $V_4$ magic labeling of some cycle related graphs.*

## 2.1 Introduction

Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$ and $(A, +)$ be an abelian group with identity element 0. Suppose $f : V(G) \to A$ be a vertex labeling and $f^* : E(G) \to A$ denote the induced edge labeling of $f$ defined by $f^*(uv) = f(u)+f(v)$ for all $uv \in E(G)$. Then $f^*$ again induces a vertex

[1]The second section of this chapter has been published in *Malaya Journal of Matematik*, Volume. 8, Number 1, 59-61, (2020).

[2]The third and fourth sections of this chapter have been published in *Malaya Journal of Matematik*, Volume. 8, Number 2, 473-477, (2020).

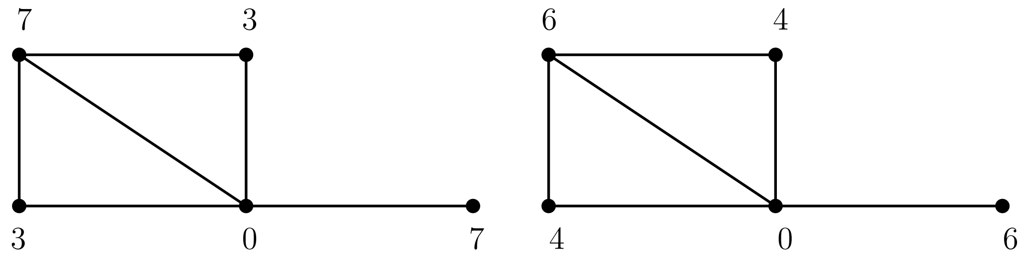


Figure 2.1: A graph $G$

labeling $f^{**} : V(G) \to A$ defined by $f^{**}(u) = \Sigma f^{*}(uv)$, where the summation is taken over all the vertices $v$ which are adjacent to $u$. A graph $G = (V(G), E(G))$ is said to be an induced $A$-magic graph and it is denoted by IMAG or simply IMG if there exists a non zero vertex labeling $f : V(G) \to A$ such that $f \equiv f^{**}$. The function $f$, so obtained is called an induced $A$-magic labeling of $G$ and it is denoted by IMAL or simply IML. This chapter discusses the induced $A$-magic labeling of some general graphs and induced $V_4$-magic labeling of some cycle related graphs which belong to the following categories:

(i) $\Gamma(A)$ := set of all induced $A$-magic graphs.

(ii) $\Gamma_k(A)$ := set of all induced $A$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k\}$, for some $k \in A \setminus \{0\}$.

(iii) $\Gamma(V_4)$ := set of all induced $V_4$-magic graphs.

(iv) $\Gamma_k(V_4)$ := set of all induced $V_4$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k\}$, for some $k \in V_4 \setminus \{0\}$.

(v) $\Gamma_{k,0}(V_4)$ := set of all induced $V_4$-magic graphs with induced magic label $f$ satisfying $f(V(G)) = \{k, 0\}$, for some $k \in V_4 \setminus \{0\}$.

Two different types of induced $\mathbb{Z}_{10}$-magic labelings of a graph $G$ are shown in Figure 2.1.

**Definition 2.1.1.** [8] *The sum of the graphs $C_n$ and $K_1$ is called a wheel graph and it is denoted by $W_n$, that is $W_n = C_n + K_1$.*

**Definition 2.1.2.** [18] *The helm $H_n$ is a graph obtained from a wheel $W_n$ by attaching a pendant edge at each vertex of the n-cycle.*

**Definition 2.1.3.** [8] *The web graph $W(2, n)$ is a graph obtained by joining the pendant points of a helm to form a cycle and adding a single pendant edge to each vertex of this outer cycle.*

**Definition 2.1.4.** [8] *A closed helm $CH_n$ is a graph obtained from a helm by joining each pendant vertex to form a cycle.*

**Definition 2.1.5.** [22] *A flower graph $Fl^n$ is a graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.*

**Definition 2.1.6.** [24] *A gear graph is a graph $G_n$ obtained from the wheel $W_n$ by adding a vertex between every pair of adjacent vertices of the $n-$cycle.*

**Definition 2.1.7.** [8] *A fan graph, denoted by $F_n$, is defined as $P_n + K_1$, where $P_n$ is a path graph on $n$ vertices.*

**Definition 2.1.8.** [8] *A flag graph is obtained by joining one vertex of $C_n$ to an extra vertex called the root and it is denoted by $Fl_n$.*

**Definition 2.1.9.** [13] *A sunflower graph is denoted by $SF_n$ and is obtained by taking a wheel with the central vertex $v_0$ and the n-cycle $v_1, v_2, v_3, \ldots, v_n$ and additional vertices $w_1, w_2, w_3, \ldots, w_n$, where $w_i$ is joined by edges to the vertices $v_i$ and $v_{i+1}$, where $i + 1$ is taken modulo $n$.*

**Definition 2.1.10.** [8] *Jelly fish graph $J(m, n)$ is obtained from a $4-$cycle $v_1v_2v_3v_4$ by joining $v_1$ and $v_3$ with an edge and appending the central vertex of $K_{1,m}$ to $v_2$ and appending the central vertex of $K_{1,n}$ to $v_4$.*

**Definition 2.1.11.** [16] *The sun graph on $n = 2p$ vertices, denoted by $Sun_n$, is the graph obtained by appending a pendant vertex to each vertex of a $p-$cycle. A broken sun graph is a connected unicyclic subgraph of a sun graph. We denote by $BS(p, q)$ the set of broken suns with $n = p + q$ vertices and with a $p-$ cycle. For $p > 2$ and $0 < q < p$, a consecutive broken sun graph, denoted by $CBSun_{p,q}$, is the graph belonging to $BS(p, q)$ such that the subgraph induced by the vertices of degree 2 is a path on $p - q$ vertices.*

**Definition 2.1.12.** *We denote by $C(n, k_1, k_2, k_3, \ldots, k_t)$ the class of all graphs obtained by identifying the apex vertices of $t$ stars $K_{1,k_i}$ $(i = 1, 2, 3, \ldots, t)$ with $t$ $(1 \leq t \leq n)$ vertices of $C_n$.*

**Definition 2.1.13.** [8] *Let $C_n^{(t)}$ denote the one-point union of $t$ cycles of length $n$. For $n = 3$ the graph $C_3^{(t)}$ is called friendship graph or Dutch 3-windmill graph.*

**Definition 2.1.14.** [19] *When $k$ copies of $C_n$ share a common edge it will form the $n-$gon book graph of $k$ pages and is denoted by $B(n, k)$.*

**Definition 2.1.15.** [19] *Let $N_2 = \{u, v\}$ be the disconnected graph of order two. The graph $C_n + N_2$, the cycle $C_n$ join $N_2$, is called bi pyramid based on $C_n$ and is denoted by $BP(n)$.*

## 2.2 Induced $A$ Magic Labeling of Graphs

**Lemma 2.2.1.** *Let $G = (V, E)$ be a graph and $f$ is an IAML of $G$. If $v_1 \in V$ is a pendant vertex adjacent to $v \in V$, then $f(v) = 0$.*

*Proof.* Let $f$ be an IAML of a graph $G$ and $v_1$ be a pendant vertex adjacent to $v$. Then $f^*(vv_1) = f(v) + f(v_1)$ and $v_1$ is a pendant vertex implies that $f^{**}(v_1) = f(v) + f(v_1)$. Also $f$ is an induced magic labeling of $G$ implies that $f(v_1) = f^{**}(v_1) = f(v) + f(v_1)$. Thus $f(v) = 0$. □

**Corollary 2.2.2.** *If $G$ has a pendant vertex, then $G \notin \Gamma_k(A)$ for any Abelian group $A$.*

*Proof.* Proof is indisputable from the Lemma 2.2.1. □

**Lemma 2.2.3.** *Let $f$ be an IAML of a graph $G$ and $wuvz$ be a path in $G$ with $w$ and $z$ are pendant vertices in $G$, then $f^*(uv) = 0$.*

*Proof.* Suppose $f$ is an IAML of a graph $G = (V, E)$ and $wuvz$ is a path in $G$ with $w$ and $z$ are pendant vertices. Then by the Lemma 2.2.1, we have $f(u) = 0 = f(v)$. Hence $f^*(uv) = 0$. □

**Theorem 2.2.4. Induced degree sum theorem**

*Let $f$ be a vertex labeling of a graph $G$. Then $f$ is an IAML of $G$, if and only if*

$$[deg\ (u) - 1]f(u) + \sum f(v) = 0, \tag{2.1}$$

*for any vertex $u \in V(G)$, where the summation is taken over all the vertices $v$ which are adjacent to $u$.*

In this case, the equation (2.1) corresponding to a vertex $u$ is called induced degree sum equation of the vertex $u$.

*Proof.* Let $f$ be an IAML of $G$ and $u$ be a vertex in $G$ with $deg\ (u) = m$. Let $v_1, v_2, v_3, \ldots, v_m$ be the vertices adjacent to $u$ in $G$. Now $f$ is an IAML if and only if $f(u) = f^{**}(u) = f^*(uv_1) + f^*(uv_2) + f^*(uv_3) + \cdots + f^*(uv_m) = mf(u) + f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_m)$.
That is if and only if $(m-1)f(u) + \sum f(v) = 0$, where $v$ is adjacent to $u$. □

**Theorem 2.2.5.** *$P_n \in \Gamma(A)$ if and only if $n$ is a multiple of 3.*

*Proof.* Suppose $n = 3m$, for some integer $m$. Let $P_n$ be the path with vertex set $V = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$. For any $a \neq 0$ in $A$, define $f : V \to A$ as

$$f(v_i) = \begin{cases} a & \text{if} \quad i = 1, 4, 7, \ldots, 3m-2 \\ 0 & \text{if} \quad i = 2, 5, 8, \ldots, 3m-1 \\ a^{-1} & \text{if} \quad i = 3, 6, 9, \ldots, 3m. \end{cases}$$

Then, $f$ is an IAML of $P_n$. Conversely, suppose $n$ is not a multiple of 3, then $n = 3m+1$ or $n = 3m+2$ for some positive integer $m$. Let $f : V \to A$ be a vertex labeling function with $f \equiv f^{**}$. Then for $1 \leq k \leq n-3$ and any path $v_k v_{k+1} v_{k+2} v_{k+3}$ in $P_n$, we have $f(v_{k+1}) = f^{**}(v_{k+1})$ implies that $f(v_k) + f(v_{k+1}) + f(v_{k+2}) = 0$. Also $f(v_{k+2}) = f^{**}(v_{k+2})$ implies that $f(v_{k+1}) + f(v_{k+2}) + f(v_{k+3}) = 0$. Therefore we should have $f(v_k) = f(v_{k+3})$. Also, since $v_2$ and $v_{n-1}$ are adjacent to the pendant vertices $v_1$ and $v_n$ respectively, we have $f(v_2) = 0$ and $f(v_{n-1}) = 0$. Let us deal with the following cases:

**Case 1 :** $n = 3m+1$.

In this context, from the above discussion we have, $0 = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3m-1}) = f(v_{n-2})$ and $0 = f(v_{n-1}) = f(v_{n-4}) = \cdots = f(v_6) = f(v_3) = 0$. Thus $f(v_3) = 0$ and $f(v_1) + f(v_3) = 0$ imply that $f(v_1) = 0$, which again implies that $0 = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3m+1}) = f(v_n.)$ Hence$f \equiv 0$, Therefore $f$ is not an IAML.

**Case 2 :** $n = 3m+2$.

In this context from the above discussion we have, $0 = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3m+2}) = f(v_n)$ and $0 = f(v_{n-1}) = f(v_{n-4}) = \cdots = f(v_4) = f(v_1)$. Thus $f(v_1) = 0$ and $f(v_1) + f(v_3) = 0$ imply that $f(v_3) = 0$, which implies $0 = f(v_3) = f(v_6) = f(v_9) = \cdots = f(v_{3m}) = f(v_{n-2})$. Hence $f \equiv 0$. Therefore, $f$ is not an IAML.

Hence if $n$ is not a multiple of 3, then $P_n \notin \Gamma(A)$ $\square$

**Theorem 2.2.6.** *Let $\{v_1, v_2, v_3, \ldots, v_{n-1}, v_n = v_0\}$ be the vertex set of $C_n$. Then for any path $v_{k-1}v_kv_{(k+1) \bmod n}$, $f$ is an IAML of $C_n$ if and only if $f(v_{k-1}) + f(v_k) + f(v_{(k+1) \bmod n}) = 0$, where $1 \leq k \leq n$. Moreover any IAML $f$ of $C_n$ satisfies $f(v_k) = f(v_{(k+3) \bmod n})$ for $1 \leq k \leq n$.*

*Proof.* For $k = 1, 2, 3, \ldots, n$, consider the path $v_{k-1}v_kv_{(k+1) \bmod n}$ in $C_n$. Observe that $f$ is an IAML of $C_n$ if and only if $f(v_k) = f^{**}(v_k)$, which holds if and only if $f(v_{k-1}) + f(v_k) + f(v_{(k+1) \bmod n}) = 0$.
Also for any $0 \leq k \leq n-1$, let $v_kv_{k+1}v_{[(k+2) \bmod n]}v_{[(k+3) \bmod n]}$, is a path in $C_n$, we have $f(v_k) + f(v_{k+1}) + f(v_{(k+2) \bmod n}) = 0$ and $f(v_{k+1}) + f(v_{(k+2) \bmod n}) + f(v_{(k+3) \bmod n}) = 0$.
Thus $f(v_k) = f(v_{(k+3) \bmod n})$. $\square$

**Corollary 2.2.7.** *$C_n \in \Gamma_k(A)$ if and only if $O(k) = 3$, where $O(k)$ denotes the order of $k$ in $A$.*

*Proof.* Consider $C_n$ with $V(C_n) = \{v_1, v_2, \ldots, v_{n-1}, v_n = v_0\}$. Suppose $C_n \in \Gamma_k(A)$, that is there exists an IAML $f$ of $C_n$ with $f(v_i) = k$ for $i = 1, 2, 3, \ldots, n$. Then by Theorem 2.2.6, we have $3k = 0$ in $A$, which implies $O(k) = 3$. Conversely, suppose $O(k) = 3$. Then consider the vertex label $f(v_i) = k$ for $i = 1, 2, 3, \ldots, n$. Since $f(v_i) = k$ for all $i$ and $O(k) = 3$, we have, $f^*(v_iv_{i+1}) = 2k$ for all $i$, and which implies $f^{**}(v_i) = f^*(v_iv_{i+1}) + f^*(v_{i-1}v_i) = 4k = k = f(v_i)$, for all $i$. Thus $f$ is an IAML of $C_n$, that is $C_n \in \Gamma_k(A)$. Hence the proof. $\square$

**Corollary 2.2.8.** *$C_n$ has a non-constant IAML if and only if $n$ is a multiple of 3.*

*Proof.* Consider $C_n$ with vertex set $\{v_1, v_2, \ldots, v_{n-1}, v_n = v_0\}$. Suppose $n = 3k$, for some integer $k$. Let $a, b, c$ be any three elements with at least two of them are

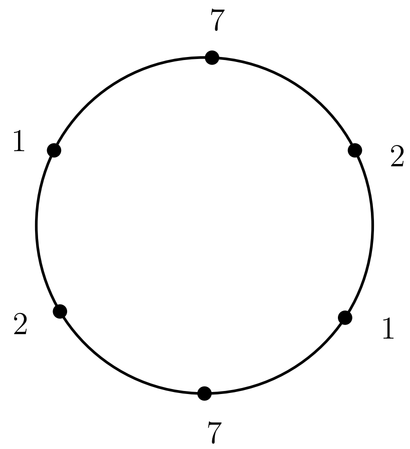


Figure 2.2: Cycle $C_6$

different in $A$, such that $a+b+c=0$, then define $f: V(C_n) \to A$ as follows.

$$f(v_i) = \begin{cases} a & \text{if} \quad i = 1, 4, 7, \ldots, 3k-2 \\ b & \text{if} \quad i = 2, 5, 8, \ldots, 3k-1 \\ c & \text{if} \quad i = 3, 6, 9, \ldots, 3k. \end{cases}$$

Then clearly $f$ is a non constant IAML of $C_n$.

Conversely, assume that $n$ is not a multiple of 3. Then either $n = 3k+1$ or $3k+2$ for some integer $k$. Let $f$ be an IAML of $C_n$ and $f(v_1) = w$.

**Case 1:** $n = 3k+1$. In this context, by the Theorem 2.2.6 we have:

$w = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3k+1}) = f(v_n) = f(v_3) = f(v_6) = f(v_9) = \cdots = f(v_{3k}) = f(v_{n-1}) = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3k-1}) = f(v_{n-2})$.

Thus $f(v_i) = w$, for $i = 1, 2, 3, \ldots, n$.

**Case 2:** $n = 3k+2$. In this context, by the Theorem 2.2.6 we have:

$w = f(v_1) = f(v_4) = f(v_7) = \cdots = f(v_{3k+1}) = f(v_{n-1}) = f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{3k-1}) = f(v_{3k+2}) = f(v_n) = f(v_0) = f(v_3) = f(v_6) = f(v_9) = \cdots = f(v_{3k}) = f(v_{n-2})$.

Thus in this case, also $f(v_i) = w$, for $i = 1, 2, 3, \ldots, n$.

Thus in either case, we have $f(v_i) = w$ for $i = 1, 2, 3, \ldots, n$. Thus if $n \not\equiv 0 \pmod 3$ then every IAML of $C_n$ is a constant IAML of $C_n$. $\square$

The Figure 2.2 represents an induced $\mathbb{Z}_{10}$-magic labeling of the cycle graph $C_6$.

**Theorem 2.2.9.** *The complete graph $K_n$ has an induced magic labeling $f$ if and only if $(n-3)f(v_1) = (n-3)f(v_2) = (n-3)f(v_3) = \cdots = (n-3)f(v_n) = -\left[f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n)\right]$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of $K_n$.*

*Proof.* For $1 \leq i, j \leq n$, we have $f(v_i) = f^{**}(v_i)$ holds if and only if $f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{i-1}) + (n-2)f(v_i) + f(v_{i+1}) + \cdots + f(v_n) = 0$, similarly the condition $f(v_j) = f^{**}(v_j)$ is equivalent to the condition $f(v_1)+f(v_2)+f(v_3)+\cdots+f(v_{j-1})+(n-2)f(v_j)+f(v_{j+1})+\cdots+f(v_n) = 0$. Thus we have $f$ is an IAML if and only if $(n-3)f(v_i) = (n-3)f(v_j) = -\left[f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n)\right]$, for $1 \leq i, j \leq n$. Hence the proof. $\square$

**Corollary 2.2.10.** *$K_n \in \Gamma_k(A)$ if and only if $O(k)$ divides $2n-3$, where $O(k)$ denotes the order of $k$ in $A$.*

*Proof.* Let $K_n$ be the complete graph with vertex set $\{v_1, v_2, v_3, \ldots, v_n\}$. We have $K_n \in \Gamma_k(A)$, means there exists an IAML $f$ with $f(v) = k$, for all $v \in V(K_n)$. Also by the Theorem 2.2.9, we have $f$ is an IAML of $K_n$ if and only if $(n-3)f(v) = -\left[f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n)\right]$, for all $v \in V(K_n)$.
Thus $K_n \in \Gamma_k(A)$ if and only if $(n-3)k = -nk$, that is if and only if $(2n-3)k = 0$, that is if and only if $O(k)$ divides $2n-3$ in $A$. This completes the proof. $\square$

**Theorem 2.2.11.** *$K_{m,n} \in \Gamma_k(A)$ if and only if $O(k)$ divides both $2m-1$ and $2n-1$, where $O(k)$ denotes the order of $k$ in $A$.*

*Proof.* Let $V(K_{m,n}) = \{v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\}$ with each $(v_iu_j) \in E(K_{m,n})$, for $1 \leq i \leq m,\ 1 \leq j \leq n$. Suppose $K_{m,n} \in \Gamma_k(A)$, then there exists an IAML say, $f$ with $f(v_iu_j) = k$, for $1 \leq i \leq m,\ 1 \leq j \leq n$. Now $f$ is an IAML of $K_{m,n}$ implies $k = f(v_1) = f^{**}(v_1) = 2nk$, since $f^*(v_1u_j) = 2k$ for $1 \leq j \leq n$, that is $(2n-1)k = 0$ in $A$, which implies $O(k)$ divides $2n-1$. Similarly by considering the equation $f(u_1) = f^{**}(u_1)$, we get $k = f(u_1) = f^{**}(u_1) = 2mk$, that is $(2m-1)k = 0$ in $A$, which implies $O(k)$ divides $2m-1$.
Conversely, suppose that $O(k)$ divides $2m-1$ and $O(k)$ divides $2n-1$. Consider the vertex label $f(v_i) = k = f(u_j)$, for $v_i, u_j \in V(K_{m,n}),\ 1 \leq i \leq m,\ 1 \leq j \leq n$. Then $f^*(v_i, u_j) = 2k$ for $1 \leq i \leq m,\ 1 \leq j \leq n$. There for $i = 1, 2, 3, \ldots, m$, $f^{**}(v_i) = \Sigma_{j=1}^{n} f^*(v_iu_j) = 2nk = k$, since $O(k)$ divides $2n-1$. Thus we have $f^{**}(v_i) = f(v_i) = k$ for $i = 1, 2, 3, \ldots, m$. In a similar way, we have $f^{**}(u_j) = f(u_j) = k$ for $j = 1, 2, 3, \ldots, n$. Hence we have $f = f^{**}$. Thus we get $K_{m,n} \in \Gamma_k(A)$. This concludes the proof. □

## 2.3 Induced $V_4$ Magic Labeling of Graphs

From this section onwards we consider the abelian group $V_4$ instead of an arbitrary abelian group $A$. By taking the Klein 4 group, $(V_4, +) = (\{0, a, b, c\}, +)$ instead of the abelian group $A$, in the definition of induced $A$ magic graphs, we can define the induced $V_4$ magic graph and induced $V_4$ magic labeling as follows:

Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : V(G) \to V_4$ be a vertex labeling and $f^* : E(G) \to V_4$ denote the induced edge labeling of $f$ defined by $f^*(uv) = f(u) + f(v)$ for all $uv \in E(G)$. Then $f^*$ again induces a vertex labeling $f^{**} : V(G) \to V_4$ defined by $f^{**}(u) = \Sigma f^*(uv)$, where the summation is taken over all the vertices $v$ which are adjacent to $u$. Then a graph $G$ is said to be an induced $V_4$-magic graph and denoted by IM$V_4$G

or simply IMG if there exists a non zero vertex labeling $f : V(G) \to V_4$ such that $f \equiv f^{**}$. The function $f$, so obtained is called an induced $V_4$-magic labeling of $G$ and denoted by IM$V_4$L or simply IML.
The "Induced degree sum theorem" corresponding to this context can be restated as follows.

**Theorem 2.3.1. Induced degree sum theorem**
*Let $f$ be any vertex labeling of a graph $G$ and $u$, be a vertex in $G$ with deg $(u) = m$. Then $f$ is an induced $V_4$ magic labeling if and only if $f(u) + \Sigma f(v) = 0$ or $\Sigma f(v) = 0$ according as deg $(u) = m$ is even or odd, where the summation is taken over all the vertices $v$ which are adjacent to $u$.*

In this case, the above equation corresponding to a vertex $u$ is called induced degree sum equation of the vertex $u$.

*Proof.* From Theorem 2.2.4, we have $f$ is an induced $V_4$magic labeling of $G$ if and only if $(m-1)f(u) + \Sigma f(v) = 0$, where $v$ is adjacent to $u$, then the result follows directly from the fact that $f(u) \in V_4$. □

**Theorem 2.3.2.** *For any graph $G$, $G \notin \Gamma_k(V_4)$.*

*Proof.* If possible, suppose $G \in \Gamma_k(V_4)$, that is there exists a non zero function $f$ such that $f : V(G) \to V_4$ with $f(v) = k$ for all $v \in V(G)$, for some $k \in V_4 \setminus \{0\}$. Then note that $f^*(e) = k + k = 0$, for all $e \in E(G)$. Thus $f^{**}(v) = 0$, for all $v \in V(G)$. That is $f \not\equiv f^{**}$, which is a contradiction. Hence our assumption is wrong. Therefore for any graph $G$, $G \notin \Gamma_k(V_4)$. □

## 2.4 Cycle Related Graphs

**Theorem 2.4.1.** *$C_n \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$.*

*Proof.* Suppose $n \equiv 0 \pmod 3$, define $f : V(C_n) \to V_4$ as follows.

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 1 \pmod 3 \\ b & \text{if} \quad i \equiv 2 \pmod 3 \\ c & \text{if} \quad i \equiv 0 \pmod 3). \end{cases}$$

Then we can prove that $f^{**} \equiv f$, that is $f$ is an IML of $C_n$. Proof of the converse part follows from Corollary 2.2.8 and Theorem 2.3.2. This completes the proof. □

**Corollary 2.4.2.** $C_n \in \Gamma_{a,0}(V_4)$ *if and only if* $n \equiv 0 \pmod 3$.

*Proof.* Suppose $n \equiv 0 \pmod 3$, define $f : V(C_n) \to V_4$ as follows:

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv \ 0, 1 \pmod 3 \\ 0 & \text{if} \quad i \equiv 2 \pmod 3). \end{cases}$$

Then we can prove that $f^{**} = f$, that is $f$ is an IML of $C_n$. Proof of the converse part follows from Corollary 2.2.8 and Theorem 2.3.2. This completes the proof. □

**Theorem 2.4.3.** *The wheel graph* $W_n \in \Gamma(V_4)$ *if and only if* $n$ *is even.*

*Proof.* Let $V(W_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $w$ is the central vertex. Suppose $n$ is even, then $n = 4k$ or $4k + 2$ for some positive integer $k$.

**Case 1:** $n = 4k$.

In this case, define $f : V(W_n) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = v_1, v_3, v_5, \ldots v_{4k-1} \\ b & \text{if} \quad v = v_2, v_4, v_6, \ldots v_{4k}. \end{cases}$$

**Case 2:** $n = 4k + 2$.

In this case, define $f : V(W_n) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = v_i. \end{cases}$$

then, in both cases we can verify that $f$ is an induced $V_4$ magic labeling of $W_n$. Conversely, suppose $n$ is an odd number, if $f$ is an induced $V_4$ magic labeling of $W_n$. Then by the induced degree sum equation of vertices in $W_n$, $f$ must satisfy the following system of equations.

$$\begin{aligned} f(v_2) + f(v_n) + f(w) &= 0 \\ f(v_1) + f(v_3) + f(w) &= 0 \\ f(v_2) + f(v_4) + f(w) &= 0 \\ f(v_3) + f(v_5) + f(w) &= 0 \\ &\vdots \\ f(v_1) + f(v_{n-1}) + f(w) &= 0 \\ f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_n) &= 0. \end{aligned}$$

From the system of equations we have, $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$, thus from the last equation we have $nf(v_i) = 0$, for $i = 1, 2, 3, \ldots, n$. Since $n$ is odd this happens only when $f(v_i) = 0$ for $i = 1, 2, 3, \ldots, n$. Using this in the first equation of the above system of equations we have $f(w) = 0$ also. Thus in this case, $f \equiv 0$. Thus $f$ is not an induced $V_4$ magic labeling of $W_n$. Hence the proof. □

**Corollary 2.4.4.** *$W_n \in \Gamma_{a,0}(V_4)$ if and only if $n$ is even.*

*Proof.* Let $V(W_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $w$ is the central vertex. Suppose $n$ is even number. Define $f : V(W_n) \to V_4$ as the following way.

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = v_i. \end{cases}$$

Then we can verify that $f$ is an induced $V_4$ magic labeling of $W_n$. The converse part follows from the Theorem 2.4.3. □

**Theorem 2.4.5.** *The helm graph $H_n \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Let $V(H_n) = \{w, v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_n\}$, where $w$ be the central vertex and $w_i$ be the pendant vertex adjacent to $v_i$, for $i = 1, 2, 3, \ldots, n$. Suppose $n$ is odd, then define $f : V(H_n) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_1, v_2, v_3, \ldots v_n \\ a & \text{if} \quad v = w, w_1, w_2, w_3, \ldots, w_n. \end{cases}$$

Then clearly $f$ is an induced $V_4$ magic labeling of $H_n$.
Conversely, suppose $n$ is an even number. If $f$ is an induced $V_4$ magic labeling of $H_n$ then by the induced degree sum equation of vertices in $H_n$, $f$ must satisfy the following system of equations.

$$\begin{aligned} f(v_i) &= 0 \text{ for } i = 1, 2, 3, \ldots, n \\ f(w) + f(w_i) &= 0 \text{ for } i = 1, 2, 3, \ldots, n \\ f(w) &= 0. \end{aligned}$$

Thus $f(v_i) = f(w_i) = f(w) = 0$, that is $f \equiv 0$. Hence $f$ is not an induced $V_4$ magic labeling. □

From the proof of Theorem 2.4.5, we have the following corollary.

**Corollary 2.4.6.** *The helm graph $H_n \in \Gamma_{a,0}(V_4)$ if and only if $n$ is odd.*

**Theorem 2.4.7.** *The web graph $W(2, n) \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$.*

*Proof.* Let $\{w, u_i, v_i, w_i : i = 1, 2, 3, \ldots, n\}$ be the vertex set of $W(2, n)$, where $w$ be the central vertex, $u_1, u_2, u_3, \ldots, u_n$ are the vertices of inner cycle, $v_1, v_2, v_3, \ldots,$ $v_n$ are the vertices of outer cycle and $w_1, w_2, w_3, \ldots, w_n$ are the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$ of $W(2, n)$.
Suppose $n \equiv 0 \pmod 3$, then define $f : V(W(2, n)) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w, v_1, v_2, v_3, \ldots, v_n \\ a & \text{if} \quad v = u_i, w_i, \text{ for } i \equiv 1 \pmod 3 \\ b & \text{if} \quad v = u_i, w_i, \text{ for } i \equiv 2 \pmod 3 \\ c & \text{if} \quad v = u_i, w_i, \text{ for } i \equiv 0 \pmod 3. \end{cases}$$

Then clearly $f$ is an induced $V_4$ magic labeling of $W(2, n)$. Conversely, suppose that $n \not\equiv 0 \pmod 3$, then $n = 3k + 1$ or $3k + 2$ for some positive integer $k$. If possible suppose $f$ is an induced $V_4$ magic labeling of $W(2, n)$ then by the induced degree sum equation of vertices of $W(2, n)$, $f$ must satisfy the following system of equations.

$$\begin{aligned} f(v_i) &= 0 \text{ for } i = 1, 2, 3, \ldots, n \\ f(u_n) + f(u_1) + f(u_2) + f(w) &= 0 \\ f(u_1) + f(u_2) + f(u_3) + f(w) &= 0 \\ f(u_2) + f(u_3) + f(u_4) + f(w) &= 0 \\ &\vdots \\ f(u_1) + f(u_{n-1}) + f(u_n) + f(w) &= 0 \\ f(u_i) + f(w_i) &= 0 \text{ for } i = 1, 2, 3, \ldots, n \\ (n-1)f(w) + \sum_{i=1}^{n} f(u_i) &= 0. \end{aligned}$$

Since $n = 3k+1$ or $3k+2$, from the above system of equations we have, $f(v_i) = 0$, $f(u_i) = f(w_i)$, $f(u_1) = f(u_2) = f(u_3) = \cdots = f(u_n)$ and $(n-1)f(w) + nf(u_i) =$

0 for $i = 1, 2, 3, \ldots, n$.

**Case 1:** $n = 3k + 1$.

**Subcase 1:** $k$ is even.

Note that $k$ is even implies $n = 3k + 1$ is odd.

Therefore the equation $(n-1)f(w) + nf(u_i) = 0$ for $i = 1, 2, 3, \ldots, n$ reduces to $f(u_i) = 0$ for $i = 1, 2, 3, \ldots, n$. Hence in this case, $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

**Subcase 2:** k is odd.

Note that $k$ is odd implies $n = 3k + 1$ is even. Thus the equation $(n-1)f(w) + nf(u_i) = 0$ for $i = 1, 2, 3, \ldots, n$ reduces to $f(w) = 0$. Thus from the system of equations we have, $f(u_i) = 0$. Hence in this case also, $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

**Case 2:** $n = 3k + 2$.

**Subcase 1:** $k$ is even.

Note that $k$ is even implies $n = 3k + 2$ is even. Thus the equation $(n-1)f(w) + nf(u_i) = 0$ for $i = 1, 2, 3, \ldots, n$ reduces to $f(w) = 0$. Thus from the system of equations we have, $f(u_i) = 0$. Hence in this case, $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

**Subcase 2:** $k$ is odd.

Note that $k$ is odd implies $n = 3k + 2$ is odd.

Therefore the equation $(n-1)f(w) + nf(u_i) = 0$ for $i = 1, 2, 3, \ldots, n$ reduces to $f(u_i) = 0$ for $i = 1, 2, 3, \ldots, n$. Hence in this case also, $f(u_i) = f(v_i) = f(w_i) = f(w) = 0$.

Hence in both cases we have $f \equiv 0$, that is $f$ is not an induced $V_4$ magic labeling.

□

**Theorem 2.4.8.** *The closed helm* $CH_n \in \Gamma(V_4)$ *for* $n$ *is odd.*

*Proof.* Let $V(CH_n) = \{w, v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_n\}$, where $w$ be the central vertex and $w_i$ be the pendant vertex adjacent to $v_i$, for $i = 1, 2, 3, \ldots, n$ in the corresponding helm $H_n$. Suppose $n$ is odd, then define $f : V(CH_n) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_1, v_2, v_3, \ldots, v_n \\ a & \text{if} \quad v = w, w_1, w_2, w_3, \ldots, w_n. \end{cases}$$

Then $f$ is an IML of $CH_n$. Hence the proof. □

**Corollary 2.4.9.** $CH_n \in \Gamma_{a,0}(V_4)$ *for* $n$ *is odd.*

*Proof.* Proof follows directly from the proof of Theorem 2.4.8. □

**Theorem 2.4.10.** *The flower graph* $Fl^n \in \Gamma(V_4)$ *for all* $n$.

*Proof.* Let $V(Fl^n) = \{w, u_i, v_i : \ i = 1, 2, 3, \ldots, n\}$, where $w$ is the central vertex, $u_1, u_2, u_3, \ldots, u_n$ are the vertices of corresponding cycle and $v_1, v_2, v_3, \ldots, v_n$ are the vertices adjacent to the central vertex $w$.

**Case 1:** n is odd.

In this case, define $f : V(Fl^n) \to V_4$ as

$$f(v) = \begin{cases} a & \text{if} \quad v = w \\ b & \text{if} \quad v = u_i \text{ for } i = 1, 2, 3, \ldots, n \\ c & \text{if} \quad v = v_i \text{ for } i = 1, 2, 3, \ldots, n. \end{cases}$$

**Case 2:** n is even.

In this case, define $f : V(Fl^n) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = u_i, v_i \text{ for } i \text{ is odd} \\ b & \text{if} \quad v = u_i, v_i \text{ for } i \text{ is even.} \end{cases}$$

Thus from both cases, we can verify that $f$ is an induced $V_4$ magic labeling of $Fl^n$. This completes the proof. $\square$

**Corollary 2.4.11.** *$Fl^n \in \Gamma_{a,0}(V_4)$ for all $n$.*

*Proof.* Let $V(Fl^n) = \{w, u_i, v_i : i = 1, 2, 3, \ldots, n\}$, where $w$ be the central vertex, $u_1, u_2, u_3, \ldots, u_n$ are the vertices of corresponding cycle and $v_1, v_2, v_3, \ldots, v_n$ are the vertices adjacent to the central vertex $w$. Define $f : V(Fl^n) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = u_i, v_i. \end{cases}$$

Then one can easily verify that $f$ is an IML of $Fl^n$. Hence the proof. $\square$

**Theorem 2.4.12.** *The gear graph $G_n \in \Gamma(V_4)$ if and only if $n$ is even.*

*Proof.* Let $V(G_n) = \{w, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$, where $w$ is the central vertex $u_1, u_2, u_3, \ldots, u_n$ are the vertices of the corresponding wheel graph $W_n$ and $v_1, v_2, v_3, \ldots, v_n$ are the remaining vertices with $u_i v_i, v_i u_{i+1} \in E(G_n)$, where $i + 1$ is taken modulo $n$.

Suppose $n$ is even, then define $f : V(G_n) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = u_i \text{ for } i \text{ is odd} \\ b & \text{if} \quad v = u_i \text{ for } i \text{ is even} \\ c & \text{if} \quad v = v_i. \end{cases}$$

Then we can easily verify that $f$ is an induced $V_4$ magic labeling of $G_n$.
Conversely, suppose that $n$ is an odd number. Then by the induced degree sum equation of vertices in $G_n$ we have: if $f$ is an induced $V_4$ magic labeling of $G_n$,

then $f$ satisfies the following system of equations.

$$\begin{aligned}
f(v_1)+f(v_2)+f(w) &= 0\\
f(v_2)+f(v_3)+f(w) &= 0\\
&\vdots\\
f(v_{n-1})+f(v_n)+f(w) &= 0\\
f(v_n)+f(v_1)+f(w) &= 0\\
f(u_1)+f(u_2)+f(u_3)+\cdots+f(u_n) &= 0\\
f(v_1)+f(u_1)+f(u_2) &= 0\\
f(v_2)+f(u_2)+f(u_3) &= 0\\
f(v_3)+f(u_3)+f(u_4) &= 0\\
&\vdots\\
f(v_n)+f(u_n)+f(u_1) &= 0.
\end{aligned}$$

Since $n$ is odd, the equations corresponding to the vertices $u_1, u_2, u_3, \ldots, u_n$ (that is the first $n$ equations) imply that $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$. Substituting these in the above system of equations, we get $f(w) = 0$. Now using the fact that $n$ is odd and $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$, the last $n$ equations in the above system of equations imply that $f(u_1) = f(u_2) = f(u_3) = \cdots = f(u_n)$. Substituting these in the equation $f(u_1) + f(u_2) + f(u_3) + \cdots + f(u_n) = 0$, we get $f(u_i) = 0$, for $i = 1, 2, 3, \ldots, n$ which implies $f(v_i) = 0$, for $i = 1, 2, 3, \ldots, n$. Hence $f \equiv 0$, that is $f$ is not an induced $V_4$ magic labeling of $G_n$. □

The Figure 2.3 represents a gear graph $G_4$ with an induced $V_4$-magic labeling.

**Theorem 2.4.13.** *The fan graph $F_n \in \Gamma(V_4)$ for $n$ is even.*

*Proof.* Suppose $n$ is an even number. We have $F_n = P_n + K_1$. Let $V(F_n) =$

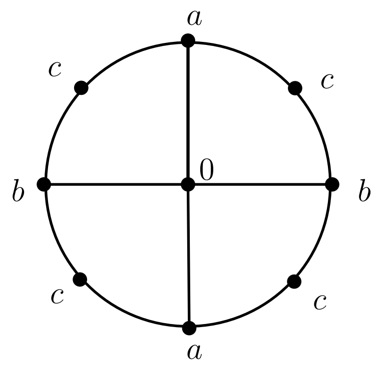


Figure 2.3: Gear graph $G_4$.

$\{w, v_1, v_2, v_3, \ldots, v_n\}$, where $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to $P_n$ and $w$ be the vertex corresponding to $K_1$. Then define $f : V(F_n) \to V_4$ as follows:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w \\ a & \text{if} \quad v = v_1, v_2, v_3, \ldots, v_n. \end{cases}$$

Then we can easily verify that $f$ is an induced $V_4$ magic labeling of $F_n$. Hence the proof follows. $\square$

**Theorem 2.4.14.** *The flag graph $Fl_n \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$.*

*Proof.* Let $V(Fl_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $v_i$ for $i = 1, 2, 3, \ldots, n$ is the vertex of corresponding cycle graph $C_n$ and $w$ is the root vertex adjacent to the vertex $v_1$.

Suppose $n \equiv 0 \pmod 3$, then define $f : V(Fl_n) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, i \equiv 1 \pmod 3 \\ a & \text{if} \quad v = v_i, i \equiv 0, 2 \pmod 3 \\ 0 & \text{if} \quad v = w. \end{cases}$$

Then we can easily verify that $f$ is an induced $V_4$ magic labeling of $Fl_n$.

Conversely, suppose that $n \not\equiv 0 \pmod 3$. If possible, suppose $f$ is an induced $V_4$ magic labeling of $Fl_n$. Then by the induced degree sum equation of vertices $w$

and $v_i$ in $Fl_n$, $f$ must satisfy the following system of equations.

$$
\begin{aligned}
f(v_1) &= 0 \\
f(v_2)+f(v_n)+f(w) &= 0 \\
f(v_2)+f(v_3) &= 0 \\
f(v_2)+f(v_3)+f(v_4) &= 0 \\
f(v_3)+f(v_4)+f(v_5) &= 0 \\
&\vdots \\
f(v_{n-2})+f(v_{n-1})+f(v_n) &= 0 \\
f(v_{n-1})+f(v_n) &= 0.
\end{aligned}
$$

Note that $n \not\equiv 0 \pmod 3$ implies that $n = 3k+1$ or $n = 3k+2$ for some integer $k$, also from the above system of equations we have $f(v_1) = f(v_{n-2}) = 0$ and $f(v_2) = f(v_3)$. Using these facts, we can prove that $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n) = 0$ and $f(w) = 0$. Thus $f$ is not an induced $V_4$ magic labeling.
Hence the proof follows. □

**Theorem 2.4.15.** *The sunflower graph $SF_n \in \Gamma(V_4)$, for $n$ is even.*

*Proof.* Suppose the given sunflower graph is obtained by taking a wheel graph with the central vertex $v_0$, the $n-$cycle $v_1, v_2, v_3, \ldots, v_n$ and additional vertices $w_1, w_2, w_3, \ldots, w_n$, where $w_i$ is joined by edges to the vertices $v_i$ and $v_{i+1}$, where $i+1$ is taken modulo $n$.
Suppose $n$ is even, then define $f : V(SF_n) \to V_4$ as

$$
f(v) = \begin{cases} 0 & \text{if } v = v_0, w_i, \quad \text{for } i = 1, 2, 3, \ldots, n \\ a & \text{if } v = v_i, \quad\quad\;\; \text{for } i = 1, 2, 3, \ldots, n. \end{cases}
$$

Then we can easily prove that $f$ is an induced $V_4$ magic labeling of $SF_n$. □

From the proof of Theorem 2.4.15, we have the following corollary.

**Corollary 2.4.16.** *If $n$ is even, then the sunflower graph $SF_n \in \Gamma_{a,0}(V_4)$.*

**Theorem 2.4.17.** *The jelly fish $J(m,n) \in \Gamma(V_4)$ for all $m$ and $n$.*

*Proof.* Consider the jelly fish graph with $V(J(m,n)) = \{v_k : k = 1,2,3,4\} \cup \{u_i : i = 1,2,3,\ldots,m\} \cup \{w_j : j = 1,2,3,\ldots,n\}$, where $v_k'$s are the vertices of corresponding $C_4$ and $u_i, w_j$ are the vertices of corresponding $K_{1,m}$ and $K_{1,n}$ respectively. Now consider the following cases:

**Case 1 :** both $m$ and $n$ are odd.

Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad u_1, w_1, v = v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 2,3,4,\ldots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 2,3,4,\ldots,n. \end{cases}$$

**Case 2 :** $m$ and $n$ are even.

Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 1,2,3,\ldots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 1,2,3,\ldots,n. \end{cases}$$

**Case 3 :** $m$ odd and $n$ even.

Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u_1, v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 2,3,4,\ldots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 1,2,3,\ldots,n. \end{cases}$$

**Case 4 :** $m$ even and $n$ odd.

Define $f : V(J(m,n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w_1, v_k, \text{ for } k = 1,2,3,4 \\ a & \text{if} \quad v = u_i, \text{ for } i = 1,2,3,\ldots,m \\ a & \text{if} \quad v = w_j, \text{ for } j = 2,3,4,\ldots,n. \end{cases}$$

In all the above cases, we can prove that $f$ is an induced magic labeling of $J(m,n)$. Hence the proof. □

**Corollary 2.4.18.** *The jelly fish $J(m,n) \in \Gamma_{a,0}(V_4)$ for all $m$ and $n$.*

**Theorem 2.4.19.** *The sun graph $Sun_n \notin \Gamma(V_4)$ for any $n$.*

*Proof.* Consider a sun graph $Sun_n$ with $\{v_1, v_2, v_3, \ldots, v_n\}$ as vertex set of the corresponding $C_n$ and $w_i,\ 1 \leq i \leq n$, be the pendant vertices attached to each $v_i, 1 \leq i \leq n$. If possible, suppose $f : V(Sun_n) \to V_4$ is an IML of $Sun_n$. Then the induced degree sum equation of $w_i$, we have $f(v_i) = 0$. Using this in the induced degree sum equation of $v_i$, we get $f(w_i) = 0$. Thus $f \equiv 0$, which is a contradiction. Hence the proof. □

**Theorem 2.4.20.** *The $CBSun_{p,q} \in \Gamma(V_4)$ if and only if $q + 3k = p - 2$ for some integer $k$.*

*Proof.* Consider a $CBSun_{p,q}$ with vertex set $\{v_i : i = 1,2,3,\ldots p\} \cup \{u_j : j = 1,2,3,\ldots,q\}$, where $u_j$ is the pendant vertices adjacent to vertices $v_j$. Suppose $q + 3k = p - 2$ for some integer $k$. Then define $f : V(CBSun_{p,q}) \to V_4$ as

$$f(v) = \begin{cases} a & \text{if} \quad v = u_1, u_q \\ 0 & \text{if} \quad v = u_2, u_3, u_4, \ldots, u_{q-1} \\ 0 & \text{if} \quad v = v_1, v_2, v_3, \ldots v_q, v_{q+3}, v_{q+6}, v_{q+9}, \ldots, v_{p-2} \\ a & \text{if} \quad v = v_{q+1}, v_{q+2}, v_{q+4}, v_{q+5}, \ldots, v_{p-3}, v_{p-1}, v_p. \end{cases}$$

Then $f$ is an IML of $CBSun_{p,q}$. Hence, in this case $CBSun_{p,q} \in \Gamma(V_4)$.

Conversely, suppose $q + 3k \neq p - 2$, that is $q + 3k = p - 1$ or $q + 3k = p$, for some integer $k$.

If possible, suppose $g : V(CBSun_{p,q}) \to V_4$ be an IML of $CBSun_{p,q}$. Then by the induced degree sum equation of the vertices $u_i$, we have $g(v_i) = 0$, for $i = 1, 2, 3, \ldots, q$. Similarly from the induced degree sum equation of the vertices $v_i$, we have the following system of equations.

$$
\begin{aligned}
g(v_p) + g(u_1) &= 0 \\
g(u_j) &= 0 \quad \text{for} \quad j = 2, 3, 4, \ldots, q-1 \\
g(v_{q+1}) + g(u_q) &= 0 \\
g(v_{q+1}) + g(v_{q+2}) &= 0 \\
g(v_{q+1}) + g(v_{q+2}) + g(v_{q+3}) &= 0 \\
g(v_{q+2}) + g(v_{q+3}) + g(v_{q+4}) &= 0 \\
&\vdots \\
g(v_{p-2}) + g(v_{p-1}) + g(v_p) &= 0 \\
g(v_{p-1}) + g(v_p) &= 0.
\end{aligned}
$$

Note that from the above system of equations, we have $g(u_q) = g(v_{q+1}) = g(v_{q+2})$. Therefore $g(v_{q+3}) = 0$ and $g(v_{p-1}) = g(v_p)$. Thus $g(v_{p-2}) = 0$.

**Case 1:** $q + 3k = p - 1$.

In this case, using above system we have: $g(v_{q+3}) = g(v_{q+6}) = g(v_{q+9}) = \cdots = g(v_{p-4}) = g(v_{p-1}) = 0$ and $g(v_{q+1}) = g(v_{q+2}) = g(v_{q+4}) = g(v_{q+5}) = \cdots = g(v_{p-5}) = g(v_{p-3}) = g(v_{p-2})$.

But $g(v_{p-2}) = 0$ implies $g(v_{q+1}) = g(v_{q+2}) = g(v_{q+4}) = g(v_{q+5}) = \cdots = g(v_{p-5}) = g(v_{p-3}) = g(v_{p-2}) = 0$.

Thus in this case, $g \equiv 0$, which is a contradiction. Hence $CBSun_{p,q} \notin \Gamma(V_4)$.

**Case 2:** $q + 3k = p$.

In this case, using above system we have: $g(v_{q+3}) = g(v_{q+6}) = g(v_{q+9}) = \cdots = g(v_{p-3}) = g(v_p) = 0$ and $g(v_{q+1}) = g(v_{q+2}) = g(v_{q+4}) = g(v_{q+5}) = \cdots = g(v_{p-4}) = g(v_{p-2}) = g(v_{p-1})$.

But $g(v_{p-2}) = 0$ implies $g(v_{q+1}) = g(v_{q+2}) = g(v_{q+4}) = g(v_{q+5}) = \cdots = g(v_{p-4}) = g(v_{p-2}) = g(v_{p-1}) = 0$.

Thus in this case, $g \equiv 0$, which is a contradiction. Hence $CBSun_{p,q} \notin \Gamma(V_4)$.

Hence the proof. □

**Theorem 2.4.21.** *The $C(n, k_1, k_2, k_3, \ldots, k_t) \in \Gamma(V_4)$ for all $n$ and $k_i$ where $i = 1, 2, 3, \ldots, t$.*

*Proof.* Let $\{v_1, v_2, v_3, \ldots, v_n\}$ are the corresponding vertices of $C_n$ and $\{v_{ij} : i = 1, 2, 3, \ldots, t, \quad j = 1, 2, 3, \ldots, k_i\}$ be the corresponding vertices of $K_{1,k_i}$. Let $f : V(C(n, k_1, k_2, k_3, \ldots, k_t)) \to V_4$. Define $f$ as in the following way.
Define $f(v_k) = 0$ for $i = 1, 2, 3, \ldots, n$. If $k_i$ is odd for some $i$, for those $i$, define $f(v_{i1}) = 0$ and $f(v_{ij}) = a$, for $j = 2, 3, 4, \ldots, k_i$ and if $k_i$ is even for some $i$, for those $i$, define $f(v_{ij}) = a$, for $j = 1, 2, 3, \ldots, k_i$.
Then we can easily prove that $f$ is an IML of $C(n, k_1, k_2, k_3, \ldots, k_t)$.
Hence the proof. □

**Theorem 2.4.22.** *$C_n \odot K_2 \in \Gamma(V_4)$ for all $n$.*

*Proof.* Let $\{v_1, v_2, v_3, \ldots, v_n\}$ are the corresponding vertices of $C_n$ and $\{u_i, w_i\}$ be the corresponding vertices of the $i^{th}$ copy of $K_2$, for $i = 1, 2, 3, \ldots, n$. Let

$f : V(C_n \odot K_2) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \qquad \text{for} \;\; i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad v = u_i, w_i, \;\; \text{for} \quad i = 1, 2, 3, \ldots, n. \end{cases}$$

Then we can easily prove that $f$ is an IML of $C_n \odot K_2$.

Hence the proof. □

**Theorem 2.4.23.** *$C_n \odot \overline{K}_m \in \Gamma(V_4)$ for all $m$ and $n$.*

*Proof.* Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{im}\}$ be the corresponding vertices of the $i^{th}$ copy of $\overline{K}_m$ for $i = 1, 2, 3, \ldots, n$. Let $f : V(C_n \odot \overline{K}_m) \to V_4$.

**Case 1:** $m$ is an even integer.

In this case, define $f$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \quad \text{for} \;\; i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad v = u_{ij}, \;\; \text{for all} \;\; i, j. \end{cases}$$

**Case 2:** $m$ is an odd integer.

In this case, define $f$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \quad \text{for} \;\; i = 1, 2, 3, \ldots, n \\ 0 & \text{if} \quad v = u_{i1}, \;\; \text{for} \;\; i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad v = u_{ij}, \;\; \text{for} \;\; i = 1, 2, 3, \ldots, n, \quad j = 2, 3, 4, \ldots, m. \end{cases}$$

Then $f$ is an IML of $C_n \odot \overline{K}_m$.

Hence the proof. □

**Theorem 2.4.24.** *$C_n \odot K_m \in \Gamma(V_4)$ for $m$ even.*

*Proof.* Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{im}\}$ be the corresponding vertices of the $i^{th}$ copy of $K_m$ for $i = 1, 2, 3, \ldots, n$.

Suppose $m$ is an even integer. Let $f : V(C_n \odot K_m) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \quad \text{for} \quad i = 1, 2, 3, \dots, n \\ a & \text{if} \quad v = u_{ij}, \ \text{for all} \quad i, j. \end{cases}$$

Then we can prove that $f$ is an IML of $C_n \odot K_m$.

Hence the proof. □

**Theorem 2.4.25.** *The friendship graph or Dutch* 3*-windmill graph* $C_3^{(t)} \in \Gamma(V_4)$ *for all* $t$.

*Proof.* Let $\{u_i, v_i, w_i\}$ be the vertex set of $i^{th}$ copy $C_3$ for $i = 1, 2, 3, \dots, t$ and the vertices $u_1, u_2, u_3, \dots, u_t$ are identified with the vertex $u$.

Let $f : V(C_3^{(t)}) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u \\ a & \text{if} \quad v = v_i, w_i \ \text{ for } \ i = 1, 2, 3, \dots, t. \end{cases}$$

Then we can prove that $f$ is an IML of $C_3^{(t)}$.

Hence the proof. □

**Theorem 2.4.26.** *The graph* $C_n^{(t)} \in \Gamma(V_4)$ *for* $n \equiv 0 \pmod 3$.

*Proof.* Let $\{u_{i1}, u_{i2}, u_{i3}, \dots, u_{in}\}$ be the vertex set of $i^{th}$ copy of $C_n$, for $i = 1, 2, 3, \dots, t$ and the vertices $u_{11}, u_{21}, u_{31}, \dots, u_{t1}$ are identified with the vertex $u$. Suppose $n \equiv 0 \pmod 3$.

**Case 1:** $t$ is an odd integer.

In this case, define $f$ as:

$$f(v) = \begin{cases} a & \text{if} \quad v = u \\ 0 & \text{if} \quad v = u_{ij}, \ \text{where} \quad j \equiv 2 \pmod 3 \ \text{ and } \ i = 1, 2, 3, \dots, t \\ a & \text{if} \quad v = u_{ij}, \ \text{where} \quad j \equiv 0, 1 \pmod 3 \ \text{ and } \ i = 1, 2, 3, \dots, t. \end{cases}$$

**Case 2:** $t$ is an even integer.

In this case, define $f$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u \\ 0 & \text{if} \quad v = u_{ij}, \text{ where } \quad j \equiv 1 \pmod 3 \quad \text{and} \quad i = 1, 2, 3, \ldots, t \\ a & \text{if} \quad v = u_{ij}, \text{ where } \quad j \equiv 0, 2 \pmod 3 \quad \text{and} \quad i = 1, 2, 3, \ldots, t. \end{cases}$$

Then one can easily verify that $f$ is an IML of $C_n^{(t)}$.
Hence $C_n^{(t)} \in \Gamma(V_4)$ for $n \equiv 0 \pmod 3$. □

**Theorem 2.4.27.** *The $n-$gon book graph $B(n,k) \in \Gamma(V_4)$ for $k$ odd and $n \equiv 0$ $(mod\ 3)$.*

*Proof.* Let $\{u_1, u_{i2}, u_{i3}, u_{i4}, \ldots, u_{i(n-1)}, u_n\}$ be the vertex set of $i^{th}$ copy of $C_n$ in $K_m^{(n)}$, where $u_1$ and $u_n$ be the common vertices.

Suppose $k$ is an odd integer and $n \equiv 0 \pmod 3$.
Let $f : V(B(n,k)) \to V_4$ be defined by

$$f(v) = \begin{cases} a & \text{if} \quad v = u_1, u_n \\ 0 & \text{if} \quad v = u_{ij}, \text{ where } \quad j \equiv 2 \pmod 3 \\ a & \text{if} \quad v = u_{ij}, \text{ where } \quad j \equiv 0, 1 \pmod 3. \end{cases}$$

Then we can easily prove that $f$ is an IML of $B(n,k)$.
Hence $B(n,k) \in \Gamma(V_4)$ for $k$ odd and $n \equiv 0 \pmod 3$. □

**Theorem 2.4.28.** *The bi pyramid $BP(n) \in \Gamma(V_4)$ for $n \equiv 0 \pmod 3$.*

*Proof.* Let $V(N_2) = \{u, v\}$ and $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $f : V(BP(n)) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u, v \\ a & \text{if} \quad v = v_i, \text{ where} \quad i \equiv 1 \pmod 3 \\ b & \text{if} \quad v = v_i, \text{ where} \quad i \equiv 2 \pmod 3 \\ c & \text{if} \quad v = v_i, \text{ where} \quad i \equiv 0 \pmod 3. \end{cases}$$

Then $f$ is an IML of $BP(n)$.
Hence $BP(n) \in \Gamma(V_4)$ for $n \equiv 0 \pmod 3$. □

**Theorem 2.4.29.** *The bi pyramid based on $C_n$, $BP(n) \in \Gamma_{a,0}(V_4)$ for $n \equiv 0 \pmod 3$.*

*Proof.* Let $V(N_2) = \{u, v\}$ and $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$. Define $f : V(BP(n)) \to V_4$ as

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u, v \\ a & \text{if} \quad v = v_i, \text{ where} \quad i \equiv 1, 2 \pmod 3 \\ 0 & \text{if} \quad v = v_i, \text{ where} \quad i \equiv 0 \pmod 3. \end{cases}$$

Then $f$ is an IML of $BP(n)$.
Hence $BP(n) \in \Gamma_{a,0}(V_4)$ for $n \equiv 0 \pmod 3$. □

**Theorem 2.4.30.** *The graph $C_n \odot C_m \in \Gamma(V_4)$ for $m$ even.*

*Proof.* Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{im}\}$ be the vertex set of $i^{th}$ copy of $C_m$, for $i = 1, 2, 3, \ldots, n$. Suppose $m$ is an even integer. Define $f : V(C_n \odot C_m) \to V_4$ as,

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \quad \text{for all } i \\ a & \text{if} \quad v = u_{ij} \quad \text{for all } i, j. \end{cases}$$

Then we can easily prove that $f$ is an IML of $C_n \odot C_m$.
Hence the proof. □

From the proof of the above theorem we have the following corollary.

**Corollary 2.4.31.** *The graph $C_n \odot C_m \in \Gamma_{a,0}(V_4)$ for $m$ even.*

# Chapter 3

# Induced $V_4$-Magic Labeling of Path and Star Related Graphs

*The first section of this chapter defines some path and star related graphs. In the second section, we discuss induced $V_4$-magic labeling of path related graphs and in the last section we deal with induced $V_4$-magic labeling of some star related graphs.*

## 3.1 Introduction

**Definition 3.1.1.** *The corona $P_n \odot K_1$ is called the comb graph $CB_n$.*

**Definition 3.1.2.** [8] *A triangular snake graph $TS_n$ is obtained from a path $v_1, v_2, v_3, \ldots, v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, 3, \ldots, n-1$.*

**Definition 3.1.3.** [8] *A double triangular snake graph $DTS_n$ consists of two triangular snake graphs that have a common path. That is, a double triangular*

[1]This chapter has been published in *South East Asian Journal of Mathematics and Mathematical Sciences*, Volume 16, No. 2, 89-102, (2020).

*snake is obtained from a path $v_1, v_2, v_3, \ldots, v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, 3, \ldots, n-1$ and to a new vertex $u_i$ for $i = 1, 2, 3, \ldots, n-1$.*

**Definition 3.1.4.** [14] *An open ladder graph $O(L_n)$, $n \geq 2$ is obtained from two paths of length $n-1$ with $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 2 \leq i \leq n-1\}$.*

**Definition 3.1.5.** [19] *Given $k$ natural numbers $a_1, a_2, a_3, \ldots, a_k$. If we connect the two vertices of $N_2 = \{u, v\}$ by $k$ parallel paths of length $a_1, a_2, a_3, \ldots, a_k$ then the resulting graph is called the generalized theta graph and is denoted by $\Theta(a_1, a_2, a_3, \ldots, a_k)$.*

**Definition 3.1.6.** [20] *The book graph $B_n$ is the graph $K_{1,n} \Box P_2$, where $K_{1,n}$ is the star with $n$ edges.*

**Definition 3.1.7.** *The bistar $B_{m,n}$ is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.*

**Definition 3.1.8.** [8] *Let $< K_{1,n} : m >$ denote the graph obtained by taking $m$ disjoint copies of $K_{1,n}$ and joining a new vertex to the centers of the $m$ copies of $K_{1,n}$.*

**Definition 3.1.9.** [8] *The $(n, k)$-banana tree $Bt(n, k)$ is the graph obtained by starting with $n$ number of $k-$stars and connecting one end vertex from each to a new vertex.*

**Definition 3.1.10.** *The graph obtained by attaching central vertices (or apex) of $n-$copies of $K_{1,n}$ by a unique vertex $u$ by $n$ distinct edges is denoted by $K^*_{1,n}$.*

**Definition 3.1.11.** [25] *The windmill graph $K_m^{(n)}$ is the graph consisting of $n$ copies of the complete graph $K_m$ with a vertex in common.*

## 3.2 Path Related Graphs

**Theorem 3.2.1.** $P_n \in \Gamma(V_4)$ *if and only if* $n \equiv 0 \pmod 3$.

*Proof.* Consider the path $P_n$ with vertex set $V = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$, where $n \equiv 0 \pmod 3$. Define $f : V(P_n) \to V_4$ as :

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1 \pmod 3 \\ 0 & \text{if} \quad i \equiv 2 \pmod 3. \end{cases}$$

Then, $f$ is an induced magic labeling of $P_n$. Hence $P_n \in \Gamma(V_4)$. Conversely, if $n \not\equiv 0 \pmod 3$ then by the Theorem 2.2.5 $P_n \notin \Gamma(V_4)$. □

From the proof of the above theorem, we have the following corollary.

**Corollary 3.2.2.** $P_n \in \Gamma_{a,0}(V_4)$ *if and only if* $n \equiv 0 \pmod 3$.

**Theorem 3.2.3.** *The comb graph* $CB_n \notin \Gamma(V_4)$ *for any* $n$.

*Proof.* Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of $CB_n$, where $v_i (1 \leq i \leq n)$ are the pendant vertices adjacent to $u_i (1 \leq i \leq n)$. If possible, suppose $f$ is an IML of the graph $CB_n$. Then from the induced degree sum equation of the vertices $v_i$ and $u_i$, we have $f(u_i) = f(v_i) = 0$. That is $f \equiv 0$, which is a contradiction. □

**Theorem 3.2.4.** *The triangular snake graph* $TS_n \in \Gamma(V_4)$ *for all* $n$.

*Proof.* Let $V(TS_n) = \{v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_{n-1}\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding path $P_n$ and $f$ be an IML of $TS_n$ with $f(v_i) = x_i$ and $f(w_j) = y_j$. Then the vertices $v_i$ and $w_j$ must satisfy the induced degree sum equation.

Note that the induced degree sum equation of $v_i$ gives the following system of equations.

$$\begin{aligned}
x_1 + x_2 + y_1 &= 0\\
x_1 + x_2 + x_3 + y_1 + y_2 &= 0\\
x_2 + x_3 + x_4 + y_2 + y_3 &= 0\\
&\vdots\\
x_{n-2} + x_{n-1} + x_n + y_{n-2} + y_{n-1} &= 0\\
x_{n-1} + x_n + y_{n-1} &= 0.
\end{aligned}$$

Similarly the induced degree sum equation of $w_j$ gives the following system of equations.

$$\begin{aligned}
x_1 + x_2 + y_1 &= 0\\
x_2 + x_3 + y_2 &= 0\\
x_3 + x_4 + y_3 &= 0\\
&\vdots\\
x_{n-2} + x_{n-1} + y_{n-2} &= 0\\
x_{n-1} + x_n + y_{n-1} &= 0.
\end{aligned}$$

By substituting the second system in the first system of equations, we get

$$\begin{aligned}
x_1 + x_2 + y_1 &= 0\\
x_1 + y_1 &= 0\\
x_2 + y_2 &= 0\\
&\vdots\\
x_{n-2} + y_{n-2} &= 0\\
x_{n-1} + x_n + y_{n-1} &= 0.
\end{aligned}$$

From the above two system of equations, one can easily conclude that

$$\begin{aligned} x_1 &= y_1 \\ x_2 = x_3 = x_4 = \cdots = x_{n-1} &= 0 \\ y_2 = y_3 = y_4 = \cdots = y_{n-2} &= 0 \\ x_n &= y_{n-1}. \end{aligned}$$

Thus to get an IML of $TS_n$, we need to define $f : V(TS_n) \to V_4$ as follows:

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, w_1 \\ 0 & \text{if} \quad v = v_i, \ \text{ for } i = 2, 3, 4, \ldots, n-1 \\ 0 & \text{if} \quad v = w_j, \ \text{for } j = 2, 3, 4, \ldots, n-2 \\ b & \text{if} \quad v = v_n, w_{n-1}. \end{cases}$$

Hence the proof. □

**Corollary 3.2.5.** *The triangular snake graph $TS_n \in \Gamma_{a,0}(V_4)$ for all $n$.*

*Proof.* Let $V(TS_n) = \{v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_{n-1}\}$, where $v_1, v_2, v_3, \ldots,$ $v_n$ are the vertices of corresponding path $P_n$. Define $f : V(TS_n) \to V_4$ as follows.

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, w_1 \\ 0 & \text{if} \quad v = v_i, \ \text{ for } i = 2, 3, 4, \ldots, n-1 \\ 0 & \text{if} \quad v = w_j, \ \text{for } j = 2, 3, 4, \ldots, n-2 \\ a & \text{if} \quad v = v_n, w_{n-1}. \end{cases}$$

Then from Theorem 3.2.4, $f$ is an IML of $TS_n$. Hence the corollary follows. □

**Theorem 3.2.6.** *The double triangular snake graph $DTS_n \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$.*

*Proof.* Consider a double triangular snake graph $DTS_n$ with vertex set $\{v_1, v_2, v_3,$

$\ldots, v_n, w_1, w_2, w_3, \ldots, w_{n-1}, u_1, u_2, u_3, \ldots, u_{n-1}\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding path $P_n$ and $w_i$, $u_i$ are the vertices attached to $v_i$ and $v_{i+1}$ for $i = 1, 2, ..., n-1$. Suppose $n \equiv 0 \pmod 3$ that is $n = 3k$, for some integer $k$. Then define $g : V(DTS_n) \to V_4$ as:

$$g(v) = \begin{cases} 0 & \text{if} \quad v = v_2, v_5, v_8, \ldots, v_{n-4}, v_{n-1} \\ a & \text{if} \quad v = v_1, v_3, v_4, v_6, v_7, \ldots, v_{n-3}, v_{n-2}, v_n \\ 0 & \text{if} \quad v = w_{3j}, \text{ for } j = 1, 2, 3, \ldots, k-1 \\ a & \text{if} \quad v = w_1, w_2, w_4, w_5, w_7, \ldots, w_{n-2}, w_{n-1} \\ 0 & \text{if} \quad v = u_{3j}, \text{ for } j = 1, 2, 3, \ldots, k-1 \\ a & \text{if} \quad v = u_1, u_2, u_4, u_5, u_7, \ldots, u_{n-2}, u_{n-1}. \end{cases}$$

Then we can easily prove that $g$ is an IML of $DTS_n$.
Conversely, suppose that $n = 3k+1$ or $n = 3k+2$ for some integer $k$. If possible, suppose $f$ is an IML of $DTS_n$. Then from the induced degree sum equation of the vertices $v_i$, we have the following system of equations.

$$\begin{aligned} f(v_2) + f(u_1) + f(w_1) &= 0 \\ f(v_1) + f(v_2) + f(v_3) + f(u_1) + f(u_2) + f(w_1) + f(w_2) &= 0 \\ f(v_2) + f(v_3) + f(v_4) + f(u_2) + f(u_3) + f(w_2) + f(w_3) &= 0 \\ f(v_3) + f(v_4) + f(v_5) + f(u_3) + f(u_4) + f(w_3) + f(w_4) &= 0 \\ &\vdots \\ f(v_{n-2}) + f(v_{n-1}) + f(v_n) + f(u_{n-2}) + f(u_{n-1}) + f(w_{n-2}) + f(w_{n-1}) &= 0 \\ f(v_{n-1}) + f(u_{n-1}) + f(w_{n-1}) &= 0. \end{aligned}$$

Similarly from the induced degree sum equation of $u_i$ and $w_i$, we get the following system of equations:

$$f(u_1) + f(v_1) + f(v_2) = 0$$

$$\begin{aligned}
f(u_2)+f(v_2)+f(v_3) &= 0\\
f(u_3)+f(v_3)+f(v_4) &= 0\\
&\vdots\\
f(u_{n-1})+f(v_{n-1})+f(v_n) &= 0.
\end{aligned}$$

and

$$\begin{aligned}
f(w_1)+f(v_1)+f(v_2) &= 0\\
f(w_2)+f(v_2)+f(v_3) &= 0\\
f(w_3)+f(v_3)+f(v_4) &= 0\\
&\vdots\\
f(w_{n-1})+f(v_{n-1})+f(v_n) &= 0.
\end{aligned}$$

By comparing the induced degree sum equations of $u_i$ and $w_i$, we get $f(u_i) = f(w_i)$, for $i = 1, 2, 3, \ldots, n-1$. Using this in the induced degree sum equations of $v_i$, we get

$$\begin{aligned}
f(v_2) &= 0\\
f(v_1)+f(v_2)+f(v_3) &= 0\\
f(v_2)+f(v_3)+f(v_4) &= 0\\
f(v_3)+f(v_4)+f(v_5) &= 0\\
&\vdots\\
f(v_{n-2})+f(v_{n-1})+f(v_n) &= 0\\
f(v_{n-1}) &= 0.
\end{aligned}$$

By solving this, we get $f(v_2) = f(v_5) = f(v_8) = \cdots = 0$, $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots$ and $f(v_n) =$

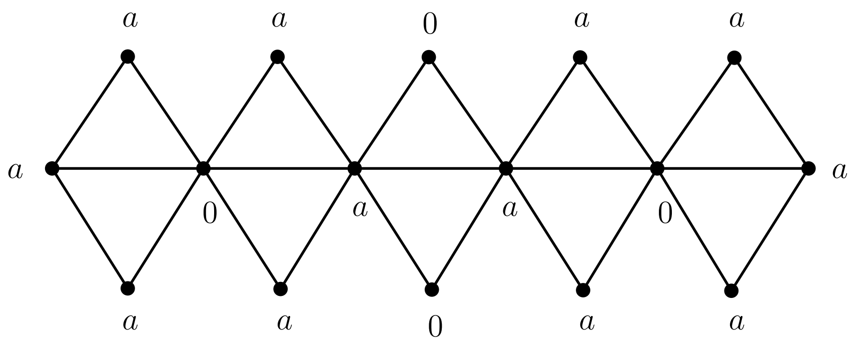


Figure 3.1: Double triangular snake graph $DTS_6$

$f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots$. By substituting this in the induced degree sum equation of $w_i$, we get $f(w_3) = f(w_6) = f(w_9) = \cdots = 0, f(w_{n-3}) = f(w_{n-6}) = f(w_{n-9}) = \cdots = 0$ and $f(w_1) = f(w_2) = f(w_4) = f(w_5) = \cdots, f(w_{n-1}) = f(w_{n-2}) = f(w_{n-4}) = f(w_{n-5}) = \cdots$.

**Case 1** : $n = 3k + 1$ for some integer $k$.

In this case, the above conclusion of the system of equations become $f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{n-2}) = 0$ and $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = f(v_3) = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots = f(v_{n-1}) = f(v_n)$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots = f(v_4) = f(v_2) = f(v_1)$. Thus $f \equiv 0$.

**Case 2** : $n = 3k + 2$ for some integer $k$.

In this case, the above conclusion of the system of equations become $f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{n-3}) = f(v_n) = 0$ and $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = f(v_4) = f(v_1) = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots = f(v_{n-2}) = f(v_{n-1})$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots = f(v_3) = f(v_2)$. Thus $f \equiv 0$.

Hence in both cases, we get $f \equiv 0$, which is a contradiction. Hence the proof. $\square$

The Figure 3.1 represents a double triangular snake graph $DTS_6$ with an induced $V_4$-magic labeling.

**Corollary 3.2.7.** *The double triangular snake graph $DTS_n \in \Gamma_{a,0}(V_4)$ if and only if $n \equiv 0 \pmod 3$.*

*Proof.* Proof follows from Theorem 3.2.6. □

**Theorem 3.2.8.** *For $n \geq 2$, the open ladder graph $O(L_n) \in \Gamma(V_4)$ for $n \equiv 0 \pmod 3$.*

*Proof.* Consider an open ladder graph $O(L_n), n \geq 2$, with the vertex set $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 2 \leq i \leq n-1\}$.
Then for $n \equiv 0 \pmod 3$, define $f : V(O(L_n)) \to V_4$ as follows.

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u_2, u_5, u_8, \ldots, u_{n-1} \\ 0 & \text{if} \quad v = v_2, v_5, v_8, \ldots, v_{n-1} \\ a & \quad\text{otherwise.} \end{cases}$$

Then $f$ is an IML of $O(L_n)$. Hence the proof follows. □

**Corollary 3.2.9.** *For $n \geq 2$, the open ladder graph $O(L_n) \in \Gamma_{a,0}(V_4)$ for $n \equiv 0 \pmod 3$.*

*Proof.* Proof follows from Theorem 3.2.8. □

**Theorem 3.2.10.** *The generalized theta graph $\Theta(a_1, a_2, a_3, \ldots, a_k) \in \Gamma(V_4)$ for $k$ is odd and $a_i \equiv 2 \pmod 3$, for $i = 1, 2, 3, \ldots, k$.*

*Proof.* Suppose $k$ is an odd integer and $a_i \equiv 2 \pmod 3$. Let $\{u, v_{i1}, v_{i2}, v_{i3}, \ldots, v_{ia_{i-1}}, v\}$ be the vertices of the $i^{th}$ path of length $a_i$.

Let $f : V(\Theta(a_1, a_2, a_3, \ldots, a_k)) \to V_4$ be defined by

$$f(v) = \begin{cases} a & \text{if} \quad v = u, v \\ a & \text{if} \quad v = v_{ij}, \text{ where } \quad j \equiv 0, 2 \pmod 3 \\ 0 & \text{if} \quad v = v_{ij}, \text{ where } \quad j \equiv 1 \pmod 3). \end{cases}$$

Then $f$ is an IML of $\Theta(a_1, a_2, a_3, \ldots, a_k)$.
Hence $\Theta(a_1, a_2, a_3, \ldots, a_k) \in \Gamma(V_4)$ for $a_i \equiv 2 \pmod 3$ and $k$ odd. □

**Theorem 3.2.11.** *The book $B_n \in \Gamma(V_4)$ for $n$ even.*

*Proof.* We have $B_n = K_{1,n} \Box P_2$. Let $V(K_{1,n}) = \{u, u_1, u_2, u_3, \ldots, u_n\}$, where $u$ be the apex vertex and $V(P_2) = \{v, w\}$. Then the vertex set of $K_{1,n} \Box P_2$ is given by $\{uv, u_1v, u_2v, u_3v, \ldots, u_nv, uw, u_1w, u_2w, u_3w, \ldots, u_nw\}$. Suppose $n$ is an even integer.
Let $f : V(B_n) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = uv, uw \\ a & \text{if} \quad v = u_iv, u_iw, \quad \text{for} \quad i = 1, 2, 3, \ldots, n. \end{cases}$$

Then we can easily prove that $f$ is an IML of $B_n$.
Hence $B_n \in \Gamma(V_4)$ for $n$ even. □

**Theorem 3.2.12.** *The graph $P_2 \Box C_n \in \Gamma(V_4)$ for $n \equiv 0 \pmod 3$.*

*Proof.* Let $V(P_2) = \{u, v\}$ and $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$. Then the $V(P_2 \Box C_n)$ is given by $\{uu_1, uu_2, uu_3, \ldots, uu_n, vu_1, vu_2, vu_3, \ldots, vu_n\}$. Suppose $n \equiv 0 \pmod 3$.
Let $f : V(P_2 \Box C_n) \to V_4$ be defined by

$$f(v) = \begin{cases} a & \text{if} \quad v = uu_i, vu_i, \quad \text{where } i \equiv 0 \pmod 3 \\ b & \text{if} \quad v = uu_i, vu_i, \quad \text{where } i \equiv 1 \pmod 3 \\ c & \text{if} \quad v = uu_i, vu_i, \quad \text{where } i \equiv 2 \pmod 3). \end{cases}$$

Then $f$ is an IML of $P_2\square C_n$.

Hence $P_2\square C_n \in \Gamma(V_4)$ for $n \equiv 0 \pmod 3$. $\square$

**Corollary 3.2.13.** *The graph $P_2\square C_n \in \Gamma_{a,0}(V_4)$ for $n \equiv 0 \pmod 3$.*

*Proof.* Let $V(P_2) = \{u, v\}$ and $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$. Then the $V(P_2\square C_n)$ is given by $\{uu_1, uu_2, uu_3, \ldots, uu_n, vu_1, vu_2, vu_3, \ldots, vu_n\}$. Suppose $n \equiv 0 \pmod 3$. Let $f : V(P_2\square C_n) \to V_4$ be defined by

$$f(v) = \begin{cases} a & \text{if} \quad v = uu_i, vu_i, \quad \text{where } i \equiv 0 \pmod 3 \\ 0 & \text{if} \quad v = uu_i, vu_i, \quad \text{where } i \equiv 1 \pmod 3 \\ a & \text{if} \quad v = uu_i, vu_i, \quad \text{where } i \equiv 2 \pmod 3). \end{cases}$$

Then one can easily verify that $f$ is an IML of $P_2\square C_n$.

Hence $P_2\square C_n \in \Gamma_{a,0}(V_4)$ for $n$ even. $\square$

**Theorem 3.2.14.** *The graph $P_n \odot K_2 \in \Gamma(V_4)$ for all $n$.*

*Proof.* Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_i, w_i\}$ be the vertex set of $i^{th}$ copy of $K_2$, for $i = 1, 2, 3, \ldots, n$.

Define $f : V(P_n \odot K_2) \to V_4$ by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \qquad i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad v = u_i, w_i, \; i = 1, 2, 3, \ldots, n. \end{cases}$$

Then we can prove that $f$ is an IML of $P_n \odot K_2$.

Hence $P_n \odot K_2 \in \Gamma(V_4)$ for all $n$. $\square$

From the proof of above theorem we have the following corollary.

**Corollary 3.2.15.** *The graph $P_n \odot K_2 \in \Gamma_{a,0}(V_4)$ for all $n$.*

**Theorem 3.2.16.** *The graph $P_n \odot \overline{K}_2 \in \Gamma(V_4)$ for all $n$.*

*Proof.* Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_i, w_i\}$ be the vertex set of $i^{th}$ copy of $\overline{K}_2$, for $i = 1, 2, 3, \ldots, n$.

Define $f : V(P_n \odot \overline{K}_2) \to V_4$ by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \quad\ \ i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad v = u_i, w_i, \ i = 1, 2, 3, \ldots, n. \end{cases}$$

Then $f$ is an IML of $P_n \odot \overline{K}_2$.
Hence $P_n \odot \overline{K}_2 \in \Gamma(V_4)$ for all $n$. $\square$

From the proof of above theorem we have the following corollary.

**Corollary 3.2.17.** *The graph $P_n \odot \overline{K}_2 \in \Gamma_{a,0}(V_4)$ for all $n$.*

**Theorem 3.2.18.** *The graph $P_n \odot \overline{K}_m \in \Gamma(V_4)$ for all $n$ and $m$.*

*Proof.* Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{im}\}$ be the vertex set of $i^{th}$ copy of $\overline{K}_m$, for $i = 1, 2, 3, \ldots, n$.

**Case 1:** $m$ is an even integer.

In this case, define $f : V(P_n \odot \overline{K}_m) \to V_4$ by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \quad\ i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad v = u_{ij}, \quad \text{for all } i, j. \end{cases}$$

**Case 2:** $m$ is an odd integer.

In this case, define $f : V(P_n \odot \overline{K}_m) \to V_4$ by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_i, \quad\ \text{for } i = 1, 2, 3, \ldots, n \\ 0 & \text{if} \quad v = u_{i1}, \quad \text{for all } i \\ a & \text{if} \quad v = u_{ij}, \quad \text{for all } i, \quad j = 2, 3, 4, \ldots, m. \end{cases}$$

Then in both cases, we can prove that $f$ is an IML of $P_n \odot \overline{K}_m$.
Hence $P_n \odot \overline{K}_m \in \Gamma(V_4)$ for all $n$ and $m$. $\square$

From the proof of above theorem we have the following corollary.

**Corollary 3.2.19.** *The graph $P_n \odot \overline{K}_m \in \Gamma_{a,0}(V_4)$ for all $n$ and $m$.*

## 3.3 Star Related Graphs

**Theorem 3.3.1.** *The complete graph $K_n \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Let $V(K_n) = \{v_1, v_2, v_3, \ldots, v_n\}$. Suppose $n$ is odd.
Define $f : V(K_n) \to V_4$ as :

$$f(v_i) = \begin{cases} 0 & \text{if} \quad i = 1 \\ a & \text{if} \quad i = 2, 3, 4, \ldots, n. \end{cases}$$

Then, $f$ is an induced magic labeling of $K_n$. Conversely, suppose $n$ is an even number. Then $deg\ (v_i) = n - 1$ is an odd number. Therefore by Theorem 2.3.1, we have $f$ is an induced magic label if and only if $f$ satisfies the following system equations:

$$\begin{aligned} f(v_2) + f(v_3) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ f(v_1) + f(v_3) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ f(v_1) + f(v_2) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ &\vdots \\ f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{n-2}) + f(v_n) &= 0 \\ f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{n-2}) + f(v_{n-1}) &= 0. \end{aligned}$$

Note that the above system of equations imply $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$ and which again implies that $(n-1)f(v_1) = 0$. That is $f(v_1) = 0$. Thus $f \equiv 0$, which is a contradiction. $\square$

The following corollary follows directly from the proof of Theorem 3.3.1.

**Corollary 3.3.2.** *$K_n \in \Gamma_{a,0}(V_4)$ if and only if $n$ is odd.*

**Theorem 3.3.3.** *For $m, n > 1$, the complete bipartite graph $K_{m,n} \in \Gamma(V_4)$ if and only if either $m$ or $n$ is odd.*

*Proof.* Let $V(K_{m,n}) = \{v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\}$, where $v_iu_j \in E(K_{m,n})$ for $i = 1, 2, 3, \ldots, m$, and $j = 1, 2, 3, \ldots, n$.

**Case 1:** $m$ and $n$ are odd.

In this case, we define $f : V(K_{m,n}) \to V_4$ as follows:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_1, u_1 \\ a & \text{if} \quad v = v_2, v_3, v_4, \ldots, v_m \\ b & \text{if} \quad v = u_2, u_3, u_4, \ldots, u_n. \end{cases}$$

**Case 2:** $m$ is odd and $n$ is even.

In this case, we define $g : V(K_{m,n}) \to V_4$ as follows:

$$g(v) = \begin{cases} 0 & \text{if} \quad v = v_1, v_2, v_3, \ldots, v_m \\ a & \text{if} \quad v = u_1, u_2, u_3, \ldots, u_n. \end{cases}$$

**Case 3:** $m$ is even and $n$ is odd.

In this case, we define $h : V(K_{m,n}) \to V_4$ as follows:

$$h(v) = \begin{cases} a & \text{if} \quad v = v_1, v_2, v_3, \ldots, v_m \\ 0 & \text{if} \quad v = u_1, u_2, u_3, \ldots, u_n. \end{cases}$$

Then in each case, we can easily verify that the vertex labeling $f, g$ and $h$ are induced magic labeling of $K_{m,n}$. Thus $K_{m,n} \in \Gamma(V_4)$ if either $m$ or $n$ is odd.

Now consider the following case:

**Case 4:** $m$ and $n$ are even.

If possible, suppose $f : V(K_{m,n}) \to V_4$ is an induced magic labeling of $K_{m,n}$. Then by Theorem 2.3.1 $f$ must satisfy the following system of equations:

$$f(v_i) + \sum_{k=1}^{n} f(u_k) = 0 \text{ for } i = 1, 2, 3, \ldots, m. \tag{3.1}$$

$$f(u_j) + \sum_{k=1}^{m} f(v_k) = 0 \text{ for } j = 1, 2, 3, \ldots, n. \tag{3.2}$$

Note that the above equations in (3.1) imply that $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$ and the equations in (3.2) imply that $f(u_1) = f(u_2) = f(u_3) = \cdots = f(u_n)$. Thus the above system reduces to:

$$\begin{aligned} f(v_1) + nf(u_1) &= 0 \\ mf(v_1) + f(u_1) &= 0. \end{aligned}$$

Since both $m$ and $n$ are even the above system implies that $f(v_1) = f(u_1) = 0$. Thus $f \equiv 0$ and which is a contradiction to our assumption. Hence in this case, $K_{m,n} \notin \Gamma(V_4)$.

□

**Corollary 3.3.4.** *For $n > 1$, the star graph $K_{1,n} \in \Gamma(V_4)$.*

*Proof.* Let $V(K_{1,n}) = \{v, v_1, v_2, v_3, \ldots, v_n\}$, where $vv_i \in E(K_{1,n})$, for $i = 1, 2, 3, \ldots, n$.

**Case 1:** $n$ is an even integer.

In this case, we define $f : V(K_{1,n}) \to V_4$ as follows:

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v \\ a & \text{if} \quad u = v_1, v_2, v_3, \dots, v_n. \end{cases}$$

**Case 2:** $n$ is an odd integer.

In this case, define $g : V(K_{1,n}) \to V_4$ as follows:

$$g(u) = \begin{cases} 0 & \text{if} \quad u = v, v_1 \\ a & \text{if} \quad u = v_2, v_3, v_4, \dots, v_n. \end{cases}$$

Then in each case, we can easily verify that the vertex labeling $f$ and $g$ are induced magic labeling of $K_{1,n}$. Thus $K_{1,n} \in \Gamma(V_4)$. for all $n > 1$. $\square$

**Corollary 3.3.5.** *For $n > 1, K_{1,n} \in \Gamma_{a,0}(V_4)$.*

**Theorem 3.3.6.** *For the bistar $B_{m,n} \in \Gamma(V_4)$ for all $m$ and $n$ with $m + n > 2$.*

*Proof.* Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \dots, v_m, u_1, u_2, u_3, \dots, u_n\}$, where $uv$, $vv_i$, $uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

**Case 1:** $m$ and $n$ are even.

In this case, we define $f : V(B_{m,n}) \to V_4$ as follows:

$$f(w) = \begin{cases} 0 & \text{if} \quad w = v, u \\ a & \text{if} \quad w = v_1, v_2, v_3, \dots, v_m \\ b & \text{if} \quad w = u_1, u_2, u_3, \dots, u_n \end{cases}$$

**Case 2:** $m$ is odd and $n$ is even.

In this case, we define $g : V(B_{m,n}) \to V_4$ as follows:

$$g(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1 \\ a & \text{if} \quad w = v_2, v_3, v_4, \ldots, v_m \\ b & \text{if} \quad w = u_1, u_2, u_3, \ldots, u_n. \end{cases}$$

**Case 3:** $m$ is even and $n$ is odd.

In this case, we define $h : V(B_{m,n}) \to V_4$ as follows:

$$h(w) = \begin{cases} 0 & \text{if} \quad w = v, u, u_1 \\ a & \text{if} \quad w = v_1, v_2, v_3, \ldots, v_m \\ b & \text{if} \quad w = u_2, u_3, u_4, \ldots, u_n. \end{cases}$$

**Case 4:** $m$ and $n$ are odd.

In this case, we define $k : V(B_{m,n}) \to V_4$ as follows:

$$k(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1, u_1 \\ a & \text{if} \quad w = v_2, v_3, v_4, \ldots, v_m \\ b & \text{if} \quad w = u_2, u_3, u_4, \ldots, u_n. \end{cases}$$

Then in each case, we can easily verify that the vertex labeling $f, g, h$ and $k$ are induced magic labeling of $B_{m,n}$. Thus $B_{m,n} \in \Gamma(V_4)$, for all $m$ and $n$. $\square$

**Corollary 3.3.7.** *For the bistar $B_{m,n} \in \Gamma_{a,0}(V_4)$ for all $m$ and $n$ with $m+n > 2$.*

*Proof.* Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\}$, where $uv$, $vv_i$, $uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

**Case 1:** $m$ and $n$ are even.

In this case, define $f : V(B_{m,n}) \to V_4$ as follows:

$$f(w) = \begin{cases} 0 & \text{if} \quad w = v, u \\ a & \text{if} \quad w = v_i, u_j, i = 1, 2, 3, \ldots, m, \quad j = 1, 2, 3, \ldots, n. \end{cases}$$

**Case 2:** $m$ is odd and $n$ is even.

In this case, define $g : V(B_{m,n}) \to V_4$ as follows:

$$g(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1 \\ a & \text{if} \quad w = v_i, u_j, \quad i = 2, 3, \ldots, m, \quad j = 1, 2, 3, \ldots, n. \end{cases}$$

**Case 3:** $m$ is even and $n$ is odd.

In this case, we define $h : V(B_{m,n}) \to V_4$ as follows

$$h(w) = \begin{cases} 0 & \text{if} \quad w = v, u, u_1 \\ a & \text{if} \quad w = v_i, u_j, \quad i = 1, 2, 3, \ldots, m, \quad j = 2, 3, \ldots, n. \end{cases}$$

**Case 4:** $m$ and $n$ are odd.

In this case, we define $k : V(B_{m,n}) \to V_4$ as follows

$$k(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1, u_1 \\ a & \text{if} \quad w = v_i, u_j, \quad i = 2, 3, \ldots, m, \quad j = 2, 3, \ldots, n. \end{cases}$$

Then in each case, we can easily verify that the vertex labeling $f, g, h$ and $k$ are induced magic labeling of $B_{m,n}$. Thus $B_{m,n} \in \Gamma_{a,0}(V_4)$, for all $m$ and $n$. $\square$

**Theorem 3.3.8.** *The graph* $< K_{1,n} : m > \in \Gamma(V_4)$ *for all* $m, n$.

*Proof.* Consider the graph $< K_{1,n} : m >$ with $\{v_i, v_{ij} : 1 \leq j \leq n\}$ as the vertex set of $i^{th}$ copy of $K_{1,n}$ with central vertex $v_i$ for $i = 1, 2, 3, \ldots, m$ and let $v$ be the unique vertex adjacent to the central vertices $v_i$ in $< K_{1,n} : m >$ . Then define $f : V(< K_{1,n} : m >) \to V_4$ as follows:

**Case 1:** $m$ and $n$ are odd.

In this case define $f$ as

$$f(u) = \begin{cases} a & \text{if} \quad u = v \\ a & \text{if} \quad u = v_{ij}, \ i = 1, 2, 3, \ldots, m, \quad j = 1, 2, 3, \ldots, n \\ 0 & \text{if} \quad u = v_i, \quad i = 1, 2, 3, \ldots, m. \end{cases}$$

**Case 2:** $m$ is odd and $n$ is even.

In this case define $f$ as

$$f(u) = \begin{cases} a & \text{if} \quad u = v \\ 0 & \text{if} \quad u = v_{i1} \\ a & \text{if} \quad u = v_{ij}, \ i = 1, 2, 3, \ldots, m, \quad j = 2, 3, 4, \ldots, n \\ 0 & \text{if} \quad u = v_i, \quad i = 1, 2, 3, \ldots, m. \end{cases}$$

**Case 3:** $m$ is even and $n$ is odd.

In this case define $f$ as

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v \\ 0 & \text{if} \quad u = v_{i1} \\ a & \text{if} \quad u = v_{ij}, \ i = 1, 2, 3, \ldots, m \quad j = 2, 3, 4, \ldots, n \\ 0 & \text{if} \quad u = v_i, \quad i = 1, 2, 3, \ldots, m. \end{cases}$$

**Case 4:** $m$ and $n$ are even.

In this case define $f$ as

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v \\ a & \text{if} \quad u = v_{ij}, \ i = 1, 2, 3, \ldots, m, \quad j = 1, 2, 3, \ldots, n \\ 0 & \text{if} \quad u = v_i, \quad i = 1, 2, 3, \ldots, m. \end{cases}$$

Then one can easily verify that the vertex label $f$ defined in all four cases are IML of $< K_{1,n} : m >$ .

Hence the proof. $\square$

From the proof of above theorem we have the following corollary.

**Corollary 3.3.9.** *The graph* $< K_{1,n} : m >\in \Gamma_{a,0}(V_4)$ *for all* $m, n$.

**Theorem 3.3.10.** *The* $(n,k)$*-banana tree* $Bt(n,k) \in \Gamma(V_4)$ *for all* $n$ *and* $k$.

*Proof.* Consider the graph $Bt(n,k)$. Let $V[Bt(n,k)]$=$\{v, v_i, v_{ij} : 1 \leq i \leq n\ , 1 \leq j \leq k\}$ and $E[Bt(n,k)]$=$\{vv_{i1}, v_iv_{ij}\} : 1 \leq i \leq n, 1 \leq j \leq k\}$.

**Case 1** : $k$ is an odd integer.

In this case, define $f : (V(Bt(n,k))) \to V_4$ by

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v, v_i, \quad \text{for } i = 1,2,3,\ldots,n \\ 0 & \text{if} \quad u = v_{i1}, \quad \text{for } i = 1,2,3,\ldots,n \\ a & \text{if} \quad u = v_{ij}, \quad \text{for } i = 1,2,3,\ldots,n,\ j = 2,3,4,\ldots,k. \end{cases}$$

**Case 2** : $k$ is an even integer.

In this case, define $f : (V(Bt(n,k))) \to V_4$ by

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v, v_i, \quad \text{for } i = 1,2,3,\ldots,n \\ 0 & \text{if} \quad u = v_{i1}, v_{i2}, \quad \text{for } i = 1,2,3,\ldots,n \\ a & \text{if} \quad u = v_{ij}, \quad \text{for } i = 1,2,3,\ldots n, j = 3,4,\ldots,k. \end{cases}$$

In both cases, we can easily verify that $f$ is an IML of $Bt(n,k)$. Hence the proof. □

**Theorem 3.3.11.** $K^*_{1,n} \in \Gamma(V_4)$ *for all* $n$.

*Proof.* Let $\{v_i, v_{i1}, v_{i2}, v_{i3}, \ldots, v_{in}\}$ be the vertex set of $i^{th}$ copy of $K_{1,n}$, for $i = 1,2,3,\ldots,n$ with $v_i$ as the central vertex. Also suppose each $v_i$ is attached to a vertex $u$ by $n$ distinct edges. Let $f : V(K^*_{1,n}) \to V_4$.

**Case 1:** $n$ is an even integer.

In this case, define $f$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u, v_i \\ a & \text{if} \quad v = v_{ij}, \quad \text{for all} \quad i, j. \end{cases}$$

**Case 2:** $n$ is an odd integer.

In this case, define $f$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u, v_i \\ 0 & \text{if} \quad v = v_{i1}, \quad \text{for} \quad i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad v = v_{ij}, \quad \text{for all} \quad i, j = 2, 3, 4, \ldots, n. \end{cases}$$

Then we can easily verify that $f$ is an IML of $K^*_{1,n}$. Thus $K^*_{1,n} \in \Gamma(V_4)$ for all $n$. Hence the proof. □

**Theorem 3.3.12.** *$K^{(n)}_m \in \Gamma(V_4)$ for all $m$ and $n$..*

*Proof.* Let $\{u, u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,(m-1)}\}$ be the vertex set of $i^{th}$ copy of $K_m$ in $K^{(n)}_m$, for $i = 1, 2, 3, \ldots, n$, where $u$ is the common vertex.

**Case (i)** $m$ is odd.

Suppose $m$ is an odd integer. Let $f : V(K^{(n)}_m) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u \\ a & \text{if} \quad v = u_{i,j} \quad \text{for all} \quad i, j. \end{cases}$$

Then we can prove that $f$ is an IML of $K^{(n)}_m$. Thus $K^{(n)}_m \in \Gamma(V_4)$ for $m$ odd.

**Case (ii)** $m$ and $n$ are even.

Suppose $m$ and $n$ are even integers. Let $f : V(K^{(n)}_m) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u \\ a & \text{if} \quad v = u_{i,j}, \quad \text{for all} \quad i, j. \end{cases}$$

Then we can prove that $f$ is an IML of $K_m^{(n)}$. Thus $K_m^{(n)} \in \Gamma(V_4)$ for $m$ and $n$ are even.

**Case (iii)** $m$ is even and $n$ is odd.

Suppose $m$ is even and $n$ is odd. Let $f : V(K_m^{(n)}) \to V_4$ be defined by

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u \\ 0 & \text{if} \quad v = u_{i,j}, \quad \text{for all} \quad i = 1, \ j = 1, 2, 3, \ldots, m-1 \\ a & \text{if} \quad v = u_{i,j}, \quad \text{for all} \quad i = 2, 3, 4, \ldots, n, \ j = 1, 2, 3, \ldots, m-1. \end{cases}$$

Then we can prove that $f$ is an IML of $K_m^{(n)}$. Thus $K_m^{(n)} \in \Gamma(V_4)$ for $m$ is even and $n$ is odd.

Hence the proof. □

From the proof of the above theorem we have the following corollary.

**Corollary 3.3.13.** $K_m^{(n)} \in \Gamma_{a,0}(V_4)$ *for all* $m$ *and* $n$.

Chapter 4

# Induced $V_4$-Magic Labeling of Subdivision and Shadow Graphs

*In this chapter, we discuss induced $V_4$-magic labeling of subdivision graphs and shadow graphs of some graph. The first section gives an introduction about subdivision graphs and then deals with the induced $V_4$-magic labeling of subdivision graphs of some general and special graphs. In the second section, we prove a pretty theorem regarding shadow graph of a graph.*

## 4.1 Subdivision Graphs

The subdivision of an edge $e = uv$ in the graph $G$ gives a new graph obtained by replacing the edge $e = uv$ by two edges $e_1 = uw$ and $e_2 = wv$. A subdivision of a graph $G$ is a graph which is denoted by $S(G)$ and is obtained from the subdivision of all edges in $G$.

**Theorem 4.1.1.** *Let $G$ be a graph with every vertex is of odd degree, then $S(G) \in \Gamma(V_4)$.*

*Proof.* Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_1, u_2, u_3, \ldots, u_m\}$ be the newly inserted vertices in $S(G)$. Let $f : V(S(G)) \to V_4$ be defined by

$$f(v) = \begin{cases} a & \text{if} \quad v = v_k, \text{ for } k = 1, 2, 3, \ldots, n \\ 0 & \text{if} \quad v = u_j, \text{ for } j = 1, 2, 3, \ldots, m. \end{cases}$$

Then we have $f^*(e_i) = a$ for all $e_i \in E(S(G))$ therefore $f^{**}(v_k) = deg\ (v_k)a = a$, since $deg\ (v_k)$ is odd. Also $f^{**}(u_j) = f^*(v_\alpha u_j) + f^*(v_\beta u_j) = a + a = 0$, where $u_j$ is inserted in the edge $v_\alpha v_\beta$. Thus $f \equiv f^{**}$. That is $f$ is an IML of $S(G)$. Hence the proof. □

From the proof of the above theorem, we have the following corollary.

**Corollary 4.1.2.** *Let $G$ be a graph with every vertex is of odd degree, then $S(G) \in \Gamma_{a,0}(V_4)$.*

**Theorem 4.1.3.** *$S(C_n) \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$.*

*Proof.* From Theorem 2.4.1, we know that $C_n \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$. Also we have $S(C_n) = C_{2n}$. Thus $S(C_n) = C_{2n} \in \Gamma(V_4)$ if and only if $2n \equiv 0 \pmod 3$, that is if and only if $n \equiv 0 \pmod 3$. Hence the proof. □

**Theorem 4.1.4.** *$S(P_n) \in \Gamma(V_4)$ if and only if $2n \equiv 1 \pmod 3$.*

*Proof.* We know that $S(P_n) = P_{2n-1}$ and $P_n \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$. Therefore $S(P_n) = P_{2n-1} \in \Gamma(V_4)$ if and only if $2n \equiv 1 \pmod 3$. Hence the proof. □

**Theorem 4.1.5.** *$S(K_n) \in \Gamma(V_4)$ for $n$ is even.*

*Proof.* Suppose $n$ is even. Then consider the complete graph $K_n$. Since $n$ is even, every vertex of $K_n$ is of odd degree. Therefore by Theorem 4.1.1, we have $S(K_n) \in \Gamma(V_4)$. □

**Corollary 4.1.6.** *$S(K_n) \in \Gamma_{a,0}(V_4)$ for $n$ is even.*

**Theorem 4.1.7.** *$S(B_{m,n}) \in \Gamma(V_4)$ if and only if $m$ and $n$ are even.*

*Proof.* Suppose $m$ and $n$ are even. Consider the bistar graph $B_{m,n}$ with vertex set $V(B_{m,n}) = \{u, v, v_i, u_j : i = 1, 2, 3, \ldots, m, j = 1, 2, 3, \ldots, n\}$, where $uv, vv_i, uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \ldots, m$, and $j = 1, 2, 3, \ldots, n$. Also let $\{v', v'_i, u'_j : i = 1, 2, 3, \ldots, m, j = 1, 2, 3, \ldots, n\}$ be the inserted vertices in $S(B_{m,n})$ corresponding to the edges $uv, vv_i$ and $uu_j$ respectively. Define $f : V(S(B_{m,n})) \to V_4$ as :

$$f(w) = \begin{cases} a & \text{if} \quad w = v, u, v_i, u_j \\ 0 & \text{if} \quad w = v', v'_i, u'_j. \end{cases}$$

Then $f$ is an IML of $S(B_{m,n})$ and thus for $m$ and $n$ are even, $S(B_{m,n}) \in \Gamma(V_4)$. Conversely, suppose that either $m$ or $n$ is odd and $f$ is an IML of $S(B_{m,n})$. Then by the induced degree sum theorem, $f$ must satisfy the following system of equations.

$$f(v'_i) = 0 \tag{4.1}$$

$$f(u'_j) = 0 \tag{4.2}$$

$$f(v_i) + f(v) = 0 \tag{4.3}$$

$$f(u_j) + f(u) = 0 \tag{4.4}$$

$$mf(v) + f(v') = 0 \tag{4.5}$$

$$nf(u) + f(v') = 0 \tag{4.6}$$

$$f(v) + f(u) + f(v') = 0. \tag{4.7}$$

From the above system of equations, (4.3) and (4.4) imply that $f(v_i) = f(v)$, $f(u_j) = f(u)$, for $i = 1, 2, 3, \ldots m$, and $j = 1, 2, 3, \ldots, n$. Then consider the following cases:

**Case 1:** $m$ and $n$ are odd.

In this case, the Equations (4.5), (4.6) and (4.7) in the above system of equations reduces to

$$\begin{aligned} f(v) + f(v') &= 0 \\ f(u) + f(v') &= 0 \\ f(v) + f(u) + f(v') &= 0. \end{aligned}$$

Above three equations imply that $f(v) = f(u) = f(v') = 0$. That is $f \equiv 0$, which is a contradiction.

**Case 2:** $m$ is odd and $n$ is even.

In this case, the Equations (4.5), (4.6) and (4.7) in the above system of equations reduces to

$$\begin{aligned} f(v) + f(v') &= 0 \\ f(v') &= 0 \\ f(v) + f(u) + f(v') &= 0. \end{aligned}$$

These equations imply that $f(v) = 0$, therefore $f(u) = 0$. That is $f \equiv 0$, which is a contradiction.

**Case 3:** $m$ is even and $n$ is odd.

In this case, the Equations (4.5), (4.6) and (4.7) in the above system of equations reduces to

$$\begin{aligned} f(v') &= 0 \\ f(u) + f(v') &= 0 \\ f(v) + f(u) + f(v') &= 0. \end{aligned}$$

These equations imply that $f(u) = 0$, therefore $f(v) = 0$. That is $f \equiv 0$, which is a contradiction.

Thus from the above three cases, we get, there exists no such IML for $S(B_{m,n})$. Hence $S(B_{m,n}) \in \Gamma(V_4)$ if and only if $m$ and $n$ are even. □

**Corollary 4.1.8.** *Let $S(B_{m,n}) \in \Gamma_{a,0}(V_4)$ if and only if $m$ and $n$ are even.*

**Theorem 4.1.9.** *For the complete graph $K_{m,n}$ we have, $S(K_{m,n}) \in \Gamma(V_4)$ for $m$ and $n$ are odd.*

*Proof.* Suppose $m$ and $n$ are odd. Since $m$ and $n$ are odd, every vertex of $K_{m,n}$ is of odd degree. Therefore by Theorem 4.1.1, we have $S(K_{m,n}) \in \Gamma(V_4)$. □

**Corollary 4.1.10.** *$S(K_{m,n}) \in \Gamma_{a,0}(V_4)$ for $m$ and $n$ are odd.*

**Theorem 4.1.11.** *$S(K_{1,n}) \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Let $V(K_{1,n}) = \{v, u_j : j = 1, 2, 3, \ldots, n\}$, where $vu_j \in E(K_{1,n})$ for $j = 1, 2, 3, \ldots, n$ and let $\{v_j : j = 1, 2, 3, \ldots, n\}$ be the inserted vertices in $S(K_n)$ corresponding to the edge $vu_j$. Suppose $n$ is odd, then define $f : V(S(K_{1,n})) \to V_4$ as

$$f(w) = \begin{cases} a & \text{if} \quad w = v, u_j \\ 0 & \text{if} \quad w = v_j. \end{cases}$$

Then $f$ gives an IML for $S(K_{1,n})$.

Conversely, suppose that $n$ is even and $f$ is an IML of $S(K_{1,n})$. Then by the induced degree sum equation of the vertices $u_j, v_j$ and $v$ the vertex labeling function $f$ must satisfies the following system of equations:

$$f(v_j) = 0 \tag{4.8}$$

$$f(v) + f(u_j) = 0 \tag{4.9}$$

$$(n-1)f(v) = 0. \tag{4.10}$$

Since $n$ is even, the Equation (4.10) implies that $f(v) = 0$. Therefore $f(u_j) = 0$. Thus $f \equiv 0$, which is a contradiction. Hence the proof. □

**Corollary 4.1.12.** *$S(K_{1,n}) \in \Gamma_{a,0}(V_4)$if and only if $n$ is odd.*

**Theorem 4.1.13.** *The subdivision of wheel graph, $S(W_n) \in \Gamma(V_4)$ for $n$ is odd.*

*Proof.* Let $V(W_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $w$ is the central vertex of $W_n$. Let $w_i$ be the inserted vertices on the edge $wv_i$ and $u_i$ be the inserted vertices on the edge $v_iv_{i+1}$, for $i = 1, 2, 3, \ldots, n$, where $i + 1$ is taken modulo $n$.
Then for $n$ odd, define $f : V(S(W_n)) \to V_4$ as

$$f(u) = \begin{cases} a & \text{if} \quad u = w, v_i \\ 0 & \text{if} \quad u = u_i, w_i. \end{cases}$$

Then, since every vertex is of odd degree, by the Theorem 4.1.1, $f$ is an IML of $S(W_n)$. Hence the proof. □

**Theorem 4.1.14.** *The subdivision of helm graph, $S(H_n) \in \Gamma(V_4)$ for $n$ even.*

*Proof.* Let $V(H_n) = \{w, v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_n\}$, where $w$ be the central vertex and $w_1, w_2, w_3, \ldots, w_n$ be the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$. Let $(\beta_i, \quad 1 \leq i \leq n)$ be the inserted vertices on the edge $(wv_i, \quad 1 \leq i \leq n)$, $(u_i, \quad 1 \leq i \leq n)$ be the inserted vertices on the edge $(v_iv_{i+1}, \quad 1 \leq i \leq n)$ and $(\alpha_i, \quad 1 \leq i \leq n)$ be the inserted vertices on the edge $(v_iw_i, \quad 1 \leq i \leq n)$.
Then for $n$ even, define $f : V(S(H_n)) \to V_4$ as:

$$f(u) = \begin{cases} a & \text{if} \quad u = v_i, w_i, \beta_i \\ 0 & \text{if} \quad u = w, u_i, \alpha_i. \end{cases}$$

Then we can easily prove that $f$ is an IML of $S(H_n)$.
Hence the proof. □

**Theorem 4.1.15.** *The subdivision of comb graph, $S(CB_n) \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Let $\{u_i, v_i : i = 1, 2, 3, \ldots, n\}$ be the vertex set of $CB_n$, where $v_i$ be the pendant vertex adjacent to $u_i$. Suppose $w_i$ and $t_j$ are inserted vertices on the edge $u_iv_i$ and $u_ju_{j+1}$, for $i = 1, 2, 3, \ldots, n$ and $j = i = 1, 2, 3, \ldots, n-1$.
Suppose $n$ is an odd number. Define $f : V(S(CB_n)) \to V_4$ as:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = w_1, w_2, w_3, \ldots, w_n \\ a & \text{if} \quad v = t_1, t_2, t_3, \ldots, t_{n-1} \\ a & \text{if} \quad v = u_i, v_i, \text{ for } i \text{ odd} \\ 0 & \text{if} \quad v = u_i, v_i, \text{ for } i \text{ even.} \end{cases}$$

Then $f$ is an IML of $S(CB_n)$.

Conversely, suppose that $n$ is an even number. If possible, suppose there exists an IML for $S(CB_n)$ say $g : V(S(CB_n)) \to V_4$.
Since $v_i$ is a pendant vertex of $S(CB_n)$ the induced degree sum equation of $v_i$ gives: $g(w_i) = 0$. Also the induced degree sum equation of $w_i$ gives:

$$g(u_i) + g(v_i) = 0 \quad \text{for } i = 1, 2, 3, \ldots, n.$$

Note that the above equation implies that $g(u_i) = g(v_i)$ for $i = 1, 2, 3, \ldots, n$
Also the induced degree sum equation of $u_i$ gives

$$\begin{aligned} g(t_1) + g(u_1) &= 0 \\ g(t_1) + g(t_2) &= 0 \\ g(t_2) + g(t_3) &= 0 \\ g(t_3) + g(t_4) &= 0 \\ &\vdots \end{aligned}$$

$$\begin{aligned} g(t_{n-2}) + g(t_{n-1}) &= 0 \\ g(t_{n-1}) + g(u_n) &= 0. \end{aligned}$$

Similarly induced degree sum equation of $t_j$ gives

$$\begin{aligned} g(t_1) + g(u_1) + g(u_2) &= 0 \\ g(t_2) + g(u_2) + g(u_3) &= 0 \\ g(t_3) + g(u_3) + g(u_4) &= 0 \\ &\vdots \\ g(t_{n-1}) + g(u_{n-1}) + g(u_n) &= 0. \end{aligned}$$

Note that the induced degree sum equation of $u_i$ implies that

$$g(u_1) = g(t_1) = g(t_2) = g(t_3) = \cdots = g(t_{n-1}) = g(u_n). \tag{4.11}$$

Since $n$ is even, by substituting this in the induced degree sum equation of $t_j$, we get

$$g(u_2) = g(u_4) = g(u_6) = \cdots = g(u_n) = 0. \tag{4.12}$$

By substituting $g(u_n) = 0$ in Equation (4.11), we get

$$g(u_1) = g(t_1) = g(t_2) = g(t_3) = \cdots = g(t_{n-1}) = g(u_n) = 0. \tag{4.13}$$

Substituting Equation( 4.13) and Equation (4.12) in the induced degree sum equation of $t_j$, we get

$$g(u_1) = g(u_3) = g(u_5) = \cdots = g(u_{n-1}) = 0. \tag{4.14}$$

Thus $g \equiv 0$, which is a contradiction to our assumption. Thus in this case,

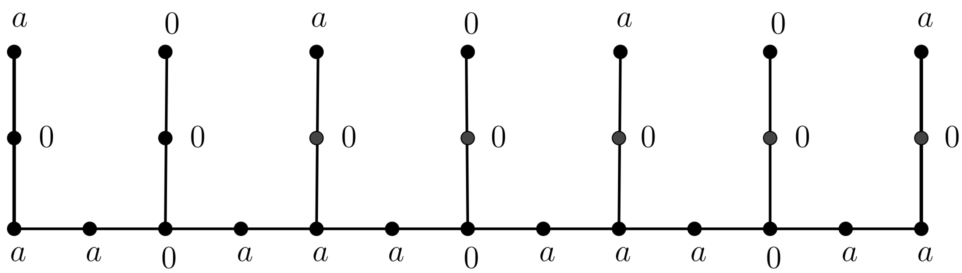


Figure 4.1: Subdivision graph of $CB_7$

$S(CB_n) \notin \Gamma(V_4)$. Hence the proof. $\square$

In the Figure 4.1 an induced $V_4$-magic labeling of the graph $S(CB_7)$ is given.

**Theorem 4.1.16.** *The subdivision of jelly fish graph $S(J(m,n)) \in \Gamma(V_4)$ for all $m$ and $n$.*

*Proof.* Consider the jelly fish graph with $V(J(m,n)) = \{v_k : k = 1,2,3,4.\} \cup \{u_i : i = 1,2,3,\ldots,m\} \cup \{w_j : j = 1,2,3,\ldots,n\}$, where $v_1, v_2, v_3, v_4$ are the vertices corresponding to $C_4$, $u_i, w_j$ are the vertices corresponding to $K_{1,m}$ and $K_{1,n}$ respectively and $\alpha_i (1 \le i \le m)$, $\beta_j (1 \le j \le n)$ be the inserted vertices on the edges $v_2u_i, v_4w_j$ respectively. Also let $\alpha, \beta, \gamma, \delta, \sigma$ be the vertices inserted on the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3$ respectively.

**Case 1:** $m$ and $n$ are odd.

In this case, Define $f : V(S(J(m,n))) \to V_4$ as :

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, v_2, v_3, v_4, u_i, w_j \\ 0 & \text{if} \quad v = \alpha_i, \beta_j, \alpha, \beta, \gamma, \delta, \sigma. \end{cases}$$

**Case 2:** ($m$ and $n$ even) or ($m$ odd and $n$ even) or ($m$ even and $n$ odd).

In this case, define $f : V(S(J(m,n))) \to V_4$ as:

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, v_3, \alpha, \beta, \gamma, \delta, \\ 0 & \text{if} \quad v = v_2, v_4, u_i, w_j, \alpha_i, \beta_j, \sigma. \end{cases}$$

Then for all cases, we can prove that $f$ is an IML of $S(J(m,n))$. Hence the proof. □

**Theorem 4.1.17.** *The subdivision of sunflower graph $S(SF_n) \in \Gamma(V_4)$, for $n$ is odd.*

*Proof.* Suppose the given sunflower graph is obtained by taking a wheel with the central vertex $v_0$ and the $n-$cycle $v_1, v_2, v_3, \ldots, v_n$ and the additional vertices $w_1, w_2, w_3, \ldots, w_n$, where $w_i$ is joined by edges to the vertices $v_i, v_{i+1}$, where $i+1$ is taken modulo $n$. Also let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be the inserted vertices on the edges $v_0v_i, v_iv_{i+,1}, v_iw_i, w_iv_{i+1}$ respectively, where $i+1$ is taken modulo $n$.
Suppose $n$ is odd, then define $f : V(S(SF_n)) \to V_4$ as :

$$f(v) = \begin{cases} a & \text{if } v = v_0, v_i, \gamma_i, \delta_i \quad \text{for } i = 1, 2, 3, \ldots, n \\ 0 & \text{if } v = w_i, \alpha_i, \beta_i \qquad \text{for } i = 1, 2, 3, \ldots, n. \end{cases}$$

Then we can easily prove that $f$ is an induced $V_4$ magic labeling of $S(SF_n)$. Hence the proof. □

**Theorem 4.1.18.** *The subdivision of gear graph, $S(G_n) \in \Gamma(V_4)$ for $n$ odd.*

*Proof.* Let $V(G_n) = \{w, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$, where $w$ is the central vertex $u_1, u_2, u_3, \ldots, u_n$ are the vertices of the corresponding wheel graph $W_n$ and $v_1, v_2, v_3, \ldots, v_n$ are the remaining vertices with $u_iv_i, v_iu_{i+1} \in E(G_n)$, where $i+1$ is taken modulo $n$. Also let $w_i, \alpha_i, \beta_i$ be the inserted vertices on the edges $wu_i, u_iv_i, v_iu_{i+i}$, where $i+1$ is taken modulo $n$.
Suppose $n$ is an odd integer. Define $f : V(S(G_n)) \to V_4$ as

$$f(v) = \begin{cases} a & \text{if } v = w, u_i, \alpha_i, \beta_i \quad \text{for } i = 1, 2, 3, \ldots, n \\ 0 & \text{if } v = w_i, v_i \qquad\quad\;\; \text{for } i = 1, 2, 3, \ldots, n. \end{cases}$$

Then we can easily prove that $f$ is an induced $V_4$ magic labeling of $S(G_n)$. Hence the proof. □

**Theorem 4.1.19.** *The subdivision of flower graph, $S(Fl^n) \in \Gamma(V_4)$ for all $n$.*

*Proof.* Let $V(Fl^n) = \{w, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$, where $w$ is the central vertex $u_1, u_2, u_3, \ldots, u_n$ are the pendant vertices of corresponding helm and $v_1, v_2, v_3, \ldots, v_n$ are the vertices adjacent to the central vertex $w$. Also let $\alpha_i, \beta_i, \gamma_i$ and $\delta_i$ be the inserted vertices on the edges $wu_i, u_iv_i, wv_i$ and $v_iv_{i+1}$ respectively, for $i = 1, 2, 3, \ldots, n$ with $i + 1$ is taken modulo $n$.

**Case 1:** $n$ is an odd number.

In this case, define $f : V(S(Fl^n)) \to V_4$ as

$$f(v) = \begin{cases} a & \text{if } v = \alpha_i, v_i \quad \text{for } i = 1, 2, 3, \ldots, n \\ b & \text{if } v = w, \beta_i \quad \text{for } i = 1, 2, 3, \ldots, n \\ c & \text{if } v = u_i, \gamma_i \quad \text{for } i = 1, 2, 3, \ldots, n \\ 0 & \text{if } v = \delta_i, \qquad \text{for } i = 1, 2, 3, \ldots, n. \end{cases}$$

**Case 2:** $n$ is an even number.

In this case, we define $f : V(S(Fl^n)) \to V_4$ as

$$f(v) = \begin{cases} a & \text{if } v = u_i, v_i, \alpha_i, \gamma_i \quad \text{for } i = 1, 2, 3, \ldots, n \\ 0 & \text{if } v = w, \beta_i, \delta_i \qquad\; \text{for } i = 1, 2, 3, \ldots, n. \end{cases}$$

Then in both cases, we can easily verify that $f$ is an induced $V_4$ magic labeling of $S(Fl^n)$. Hence the proof. □

## 4.2 Shadow Graphs

**Definition 4.2.1.** *The shadow graph $Sh(G)$ of a connected graph $G$ is constructed by taking 2 copies $G_1$ and $G_2$ of $G$ and joining each vertex $u$ in $G_1$ to the neighbours of the corresponding vertex $v$ in $G_2$.*

**Theorem 4.2.2.** *For any graph $G$, $Sh(G) \notin \Gamma(V_4)$.*

*Proof.* If possible, suppose $f : V(Sh(G)) \to V_4$ be an IML of $Sh(G)$.

Suppose $\{v_1, v_2, v_3, \dots, v_n, u_1, u_2, u_3, \dots, u_n\}$ be the vertex set of $Sh(G)$, where $u_i$ is the corresponding vertex of $v_i$ in $Sh(G)$, for $i = 1, 2, 3, \dots, n$. Then note that if $N(v_i) = \{v_{i1}, v_{i2}, v_{i3}, \dots, v_{im}\}$ in the graph $G$, then $N(v_i) = N(u_i) = \{v_{i1}, v_{i2}, v_{i3}, \dots, v_{im}, u_{i1}, u_{i2}, u_{i3}, \dots, u_{im}\}$ in $Sh(G)$. Then note that the induced degree sum equation of the vertices $v_i$ and $u_i$ gives:

$$f(v_i) + \sum_{j=1}^{m} f(v_{ij}) + \sum_{j=1}^{m} f(u_{ij}) = 0. \tag{4.15}$$

$$f(u_i) + \sum_{j=1}^{m} f(v_{ij}) + \sum_{j=1}^{m} f(u_{ij}) = 0. \tag{4.16}$$

Equation (4.15) and Equation (4.16) imply that $f(v_i) = f(u_i)$.

Since $v_i$ and $u_i$ are arbitrary vertices of $Sh(G)$, we have $f(v_i) = f(u_i)$, for all $i = 1, 2, 3, \dots, n$.

Thus Equation (4.15) and Equation (4.16) imply that $f(v_i) = f(u_i) = 0$, since $f(v_{ij}) = f(u_{ij})$.

Since $v_i$ and $u_i$ are arbitrary vertices of $Sh(G)$, we have $f(v_i) = f(u_i) = 0$ for all pairs of vertices $v_i$ and $u_i$ in $Sh(G)$. Hence $f \equiv 0$, which is a contradiction. This completes the proof. □

Chapter 5

# Induced $V_4$-magic Labeling of Middle and Line Graphs

*This chapter discusses the induced $V_4$-magic labeling of middle graph and line graph of some graphs. The first section gives an introduction about middle graphs and then deals with the induced $V_4$-magic labeling of middle graphs. In the second section, we discuss the basic idea about line graph of a graph and the induced $V_4$-magic labeling of line graphs of some graphs.*

## 5.1 Middle Graphs

The concept of middle graph was introduced by J. Akiyama, T. Hamada and I. Yoshimura [1] in 1974.

**Definition 5.1.1.** *The middle graph of a graph $G$, denoted by $M(G)$, is the graph obtained from $G$ by inserting a new vertex into every edge of $G$ and by joining those pairs of these new vertices with edges which lie on adjacent edges*

[1]The first section of this chapter has been accepted for publication in the journal of *Advances and Applications in Discrete Mathematics*.

*of* $G$.

**Theorem 5.1.2.** *Let* $G$ *be graph with every vertex is of odd degree, then* $M(G) \in \Gamma(V_4)$.

*Proof.* Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_1, u_2, u_3, \ldots, u_m\}$ be the inserted vertices in $M(G)$. Let $f : V(M(G)) \to V_4$ be defined by:

$$f(v) = \begin{cases} a & \text{if} \quad v = v_k, \ \text{for } k = 1, 2, 3, \ldots, n \\ 0 & \text{if} \quad v = u_j, \ \text{for } j = 1, 2, 3, \ldots, m. \end{cases}$$

Then we have $f^*(e) = a$ for all $e \in E(M(G))$. Therefore $f^{**}(v_k) = \ deg\ (v_k)a = a$ since $deg\ (v_k)$ is odd. Also $f^{**}(u_j) = f^*(v_\alpha u_j) + f^*(v_\beta u_j) = 0$, where $u_j$ is inserted on the edge $v_\alpha v_\beta$. Thus $f \equiv f^{**}$. That is $f$ is an IML of $M(G)$.
This completes the proof. ☐

**Theorem 5.1.3.** *Let* $P_2$ *be the path with 2 vertices, then* $M(P_2) \in \Gamma(V_4)$.

*Proof.* Consider $P_2$, we have $M(P_2) = P_3$. Let $V(M(P_2)) = \{u, v, w\}$ with $E(M(P_2)) = \{uv, vw\}$. Define $f : V(M(P_2)) \to V_4$ as

$$f(t) = \begin{cases} a & \text{if} \quad t = u, w \\ 0 & \text{if} \quad t = v \end{cases}$$

Then $f^*(uv) = a$ and $f^*(vw) = a$, therefore $f^{**}(u) = a$, $f^{**}(v) = a + a = 0$ and $f^{**}(w) = a$. Thus $f \equiv f^{**}$. That is $f$ is an IML of $M(P_2)$. Hence $M(P_2) \in \Gamma(V_4)$. ☐

**Theorem 5.1.4.** *Let* $P_n$ *be the path with* $n$ *vertices, then* $M(P_n) \notin \Gamma(V_4)$ *for any* $n \geq 3$.

*Proof.* Suppose $n \geq 3$. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of $P_n$ and $u_j$ be the inserted vertex on the edge $v_j v_{j+1}$ in $M(P_n)$, for $j = 1, 2, 3, \ldots, n-1$. If possible, suppose $f : V(M(P_n)) \to V_4$ is an IML of $M(P_n)$ with $f(v_k) = x_k$, for $k = 1, 2, 3, \ldots, n$ and $f(u_j) = y_j$, for $j = 1, 2, 3, \ldots, n-1$. Then by the induced degree sum equation of each vertex in $M(P_n)$, we have the following set of equations.

The induced degree sum equation of $v_1, v_2, v_3, \ldots, v_n$ in $M(P_n)$, give

$$\begin{aligned}
y_1 &= 0 \\
y_1 + y_2 + x_2 &= 0 \\
y_2 + y_3 + x_3 &= 0 \\
y_3 + y_4 + x_4 &= 0 \\
&\vdots \\
y_{n-3} + y_{n-2} + x_{n-2} &= 0 \\
y_{n-2} + y_{n-1} + x_{n-1} &= 0 \\
y_{n-1} &= 0.
\end{aligned}$$

Similarly the induced degree sum equation of $u_1, u_2, u_3, \ldots, u_{n-1}$ in $M(P_n)$, give

$$\begin{aligned}
y_2 + x_1 + x_2 &= 0 \\
y_1 + y_2 + y_3 + x_2 + x_3 &= 0 \\
y_2 + y_3 + y_4 + x_3 + x_4 &= 0 \\
y_3 + y_4 + y_5 + x_4 + x_5 &= 0 \\
&\vdots \\
y_{n-3} + y_{n-2} + y_{n-1} + x_{n-2} + x_{n-1} &= 0 \\
y_{n-2} + x_{n-1} + x_n &= 0.
\end{aligned}$$

By substituting the first set of equation in the second, we get the following equations.

$$\begin{aligned} y_1 &= 0 \\ y_{n-1} &= 0 \\ y_k + x_k &= 0, \text{ for } k = 2, 3, 4, \ldots, n-1. \end{aligned}$$

But these equations with first system of equations implies that $y_k = 0$ for $k = 2, 3, 4, \ldots, n-2$, thus $x_k = 0$ also for $k = 2, 3, \ldots, n-1$. Using these in the induced degree sum equation of the vertices $u_1$ and $u_{n-1}$, we get $x_1 = 0$ and $x_n = 0$. Thus we have $x_1 = x_2 = x_3 = \cdots = x_n = 0$ and $y_1 = y_2 = y_3 = \cdots = y_{n-1} = 0$. That is $f \equiv 0$, which means our assumption that $f$ is an IML is wrong. Hence $M(P_n) \notin \Gamma(V_4)$, for any $n \geq 3$. □

**Theorem 5.1.5.** *For the star graph $K_{1,n}$, $M(K_{1,n}) \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Consider the star graph $K_{1,n}$, where $n$ is an odd number. Then note that each vertex of $K_{1,n}$ is of odd degree, then by Theorem 5.1.2, we get $M(K_{1,n}) \in \Gamma(V_4)$.

Conversely, Suppose that $n$ is an even number. Let $\{v, v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of $K_{1,n}$, with $v$ as the central vertex $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $u_1, u_2, u_3, \ldots, u_n$ be the inserted vertices on the edges $vv_1, vv_2, vv_3, \ldots, vv_n$ respectively in $M(K_{1,n})$.

If possible, suppose $f : V(M(K_{1,n})) \to V_4$ is an IML with $f(v) = x, f(v_j) = x_j$ and $f(u_j) = y_j$ for $j = 1, 2, 3, \ldots, n$.

Then the induced degree sum equation of the vertices $v_j$ in $M(K_{1,n})$, imply that $y_j = 0$.

Using the fact that $y_j = 0$ in the induced degree sum equation of the vertices $u_j$ and the vertex $v$ in $M(K_{1,n})$ we get $x + x_j = 0$ and $(n-1)x = 0$. But we have supposed that $n$ is an even number, therefore the equation $(n-1)x = 0$ reduces to $x = 0$, thus $x + x_j = 0$ implies $x_j = 0$. Thus we have $f \equiv 0$ and our assumption that $f$ is an IML is wrong. Therefore for $n$ even, $M(K_{1,n}) \notin \Gamma(V_4)$. Hence the proof. □

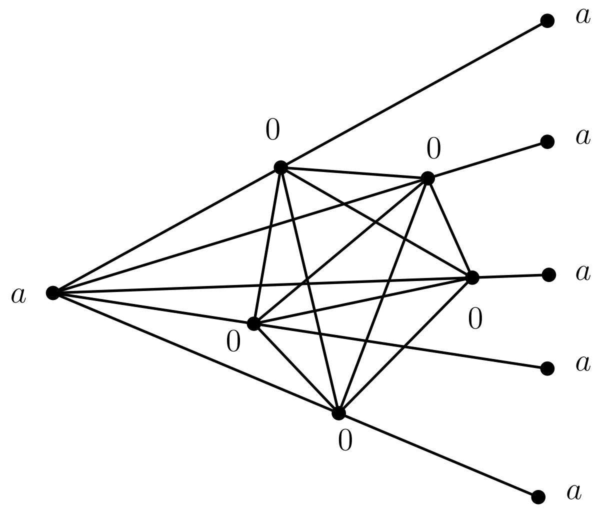


Figure 5.1: Middle graph of $K_{1,5}$

In the Figure 5.1 an induced $V_4$-magic labeling of the graph $M(K_{1,5})$ is given.

**Theorem 5.1.6.** *For the bistar $B_{m,n}$, $M(B_{m,n}) \in \Gamma(V_4)$ if and only if $m$ and $n$ are even.*

*Proof.* Let $\{u, u_1, u_2, u_3, \ldots, u_m, v, v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of $B_{m,n}$ with edge set $\{uv, uu_i, vv_j : i = 1, 2, 3, \ldots, m, \; j = 1, 2, 3, \ldots, n\}$. Let $\alpha_i$ be the inserted vertex on the edge $uu_i$, $\beta_j$ be the inserted vertex on the edge $vv_j$ and $\alpha$ be the inserted vertex on the edge $uv$.

Suppose $m$ and $n$ are even, then we have every vertex of $B_{m,n}$ is of odd degree, therefore by Theorem 5.1.2, we get $M(B_{m,n}) \in \Gamma(V_4)$.

To establish the converse part, suppose that both $m$ and $n$ are not even. If possible, suppose $f : V(M(B_{m,n})) \to V_4$ is an IML. Consider the induced degree sum equation of each vertices in $M(B_{m,n})$. Since $u_i$ and $v_j$ are pendant vertices of $M(B_{m,n})$, we get $f(\alpha_i) = f(\beta_j) = 0$. Since $f(\alpha_i) = f(\beta_j) = 0$, the induced degree sum equation of $\alpha_i$, $\beta_j$, $u$, $v$ and $\alpha$ imply that:

$$\begin{aligned}
f(u) + f(u_i) + f(\alpha) &= 0 \text{ for } i = 1, 2, 3, \ldots, m && (5.1)\\
f(v) + f(v_j) + f(\alpha) &= 0 \text{ for } j = 1, 2, 3, \ldots, n && (5.2)\\
mf(u) + f(\alpha) &= 0 && (5.3)\\
nf(v) + f(\alpha) &= 0 && (5.4)\\
f(u) + f(v) + (m + n - 1)f(\alpha) &= 0. && (5.5)
\end{aligned}$$

Now suppose the following cases:

**Case 1:** $m$ and $n$ are odd.

Note that if $m$ and $n$ are odd, then Equations (5.3) and (5.4) reduce to

$$\begin{aligned}
f(u) + f(\alpha) &= 0\\
f(v) + f(\alpha) &= 0.
\end{aligned}$$

That is $f(u) = f(v) = f(\alpha)$. Therefore, since $m$ and $n$ are odd Equation (5.5) reduces to $f(\alpha) = 0$. Therefore $f(u) = f(v) = f(\alpha) = 0$. Thus Equations (5.1) and (5.2) imply that $f(u_i) = f(v_j) = 0$. Hence in this case $f \equiv 0$.

**Case 2:** $m$ is odd and $n$ is even.

Note that if $m$ is odd and $n$ is even, then the Equations (5.3) and (5.4) reduce to

$$\begin{aligned} f(u) + f(\alpha) &= 0 \\ f(\alpha) &= 0. \end{aligned}$$

That is $f(u) = f(\alpha) = 0$. Therefore Equation (5.5) implies $f(v) = 0$. Thus Equations (5.1) and (5.2) imply that $f(u_i) = f(v_j) = 0$. Hence in this case $f \equiv 0$.

**Case 3:** $m$ is even and $n$ is odd.

Note that if $m$ is even and $n$ is odd, then Equations (5.3) and (5.4) reduce to

$$\begin{aligned} f(\alpha) &= 0 \\ f(v) + f(\alpha) &= 0. \end{aligned}$$

That is $f(v) = f(\alpha) = 0$. Therefore Equation (5.5) implies $f(u) = 0$. Thus Equations (5.1) and (5.2) imply that $f(u_i) = f(v_j) = 0$. Hence in this case $f \equiv 0$.

From the above three cases, we get $f \equiv 0$, which is a contradiction to our assumption. Hence the proof of the converse part follows. □

**Theorem 5.1.7.** *For the complete bipartite graph $K_{m,n}$, we have $M(K_{m,n}) \in \Gamma(V_4)$ for $m$ and $n$ are odd.*

*Proof.* Suppose $m$ and $n$ are odd integers. Then consider a complete bipartite graph $K_{m,n}$. If $m$ and $n$ are odd, then each vertex in $K_{m,n}$ has odd degree, therefore by Theorem 5.1.2, $M(K_{m,n}) \in \Gamma(V_4)$.

Hence the proof follows. □

**Theorem 5.1.8.** *For the complete graph $K_n$, we have $M(K_n) \in \Gamma(V_4)$ for $n$ is even.*

*Proof.* Suppose $n$ is an even number. Then note that every vertex is of odd degree in $K_n$, therefore by Theorem 5.1.2, $M(K_n) \in \Gamma(V_4)$ for $n$ even.
This completes the proof. □

**Theorem 5.1.9.** *$M(CB_n) \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of $CB_n$, where $v_i$ is the pendant vertex adjacent to $u_i$, for $i = 1, 2, 3, \ldots, n$. Also let $\{\alpha_i : i = 1, 2, 3, \ldots, n\}$ and $\{\beta_i : i = 1, 2, 3, \ldots, n-1\}$ be the inserted vertices on the edges $u_iv_i$ for $i = 1, 2, 3, \ldots, n$ and $u_iu_{i+1}$ for $i = 1, 2, 3, \ldots, n-1$ respectively in the graph $M(CB_n)$. Suppose $n$ is an odd integer. Define $f : V(M(CB_n)) \to V_4$ as follows:

$$f(v) = \begin{cases} a & \text{if} \quad v = u_1, u_3, u_5, \ldots, u_{n-2}, u_n \\ 0 & \text{if} \quad v = u_2, u_4, u_6, \ldots, u_{n-3}, u_{n-1} \\ 0 & \text{if} \quad v = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \\ a & \text{if} \quad v = \beta_1, \beta_2, \beta_3, \ldots, \beta_{n-1} \\ 0 & \text{if} \quad v = v_1, v_2, v_4, v_6, \ldots, v_{n-3}, v_{n-1}, v_n \\ a & \text{if} \quad v = v_3, v_5, v_7, \ldots, v_{n-4}, v_{n-2}. \end{cases}$$

Then we can easily prove that $f \equiv f^{**}$. Therefore $f$ is an IML of $M(CB_n)$.

Conversely, suppose that $n$ is an even number. If possible, suppose that $g : V(M(W_n)) \to V_4$ is an IML. Then the induced degree sum equation of $v_i$ gives

$$g(\alpha_1) = g(\alpha_2) = g(\alpha_3) = \cdots = g(\alpha_n) = 0. \tag{5.6}$$

The induced degree sum equation of $\alpha_i$ gives

$$g(u_1) + g(v_1) + g(\beta_1) \quad = \quad 0$$

$$\begin{aligned}
g(u_2)+g(v_2)+g(\beta_1)+g(\beta_2) &= 0\\
g(u_3)+g(v_3)+g(\beta_2)+g(\beta_3) &= 0\\
&\vdots\\
g(u_{n-1})+g(v_{n-1})+g(\beta_{n-2})+g(\beta_{n-1}) &= 0\\
g(u_n)+g(v_n)+g(\beta_{n-1}) &= 0.
\end{aligned}$$

The induced degree sum equation of $u_i$ gives

$$\begin{aligned}
g(u_1)+g(\beta_1) &= 0\\
g(\beta_1)+g(\beta_2) &= 0\\
g(\beta_2)+g(\beta_3) &= 0\\
&\vdots\\
g(\beta_{n-2})+g(\beta_{n-1}) &= 0\\
g(\beta_{n-1})+g(u_n) &= 0.
\end{aligned}$$

The induced degree sum equation of $\beta_i$ gives

$$\begin{aligned}
g(u_1)+g(u_2)+g(\beta_2) &= 0\\
g(u_2)+g(u_3)+g(\beta_1)+g(\beta_2)+g(\beta_3) &= 0\\
g(u_3)+g(u_4)+g(\beta_2)+g(\beta_3)+g(\beta_4) &= 0\\
&\vdots\\
g(u_{n-2})+g(u_{n-1})+g(\beta_{n-3})+g(\beta_{n-2})+g(\beta_{n-1}) &= 0\\
g(u_{n-1})+g(u_n)+g(\beta_{n-2}) &= 0.
\end{aligned}$$

Using the above system of equations one can easily prove that $g \equiv 0$, which is a contradiction. Thus $M(CB_n) \notin \Gamma(V_4)$ for $n$ is even. Hence the proof. □

**Theorem 5.1.10.** *Let $C_n$ be the cycle with $n$ vertices, then $M(C_n) \notin \Gamma(V_4)$.*

*Proof.* Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of $C_n$ and $u_i$ be the inserted vertex on the edge $v_iv_{i+1}$ in $M(C_n)$. If possible, suppose $f : V(M(C_n)) \to V_4$ is an IML with $f(v_k) = x_k$ and $f(u_k) = y_k$, for $k = 1, 2, 3, \ldots, n$. Then by the induced degree sum equation of each vertex in $M(C_n)$, we have the following set of equations.

The induced degree sum equations of $v_1, v_2, v_3, \ldots, v_n$ in $M(C_n)$ give

$$\begin{aligned} y_n + y_1 + x_1 &= 0 \\ y_1 + y_2 + x_2 &= 0 \\ y_2 + y_3 + x_3 &= 0 \\ &\vdots \\ y_{n-1} + y_n + x_n &= 0. \end{aligned}$$

Also the induced degree sum equations of $u_1, u_2, u_3, \ldots, u_n$ in $M(C_n)$ imply that

$$\begin{aligned} y_n + y_1 + y_2 + x_1 + x_2 &= 0 \\ y_1 + y_2 + y_3 + x_2 + x_3 &= 0 \\ y_2 + y_3 + y_4 + x_3 + x_4 &= 0 \\ &\vdots \\ y_{n-1} + y_n + y_1 + x_n + x_1 &= 0. \end{aligned}$$

By substituting the first set of equations in the second, we get the following equations: $x_k + y_k = 0$ for $k = 1, 2, 3, \ldots, n$, which implies $x_k = y_k$, for $k = 1, 2, 3, \ldots, n$. Thus from the first set of equations, we get $y_k = 0$, for $k = 1, 2, 3, \ldots, n$ which implies $x_k = 0$, for $k = 1, 2, 3, \ldots, n$. Thus we have $f \equiv 0$. Therefore our assumption is wrong. Hence $M(C_n) \notin \Gamma(V_4)$. $\square$

**Theorem 5.1.11.** *$M(W_n) \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Let $V(W_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $w$ is the central vertex and $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of the corresponding $n-$ cycle. Let $\{w_k : k = 1, 2, 3, \ldots, n\}$ be the inserted vertices on the edges $wv_k$, and $\{u_k : k = 1, 2, 3, \ldots, n\}$ be the inserted vertices on the edges $v_k v_{k+1}$ for $k = 1, 2, 3, \ldots, n$ where $k+1$ is taken modulo $n$ in the graph $M(W_n)$.

Suppose $n$ is odd, then every vertex of $W_n$ is of odd degree, therefore by Theorem 5.1.2, $M(W_n)$ is an induced magic graph.

Conversely, suppose $n$ is an even number. If possible, suppose $f : V(M(W_n)) \to V_4$ is an IML of $M(W_n)$. Let $f(v_k) = x_k$, $f(u_k) = y_k$, $f(w_k) = z_k$ and $f(w) = x$.

Then the induced degree sum equation of $v_k$, $u_k$, $w_k$ and $w$ gives

$$y_{k-1} + y_k + z_k = 0. \quad (5.7)$$

$$y_{k-1} + y_k + y_{k+1} + x_k + x_{k+1} + z_k + z_{k+1} = 0. \quad (5.8)$$

$$x + x_k + y_{k-1} + y_k + z_1 + z_2 + \cdots + z_{k-1} + z_{k+1} + \cdots + z_n = 0. \quad (5.9)$$

$$x + z_1 + z_2 + z_3 + \cdots + z_n = 0. \quad (5.10)$$

Note that Equation (5.9) and (5.10) imply that

$$x_k + y_{k-1} + y_k + z_k = 0. \quad (5.11)$$

Thus the Equation (5.7) and (5.11) imply that $x_k = 0$.
Therefore Equation (5.8) and (5.11) imply that $y_k = z_k$.
Thus Equation (5.7) again implies that $y_k = 0$, therefore $z_k = 0$ and Equation (5.10) imply that $x = 0$. Thus we have $x_k = y_k = z_k = x = 0$. Hence $f \equiv 0$, therefore $f$ is not an IML. Since $f$ is arbitrary, there exists no such IML for $M(W_n)$. Hence $M(W_n) \notin \Gamma(V_4)$, for $n$ is even. □

**Theorem 5.1.12.** *$M(H_n) \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* Let $V(H_n) = \{w, v_i, u_i : i = 1, 2, 3, \ldots, n\}$, where $w$ be the central vertex and $u_1, u_2, u_3, \ldots, u_n$ be the pendant vertices adjacent to $v_1, v_2, v_3, \ldots, v_n$. Also let $w_k$ be the inserted vertices on the edges $wv_k$, $\alpha_k$ be the inserted vertices on the edges $v_kv_{k+1}$ and $\beta_k$ be the inserted vertices on the edges $v_ku_k$ for $k = 1, 2, 3, \ldots, n$, where $k+1$ is taken modulo $n$.
Suppose $n$ is odd. Define $f : V(M(H_n)) \to V_4$ as follows.

$$f(v) = \begin{cases} a & \text{if} \quad v = v_k, w_k, \quad \text{for } k = 2, 3, \ldots, n \\ 0 & \text{if} \quad v = v_1, w_1, w, u_k, \alpha_k, \beta_k. \end{cases}$$

Then we can easily prove that $f$ is an IML of $M(H_n)$.
Conversely, suppose that $n$ is an even number. If possible, suppose $g : V(M(H_n)) \to V_4$ is an IML. Then the induced degree sum equation of $u_k$, $\beta_k$, $v_k$,$w_k$, $w$ and $\alpha_k$ give

$$g(\beta_k) = 0. \tag{5.12}$$

$$g(v_k) + g(u_k) + g(\alpha_{k-1}) + g(\alpha_k) + g(w_k) = 0. \tag{5.13}$$

$$g(\alpha_{k-1}) + g(\alpha_k) + g(w_k) + g(v_k) = 0. \tag{5.14}$$

$$\sum_{i=1}^{n} g(w_i) + g(\alpha_{k-1}) + g(\alpha_k) + g(v_k) + g(w) = 0. \tag{5.15}$$

$$g(w) + \sum_{i=1}^{n} g(w_i) = 0. \tag{5.16}$$

$$\begin{aligned} g(v_k) + g(v_{k+1}) + g(w_k) + g(w_{k+1}) & \\ + g(\alpha_{k-1}) + g(\alpha_k) + g(\alpha_{k+1}) &= 0. \end{aligned} \tag{5.17}$$

Note that Equation (5.13) and Equation (5.14) imply that $g(u_k) = 0$ and Equation (5.15) and Equation (5.16) imply that

$$g(\alpha_{k-1}) + g(\alpha_k) + g(v_k) = 0. \tag{5.18}$$

Therefore Equation (5.14) implies that $g(w_k) = 0$. Thus Equation (5.16) again implies that $g(w) = 0$. Since $g(\alpha_{k-1})+g(\alpha_k)+g(v_k) = 0$ and $g(w_k) = 0$, Equation (5.17) implies that $g(v_k) = g(\alpha_k)$. Thus Equation (5.18) implies that $g(\alpha_k) = 0$, therefor $g(v_k) = 0$. Thus we have $g(\beta_k) = g(u_k) = g(w_k) = g(w) = g(\alpha_k) = g(v_k) = 0$. Hence $g \equiv 0$. Since the function $g$ is arbitrary, we get $M(H_n) \notin \Gamma(V_4)$, for $n$ is even. Hence the Proof. □

**Theorem 5.1.13.** *$M(Fl_n) \in \Gamma(V_4)$ for all $n$.*

*Proof.* Let $V(Fl_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding cycle graph $C_n$ and $w$ is the root vertex adjacent to the vertex $v_1$. Also let $w_1$ be the inserted vertex on the edge $v_1w$ and $u_1, u_2, u_3, \ldots, u_n$ be the inserted vertices on the edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_nv_1$ respectively in the graph $M(Fl_n)$.

Then define $f : V(M(Fl_n)) \to V_4$ as follows:

$$f(v) = \begin{cases} a & \text{for } v = u_1, u_n, v_2, v_n \\ 0 & \text{otherwise.} \end{cases}$$

Then we can easily prove that $f$ is an IML of $M(Fl_n)$.

Hence the proof. □

**Theorem 5.1.14.** *$M(Sun_n) \in \Gamma(V_4)$ for all $n$.*

*Proof.* Note that in a sun graph $Sun_n$, every vertex is of odd degree, therefore by Theorem 5.1.2, $M(Sun_n) \in \Gamma(V_4)$ for all $n$. □

## 5.2 Line Graphs

**Definition 5.2.1.** [12] *Let $G$ be a graph, then the line graph of $G$ is denoted by $L(G)$ and it is a graph whose vertex set is in $1-1$ correspondence with the* *edge set of $G$ and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges of $G$ are adjacent in $G$.*

**Theorem 5.2.2.** *$L(C_n) \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$.*

*Proof.* We know that $L(C_n) = C_n$ and by Theorem 2.4.1, we have $C_n \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$. Hence the proof. □

**Theorem 5.2.3.** *For $n > 1, L(P_n) \in \Gamma(V_4)$ if and only if $n \equiv 1 \pmod 3$.*

*Proof.* We know that $L(P_n) = P_{n-1}$ and by Theorem 3.2.1, we have $P_n \in \Gamma(V_4)$ if and only if $n \equiv 0 \pmod 3$. Hence $L(P_n) = P_{n-1} \in \Gamma(V_4)$ if and only if $n - 1 \equiv 0 \pmod 3$. Hence the proof. □

**Theorem 5.2.4.** *For $n > 1, L(K_{1,n}) \in \Gamma(V_4)$ if and only if $n$ is odd.*

*Proof.* We know that $L(K_{1,n}) = K_n$ and by Theorem 3.3.1, we have $K_n \in \Gamma(V_4)$ if and only if $n$ is odd. Hence $L(K_{1,n}) = K_n \in \Gamma(V_4)$ if and only if $n$ is odd. Hence the proof. □

**Theorem 5.2.5.** *For the bistar graph $B_{m,n}$ we have $L(B_{m,n}) \in \Gamma(V_4)$ for all $m$ and $n$.*

*Proof.* Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\}$, where $e = uv$, $\alpha_i = vv_i,\ \beta_j = uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. Then we have $V(L(B_{m,n})) = \{e, \alpha_i, \beta_j :\ i = 1, 2, 3, \ldots, m,\ j = 1, 2, 3, \ldots, n\}$.

**Case 1:** $m$ is odd and $n$ is even.

In this case, define $f : V(L(B_{m,n})) \to V_4$ as follows.

$$f(u) = \begin{cases} 0 & \text{if} \quad u = e, \alpha_i, \ \text{for } i = 1, 2, 3, \ldots, m \\ a & \text{if} \quad u = \beta_j, \qquad \text{for } j = 1, 2, 3, \ldots, n. \end{cases}$$

**Case 2:** $m$ is even and $n$ is odd.

In this case, define $g : V(L(B_{m,n})) \to V_4$ as follows:

$$g(u) = \begin{cases} 0 & \text{if} \quad u = e, \beta_j, \text{ for } j = 1, 2, 3, \ldots, n \\ a & \text{if} \quad u = \alpha_i, \quad \text{for } i = 1, 2, 3, \ldots, m. \end{cases}$$

**Case 3:** $m$ and $n$ are odd.

In this case, define $h : V(L(B_{m,n})) \to V_4$ as follows:

$$h(u) = \begin{cases} 0 & \text{if} \quad u = e \\ a & \text{if} \quad u = \beta_j, \quad \text{for } j = 1, 2, 3, \ldots, n \\ a & \text{if} \quad u = \alpha_i, \quad \text{for } i = 1, 2, 3, \ldots, m. \end{cases}$$

**Case 4:** $m$ and $n$ are even.

In this case, define $k : V(L(B_{m,n})) \to V_4$ as follows:

$$k(u) = \begin{cases} 0 & \text{if} \quad u = e \\ a & \text{if} \quad u = \beta_j, \quad \text{for } j = 1, 2, 3, \ldots, n \\ a & \text{if} \quad u = \alpha_i, \quad \text{for } i = 1, 2, 3, \ldots, m. \end{cases}$$

Then we can easily prove that the vertex labeling functions $f, g, h$ and $k$ are IML for $L(B_{m,n})$. Thus for all $m$ and $n$, we have $L(B_{m,n})$ is an induced magic graph. Hence the proof. □

**Theorem 5.2.6.** *For the sun graph $Sun_n$, we have $L(Sun_n) \notin \Gamma(V_4)$ for any $n$.*

*Proof.* Consider a sun graph $Sun_n$ with $\{v_1, v_2, v_3, \ldots, v_n\}$ as vertex set of the corresponding $C_n$ and $w_i,\ 1 \leq i \leq n$, be the pendant vertices attached to each $v_i,\ 1 \leq i \leq n$. Let $\alpha_i = v_i v_{i+1}$ and $\beta_i = v_i w_i$ be the edges in $Sun_n$. Then the vertices of $L(Sun_n)$ is given by $\alpha_i = v_i v_{i+1}$ and $\beta_i = v_i w_i$ for $i = 1, 2, 3, \ldots, n$ and $i + 1$ is taken modulo $n$.

If possible, suppose there exists an IML of $L(Sun_n)$ say $f : V(L(Sun_n)) \to V_4$. Then the induced degree sum equation of the vertices $\alpha_i$ and $\beta_i$ are given by:

$$f(\alpha_{i-1}) + f(\alpha_i) + f(\alpha_{i+1}) + f(\beta_i) + f(\beta_{i+1}) = 0. \tag{5.19}$$

$$f(\alpha_{i-1}) + f(\alpha_i) + f(\beta_i) = 0. \tag{5.20}$$

Substituting Equation (5.20) in (5.19) we get $f(\alpha_i) = f(\beta_i)$ for $i = 1, 2, 3, \ldots, n$. Therefore Equation (5.20) again implies that $f(\alpha_i) = 0$ for $i = 1, 2, 3, \ldots, n$. Thus we have $f \equiv 0$. Since the function $f$ is arbitrary, we get $L(sun_n) \notin \Gamma(V_4)$ for any $n$. Hence the proof. □

**Theorem 5.2.7.** *For the comb graph $CB_n$, we have $L(CB_n) \notin \Gamma(V_4)$ for any $n$.*

*Proof.* Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of $CB_n$, where $v_i$ $(1 \leq i \leq n)$ are the pendant vertices adjacent to $u_i$ $(1 \leq i \leq n)$. Also let $\alpha_i = u_i v_i,\ i = 1, 2, 3, \ldots, n$ and $\beta_k = u_k u_{k+1},\ \ k = 1, 2, 3, \ldots, n-1$ be the edges in $CB_n$. Then $\{\alpha_i,\ \beta_k\ :\ i = 1, 2, 3, \ldots, n,\ k = 1, 2, 3, \ldots, n-1\}$ is the vertex set of $L(CB_n)$. If possible, suppose $f$ is an IML of the graph $L(CB_n)$. Then from the induced degree sum equation of the vertices $\alpha_i$, we get the following system of equations.

$$\begin{aligned}
f(\beta_1) &= 0 \\
f(\beta_1) + f(\beta_2) + f(\alpha_2) &= 0 \\
f(\beta_2) + f(\beta_3) + f(\alpha_3) &= 0 \\
f(\beta_3) + f(\beta_4) + f(\alpha_4) &= 0 \\
&\vdots \\
f(\beta_{n-2}) + f(\beta_{n-1}) + f(\alpha_{n-1}) &= 0 \\
f(\beta_{n-1}) &= 0.
\end{aligned}$$

Similarly, from the induced degree sum equation of the vertices $\beta_k$, we get the following system of equations.

$$
\begin{aligned}
f(\alpha_1)+f(\alpha_2)+f(\beta_2) &= 0\\
f(\alpha_2)+f(\alpha_3)+f(\beta_1)+f(\beta_2)+f(\beta_3) &= 0\\
f(\alpha_3)+f(\alpha_4)+f(\beta_2)+f(\beta_3)+f(\beta_4) &= 0\\
f(\alpha_4)+f(\alpha_5)+f(\beta_3)+f(\beta_4)+f(\beta_5) &= 0\\
&\vdots\\
f(\alpha_{n-2})+f(\alpha_{n-1})+f(\beta_{n-3})+f(\beta_{n-2})+f(\beta_{n-1}) &= 0\\
f(\alpha_{n-1})+f(\alpha_n)+f(\beta_{n-2}) &= 0.
\end{aligned}
$$

By solving the above two system of equations one can easily prove that $f(\alpha_i)=0$ and $f(\beta_k)=0$. That is $f\equiv 0$, which is a contradiction to our assumption. Thus $L(CB_n)\notin\Gamma(V_4)$ for any $n$. Hence the proof. $\square$

**Theorem 5.2.8.** *For the wheel graph $W_n$, we have $L(W_n)\notin\Gamma(V_4)$ for $n$ is odd.*

*Proof.* Let $V(W_n)=\{w,v_1,v_2,v_3,\ldots,v_n\}$, where $w$ is the central vertex. Also let $\alpha_i=wv_i$ and $\beta_i=v_iv_{i+1}$ for $i=1,2,3,\ldots,n$ be the edges in $W_n$, where $i+1$ is taken modulo $n$. Then we have $V(L(W_n))=\{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n,\beta_1,\beta_2,\beta_3,\ldots,\beta_n\}$.

Suppose $n$ is odd. If possible suppose $f:V(L(W_n))\to V_4$ be an IML of $L(W_n)$.

Then from the induced degree sum equation of the vertices $\beta_i$, we get,

$$
\begin{aligned}
f(\alpha_n)+f(\alpha_1)+f(\beta_n)+f(\beta_1)+f(\beta_2) &= 0\\
f(\alpha_1)+f(\alpha_2)+f(\beta_1)+f(\beta_2)+f(\beta_3) &= 0\\
f(\alpha_2)+f(\alpha_3)+f(\beta_2)+f(\beta_3)+f(\beta_4) &= 0\\
&\vdots\\
f(\alpha_{n-1})+f(\alpha_n)+f(\beta_{n-1})+f(\beta_n)+f(\beta_1) &= 0.
\end{aligned}
$$

Similarly, since $n$ is odd, from the induced degree sum equation of the vertices $\alpha_i$, we get,

$$\begin{aligned}
f(\alpha_1)+f(\alpha_2)+f(\alpha_3)+\cdots+f(\alpha_n)+f(\beta_1)+f(\beta_2) &= 0\\
f(\alpha_1)+f(\alpha_2)+f(\alpha_3)+\cdots+f(\alpha_n)+f(\beta_2)+f(\beta_3) &= 0\\
f(\alpha_1)+f(\alpha_2)+f(\alpha_3)+\cdots+f(\alpha_n)+f(\beta_3)+f(\beta_4) &= 0\\
&\vdots\\
f(\alpha_1)+f(\alpha_2)+f(\alpha_3)+\cdots+f(\alpha_n)+f(\beta_n)+f(\beta_1) &= 0.
\end{aligned}$$

Since $n$ is odd, the above system of equations implies that

$$f(\beta_1)=f(\beta_2)=f(\beta_3)=\cdots=f(\beta_n)=\beta \text{ (say)}.$$

Therefore the above two system of equations reduce to,

$$\begin{aligned}
f(\alpha_n)+f(\alpha_1)+\beta &= 0\\
f(\alpha_1)+f(\alpha_2)+\beta &= 0\\
f(\alpha_2)+f(\alpha_3)+\beta &= 0\\
&\vdots\\
f(\alpha_{n-1})+f(\alpha_n)+\beta &= 0.
\end{aligned}$$

and $f(\alpha_1)+f(\alpha_2)+f(\alpha_3)+\cdots+f(\alpha_n)=0$.

Since $n$ is odd, the above system of equations implies that $f(\alpha_1)=f(\alpha_2)=f(\alpha_3)=\cdots=f(\alpha_n)=\alpha$ (say) and $\alpha=0$. That is $f(\alpha_1)=f(\alpha_2)=f(\alpha_3)=\cdots=f(\alpha_n)=0$. Using this fact in the last system of equations we get $\beta=0$. That is $f(\beta_1)=f(\beta_2)=f(\beta_3)=\cdots=f(\beta_n)=0$. Thus $f\equiv 0$, which is not admissible. Thus there exist no such IML for $L(W_n)$.

Hence the proof. $\square$

**Theorem 5.2.9.** *For the triangular snake graph $TS_n$ we have $L(TS_n) \in \Gamma(V_4)$ for all $n$.*

*Proof.* Let $V(TS_n) = \{v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_{n-1}\}$, where $v_i'$s are the vertices of corresponding path $P_n$. Also let $\alpha_i = v_i w_i$, $\beta_i = v_i v_{i+1}$ and $\gamma_i = v_{i+1} w_i$, where $i = 1, 2, 3, \ldots, n-1$ be the edges in $TS_n$. Then $V(L(TS_n))$ consists of the vertices $\alpha_i$, $\beta_i$ and $\gamma_i$ for $i = 1, 2, 3, \ldots, n-1$. Define $f : V(L(TS_n)) \to V_4$ as follows:

$$f(u) = \begin{cases} a & \text{if} \quad u = \beta_1, \beta_{n-1}, \gamma_1, \alpha_{n-1} \\ 0 & \text{if} \quad u = \beta_i, \quad \text{for} \quad i = 2, 3, 4, \ldots, n-2 \\ 0 & \text{if} \quad u = \gamma_i, \quad \text{for} \quad i = 2, 3, 4, \ldots, n-1 \\ 0 & \text{if} \quad u = \alpha_i, \quad \text{for} \quad i = 1, 2, 3, \ldots, n-2. \end{cases}$$

Then we can prove that $f$ is an IML of $L(TS_n)$. Thus $L(TS_n) \in \Gamma(V_4)$ for all $n$. Hence the proof. □

**Theorem 5.2.10.** *For the gear graph $G_n$, we have $L(G_n) \in \Gamma(V_4)$ for all $n$.*

*Proof.* Let $V(G_n) = \{w, u_i, v_i : i = 1, 2, 3, \ldots, n\}$, where $w$ is the central vertex, $u_1, u_2, u_3, \ldots, u_n$ are the vertices of the corresponding wheel graph $W_n$ and $v_1, v_2, v_3, \ldots, v_n$ are the remaining vertices with $\alpha_i = u_i v_i$, $\beta_i = v_i u_{i+1}$, $\gamma_i = w u_i \in E(G_n)$, where $i+1$ is taken modulo $n$. Then the vertex set of $L(G_n)$ consists of the vertices $\alpha_i = u_i v_i$, $\beta_i = v_i u_{i+1}$ and $\gamma_i = w u_i$ for $i = 1, 2, 3, \ldots, n$, where $i+1$ is taken modulo $n$. Define $f : V(L(G_n)) \to V_4$ as follows:

$$f(u) = \begin{cases} a & \text{if} \quad u = \alpha_i, \quad \text{for} \quad i = 1, 2, 3, \ldots, n \\ a & \text{if} \quad u = \beta_i, \quad \text{for} \quad i = 1, 2, 3, \ldots, n \\ 0 & \text{if} \quad u = \gamma_i, \quad \text{for} \quad i = 1, 2, 3, \ldots, n. \end{cases}$$

Then we can easily prove that $f$ is an IML of $L(G_n)$. Thus $L(G_n) \in \Gamma(V_4)$ for all $n$. Hence the proof. □

**Theorem 5.2.11.** *Consider the flag graph $Fl_n$. Then $L(Fl_n) \in \Gamma(V_4)$ if and only if $n \equiv 2 \pmod 3$.*

*Proof.* Let $V(Fl_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding cycle graph $C_n$ and $w$ is the root vertex adjacent to the vertex $v_1$. Also suppose $e_i = v_i v_{i+1}$ and $e = v_1 w$ are the edges in $Fl_n$. Therefore we can take the vertex set of $L(Fl_n)$ as $\{e, e_1, e_2, e_3, \ldots, e_n\}$.
Suppose $n \equiv 2 \pmod 3$, then define $f : V(L(Fl_n)) \to V_4$ as :

$$f(v) = \begin{cases} 0 & \text{if} \quad v = e_i,\ i \equiv 0 \pmod 3 \\ a & \text{if} \quad v = e_i,\ i \equiv 1, 2 \pmod 3 \\ 0 & \text{if} \quad v = e. \end{cases}$$

Then we can easily verify that this $f$ is an induced $V_4$ magic labeling of $L(Fl_n)$.
To prove the converse part, consider the case $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$.
If possible, suppose there exists an IML say $g : V(L(Fl_n)) \to V_4$. Then from the induced degree sum equation of the vertices $e$ and $e_i$, we have the following set of equations:

$$\begin{aligned} g(e_1) + g(e_n) + g(e) &= 0 \\ g(e_n) + g(e_2) + g(e) &= 0 \\ g(e_1) + g(e_2) + g(e_3) &= 0 \\ g(e_2) + g(e_3) + g(e_4) &= 0 \\ &\vdots \\ g(e_{n-2}) + g(e_{n-1}) + g(e_n) &= 0 \\ g(e_{n-1}) + g(e_1) + g(e) &= 0. \end{aligned}$$

Now consider the following cases

**Case (i)** $n \equiv 0 \pmod 3$.

By solving the above system, we get $g(e_1) = g(e_2) = g(e_4) = g(e_5) = g(e_7) = \cdots = g(e_{n-1})$ and $g(e_3) = g(e_6) = g(e_9) = \cdots = g(e_n) = 0$. Similarly $g(e_n) = g(e_{n-1}) = g(e_{n-3}) = g(e_{n-4}) = g(e_{n-6}) = \cdots = g(e_3) = g(e_2)$ and $g(e_{n-2}) = g(e_{n-5}) = \cdots = g(e_4) = g(e_1) = 0$. That is $g(e_i) = 0$, for $i = 1, 2, 3, \ldots, n$ and hence $g(e) = 0$.

Therefore in this case, $g \equiv 0$.

**Case(ii)** $n \equiv 1 \pmod 3$.

By solving the above system, we get $g(e_1) = g(e_2) = g(e_4) = g(e_5) = \cdots = g(e_{n-2}) = g(e_n)$ and $g(e_3) = g(e_6) = g(e_9) = \cdots = g(e_{n-1}) = 0$. Similarly $g(e_n) = g(e_{n-1}) = g(e_{n-3}) = g(e_{n-4}) = g(e_{n-6}) = \cdots = g(e_3) = g(e_1)$ and $g(e_{n-2}) = g(e_{n-5}) = \cdots = g(e_5) = g(e_2) = 0$. That is $g(e_i) = 0$, for $i = 1, 2, 3, \ldots, n$ and hence $g(e) = 0$.

Thus in this case, $g \equiv 0$.

Thus in both the cases we proved that $g$ is not an IML of $L(Fl_n)$. Thus $L(Fl_n) \in \Gamma(V_4)$ if and only if $n \equiv 2 \pmod 3$.

Hence the proof. $\square$

Chapter 6

# Edge Induced $V_4-$ Magic Labeling of Graphs

*This chapter brings to light a new concept of labeling, and we call it as edge induced $V_4$-magic labeling of Graphs. In the first section, we give the definition of the idea and in the second section we give some main results about it. Third and fourth sections deal with edge induced $V_4$-magic labeling of some graphs and special graphs respectively.*

## 6.1 Introduction

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0 and $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \to V_4$ denote the induced vertex labeling of $f$ defined by $f^+(u) = \sum\limits_{uv \in E(G)} f(uv)$ for all $u \in (V(G)$. Then $f^+$ again induces an edge labeling $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$. Then a graph $G = (V(G), E(G))$ is said to be an edge induced

$V_4$-magic graph or simply edge induced magic graph if $f^{++}(e)$ is a constant for all $e \in E(G)$. If this constant is $x$, then $x$ is said to be the induced edge sum of the graph $G$. The function $f$ so obtained is called an edge induced $V_4$-magic labeling of $G$ or simply edge induced magic labeling of $G$ and it is denoted by EIM$V_4$L or simply EIML.

Figure 6.1 and Figure 6.2 represent edge induced $V_4$ magic labeling of graphs $G_1$ and $G_2$ with induced edge sums 0 and $a$ respectively.

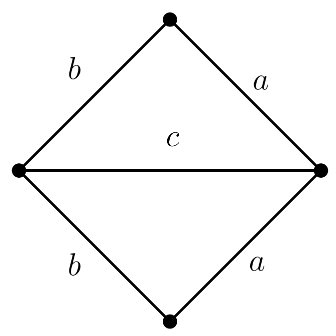


Figure 6.1: Graph $G_1$

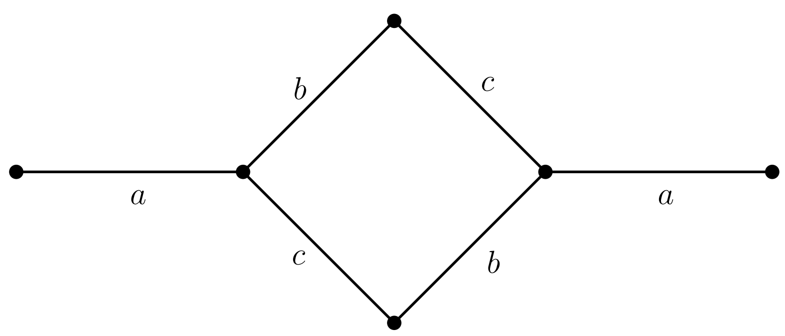


Figure 6.2: Graph $G_2$

This chapter discusses the concept of edge induced $V_4$-magic labeling of some graphs which belongs to the following categories:

(i) $\sigma_a(V_4)$ := Set of all edge induced $V_4$-magic graphs with edge induced magic labeling $f$ satisfying $f^{++}(u) = a$ for all $u \in V$.

(ii) $\sigma_0(V_4)$ := Set of all edge induced $V_4$-magic graphs with edge induced magic labeling $f$ satisfying $f^{++}(u) = 0$ for all $u \in V$.

(iii) $\sigma(V_4) := \sigma_a(V_4) \bigcap \sigma_0(V_4)$.

## 6.2 Main Results

**Theorem 6.2.1.** *Let $G = (V, E)$ be a graph with either each vertex is of odd degree or even degree then $G \in \sigma_0(V_4)$.*

*Proof.* Let $G$ be a graph with $deg(v_i) = r_i$ for $v_i \in V, \quad i = 1, 2, 3, \ldots, n$.

**Case 1:** $r_i$ is odd.

In this case, define $f : E \to V_4 \setminus \{0\}$ as $f(e) = a$ for all $e \in E$. Then $f^+(u_i) = deg\ (u_i)a = r_i a = a$. Thus $f^{++}(e) = 0$ for all $e \in E$.

**Case 2:** $r_i$ is even.

In this case, define $f : E \to V_4 \setminus \{0\}$ as $f(e) = a$ for all $e \in E$. Then $f^+(u_i) = deg\ (u_i)a = r_i a = 0$. Thus $f^{++}(e) = 0$ for all $e \in E$.

Thus in both cases $f^{++} \equiv 0$. Therefore $G \in \sigma_0(V_4)$.
Hence the proof. □

**Theorem 6.2.2.** *Let $G = (V, E)$ be a magic graph. Then $G \in \sigma_0(V_4)$.*

*Proof.* Suppose $G = (V, E)$ is a magic graph. then there exists a magic labeling say $f : E \to V_4 \setminus \{0\}$ such that $f^+ : V \to V_4$ is a constant function. Then $f^{++} \equiv 0$. Hence $G \in \sigma_0(V_4)$. □

**Theorem 6.2.3.** *Let $G = (V, E)$ be a graph with $uv \in E$ and $f : E \to V_4 \setminus \{0\}$ be an edge label of $G$ then $f^{++}(uv) = \sum_{u\alpha \in E} f(u\alpha) + \sum_{\beta v \in E} f(\beta v)$, where $\alpha \neq v$ and $\beta \neq u$.*

*Proof.* Let $f : E \to V_4 \setminus \{0\}$ be an edge label of $G$, then $f^{+}(u) = \sum_{u\alpha \in E} f(u\alpha)$ for all $u \in V$. Thus we have:

$$\begin{aligned}
f^{++}(uv) &= f^{+}(u) + f^{+}(v) \\
&= \sum_{u\alpha \in E} f(u\alpha) + \sum_{\beta v \in E} f(\beta v) \\
&= \sum_{u\alpha \in E, v \neq \alpha} f(u\alpha) + f(uv) + \sum_{\beta v \in E, u \neq \beta} f(\beta v) + f(uv) \\
&= \sum_{u\alpha \in E, v \neq \alpha} f(u\alpha) + \sum_{\beta v \in E, u \neq \beta} f(\beta v), \quad \text{(Since } f(uv) \in V_4\text{).}
\end{aligned}$$

□

**Theorem 6.2.4. Induced edge sum theorem.**
*For any graph $G$, $f$ is an edge induced $V_4$-Magic labeling of $G$ if and only if the induced edge sum*

$$x = f^{++}(uv) = \sum_{u\alpha \in E,\ \alpha \neq v} f(u\alpha) + \sum_{\beta v \in E,\ \beta \neq u} f(\beta v),\ \textit{for all } (u, v) \in E \qquad (6.1)$$

The Equation (6.1) corresponding to an edge $uv$ in $G$, is called induced edge sum equation of the edge $uv$.

*Proof.* Proof follows from the definition of edge induced magic labeling and Theorem 6.2.3. □

## 6.3 Edge Induced $V_4$ Magic Labeling of Some Graphs

**Theorem 6.3.1.** *$P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$.*

*Proof.* Consider the path $P_2 : v_1e_1v_2$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_1) = x$, for some $x \in V_4 \setminus \{0\}$. Then $f^+(v_1) = f^+(v_2) = x$. Therefore $f^{++}(e_1) = 0$. Hence $P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$. □

**Corollary 6.3.2.** *$P_2 \notin \sigma(V_4)$.*

*Proof.* Proof follows from Theorem 6.3.1. □

**Theorem 6.3.3.** *$P_3 \in \sigma_a(V_4)$ and $P_3 \notin \sigma_0(V_4)$.*

*Proof.* Consider the path $P_3 : v_1e_1v_2e_2v_3$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_1) = x_1, f(e_2) = x_2$ for some $x_1, x_2 \in V_4 \setminus \{0\}$. Then $f^+(v_1) = x_1, f^+(v_2) = x_1 + x_2, f^+(v_3) = x_2$. Therefore $f^{++}(e_1) = x_2, f^{++}(e_2) = x_1$. Then $P_3 \in \sigma_0(V_4)$ or $P_3 \in \sigma_a(V_4)$ accordingly $x_1 = x_2 = 0$ or $x_1 = x_2 = a$. Since $x_1, x_2 \in V_4 \setminus \{0\}$, $x_1 = x_2 = 0$ is not possible. Therefore $P_2 \notin \sigma_0(V_4)$. Therefore if we take $x_1 = x_2 = a$ then $f$ is an EIML of $P_3$. Thus $P_3 \in \sigma_a(V_4)$. Hence the proof. □

**Corollary 6.3.4.** *$P_3 \notin \sigma(V_4)$.*

*Proof.* Clearly the proof follows from Theorem 6.3.3. □

**Theorem 6.3.5.** *$P_4 \in \sigma_a(V_4)$ and $P_4 \notin \sigma_0(V_4)$.*

*Proof.* Consider the path $P_4 : v_1e_1v_2e_2v_3e_3v_4$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_1) = x_1, f(e_2) = x_2, f(e_3) = x_3$ for some $x_1, x_2, x_3 \in V_4 \setminus \{0\}$. Then $f^+(v_1) = x_1, f^+(v_2) = x_1 + x_2, f^+(v_3) = x_2 + x_3, f^+(v_4) = x_3$. Therefore

$f^{++}(e_1) = x_2, f^{++}(e_2) = x_1 + x_3, f^{++}(e_3) = x_2$. Then $P_4 \in \sigma_0$ or $P_4 \in \sigma_a$ accordingly $x_2 = x_1 + x_3 = 0$ or $x_2 = x_1 + x_3 = a$. But $x_2 = 0$ is not possible. Therefore $P_4 \notin \sigma_0(V_4)$. Thus if we take $x_1 = b, x_2 = a, x_3 = c$, then $f$ is an EIML of $P_4$. Hence the proof. □

**Corollary 6.3.6.** $P_4 \notin \sigma(V_4)$.

*Proof.* Proof follows from Theorem 6.3.5. □

**Theorem 6.3.7.** *$P_n$ is not an edge induced magic graph for any $n \geq 5$.*

*Proof.* Suppose that $n \geq 5$. Consider the path $P_n : v_1e_1v_2e_2v_3e_3 \cdots v_{n-1}e_nv_n$. Let $f : E \to V_4 \setminus \{0\}$ be defined by $f(e_i) = x_i$ for some $x_i \in V_4 \setminus \{0\}$ for $i = 1, 2, 3, \ldots, n-1$. Then $f^+(v_1) = x_1, f^+(v_2) = x_1 + x_2, f^+(v_3) = x_2 + x_3, f^+(v_4) = x_3 + x_4$ and so on. Therefore $f^{++}(e_1) = x_2, f^{++}(e_2) = x_1 + x_3, f^{++}(e_3) = x_2 + x_4$. Now if possible, suppose $f$ is an EIML of $P_n$. Then we have $f^{++}(e_1) = x_2 = x_2 + x_4 = f^{++}(e_3)$, which implies $x_4 = 0$, which is a contradiction to our assumption. Hence $P_n \notin \sigma_0(V_4)$ and $P_n \notin \sigma_a(V_4)$ for $n \geq 5$.
Hence the proof. □

**Corollary 6.3.8.** $P_n \notin \sigma(V_4)$ *for any $n$.*

*Proof.* Proof of the corollary follows from Corollary 6.3.2, Corollary 6.3.4, Corollary 6.3.6 and Theorem 6.3.7. □

**Theorem 6.3.9.** $C_n \in \sigma_0(V_4)$ *for all $n$.*

*Proof.* We can observe that the proof follows from Theorem 6.2.1. □

**Theorem 6.3.10.** $C_n \in \sigma_a(V_4)$ *if and only if $n$ is a multiple of* 4.

*Proof.* Consider the cycle graph defined by $C_n := v_1e_1v_2e_2v_3e_3 \cdots v_{n-1}e_{n-1}v_ne_nv_1$. Suppose $n$ is a multiple of 4, say $n = 4k$, for some integer $k$. Define $f : E(C_n) \to$

$V_4 \setminus \{0\}$ as

$$f(e_j) = \begin{cases} b & \text{for} \quad j = 1, 5, 9, \ldots, 4k-3, \quad 2, 6, 10, \ldots, 4k-2 \\ c & \text{for} \quad j = 3, 7, 11, \ldots, 4k-1, \quad 4, 8, 12, \ldots, 4k. \end{cases}$$

Then we can prove that $f^{++}(e_j) = a$ for $j = 1, 2, 3, \ldots, n$. That is $f$ is an EIML of $C_n$. Therefore in this case, $C_n \in \sigma_a(V_4)$.

Conversely, suppose that $n$ is not a multiple of 4. Then $n = 4k+1$ or $n = 4k+2$ or $n = 4k+3$ for some integer $k$. If possible, suppose $f$ is an EIML of $C_n$ with $f(e_i) = x_i$ for $i = 1, 2, 3, \ldots, n$. Then from the induced edge sum equation of each edge, we get

$$x_n + x_2 = x_1 + x_3 = x_2 + x_4 = x_3 + x_5 = \cdots = x_{n-1} + x_1. \tag{6.2}$$

**Case 1:** $n = 4k+1$.

In this case, Equation (6.2) implies that $x_1 = x_5 = x_9 = \cdots = x_n = x_4 = x_8 = x_{12} = \cdots = x_{n-1} = x_3 = x_7 = x_{11} = \cdots = x_{n-2} = x_2 = x_6 = x_{10} = \cdots = x_{n-3}$. Thus in this case if we let $x_i = f(e_i) = a$ for all $i$ then $f^{++}(e_i) = 0$ for all $i$. Hence $C_n \notin \sigma_a(V_4)$.

**Case 2:** $n = 4k+2$.

In this case, Equation (6.2) implies that $x_1 = x_5 = x_9 = \cdots = x_{n-1} = x_3 = x_7 = x_{11} = \cdots = x_{n-3}$ and $x_2 = x_6 = x_{10} = \cdots = x_n = x_4 = x_8 = x_{12} = \cdots = x_{n-2}$. Then if we let $f(e_1) = a$ and $f(e_2) = b$ then $f^{+}(v_j) = c$ for all $j$. Thus $f^{++}(e_i) = 0$ for all $i$. Hence $C_n \notin \sigma_a(V_4)$

**Case 3:** $n = 4k+3$.

In this case, Equation (6.2) implies that $x_1 = x_5 = x_9 = \cdots = x_{n-2} = x_2 = x_6 = x_{10} = \cdots = x_{n-1} = x_3 = x_7 = x_{11} = \cdots = x_n = x_4 = x_8 = x_{12} = \cdots = x_{n-3}$. Thus in this case, if we let $x_i = f(e_i) = a$ for all $i$ then

$f^{++}(e_i) = 0$ for all $i$. Hence $C_n \notin \sigma_a(V_4)$.

Thus from all the three cases above, we have $C_n \notin \sigma_a(V_4)$. Thus $C_n \in \sigma_a(V_4)$ if and only if $n$ is a multiple of 4.

Hence the proof. □

**Corollary 6.3.11.** *$C_n \in \sigma(V_4)$ if and only if $n$ is a multiple of 4.*

*Proof.* Proof follows from Theorem 6.3.9 and Theorem 6.3.10. □

**Theorem 6.3.12.** *Consider the star graph $K_{1,n}$, then we have the following.*

**(i)** *$K_{1,n} \in \sigma_0(V_4)$ if and only if $n$ is odd.*

**(ii)** *$K_{1,n} \in \sigma_a(V_4)$ if and only if $n$ is even.*

*Proof.* Consider $K_{1,n}$ with vertex set $\{v, v_1, v_2, v_3, \ldots, v_n\}$, where $vv_i \in E(K_{1,n})$ for $i = 1, 2, 3, \ldots, n$. Let $f$ be an edge label of $K_{1,n}$, with $f(vv_i) = x_i$, then from the induced edge sum equation of each edge we have the equation:

$$x_2+x_3+x_4+\cdots+x_n = x_1+x_3+x_4+\cdots+x_n = \cdots = x_1+x_2+x_3+\cdots+x_{n-1}. \quad (6.3)$$

Thus we have $f$ is an EIML of $K_{1,n}$ if and only if $x_1 = x_2 = x_3 = \cdots = x_n$.

**case (1)** $n$ is an odd integer.

Let $f(vv_i) = x_i = a$, then $f^+(v) = na = a$ and $f^+(v_i) = a$. Thus $f^{++}(vv_i) = a + a = 0$ for all $i$. Hence in this case, we can conclude that $K_{1,n} \in \sigma_0(V_4)$ and $K_{1,n} \notin \sigma_a(V_4)$.

**case (2)** $n$ is an even integer.

Let $f(vv_i) = x_i = a$, then $f^+(v) = na = 0$ and $f^+(v_i) = a$. Thus $f^{++}(vv_i) = 0 + a = a$ for all $i$. Hence in this case, we can conclude that $K_{1,n} \in \sigma_a(V_4)$ and $K_{1,n} \notin \sigma_0(V_4)$.

Hence the proof. $\square$

From the Theorem 6.3.12, we have the following corollary.

**Corollary 6.3.13.** *$K_{1,n} \notin \sigma(V_4)$ for any $n$.*

**Theorem 6.3.14.** *Consider the bipartite graph $K_{m,n}$, then we have then we have the following.*

**(i)** *$K_{m,n} \in \sigma_0(V_4)$ for $m+n$ is even.*

**(ii)** *$K_{m,n} \in \sigma_a(V_4)$ for $m+n$ is odd.*

*Proof.* Let $V(K_{m,n}) = \{v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\}$, where $v_i u_j \in E(K_{m,n})$ for $i = 1, 2, 3, \ldots m$ and $j = 1, 2, 3, \ldots, n$. Let $f : E(K_{m,n}) \to V_4 \setminus \{0\}$ be defined by $f(v_i u_j) = a$ for all $v_i u_j \in E(K_{m,n})$.

**Case 1:** $m+n$ is even.

**subcase (i)** $m$ and $n$ are odd.

In this case, we get $f^+(v_i) = na = a$ and $f^+(u_j) = ma = a$. Thus $f^{++}(v_i u_j) = 0$ for all $i$ and $j$.

**subcase (ii)** $m$ and $n$ are even.

In this case, we get $f^+(v_i) = na = 0$ and $f^+(u_j) = ma = 0$. Thus $f^{++}(v_i u_j) = 0$ for all $i$ and $j$.

Thus, if $m+n$ is even then $K_{m,n} \in \sigma_0(V_4)$.

**Case 2:** $m+n$ is odd.

**subcase (i)** $m$ is even $n$ is odd.

In this case, we get $f^+(v_i) = na = a$ and $f^+(u_j) = ma = 0$. Thus $f^{++}(v_i u_j) = a + 0 = a$ for all $i$ and $j$.

**subcase (ii)** $m$ is odd $n$ is even.

In this case, we get $f^+(v_i) = na = 0$ and $f^+(u_j) = ma = a$. Thus $f^{++}(v_i u_j) = 0 + a = a$ for all $i$ and $j$.

Thus if $m+n$ is even, then $K_{m,n} \in \sigma_a(V_4)$.

Hence the proof. □

**Theorem 6.3.15.** *Consider bistar graph $B_{m,n}$, then then we have the following.*

**(i)** $B_{m,n} \in \sigma_0(V_4)$ *if and only if* $m$ *and* $n$ *are even.*

**(ii)** $B_{m,n} \in \sigma_a(V_4)$ *if and only if* $m$ *and* $n$ *are odd.*

*Proof.* Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \ldots, v_m, u_1, u_2, u_3, \ldots, u_n\}$, where $uv$, $vv_i$, $uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. Let $f : E(B_{m,n}) \to V_4 \setminus \{0\}$ be an edge label defined as follows:

$$f(e) = \begin{cases} \alpha & \text{if} \quad e = uv \\ x_i & \text{if} \quad e = vv_1, vv_2, vv_3, \ldots, vv_m \\ y_j & \text{if} \quad e = uu_1, uu_2, uu_3, \ldots, uu_n. \end{cases}$$

Then by considering the induced edge sum equation of the edges $vv_i$ we have:

$$\begin{aligned} \alpha + x_2 + x_3 + x_4 + \cdots + x_m &= \alpha + x_1 + x_3 + x_4 + \cdots + x_m \\ &= \alpha + x_1 + x_2 + x_3 + \cdots + x_m \\ &\vdots \\ &= \alpha + x_2 + x_3 + x_4 + \cdots + x_{m-1}. \end{aligned} \tag{6.4}$$

In the light of Equation (6.4), we have $x_1 = x_2 = x_3 = \cdots = x_m = \beta$ (say). Similarly by considering the induced edge sum equation of the edges $uu_j$ one can easily prove that $y_1 = y_2 = y_3 = \cdots = y_n = \gamma$ (say). Thus the induced edge sum equations of the edge $vv_i$ and $uu_j$ are given by $\alpha + (m-1)\beta = \alpha + (n-1)\gamma$. Also the induced edge sum equation of the edge $uv$ is given by $x_1 + x_2 + x_3 + \cdots + x_m + y_1 + y_2 + y_3 + \cdots + y_n = m\beta + n\gamma$.

Thus $f$ is an edge induced magic label with induced edge sum $x$ if and only if

$$x = \alpha + (m-1)\beta = \alpha + (n-1)\gamma = m\beta + n\gamma. \tag{6.5}$$

**Case 1:** $m$ and $n$ are even.

In this case, Equation (6.5) becomes $x = \alpha + \beta = \alpha + \gamma = 0$. Thus $\alpha = \beta = \gamma$, and the induced edge sum $x = 0$.

Hence in this case, $B_{m,n} \in \sigma_0(V_4)$.

**Case 2:** $m$ and $n$ are odd.

In this case, Equation (6.5) becomes $x = \alpha = \beta + \gamma$. Thus in this we can choose $\beta = b$ and $\gamma = c$ then $\alpha = a$ and which implies that the induced edge sum becomes $x = a$.

Hence in this case, $B_{m,n} \in \sigma_a(V_4)$.

**Case 3:** $m$ is even and $n$ is odd.

In this case, Equation (6.5) becomes $x = \alpha + \beta = \alpha = \gamma$ which implies that $\beta = 0$. That is $f(vv_i) = x_i = \beta = 0$, which a contradiction to the choice of $f$. Therefore $B_{m,n} \notin \sigma_0(V_4)$ and $B_{m,n} \notin \sigma_a(V_4)$.

**Case 4:** $m$ is odd and $n$ is even.

In this case, Equation (6.5) becomes $x = \alpha = \alpha + \gamma = \beta$ which implies $\gamma = 0$. That is $f(uu_j) = y_j = \gamma = 0$, which a contradiction to the choice of $f$. Therefore $B_{m,n} \notin \sigma_0(V_4)$ and $B_{m,n} \notin \sigma_a(V_4)$.

Hence the proof. $\square$

**Theorem 6.3.16.** *Let $K_n$ be the complete graph with $n$ vertices, then $K_n \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Consider the complete graph $K_n$. Let $f : E(K_n) \to V_4 \setminus \{0\}$ be an edge label with $f(e) = a$ for all $e \in E(K_n)$. Then the induced vertex label $f^+$ becomes $f^+(u) = (n-1)a$, for all $u \in V(K_n)$. Using this, we have the induced edge label $f^{++}$ becomes $f^{++}(e) = 2(n-1)a = 0$, for all $e \in E(K_n)$. Thus $K_n \in \sigma_0(V_4)$ for all $n$. Hence the proof. □

## 6.4 Edge Induced $V_4$ Magic Labeling of Some Special Graphs

**Theorem 6.4.1.** *The sun graph $Sun_n \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Since every vertex is of odd degree, by Theorem 6.2.1 the theorem follows. □

**Theorem 6.4.2.** *The sun graph $Sun_n \in \sigma_a(V_4)$ for $n$ is even.*

*Proof.* Consider a sun graph $Sun_n$ with $\{v_1, v_2, v_3, \ldots, v_n\}$ as vertex set of the corresponding $C_n$ and $w_i, 1 \leq i \leq n$, be the pendant vertices attached to each $v_i, 1 \leq i \leq n$. Let $f : E(Sun_n) \to V_4 \setminus \{0\}$ be defined by

$$f(e) = \begin{cases} b & \text{if} \quad e = v_1v_2, v_3v_4, v_5v_6, \ldots, v_{n-1}v_n \\ c & \text{if} \quad e = v_2v_3, v_4v_5, v_6v_7, \ldots, v_nv_1 \\ b & \text{if} \quad e = v_1w_1, v_3w_3, v_5w_5, \ldots, v_{n-1}w_{n-1} \\ c & \text{if} \quad e = v_2w_2, v_4w_4, v_6w_6, \ldots, v_nw_n. \end{cases}$$

Then we can easily prove that $f^{++}(e) = a$ for all $e \in E(Sun_n)$. That is $Sun_n \in \sigma_a(V_4)$. Hence the proof. □

**Corollary 6.4.3.** *The sun graph $Sun_n \in \sigma(V_4)$ if and only if $n$ is even.*

*Proof.* Proof follows from Theorem 6.4.1 and Theorem 6.4.2. □

**Theorem 6.4.4.** *The comb graph $CB_n$ is not an edge induced magic graph, for any $n$.*

*Proof.* Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of $CB_n$, where $v_i (1 \leq i \leq n)$ are the pendant vertices adjacent to $u_i (1 \leq i \leq n)$. If possible, suppose $f : E(CB_n) \to V_4 \setminus \{0\}$ is an induced edge label of $CB_n$. Then using the induced edge sum equation of the edges $u_1v_1$ and $u_2v_2$ we get, $f^{++}(u_1v_1) = f^{++}(u_2v_2)$, which implies $f(u_1u_2) = f(u_1u_2) + f(u_2u_3)$. That is $f(u_2u_3) = 0$, which is a contradiction. Thus $CB_n$ is not an edge induced magic graph.
Hence the proof. □

**Theorem 6.4.5.** *The wheel graph $W_n \in \sigma_0(V_4)$ for $n$ is odd.*

*Proof.* Suppose $n$ is odd. Then, since every vertex of $W_n$ is of odd degree, by Theorem 6.2.1 the proof follows. □

**Theorem 6.4.6.** *Let $J(m, n)$ be the jelly fish graph then we have the following.*

**(i)** *$J(m, n) \in \sigma_0(V_4)$ if and only if $m$ and $n$ are of same parity.*

**(ii)** *$J(m, n) \notin \sigma_a(V_4)$ for any $m$ and $n$.*

*Proof.* Consider the jelly fish graph with $V(J(m, n)) = \{v_k : k = 1, 2, 3, 4\}$ $\cup\{u_i : i = 1, 2, 3, \dots, m\} \cup \{w_j : j = 1, 2, 3, \dots, n\}$, where $v_k'$s are the vertices of $C_4$ and $u_i, w_j$ are the vertices of corresponding $K_{1,m}$ and $K_{1,n}$ respectively. Let $f : E(J(m, n)) \to V_4 \setminus \{0\}$ be an edge induced magic label with $f(v_1v_2) = x_1, f(v_1v_4) = x_2, f(v_3v_4) = x_3, f(v_2v_3) = x_4, f(v_1v_3) = x_5, f(v_2u_i) = e_i$, and $f(v_4w_j) = y_j$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

Using the induced edge sum equation of the edges $v_2u_i$, we get

$$\begin{aligned} e_2 + e_3 + e_4 + \cdots + e_m + x_1 + x_4 &= e_1 + e_3 + e_4 + \cdots + e_m + x_1 + x_4 \\ &= e_1 + e_2 + e_4 + \cdots + e_m + x_1 + x_4 \\ &\vdots \\ &= e_1 + e_2 + e_3 + \cdots + e_{m-1} + x_1 + x_4. \end{aligned}$$

The above equations imply that $e_1 = e_2 = e_3 = \cdots = e_m = \alpha$ (say). Thus the induced edge sum equation of $v_2u_i$ reduces to $(m-1)\alpha + x_1 + x_4$.

In a similar way, by considering the induced edge sum equation of the edges $v_4w_j$, we get $y_1 = y_2 = y_3 = \cdots = y_n = \beta$ (say). Thus the induced edge sum equation of $v_4w_j$ reduces to $(n-1)\beta + x_2 + x_3$.

Now consider the induced edge sum equation of the edges $v_1v_2$ and $v_2v_3$, then we get $m\alpha + x_2 + x_4 + x_5 = m\alpha + x_1 + x_3 + x_5$ which implies $x_2 + x_4 = x_1 + x_3$. Similarly by considering the induced edge sum equation of $v_1v_4$ and $v_3v_4$, we get $n\beta + x_1 + x_3 + x_5 = n\beta + x_2 + x_4 + x_5$. Also from the induced edge sum equation of $v_1v_3$, we get its induced edge sum equal to $x_1 + x_2 + x_3 + x_4 = 0$, since $x_2 + x_4 = x_1 + x_3$.

Thus from the above discussion we have the induced edge sum is given by:

$$\begin{aligned} x &= (m-1)\alpha + x_1 + x_4 = (n-1)\beta + x_2 + x_3 \\ &= m\alpha + x_2 + x_4 + x_5 = m\alpha + x_1 + x_3 + x_5 \qquad (6.6) \\ &= n\beta + x_1 + x_3 + x_5 = n\beta + x_2 + x_4 + x_5 = 0. \end{aligned}$$

Since the induced sum is 0, we have $J(m,n) \notin \sigma_a(V_4)$ for any $m, n$.

Now consider the following cases.

**Case 1 :** $m$ and $n$ are even.

In this case, equation (6.6) becomes

$$x = \alpha + x_1 + x_4 = \beta + x_2 + x_3 = x_2 + x_4 + x_5 = x_1 + x_3 + x_5 = 0. \quad (6.7)$$

Choose $\alpha = \beta = x_5 = c,\ x_1 = x_2 = a,\ x_3 = x_4 = b$, then above Equation (6.7) follows. Thus in this case, $J(m,n) \in \sigma_0(V_4)$

**Case 2 :** $m$ and $n$ are odd.

In this case, equation (6.6) becomes

$$\begin{aligned} x &= x_1 + x_4 = x_2 + x_3 \\ &= \alpha + x_2 + x_4 + x_5 = \alpha + x_1 + x_3 + x_5 \qquad (6.8) \\ &= \beta + x_1 + x_3 + x_5 = \beta + x_2 + x_4 + x_5 = 0. \end{aligned}$$

Choose $\alpha = \beta = x_5 = a,\ x_1 = x_2 = x_3 = x_4 = b$, then above Equation (6.8) follows. Thus in this case, $J(m,n) \in \sigma_0(V_4)$

**Case 3 :** $m$ odd and $n$ even.

In this case, Equation (6.6) becomes

$$\begin{aligned} x &= x_1 + x_4 = \beta + x_2 + x_3 \\ &= \alpha + x_2 + x_4 + x_5 = \alpha + x_1 + x_3 + x_5 \\ &= x_1 + x_3 + x_5 = x_2 + x_4 + x_5 = 0. \end{aligned}$$

Note that above equations imply that $\alpha = e_i = f(v_2u_i) = 0$, which is not admissible. Thus in this case, $J(m,n) \notin \sigma_0(V_4)$.

**Case 4 :** $m$ even and $n$ odd.

In this case, Equation (6.6) becomes

$$x = \alpha + x_1 + x_4 = x_2 + x_3$$

$$\begin{aligned} &= x_2 + x_4 + x_5 = x_1 + x_3 + x_5 \\ &= \beta + x_1 + x_3 + x_5 = \beta + x_2 + x_4 + x_5 = 0. \end{aligned}$$

Note that above equations imply that $\beta = y_j = f(v_4 w_j) = 0$, which is not admissible. Thus in this case, $J(m, n) \notin \sigma_0(V_4)$.

Thus $J(m, n) \in \sigma_0(V_4)$ if and only if $m$ and $n$ are of same parity.
Hence the proof. □

**Theorem 6.4.7.** *The triangular snake graph $TS_n \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Since every vertex of $TS_n$ is of even degree, by Theorem 6.2.1 the proof follows. □

**Theorem 6.4.8.** *The open ladder graph $O(L_n) \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Since every vertex of $O(L_n)$ is of odd degree, by Theorem 6.2.1 the proof follows. □

Chapter 7

# Edge Induced $V_4-$ Magic Labeling of Subdivision Graphs and Line Graphs

*This chapter discusses the edge induced $V_4$-magic labeling of subdivision graph and line graph of some graphs. The first section deals with the edge induced $V_4$-magic labeling of subdivision graphs of some graphs. In the second section, we discuss the edge induced $V_4$-magic labeling of line graphs of some graphs.*

## 7.1 Subdivision Graphs

**Theorem 7.1.1.** *Let $G$ be graph with every vertex is of odd degree, then $S(G) \in \sigma_a(V_4)$.*

*Proof.* Suppose $G$ is a graph with every vertex is of odd degree. Then define $f : E(S(G)) \to V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(G))$.
Let $uv \in E(G)$ and $\alpha$ be the inserted vertex on the edge $uv$ in $S(G)$. Then $f(u\alpha) = f(v\alpha) = a$. Therefore $f^+(u) = f^+(v) = deg\ (u)a = a$, since $deg\ (u)$

is odd and $f^+(\alpha) = deg\ (\alpha)a = 0$, since $deg\ (\alpha) = 2$. Thus $f^{++}(u\alpha) = a$ and $f^{++}(v\alpha) = a$. Since $uv$ is an arbitrary edge in $S(G)$, we can conclude that $f^{++}(e) = a$ for all $e \in S(G)$. Thus $S(G) \in \sigma_a(V_4)$.
Hence the proof. □

**Theorem 7.1.2.** *Let $G$ be graph with every vertex is of even degree, then $S(G) \in \sigma_0(V_4)$.*

*Proof.* Suppose $G$ is a graph with every vertex is of even degree. Then define $f : E(S(G)) \to V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(G))$.

Let $uv \in E(G)$ and $\alpha$ be the inserted vertex on the edge $uv$ in $S(G)$. Then $f(u\alpha) = f(v\alpha) = a$. Therefore $f^+(u) = f^+(v) = deg\ (u)a = 0$, since $deg\ (u)$ is even and $f^+(\alpha) = deg\ (\alpha)a = 0$, since $deg\ (\alpha) = 2$. Thus $f^{++}(u\alpha) = 0$ and $f^{++}(v\alpha) = 0$. Since $uv$ is an arbitrary edge in $S(G)$, we can conclude that $f^{++}(e) = 0$ for all $e \in S(G)$. Thus $S(G) \in \sigma_0(V_4)$.
Hence the proof. □

**Theorem 7.1.3.** $S(P_2) \in \sigma_a(V_4)$ *and* $S(P_2) \notin \sigma_0(V_4)$.

*Proof.* Since $S(P_2) = P_3$ proof follows directly from Theorem 6.3.3. □

**Theorem 7.1.4.** $S(P_n) \notin \sigma_a(V_4)$ *and* $S(P_n) \notin \sigma_0(V_4)$ *for any* $n \geq 3$.

*Proof.* Note that $S(P_n) = P_{2n-1}$ and if $n \geq 3$ then $2n \geq 5$, therefore the proof follows directly from Theorem 6.3.7. □

**Theorem 7.1.5.** $S(C_n) \in \sigma_0(V_4)$ *for all* $n$.

*Proof.* Since $S(C_n) = C_{2n}$, proof follows from Theorem 6.3.9. □

**Theorem 7.1.6.** $S(C_n) \in \sigma_a(V_4)$ *if and only if* $n$ *is even.*

*Proof.* Since $S(C_n) = C_{2n}$ proof follows directly from Theorem 6.3.10. □

**Corollary 7.1.7.** *$S(C_n) \in \sigma(V_4)$ if and only if $n$ is even.*

*Proof.* The proof follows from Theorem 7.1.5 and Theorem 7.1.6. □

**Theorem 7.1.8.** *For the star graph $K_{1,n}$, we have the following.*

**(i)** *$S(K_{1,n}) \in \sigma_a(V_4)$ if and only if $n$ is odd.*

**(ii)** *$S(K_{1,n}) \notin \sigma_0(V_4)$ for any $n$.*

*Proof.* Consider $K_{1,n}$ with vertex set $\{v, v_1, v_2, v_3, \ldots, v_n\}$, where $vv_i \in E(K_{1,n})$ for $i = 1, 2, 3, \ldots n$. Let $u_i$ be the inserted vertices on the edge $vv_i$ for $i = 1, 2, 3, \ldots, n$ in $S(K_{1,n})$.

Let $f : E(S(K_{1,n})) \to V_4 \setminus \{0\}$ with $f(vu_i) = x_{i1}$, and $f(u_iv_i) = x_{i2}$ for $i = 1, 2, 3, \ldots, n$. Then from the induced edge sum equation of each edge we have the following equation.

$$\begin{aligned}
x_{11} = x_{21} = x_{31} = \cdots = x_{n1} &= x_{21} + x_{31} + x_{41} + \cdots + x_{n1} + x_{12} \\
&= x_{11} + x_{31} + x_{41} + \cdots + x_{n1} + x_{22} \\
&= x_{11} + x_{21} + x_{31} + \cdots + x_{n1} + x_{32} \\
&\vdots \\
&= x_{11} + x_{31} + x_{41} + \cdots + x_{n-11} + x_{n2}.
\end{aligned}$$

Let $x = x_{11} = x_{21} = x_{31} = \cdots = x_{n1}$ then above equations become

$$\begin{aligned}
x &= (n-1)x + x_{12} \\
&= (n-1)x + x_{22} \\
&= (n-1)x + x_{32} \\
&\vdots \\
&= (n-1)x + x_{n2}.
\end{aligned}$$

Note that the above system implies that $x_{12} = x_{22} = x_{32} = \cdots = x_{n2} = y$(say). Then the above system of equations reduces to $x = (n-1)x + y$.

$$\text{That is } (n-2)x + y = 0.$$

**Case (i)** $n$ is an odd integer.

In this case, the equation $(n-2)x+y = 0$ reduces to $x+y = 0$, that is $x = y$. Thus by taking $x = y = a$ that is, by defining $f : E(S(K_{1,n})) \to V_4 \setminus \{0\}$ as $f(e_i) = a$, for all $e_i \in E(S(K_{1,n}))$ we can prove that $S(K_{1,n}) \in \sigma_a(V_4)$.

**Case (ii)** $n$ is an even integer.

In this case, the equation $(n-2)x + y = 0$ reduces to $y = 0$. That is $f(u_iv_i) = x_{i2} = 0$, which is a contradiction to the choice for $f$. Therefore, in this case, $S(K_{1,n})$ is not an edge induced magic graph.

Note that $S(K_{1,n}) \in \sigma_0(V_4)$ only when $x = 0$. But $x = 0$ is not possible. Therefore $S(K_{1,n}) \notin \sigma_0(V_4)$ for any $n$.

Hence the proof. □

**Theorem 7.1.9.** *For the bistar graph $B_{m,n}$, $S(B_{m,n})$ is not an edge induced magic graph for any $m$ and $n$.*

*Proof.* Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \dots, v_m, u_1, u_2, u_3, \dots, u_n\}$, where $uv, vv_i$, $uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$. Also let $w_i$, $t_j$ and $w$ be the inserted vertices on the edge $vv_i$, $uu_j$ and $uv$ respectively for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$ in the graph $S(B_{m,n})$.

Let $f : E(S(B_{m,n})) \to V_4 \setminus \{0\}$ with $f(vw) = \gamma$, $f(wu) = \delta$, $f(vw_i) = x_i$, $f(w_iv_i) = \alpha_i$, $f(ut_j) = y_j$ and $f(t_ju_j) = \beta_j$, then by considering the induced edge sum equation of each edge we have the following equations.

The induced edge sum equation of the edges $w_iv_i$ gives: $x_1 = x_2 = x_3 = \cdots = x_m = x$ (say). Similarly the induced edge sum equation of the edges $t_ju_j$ gives:

$y_1 = y_2 = y_3 = \cdots = y_n = y$ (say).
The induced edge sum equation of the edges $vw_i$ gives:

$$\begin{aligned}
\alpha_1 + \gamma + (m-1)x &= \alpha_2 + \gamma + (m-1)x \\
&= \alpha_3 + \gamma + (m-1)x \\
&\vdots \\
&= \alpha_m + \gamma + (m-1)x.
\end{aligned}$$

Note that above system of equations imply that $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_m = \alpha$ (say). Thus each induced edge sum in above system reduces to $\alpha+\gamma+(m-1)x$. Similarly by considering the induced edge sum equation of the edges $ut_j$, we get the induced induced edge sum is $\beta + \delta + (n-1)y$, where $\beta = \beta_1 = \beta_2 = \beta_3 = \cdots = \beta_n$.
Also we get, the induced edge sum of the edges $vw$ is $mx + \delta$ and the edge sum of the edge $wu$ is $ny + \gamma$.
Thus the edge sum equation of the graph $S(B_{m,n})$ is given by:

$$x = y = \alpha + \gamma + (m-1)x = \beta + \delta + (n-1)y = mx + \delta = ny + \gamma. \qquad (7.1)$$

**Case 1:** $m$ and $n$ are even integers.

In this case, Equation (7.1) becomes

$$x = y = \alpha + \gamma + x = \beta + \delta + y = \delta = \gamma.$$

Therefore $x = \gamma$, which implies that $\alpha = 0$, which is not possible.

Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

**Case 2:** $m$ and $n$ are odd integers.

In this case, Equation (7.1) becomes

$$x = y = \alpha + \gamma = \beta + \delta = x + \delta = y + \gamma.$$

Therefore $x = x + \delta$ which implies that $\delta = 0$, which is not possible.

Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

**Case 3:** $m$ is even and $n$ is odd.

In this case, Equation (7.1) becomes

$$x = y = \alpha + \gamma + x = \beta + \delta = \delta = y + \gamma.$$

Therefore $\beta + \delta = \delta$ which implies that $\beta = 0$, which is not possible.

Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

**Case 4:** $m$ is odd and $n$ is even.

In this case, Equation (7.1) becomes

$$x = y = \alpha + \gamma = \beta + \delta + y = x + \delta = \gamma.$$

Therefore $\alpha + \gamma = \gamma$ which implies that $\alpha = 0$, which is not possible.

Hence in this case, $B_{m,n}$ is not an edge induced magic graph.

Thus in all cases, we get $S(B_{m,n})$ is not an edge induced magic graph.

Hence the proof. □

**Theorem 7.1.10.** *For the the complete graph $K_n$ with $n$ vertices, we have the following.*

**(i)** $S(K_n) \in \sigma_0(V_4)$ *for $n$ odd.*

**(ii)** $S(K_n) \in \sigma_a(V_4)$ *for $n$ even.*

*Proof.* Consider the subdivision graph of the complete graph $S(K_n)$. Let $vu$ an edge in $S(K_n)$, where $v \in V(K_n)$ and $u$ be an inserted vertex in $S(K_n)$. Define $f : E(S(K_n)) \to V_4 \setminus \{0\}$ by $f(e) = a$ for all $e \in E(S(K_n))$. Then $f^+(v) = (n-1)a$ and $f^+(u) = a + a = 0$. Therefore $f^{++}(vu) = (n-1)a$. Since the vertices $u$ and $v$ are arbitrary, we have $f^{++}(vu)$ is a constant.

**Case (i)** $n$ is an odd integer.

In this case, $f^{++}(vu) = (n-1)a = 0$. Therefore $S(K_n) \in \sigma_0(V_4)$.

**Case (ii)** $n$ is an even integer.

In this case, $f^{++}(vu) = (n-1)a = a$. Therefore $S(K_n) \in \sigma_a(V_4)$.

Hence the proof. □

**Theorem 7.1.11.** *For the sun graph $Sun_n$, we have $S(Sun_n) \in \sigma_a(V_4)$ for all $n$.*

*Proof.* Let $\{u_i, v_i : i = 1, 2, 3, \ldots, n\}$ be the vertex set of $CB_n$, where $v_i$ are the pendant vertex adjacent to $u_i$. Also let $t_i$ and $w_i$, be the inserted vertices on the edge $u_iu_{i+1}$, $u_iv_i$, for $i = 1, 2, 3, \ldots, n$ and $i + 1$ is taken modulo $n$.
Suppose $f : E(S(Sun_n)) \to V_4 \setminus \{0\}$ is an edge induced magic label of $Sun_n$ with $f(u_it_i) = e_i$, $f(t_iu_{i+1}) = \alpha_i$, $f(u_iw_i) = \beta_i$ and $f(w_iv_i) = \gamma_i$.
Then using the induced edge sum equation of the edges $w_iv_i$, we get

$$\beta_1 = \beta_2 = \beta_3 = \cdots = \beta_n = \beta \text{ (say)}. \tag{7.2}$$

By the induced edge sum equation of the edges $u_it_i$, we get

$$\alpha_n + \alpha_1 + \beta = \alpha_1 + \alpha_2 + \beta = \alpha_2 + \alpha_3 + \beta = \cdots = \alpha_{n-1} + \alpha_n + \beta. \tag{7.3}$$

By the induced edge sum equation of the edges $t_iu_{i+1}$, we get

$$e_1 + e_2 + \beta = e_2 + e_3 + \beta = e_3 + e_4 + \beta = \cdots = e_n + e_1 + \beta. \tag{7.4}$$

By the induced edge sum equation of the edges $u_iw_i$, we get

$$\alpha_n + e_1 + \gamma_1 = \alpha_1 + e_2 + \gamma_2 = \alpha_2 + e_3 + \gamma_3 = \cdots = \alpha_{n-1} + e_n + \gamma_n. \tag{7.5}$$

**Case (i)** $n$ is an odd integer.

In this case, Equation (7.3) and Equation(7.4) implies that

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = \alpha\ (say).$$
$$e_1 = e_2 = e_3 = \cdots = e_n = e\ (say).$$

Therefore equation (7.5) implies that

$$\gamma_1 = \gamma_2 = \gamma_3 = \cdots = \gamma_n = \gamma\ (say).$$

Therefore in this case, the induced edge sum equation of the graph $S(Sun_n)$ is given by:

$$\beta = 2\alpha + \beta = 2e + \beta = \alpha + e + \gamma.$$

Since $\alpha \in V_4$, the above equation reduces to $\beta = \alpha + e + \gamma$.

Therefore in this case, if we choose $\alpha = e = b$ and $\beta = \gamma = a$, then we can easily prove that $S(Sun_n) \in \sigma_a(V_4)$.

**Case (ii)** $n$ is an even integer.

In this case, Equation (7.3) implies

$$\alpha_1 = \alpha_3 = \alpha_5 = \cdots = \alpha_{n-1} = x_1\ (say).$$
$$\alpha_2 = \alpha_4 = \alpha_6 = \cdots = \alpha_n = x_2\ (say).$$

Also in this case Equation (7.4) implies

$$e_1 = e_3 = e_5 = \cdots = e_{n-1} = y_1 \ (say).$$
$$e_2 = e_4 = e_6 = \cdots = e_n = y_2 \ (say).$$

Therefore Equation (7.5) reduces to

$$x_2+y_1+\gamma_1 = x_1+y_2+\gamma_2 = x_2+y_1+\gamma_3 = x_1+y_2+\gamma_4 = \cdots = x_1+y_2+\gamma_n. \tag{7.6}$$

Note that Equation (7.6) implies that

$$\gamma_1 = \gamma_3 = \gamma_5 = \cdots = \gamma_{n-1} = z_1 \ (say).$$
$$\gamma_2 = \gamma_4 = \gamma_6 = \cdots = \gamma_n = z_2 \ (say).$$

Therefore in this case, the induced edge sum equation of the graph $S(Sun_n)$ is given by:

$$\beta = x_1 + x_2 + \beta = y_1 + y_2 + \beta = x_2 + y_1 + z_1 = x_1 + y_2 + z_2.$$

Therefore in this case, if we choose $x_1 = x_2 = y_1 = y_2 = b$ and $\beta = z_1 = z_2 = a$ then we can easily prove that $f^{++}(e) = a$ for all $e \in E(S(Sun_n))$. Thus $S(Sun_n) \in \sigma_a(V_4)$.

Hence the proof. □

**Theorem 7.1.12.** *For the comb graph $CB_n$, we have $S(CB_n)$ is not an edge induced magic graph, for any $n$.*

*Proof.* Let $\{u_i, v_i : 1, 2, 3, \ldots, n\}$ be the vertex set of $CB_n$, where $v_i$ is the pendant vertex adjacent to $u_i$. Let $w_i$ and $t_j$ be the inserted vertices in the edges $u_iv_i$ and $u_ju_{j+1}$ for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n-1$ respectively . If possible,

suppose $f : E(S(CB_n)) \to V_4 \setminus \{0\}$ is an edge induced magic label of $S(CB_n)$. Then using the induced edge sum equation of the edges $v_1w_1$ and $u_1t_1$, we get $f(u_1w_1) = f(t_1u_2) + f(u_1w_1)$. That is $f(t_1u_2) = 0$, which is a contradiction. Hence $S(CB_n)$ is not an edge induced magic graph, for any $n$.
Hence the proof. □

**Theorem 7.1.13.** *Let $J(m,n)$ be the jelly fish graph. Then $S(J(m,n)) \in \sigma_a(V_4)$ for $m$ and $n$ are of same parity.*

*Proof.* Consider the jelly fish graph with $V(J(m,n)) = \{v_k : k = 1,2,3,4\} \cup \{u_i : i = 1,2,3,\ldots,m\} \cup \{w_j : j = 1,2,3,\ldots,n\}$, where $v_k'$s are the vertices of corresponding $C_4$, $u_i, w_j$ are the vertices of corresponding $K_{1,m}$ and $K_{1,n}$ respectively and $\alpha_i$ $(1 \leq i \leq m), \beta_j$ $(1 \leq j \leq n)$ be the inserted vertices on the edges $v_2u_i, v_4w_j$ respectively and $\alpha, \beta, \gamma, \delta, \mu$ be the vertices inserted on the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3$ respectively.

**Case (i)** $m$ and $n$ are even integers.

In this case, define $f : E(S(J(m,n))) \to V_4 \setminus \{0\}$ by

$$f(e) = \begin{cases} a & \text{if} \quad e = \alpha_i u_i,\ v_2\alpha_i,\ v_4\beta_j,\ \beta_j w_j,\ v_1\mu,\ v_3\mu \\ b & \text{if} \quad e = v_2\alpha,\ \alpha v_1,\ v_1\delta,\ \delta v_4 \\ c & \text{if} \quad e = v_4\gamma,\ \gamma v_3,\ v_3\beta,\ \beta v_2. \end{cases}$$

**Case (ii)** $m$ and $n$ are $n$ odd integers.

In this case, define $f : E(S(J(m,n))) \to V_4 \setminus \{0\}$ by

$$f(e) = \begin{cases} a & \text{if} \quad e = v_2\alpha,\ \alpha v_1,\ v_3\beta,\ \beta v_2,\ v_2\alpha_i,,\ v_4\beta_j,\ \alpha_i u_i,\ \beta_j w_j \\ b & \text{if} \quad e = v_1\delta,\ \delta v_4,\ v_4\gamma,\ \gamma v_3,,\ v_1\mu,\ v_3\mu. \end{cases}$$

Then in both cases we can verify that $f^{++}(e) = a$ for all $e \in E(S(J(m,n)))$. That is $f$ is an EIML of $S(J(m,n))$. Thus in both cases $S(J(m,n)) \in \sigma_a(V_4)$.

Hence the proof. $\square$

**Theorem 7.1.14.** *For the wheel graph $W_n$, we have $S(W_n) \in \sigma_a(V_4)$ for $n$ is odd.*

*Proof.* Suppose $n$ is odd. Since every vertex is of odd degree, the proof follows from Theorem 7.1.1. $\square$

**Theorem 7.1.15.** *For the flag graph $Fl_n$, we have the following.*

**Case (i)** $S(Fl_n) \notin \sigma_0(V_4)$ *for any $n$.*

**Case (ii)** $S(Fl_n) \in \sigma_a(V_4)$ *if and only if $n$ is odd.*

*Proof.* Let $V(Fl_n) = \{v, v_1, v_2, v_3, \ldots, v_n\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding cycle graph $C_n$ and $v$ is the root vertex adjacent to the vertex $v_1$. Also let $u$ be the inserted vertex on the edge $v_1v$ and $u_1, u_2, u_3, \ldots, u_n$ be the inserted vertices on the edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_nv_1$ respectively in the graph $S(Fl_n)$.

If possible, let $g : E(S(Fl_n)) \to V_4 \setminus \{0\}$ be an edge label with $g^{++}(e) = 0$ for all edge in $S(Fl_n)$. Then consider the induced edge sum of the edge $uv$. Note that $g^{++}(uv) = g(uv_1)$. Therefore $g(uv_1) = 0$, which is a contradiction and it proves (i).

Suppose $n$ is an odd integer. In this case, define $f : E(S(Fl_n)) \to V_4 \setminus \{0\}$ as follows.

$$f(e) = \begin{cases} a & \text{if} \quad e = uv, uv_1 \\ b & \text{if} \quad e = u_1v_1, u_3v_3, u_5v_5, \ldots u_{n-2}v_{n-2}, u_nv_n \\ c & \text{if} \quad e = u_2v_2, u_4v_4, u_6v_6, \ldots u_{n-3}v_{n-3}, u_{n-1}v_{n-1} \\ b & \text{if} \quad e = u_1v_2, u_3v_4, u_5v_6, \ldots u_{n-2}v_{n-1}, u_nv_1 \\ c & \text{if} \quad e = u_2v_3, u_4v_5, u_6v_7, \ldots u_{n-3}v_{n-2}, u_{n-1}v_n. \end{cases}$$

Then $f^{++}(e) = a$ for all $e \in E(S(Fl_n))$. Thus $S(Fl_n) \in \sigma_a(V_4)$.

To prove the converse part, suppose $n$ is an even integer. If possible, let $h : E(S(Fl_n)) \to V_4 \setminus \{0\}$ be an edge label with $h^{++}(e) = a$ for all edge in $S(Fl_n)$. Consider the induced edge sum of the edge $uv$. We have $h^{++}(uv) = h(uv_1)$. Similarly if we let $h(u_i v_{i+1}) = y_i$, for $i = 1, 2, 3, \ldots, n$ with $i+1$ is taken modulo $n$. Then the induced edge sum of the edges $v_i u_i$ for $i = 1, 2, 3, \ldots, n$ gives

$$y_n + y_1 + h(uv_1) = y_1 + y_2 = y_2 + y_3 = \cdots = y_{n-1} + y_n. \tag{7.7}$$

Since $n$ is an even integer the above equation implies that $y_1 = y_3 = y_5 = \cdots = y_{n-1} = x$ (say) and $y_2 = y_4 = y_6 = \cdots = y_n = y$ (say). Thus the Equation (7.7) reduces to $x + y + h(uv_1) = x + y$, which implies that $h(uv_1) = 0$, which is not admissible. Hence there exists no such edge label $h$. Hence if $n$ is an even integer, then $S(Fl_n) \notin \sigma_a(V_4)$.
Hence the proof. □

**Corollary 7.1.16.** *$S(Fl_n) \in \sigma(V_4)$ if and only if $n$ is odd.*

*Proof.* Proof follows from the above Theorem 7.1.15. □

**Theorem 7.1.17.** *For the the triangular snake graph $TS_n$, we have $S(TS_n) \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Since every vertex is of even degree, the proof follows from Theorem 7.1.2. □

## 7.2 Line Graphs

**Theorem 7.2.1.** *$L(P_3) \in \sigma_0(V_4)$ and $L(P_3) \notin \sigma_a(V_4)$.*

*Proof.* Since $L(P_3) = P_2$, proof follows directly from Theorem 6.3.1. $\square$

**Corollary 7.2.2.** *$L(P_3) \notin \sigma(V_4)$.*

*Proof.* Proof follows from Theorem 7.2.1. $\square$

**Theorem 7.2.3.** *$L(P_4) \in \sigma_a(V_4)$ and $L(P_4) \notin \sigma_0(V_4)$.*

*Proof.* Since $L(P_4) = P_3$ proof follows directly from Theorem 6.3.3. $\square$

**Corollary 7.2.4.** *$L(P_4) \notin \sigma(V_4)$.*

*Proof.* Proof follows from Theorem 7.2.3. $\square$

**Theorem 7.2.5.** *$L(P_5) \in \sigma_a(V_4)$ and $L(P_5) \notin \sigma_0(V_4)$.*

*Proof.* Since $L(P_5) = P_4$, proof follows directly from Theorem 6.3.5. $\square$

**Corollary 7.2.6.** *$L(P_5) \notin \sigma(V_4)$.*

*Proof.* Proof follows from the Theorem 7.2.5. $\square$

**Theorem 7.2.7.** *$L(P_n)$ is not an edge induced magic graph for any $n \geq 6$.*

*Proof.* Note that $L(P_n) = P_{n-1}$. Thus if $n \geq 6$, then $L(P_n) = P_{n-1}$ and $n-1 \geq 5$, therefore the proof follows from Theorem 6.3.7. $\square$

**Corollary 7.2.8.** *$L(P_n) \notin \sigma(V_4)$ for any $n \geq 3$.*

*Proof.* Proof of the corollary follows from Corollary 7.2.2, Corollary 7.2.4, Corollary 7.2.6 and Theorem 7.2.7. $\square$

**Theorem 7.2.9.** *$L(C_n) \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Since $L(C_n) = C_n$, the proof follows from Theorem 6.3.9. $\square$

**Theorem 7.2.10.** *$L(C_n) \in \sigma_a(V_4)$ if and only if $n$ is a multiple of* 4.

*Proof.* Since $L(C_n) = C_n$, the proof follows from Theorem 6.3.10. □

**Corollary 7.2.11.** *$L(C_n) \in \sigma(V_4)$ if and only if $n$ is a multiple of 4.*

*Proof.* The proof follows from Theorem 7.2.9 and Theorem 7.2.10. □

**Theorem 7.2.12.** *Let $K_{1,n}$ be the star graph, then $L(K_{1,n}) \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Since $L(K_{1,n}) = K_n$, the proof follows from Theorem 6.3.16. □

**Theorem 7.2.13.** *Let $CB_n$ be the comb graph, then we have the following.*

**(i)** *$L(CB_n) \notin \sigma_0(V_4)$ for any $n$.*

**(ii)** *$L(CB_n) \notin \sigma_a(V_4)$ for any $n$.*

*Proof.* Let $\{u_i, v_i : i = 1, 2, 3, \ldots, n\}$ be the vertex set of $CB_n$, where $u_i$ is the pendant vertex adjacent to $v_i$. Also let $w_i = u_i v_i,\ i = 1, 2, 3, \ldots, n$ and $t_k = u_k u_{k+1},\ \ k = 1, 2, 3, \ldots, n-1$ be the edges in $CB_n$. Then $\{w_i,\ t_k\ :\ i = 1, 2, 3, \ldots, n,\ k = 1, 2, 3, \ldots, n-1\}$ are the vertices of $L(CB_n)$.

**Proof of (i)** .

If possible, suppose $L(CB_n) \in \sigma_0(V_4)$ for some $n$. Then there exists an EIML say $f : E(L(CB_n)) \to V_4 \setminus \{0\}$ with $f^{++}(e) = 0$ for all $e \in E(L(CB_n))$.

Let $f(t_i w_i) = \alpha_i$, $f(t_i w_{i+1}) = \beta_i$, for $i = 1, 2, 3, \ldots, n-1$ and $f(t_j t_{j+1}) = \gamma_j$, for $j = 1, 2, 3, \ldots, n-2$. Then the induced edge sum equation of the edges $t_i w_i$ gives the equation,

$$\gamma_1 + \beta_1 \quad = \quad \gamma_1 + \beta_1 + \gamma_2 + \beta_2$$

$$\begin{aligned} &= \gamma_2 + \beta_2 + \gamma_3 + \beta_3 \\ &\cdots \\ &= \gamma_{n-3} + \beta_{n-3} + \gamma_{n-2} + \beta_{n-2} \\ &= \gamma_{n-2} + \beta_{n-2} + \beta_{n-1}. \end{aligned}$$

But since $f^{++} \equiv 0$, we get $\gamma_1 + \beta_1 = 0$, using this fact in the above system of equations, we get

$$\gamma_1 + \beta_1 = \gamma_2 + \beta_2 = \gamma_3 + \beta_3 = \cdots = \gamma_{n-2} + \beta_{n-2} = 0.$$

That is $\gamma_i = \beta_i$ for $i = 1, 2, 3, \ldots, n-2$. Thus using $\gamma_{n-2} = \beta_{n-2}$ in the equation $\gamma_{n-2} + \beta_{n-2} + \beta_{n-1} = 0$, we get $\beta_{n-1} = 0$. That is $f(t_{n-1}w_n) = 0$, which is a contradiction. Hence our assumption is wrong, that is $L(CB_n) \notin \sigma_0(V_4)$ for all $n$.

**Proof of (ii)** .

If possible, suppose $L(CB_n) \in \sigma_a(V_4)$ for some $n$. Then there exists an EIML say $g : E(L(CB_n)) \to V_4 \setminus \{0\}$ with $g^{++}(e) = a$ for all $e \in E(L(CB_n))$.

Let $g(t_iw_i) = \alpha_i$, $g(t_iw_{i+1}) = \beta_i$, for $i = 1, 2, 3, \ldots, n-1$ and $g(t_jt_{j+1}) = \gamma_j$, for $j = 1, 2, 3, \ldots, n-2$.

Then the induced edge sum of the edge $t_1w_1$ and $t_2w_2$ gives $\gamma_1 + \beta_1 = \gamma_1 + \beta_1 + \gamma_2 + \beta_2 = a$. Thus we get $\gamma_1 + \beta_1 = a$ and $\gamma_2 + \beta_2 = 0$.

Similarly the induced edge sum of the edges $t_1w_2$ and $t_1t_2$ gives $\alpha_1 + \alpha_2 + \gamma_1 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_2$. Since $\gamma_2 + \beta_2 = 0$ the above equation reduces to $\alpha_1 + \alpha_2 + \gamma_1 = \alpha_1 + \alpha_2 + \beta_1$ and which implies that $\gamma_1 = \beta_1$. That is $\gamma_1 + \beta_1 = 0$, which is contradiction. Hence our assumption is wrong, that is $L(CB_n) \notin \sigma_a(V_4)$ for any $n$.

Hence the Proof. $\square$

**Theorem 7.2.14.** *For the flag graph $Fl_n$, we have $L(Fl_n) \notin \sigma_a(V_4)$ for any $n$.*

*Proof.* Let $V(Fl_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding cycle graph $C_n$ and $w$ is the root vertex adjacent to the vertex $v_1$. Also suppose $e_i = v_iv_{i+1}$ and $e = v_1w$ are the edges in $Fl_n$. Therefore we can take the vertex set of $L(Fl_n)$ equal to $\{e, e_1, e_2, e_3, \ldots, e_n\}$.
If possible, suppose $L(Fl_n) \in \sigma_a(V_4)$ for some $n$. Then there exists an EIML say $f : E(L(Fl_n)) \to V_4 \setminus \{0\}$ with $f^{++}(e) = a$ for all $e \in E(L(Fl_n))$.
Let $f(e_ie_{i+1}) = \alpha_i$, for $i = 1, 2, 3, \ldots, n$ with $i+1$ is taken modulo $n$, $f(e_1e) = \alpha$ and $f(e_ne) = \beta$. Then the induced edge sum equation of the edges $e_ne_1$, $e_1e$, and $e_ne$ gives the equation:

$$\alpha_{n-1} + \alpha_1 + \alpha + \beta = \alpha_n + \alpha_1 + \beta = \alpha_{n-1} + \alpha_n + \alpha = a.$$

but $\alpha_n + \alpha_1 + \beta = \alpha_{n-1} + \alpha_n + \alpha$ implies $\alpha + \beta = \alpha_1 + \alpha_{n-1}$, thus $\alpha_{n-1} + \alpha_1 + \alpha + \beta = 0$, which is a contradiction. Hence our assumption is wrong, that is $L(Fl_n) \notin \sigma_a(V_4)$ for any $n$.
Hence the Proof. $\square$

**Theorem 7.2.15.** *For the sun graph $Sun_n$, we have $L(Sun_n) \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Note that in $L(Sun_n)$ every vertex is of even degree. Therefore by Theorem 6.2.1, we have $L(Sun_n) \in \sigma_0(V_4)$ for all $n$. $\square$

**Theorem 7.2.16.** *Consider bistar graph $B_{m,n}$, then $L(B_{m,n}) \in \sigma_0(V_4)$ for $m$ and $n$ are even.*

*Proof.* Note that for $m$ and $n$ are even, every vertex in $L(B_{m,n})$ is of even degree. Therefore by Theorem 6.2.1, we have $L(B_{m,n}) \in \sigma_0(V_4)$ for all $n$. $\square$

**Theorem 7.2.17.** *For the triangular snake graph $TS_n$, we have $L(TS_n) \in \sigma_0(V_4)$ for all $n$.*

*Proof.* Since every vertex in $L(TS_n)$ is of even degree, by Theorem 6.2.1 the proof follows. $\square$

Chapter 8

# Conclusion and Further Scope of Research

*A summary of the thesis is given in the first section of the chapter. The following section includes some guidelines for a researcher to explore more areas.*

## 8.1 Summary of the Thesis

In this thesis, we introduced three types of graph labelings namely induced $A$-magic labeling, induced $V_4$-magic labeling and edge induced $V_4$-magic labeling. In the first part of the work, we discussed the induced $A$-magic labeling of some general graphs and induced $V_4$-magic labeling of cycle related, path related and star related graphs. Finally we discussed the induced $V_4$-magic labeling of Subdivision graph, Shadow graph, Middle graph and Line graph of some general and special graphs.

The thesis also introduced the concept of edge induced $V_4$-magic labeling of graphs and give the necessary and sufficient conditions for some general graphs like path $P_n$, cycle $C_n$, complete graph $K_n$ and the complete bipartite graph

$K_{m,n}$ and some more graphs having edge induced $V_4$-magic labeling. The thesis concluded with the study of edge induced $V_4$-magic labeling of Subdivision graph and Line graphs of some general and special graphs.

## 8.2 Further Scope of Research

(i) Study the Necessary and Sufficient conditions of induced $V_4$-magic labeling of some more graphs.

(ii) Examine the Necessary and Sufficient conditions of Induced $V_4$-magic labeling of operation of two graphs.

(iii) Investigate induced $V_4$-magic labeling of total graphs of some special graphs.

(iv) Study the Necessary and Sufficient conditions of edge induced $V_4$-magic labeling of some more graphs.

(v) Examine the Necessary and Sufficient conditions of edge induced $V_4$-magic labeling of operation of two graphs.

(vi) Investigate edge induced $V_4$-magic labeling of total graphs of some special graphs.

# Bibliography


[1] Akiyama. J., Hamada.T., and Yoshimura.Y., *Miscellaneous properties of middle graphs*, TRU Math. 10, 41-53, (1974).

[2] A. Krishnaa, *Some Applications of Labelled Graphs* International Journal of Mathematics Trends and Technology (IJMTT) –Volume 37, Number 3, September, (2016).

[3] A. Rosa, *On certain valuations of the vertices of a graph, Theory of Graph* (International Symposium, Rome, July 1996), Gordon and Breach, N. Y. and Dunod Paris, 349-355, (1967).

[4] Frank Harary, *Graph theory*, Addison-Wesley Publishing Company, Reading, MA, (1972).

[5] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, Mc Graw - Hill, Boston, (2005).

[6] John B. Fraleigh, *A first course in abstract algebra*, seventh edition, Pearson.

[7] John Clark and D. A. Holton, *A First Look at Graph Theory*, World Scientific, (1991).

[8] Joseph A. Gallian, *A dynamic survey of graph labeling.* The Electronics Journal of Combinatorics, 17, (2014).

[9] Joseph A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, Twenty-first edition, December 21, (2018).

[10] Koilraj S. and Ayyaswamy S. K., *Labeling of Graphs*, Ph.D thesis submitted to Bharatidasan university, Tiruchirappalli.

[11] Rajesh Kanna M. R., Pradeep Kumar R. and Jagadeesh R., *Computation of Topological Indices of Dutchwindmill Graph*, Open journal of Discrete Mathematics, 674-81, (2016).

[12] R. Balakrishnan and K. Ranganathan, *A text book of graph theory.* Springer-Verlag, NewYork, (2012).

[13] Ponraj R. et.al, *Radio mean labeling of a graph,* AKCE International Journal of Graphs and Combinatorics, 12, 224-228, (2015).

[14] P. Sumathi and A. Rathi, *Quotient Labeling of Some Ladder Graphs,* American Journal of Engineering Research (AJER) Volume-7, Issue-12, 38-42, (2018).

[15] R. L. Graham and N. J. A. Sloane, *On additive bases and harmonious graphs*, SIAM J. Alg. Discrete Math., 1, 382-404, (1980).

[16] Romain Boulet, *Spectral characterizations of sun graphs and broken sun graphs*, Discrete Mathematics and Theoretical Computer Science, Vol. 11, No. 2, 149-160, (2009).

[17] S. Arumukam and S. Ramachandran, *An introduction to graph theory.* Scitech Publications (India) Pvt. Ltd.

[18] Seoud M. A. and Youssef M. Z., *Harmonious labellings of helms and related graphs,* unpublished.

[19] S. M. Lee et. al, *On the $V_4$-magic graphs, Comgressus Numerantium*, 156, 59-67, (2002).

[20] Swaminathan V. and Jeyanthi P., *Super edge-magic strength of re crackers, banana trees and unicyclic graphs,* Discrete Math., 306, 1624-1636, (2006).

[21] Tom M. Apostol, *Introduction to analytic number theory.* Springer International, Student Edition.

[22] Vandana P. T. and Anil Kumar V., *$V_4$-magic labelings of wheel related graphs*, British journal of Mathematics and Computer science, 8(3), 189-219, (2015).

[23] Vandana P. T. and Anil Kumar V., *$V_4$-magic labelings of some graphs,* British journal of Mathematics and Computer science, 11(5), 1-20, (2015).

[24] Vaithilingam K., *Difference labeling of some graphs families*, International Journal of Mathematics and Statistics Invention (IJMSI), Vol.2, Issue 6, 37-43, June (2014).

[25] Yixiao Liu, Zhiping Wang, *The Rainbow Connection of Windmill and Corona Graph*, Applied Mathematical Sciences, Vol. 8, no. 128, 6367 - 6372, (2014).

# APPENDIX I

## List of Publications

1. K. B. Libeeshkumar and V. Anil Kumar, *Induced Magic Labeling of Some Graphs,* Malaya Journal of Matematik, Volume. 8, Number 1, 59-61, (2020).

2. K. B. Libeeshkumar and V. Anil Kumar, *Induced* $V_4-$*magic labeling of cycle related graphs,* Malaya Journal of Matematik, Volume. 8, Number 2, 473-477, (2020).

3. K. B. Libeeshkumar and V. Anil Kumar, *Induced* $V_4-$ *Magic Labeling of Some Star and Path Related Graphs,* South East Asian Journal of Mathematics and Mathematical Sciences, Volume 16, No. 2, 89-102, (2020).

4. K. B. Libeeshkumar and V. Anil Kumar, *Induced* $V_4-$*Magic Labeling of Middle Graphs,* Advances and Applications in Discrete Mathematics (Accepted for publication).

5. K. B. Libeeshkumar and V. Anil Kumar, *Induced* $V_4-$*Magic Labeling of Line Graphs,* Advances and Applications in Mathematical Sciences, Communicated.

6. K. B. Libeeshkumar and V. Anil Kumar, *Induced $V_4-$Magic Labeling of some Subdivision Graphs,* National Conference on Recent Frontiers in Fractional Calculus Theory and its Applications, Communicated.

7. K. B. Libeeshkumar and V. Anil Kumar, *Edge Induced $V_4-$ Magic Labeling of Graphs,* Malaya Journal of Matematik, Communicated.

# Index