C 63447 (Pages : 3) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Elective—ADVANCED COMPLEX ANALYSIS

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. (a) Prove that every meromorphic function can be expressed as the quotient of two entire functions.

   (b) Discuss the convergence of $\prod_{k=1}^{\infty}\left\{1-\frac{1}{k(k+3)}\right\}$.

   (c) Prove that a non-constant entire function of finite order assumes every complex value with only one possible exception.

   (d) Prove that a sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to f if and only if $\{f_n\}$ converges to f uniformly on all compact subsets of G.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

2. (a) Let $\{b_k\}_{k=1}^{\infty}$ be a sequence of distinct points having no accumulation point in the finite complex plane $\mathbb{C}$, and $\{P_k(2)\}_{k=1}^{\infty}$ be a sequence of polynomials without constant terms. Prove that there exists a meromorphic function $f(z)$ having the singular part $P_k\left(\frac{1}{z-b_k}\right)$ at $b_k (k \in N)$ and no other singularity in the finite complex plane $\mathbb{C}$.

   (b) Prove that $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$.

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3. (a) Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of distinct points having no finite accumulation point and $\{\mathcal{G}_j\}_{j=1}^{\infty}$ a completely arbitrary sequence of complex numbers (with repetitions permitted). Prove that there exists an entire function $f(z)$ such that $f(a_j) = \mathcal{G}_j$ where $j \in \mathbb{N}$.

   (b) State and prove the Cauchy criterion for the convergence of an infinite product.

4. (a) Let $f(z)$ be a bounded analytic function, not identically zero in the unit disc D. If $\{a_n\}_{n=1}^{\infty}$ is the sequence of roots of $f(z)$ in the unit disc D, each repeated according to their multiplicities, then prove that the product $\prod_{n=1}^{\infty} |a_n|$ is convergent.

   (b) Prove that $|1 - E_n(z)| \leq |z|^{n+1}$ where $z \in \overline{D}(0,1)$.

Unit II

5. State and prove Hadamard factorization theorem.

6. (a) Suppose $f(z)$ is a function continuous on Γ, where Γ consists of a finite number of curves $C_k (k = 1, 2, \ldots, m)$. Let k be a compact set that does not intersect Γ. Then for any $\in > 0$, prove that there exists a rational function $\phi(z)$ having all its poles on Γ and satisfying $\left| \frac{1}{2\pi i} \int_{\Gamma} \frac{f\mathcal{G}}{\mathcal{G} - z} d\mathcal{G} - \phi(z) \right| < \in$ for all $z \in K$.

   (b) Let K be a compact subset of $\mathbb{C}$ and a and b be two distinct arbitrary points that are in the same component of $\hat{C} - K$, $a \neq \infty$. Prove that any rational function $R_0(z)$ having its only pole at a can be uniformly approximated on K by rational functions whose only pole is at b.

7. (a) Let $f(z)$ be any power series at a point 'a' and let Γ be an arc joining 'a' to 'b'. If $f(z)$ can be continued along the arc Γ, prove that the analytic continuation of $f(z)$ at the point b along the arc Γ is unique.

   (b) For any region $\Omega \subset \hat{C}$, prove that there exists a function $f(z) \in H(\Omega)$ having the boundary $\partial\Omega$ of the region Ω as its natural boundary.

Unit III

8. (a) State and prove Schwarz reflection principle.

   (b) Suppose $\mathcal{F} \subset C(G, \Omega)$ is equicontinuous at each point of Ω. Prove that $\mathcal{F}$ is equicontinuous over each compact subset of G.

9. (a) Suppose that $\mathcal{F} \subset C(G, \Omega)$ satisfies the following two conditions :

   (i) For each z in G, $\{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω.

   (ii) $\mathcal{F}$ is equicontinuous at each point of G.

   Prove that $\mathcal{F}$ is normal in $C(G, \Omega)$.

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(b) If $f_n : G \to \mathbb{C}$ is analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ then prove that $f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z)$.

10. (a) A family $\mathcal{F}$ in H(G) is normal if and only if $\mathcal{F}$ is locally bounded.

(b) Prove that H(G) $\cup\{\infty\}$ is closed in $C(G, \mathbb{C}_\infty)$.

(4 × 16 = 64 marks)

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C 63445 (Pages : 2) Name..................................

Reg. No..............................

FOURTH SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Elective—PROBABILITY THEORY

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. (a) If $\{E_n\}$ is a sequence of events, define (i) lim inf E_n, (ii) lim sup E_n, and (iii) lim E_n. If $E_n = \left(2 - \frac{1}{n}, 2 + \frac{1}{n}\right)$ for n odd and $E_n = \phi$ for n even, discuss about the convergence of $\{E_n\}$.

(b) Show that any Borel function of a random variable X is again a random variable.

(c) Explain the term Lebesgue-measurable sets.

(d) Define discontinuities of a distribution function. State Jordan decomposition theorem on distribution functions.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

II. (a) Given a collection of events denoted by G. Let A be the class of all events constructed from G by finite unions and complimentations. Let B be the class of all events constructed from G by finite intersections and complimentations. Then prove or disprove "A = B".

(b) Define a Borel set. Let E be a Borel set in R. Define $E(x) = \{a + x; a \in E\}$ for $x \in R$. Then show that both $E(x)$ and its compliment $E'(x)$ are Borel sets.

III. (a) Show that a necessary and sufficient condition that a given function to be measurable is that its positive and negative parts are measurable.

(b) Given a function X defined on a sample space is such that $|X|$ is a random variable. Then prove or disprove : "X is also a random variable".

IV. (a) Define Lebesgue-Stieljes measure. Show that every distribution function of a random variable induces a Lebesgue-Stieljes measure.

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(b) Suppose $X_1, X_2, \ldots, X_n$ are the observations drawn independently from a distribution with distribution function $F(x)$. If the observations are ordered as $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$, then derive the distribution function $F_{\gamma:n}(x)$ of $X_{\gamma:n}$ for $1 \le \gamma \le n$.

Unit II

V. (a) Define moments of a random variable. If X is a random variable and $E(X) < \infty$, then show that

$$E(X) = \int_0^{\infty} [1 - F(x)]dx - \int_{-\infty}^{0} F(x)dx.$$

(b) State and prove Holder's and Jensen's inequalities on expectation.

VI. (a) Define Lr space. Show that it is a complete metric space.

(b) Define convergence in rth mean of a sequence of random variables. If $\{X_n\}$ is a sequence of random variables such that $X_n \xrightarrow{r} X$, then show that $X_n \xrightarrow{s} X$ for every positive integer s such that $s < r$.

VII. (a) Define characteristic function of a random variable. State and prove some of the basic properties of characteristic functions.

(b) State and establish the inversion formula connecting the distribution function and characteristic function of a random variable.

Unit III

VIII. (a) Let X_1, X_2, X_3 be three random variables with probability density functions f_1, f_2, f_3 respectively. Let the joint pdf of X_1, X_2 and X_3 be $f(x_1, x_2, x_3)$. Then prove that X_1, X_2, X_3 are independently distributed if and only if $f(x_1, x_2, x_3) = f_1(x_1) f_2(x_2) f_3(x_3)$.

(b) State and prove Kolmogorov 0-1 law.

IX. (a) State and prove Borel-Cantelli lemma.

(b) State and establish Kolmogorov inequality.

X. (a) Distinguish between law of large numbers and central limit theorems. Explain if central limit theorem holds for the sequence $\{X_n\}$ of independent random variables with X_n is normal having 0 mean and $\text{Var}(X_n) = 2^{-n}$.

(b) State and prove Lindeberg-Feller Central Limit Theorem.

(4 × 16 = 64 marks)

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C 63444 (Pages : 2) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

OPERATIONS RESEARCH (Elective 9)

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. (a) Define strongly connected graph. Give example of a graph that is strongly connected and another graph that is not strongly connected.

(b) A factory can manufacture two products A and B. The profit on a unit of A is Rs. 80 and of B is Rs. 40. The maximum demand for A is 6 units per week and of B it is 8 units. The manufacturer has set up a goal of achieving a profit of Rs. 640 per week. Formulate the problem as goal programming.

(c) Discuss the generalization of geometric programming problem through Kuhn-Tucker theory.

(d) Describe the method of axial directions in optimization problems.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

II. (a) Define potential difference in an arc of a graph. Prove that the potential difference in a cycle is zero.

(b) Show that if $\{x_i\}$ and $\{y_i\}$ are two flows in a graph, then $\{ax_i + by_i\}$, where a and b are real constants, is also a flow.

III. (a) Prove that the maximum flow in a graph is equal to the minimum of the capacities of all possible cuts in it.

(b) Find the maximum on negative flow in the network described below, arc (v_i, v_j) being denoted as (j, k). v_a is the source and v_b is the sink :

Arc ...	$(a, 1)$	$(a, 2)$	$(1, 2)$,	$(1, 3)$	$(1, 4)$	$(2, 4)$	$(3, 2)$	$(3, 4)$	$(4, 3)$	$(3, b)$	$(4, b)$
Capacity ...	8	10	3	4	2	8	3	4	2	10	9

Turn over

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IV. (a) Discuss the concept of parametric linear programming.

(b) Minimize $f = x_1 - 2x_2 + x_3$ subject to $2x_1 - x_2 + 2x_3 \leq 2 + 2\lambda, x_1 - x_2 \leq 3 + \lambda,$ $x_1 + 2x_2 - 2x_3 \leq 4 - 4\lambda, x_1, x_2, x_3 \geq 0.$

Unit II

V. (a) If F(X, Y) has a saddle point (X_0, Y_0) for every $Y \geq 0$, then with usual notations, prove that $G(X_0) \leq 0, Y_0' G(X_0) = 0.$

(b) Using Kuhn-Tucker conditions, find the minimum of $f(X) = (x_1 + 1)^2 + (x_2 - 2)^2$ subject to $g_1(X) = x_1 - 2 \leq 0$, $g_2(X) = x_2 - 1 \leq 0$, $x_1 \geq 0$, $x_2 \geq 0$.

VI. Solve the quadratic programming :

Minimize $-6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$ subject to $x_1 + x_2 \leq 2$, $x_1 \geq 0$, $x_2 \geq 0$.

VII. Use geometric programming method to minimize $f(X) = \frac{c_1}{x_1x_2x_3} + c_2x_2x_3 + c_3x_1x_3 + c_4x_1x_2$ where $c_i > 0, x_j > 0, i = 1,2,3,4, j = 1,2,3.$

Unit III

VIII. (a) What is meant by dynamic programming ? Illustrate the concept using an example.

(b) Minimize $u_1^2 + u_2^2 + u_3^2$ subject to $u_1 + u_2 + u_3 \geq 10; u_1, u_2, u_3 \geq 0.$

IX. (a) Describe the computational economy in Dynamic programming.

(b) Find the maximum of $f(x) = -0.55 + 3x - x^2$ by Rosenbrock algorithm starting from $x = 0$, $h = 1$.

X. (a) Discuss the line search methods in multidimensional search.

(b) Define unimodal function. Prove that the function $f(x) = 2 - x, 1 \leq x \leq 2$ is unimodal in (0, 2).

(4 × 16 = 64 marks)

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C 63443 (Pages : 2) Name..................................

Reg. No..............................

FOURTH SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

FLUID DYNAMICS—Elective 8

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Show that the magnitude of the vorticity multiplied by the cross-sectional area is constant along the filament.
2. Show that the stream function is constant along a streamline.
3. What is cavitation ?
4. Obtain the complex potential for a simple source.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 marks.

Unit I

I. (a) Obtain the equation of continuity for a liquid of an irrotational motion.

(b) Show that $u = -\dfrac{2xyz}{\left(x^2+y^2\right)^2}, v = \dfrac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}, w = \dfrac{y}{x^2+y^2}$ are the velocity-components of a possible fluid motion. Is this motion irrotational.

II. (a) Establish the energy equation, when the forces are conservative.

(b) State and prove Kelvin's minimum energy theorem.

III. (a) Obtain vector expression for velocity and vorticity.

(b) Show that the velocity potential

$$\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

gives a possible motion, and determine the form of the streamlines.

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Unit II

IV. (a) Discuss the streaming motion past a circular cylinder.

(b) Verify that the velocity potential $\phi = u\left(r + \frac{a^2}{r}\right)\cos\theta$ represents a streaming motion past a finxed circular cylinder.

V. (a) State and prove Blasius's theorem.

(b) Discuss the motion represented by $w = \frac{1}{2}\frac{ua^3}{z^2}$, and show that the streamlines are lemniscates.

VI. (a) Describe the geometrical construction of Joukowski aerofoils.

(b) Write a short note on aerofoils.

Unit III

VII. (a) Determine the effect on a wall of a source parallel to the wall.

(b) If there is a source at $(a, 0)$ and $(-a, 0)$ and sinks at $(0, a)$, $(0, -a)$, all of equal strength, show that the circle through these four points is a streamline.

VIII. (a) Show that if we map the z-plane on the ξ-plane by a conformal transformation $\xi = f(z)$, a source in the z-plane will transform into a corresponding point of the ξ-plane.

(b) Prove that in conformal transformation a doublet will transform into a doublet, but that the strength will differ.

IX. (a) What is Stoke's stream function ?

(b) Verify the $\psi = \left(\frac{A}{r^2}\cos\theta + Br^2\right)\sin^2\theta$ is a possibe form of Stoke's stream function, and find the corresponding velocity potential.

(4 × 16 = 64 marks)

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C 63439 **(Pages : 2)** **Name..................................**

Reg. No................................

FOURTH SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Elective 2—ALGEBRAIC NUMBER THEORY

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Show that if $\{\alpha_1,\ldots,\alpha_n\}$ is any Q-basis of a number field $K = Q(\theta)$ of degree n, then $\Delta[\alpha_1,\ldots,\alpha_n] = \det(T(\alpha_i\alpha_j))$.

2. Let $K = Q(\xi)$ where $\xi = e^{\frac{2\pi i}{p}}$ for a rational prime p. In the ring of integer $Z[\xi]$, show that $\alpha \varepsilon Z[\xi]$ is a unit iff $N_k(\alpha) = \pm 1$.

3. Show that the ring of integers D in a number field K is noetherian.

4. Sketch the lattice in R^2 generated by (– 1, 2) and (2, 2).

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

I. (a) Show that every subgroup H of a free abelian group G of rank n in free of rank $s \leq n$.

(b) Find the order of the grou G/H where G is free abelian with Z-basis x, y, z and H is generated by $41x + 32y - 999z, 16y + 3z, 2y + 111z$.

II. (a) Show that if k is a number field then $K = Q(\theta)$ for some algebraic number θ.

(b) Express $Q(\sqrt{2}, \sqrt[3]{5})$ in the form $Q(\theta)$.

III. (a) Show that every number field K possesses an integral basis.

(b) Let d be a square free rational integer and $d \equiv 1 \pmod 4$. Find an integral basis and discriminant for $Q(\sqrt{d})$.

Turn over

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Unit II

IV. (a) Find the group of units u of the integers in $Q\left(\sqrt{-3}\right)$.

(b) Show that factorization into irreducibles is not unique in the ring of integers of $Q\left(\sqrt{-10}\right)$.

V. (a) Show that the ring of integers of $Q\left(\sqrt{-2}\right)$ is Eluclidean.

(b) Show that every Euclidean domain is a principal ideal domain.

VI. (a) Let D be the ring of integers of a number field k of degree n. Show that if a and b are non-zero ideals of D, then $N(ab) = N(a)\,N(b)$.

(b) In $Z\left[\sqrt{-5}\right]$, define the ideals $p = \langle 2, 1+\sqrt{-5}\rangle$ and $q = \langle 3, 1+\sqrt{-5}\rangle$. Determine pq.

Unit III

VII. (a) Show that if X is a bounded subset of R^n and $v(X)$ exists and if $v\left(r(X)\right) \neq v(X)$, then r/X is not injective.

(b) State and prove Minkowski's theorem.

VIII. (a) Let $K = Q(\theta)$ be a number field of degree n, where θ is an algebraic integer. Explain the concept of the vectors space L^{st} and show that the σ-map is a ring homomorphism from K into L^{st}.

(b) Let $K = Q(\theta)$ where $\theta \varepsilon R$ and $\theta^3 = 2$. What is the σ-map in this case ? Show that this σ-map is injective.

IX. (a) Show that the equation $x^4 + y^4 = z^2$ has no integer solutions with $x, y, z \neq 0$.

(b) Let $K = Q(\xi)$ where $\xi = e^{\frac{2\pi i}{p}}$ for an odd prime p. Show that the only roots of unity in K are $\pm\xi^s$ for integers s.

(4 × 16 = 64 marks)

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C 63437 (Pages : 2) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper XVI—FUNCTIONAL ANALYSIS—II

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. (a) Let X be a Banach space. Show that $A \in BL(X)$ is invertible if and only if A is bijective.

   (b) Let X be a Banach space. Prove that if X is finite dimensional and strictly convex then X is uniformly convex.

   (c) Show that pointwise limit of a sequence of compact operators is compact.

   (d) Let H be a Hilbert space and $A \in BL(H)$. Show that A is normal if and only if $\|A\| = \|A^*\|$ for all $x \in H$.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

2. (a) Let X be a Banach space. Prove that the set of all invertible operators is open in BL(X) and the map $A \mapsto A^{-1}$ is continuous in this set.

   (b) Let X be a non-zero Banach space over C and $A \in BL(X)$. Prove that $\sigma(A)$ is a non-empty subset of C.

3. (a) Let X be a non-zero Banach space over C and $A \in BL(X)$. Then show that the spectral radius

   $r_\sigma(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}$.

   (b) Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of c_{00} with norm $\| \; \|_p$ is linearly isometric to l^q.

4. (a) Define uniform convexity of a normed space. Show that if X is finite dimensional and strictly convex then X is uniformly convex.

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(b) Let X be a Banach space which is uniformly convex in some equivalent norm. Show that X is reflexive.

Unit II

5. (a) Define a compact linear map. Let X and Y the Banach spaces and $F : X \to Y$ be linear. Prove that F is compact if and only if for every sequence $\{x_n\}$ in X, $(F(x_n))$ has a convergent subsequence.

(b) Let X be a normed space and $A \in CL(X)$. Show that every non-zero spectral value of A is an eigenvalue of A.

6. (a) Let X and Y be normed spaces and $F \in BL(X,Y)$. If $F \in CL(X,Y)$ then show that $F' \in CL(Y', X')$.

(b) Let X be an inner product space and $E \subset X$ convex how that there exists at most one best approximation from E to any $x \in X$.

7. (a) State and prove Riesz representation theorem.

(b) Prove that every Hilbert space is reflexive.

Unit III

8. (a) Let H be a Hilbert space and $A \in BL(H)$. If x_1 and x_2 are eigenvectors of A corresponding to distinct eigen values, then show that $x_1 \perp x_2$.

(b) Let $A \in BL(H)$ be self adjoint. Prove that A or $-A$ is a positive operator if and only if $|\langle A(x), y\rangle|^2 \le \langle A(x), y\rangle \langle A(y), x\rangle$ for all $x, y \in H$.

9. (a) Let $T : l^2 \to l^2$ defined by $T(x_1, x_2, \ldots) = \left(x_1, \frac{x_2}{2} \ldots\right)$. Show that T is a Hilbert Schdmit operator.

(b) Let H be a Hilbert space and $A \in BL(H)$. Then show that $k \in \sigma(A)$ if and only if $\bar{k} \in \sigma(A^*)$.

10. State and prove the spectral theorem for compact self adjoint operators.

(4 × 16 = 64 marks)

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C 2716 (Pages : 3) Name.......................................

Reg. No..................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

M.Sc. Mathematics

MAT 4E 10—NON-LINEAR PROGRAMMING

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Determine whether the function f given by

$$f(x_1, x_2, x_3) = 4x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + x_1x_3 - 3x_1 - 2x_2 + 15$$

is convex or not.

2. Define Lagrangian function and saddle point of a programming problem.

3. Explain the terms (a) Mixed strategy ; and (b) Pure strategy related to matrix game.

4. Find the saddle points and the game value of the pay-off matrix $\begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 5 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix}$.

(4 × 4 = 16 marks)

Part B

Answer any **two** *questions.*
Each question carries 8 marks.

5. Write Kuhn-Tucker conditions for the problem :

Minimize $f(x) = -x_1 - x_2 - x_3 + \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

subject to $x_1 + x_2 + x_3 - 1 \le 0$

$4x_1 + 2x_2 \le \frac{7}{3}$

$x_1, x_2, x_3 \ge 0.$

Turn over

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6. Solve the dynamic programming problem : Minimize $x^2 + y^2 + z^2$

subject to the constraints $xyz \le 6$, x, y, z are integers.

7. Solve the game, whose pay-off matrix is given below, by the principle of domination :

$$\begin{array}{c} \quad B \\ A \begin{bmatrix} 4 & 7 & 0 & 2 & 1 & 1 \\ 4 & 3 & 1 & 3 & 2 & 2 \\ 4 & 3 & 7 & -5 & 1 & 2 \\ 4 & 3 & 4 & -1 & 2 & 2 \\ 4 & 3 & 3 & -2 & 2 & 2 \end{bmatrix}. \end{array}$$

(2 × 8 = 16 marks)

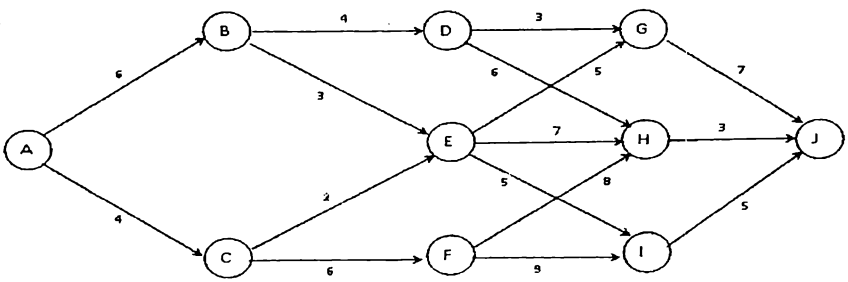
Part C

Answer **either** *A* **or** *B of each question.*
Each question carries 24 marks.

8. (A) Solve by the method of quadratic programming :

$$\text{Minimize } f(x) = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$$
$$\text{subject to} \quad 2x_1 + x_2 \le 6$$
$$x_1 - 4x_2 \le 0$$
$$x_1, \quad x_2 \ge 0.$$

(B) Following figure shows a network of cities spread over a state. A company has to transport some goods from city A to city J. The cost of transportation between different cities is given along the line connected in nodes. A node represents a city. Find the optimal root connecting A to J using dynamic programming.



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9. (A) (i) Explain dynamic programming problems with single additive constraint, additive the separable return.

(ii) State and prove the fundamental theorem of rectangular games.

(B) (i) Explain serial multi-stage model in dynamic programming.

(ii) Using graphical method solve the game, whose pay-off matrix is

		B			
		I	II	III	IV
A	I	1	3	− 3	7
	II	2	5	4	− 6

(2 × 24 = 48 marks)

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C 2715 (Pages : 3) Name..

Reg. No..

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

M.Sc. Mathematics

MAT 4E 05—ALGEBRAIC TOPOLOGY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. State Jordan Curve Theorem.

2. Write the closure of the 2-simplex $\sigma = < a_0 \, a_1 \, a_2 >$.

3. Define chain mapping between complexes and give an example.

4. Prove that : $\mathbb{R}^3$ and $\mathbb{R}^2$ are not homeomorphic ?

5. Give an example of a path connected space.

6. Define degree of a loop in S^1.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 *marks.*

7. Prove that a simplex σ is the smallest convex set which contains all vertices of σ.

8. Prove that a set $A = \{a_0, a_1 .....\}$ of points in $\mathbb{R}^n$ is geometrically independent if and only if the set of vectors $\{a_1 - a_2, ..., a_k - a_0\}$ is linearly independent.

9. Suppose that the complexes K_1 and K_2 have the same simplexes but different orientations. How are chain groups $C_p(K_1)$ and $C_p(K_2)$ related ?

Turn

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10. Prove that every simplicial mappings $\phi : |K| \rightarrow |L|$ is continuous.

11. Give an example of a simply connected space which is not contractible with justification.

12. If K is an oriented complex , $B_p(K), Z_p(K)$ are the p dimensional boundary group and cyclic group of K respectively, then prove that $B_p(K) \subset Z_p(K)$ for each integer p such that $0 \leq p \leq n$, where n is the dimension of K.

13. Prove that a discrete space X is contractible if and only if X has only one point.

14. State and prove the Generalized Covering Path Property.

(5 × 4 = 20 marks)

Part C

Answer **either** *A* **or** *B of each of the following questions.*
Each question carries 16 *marks.*

15. A (a) Let K be an oriented complex, σ^p an oriented p-simplex of K and σ^{p-2} a $(p-2)$-face of σ^p. Then prove that $\sum \left[\sigma^p, \sigma^{p-1}\right]\left[\sigma^{p-1}, \sigma^{p-2}\right] = 0,\ \sigma^p \in K$.

(b) If K is an oriented complex and $p \geq 2$, then prove that the composition $\delta\delta : C_p(K) \rightarrow C_{p-2}(K)$ in the diagram $C_p(K) \rightarrow^{\delta} C_{p-1}(K) \rightarrow^{\delta} C_{p-2}(K)$ is the trivial homomorphism.

B (a) Let K be a geometric complex with two orientations, and let K_1, K_2 denote the resulting oriented geometric complexes. Then prove that the homology groups $H_p(K_1)$ and $H_p(K_2)$ are isomorphic for each dimension p.

(b) Let K be a complex with r combinatorial components. Then prove that $H_o(K)$ is isomorphic to the direct sum of r copies of the group $\mathbb{Z}$ of integers.

16. A (a) If S is a simple polyhedron with V vertices, E edges, and F faces, then prove that $V - E + F = 2$.

(b) Show that an n-pseudomanifold K is orientable if and only if the n^{th} homology group $H_n(K)$ is not the trivial group.

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B (a) For any complex K, show that $\lim_{s \to \infty} \text{mesh } K^{(s)} = 0$.

(b) Let K and L be polyhedra with triangulations K and L respectively and $f : |K| \to |L|$ a continuous function such that K is star related to L relative to f. Then show that f has a simplicial approximation $g : |K| \to |L|$.

17. A (a) Show that two loops α and β in S^1 with base point 1 are equivalent if and only if they have the same degree.

(b) Show that the fundamental group $\pi_1(S^1)$ is isomorphic to the group $\mathbb{Z}$ of integers under addition.

B (a) If A is a deformation retract of a space X and x_0 is a point of A, then prove that $\pi_1(X, x_o)$ is isomorphic to $\pi_1(A, x_o)$.

(b) Let X be a space for which there is an open cover $\{V_i\}$ of X such that :

(i) $\cap V_i \neq \theta$,

(ii) Each V_i is simply connected, and

(iii) For $i \neq j$, $V_i \cap V_j$ is path connected. Then show that X is simply connected.

(3 × 16 = 48 marks)

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C 2714 (Pages : 2) Name.......................................

Reg. No..................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

M.Sc. Mathematics

MAT 4E 04—ALGEBRAIC GRAPH THEORY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Obtain one spanning subgraph and one induced subgraph of Petersen graph.
2. What do you mean by proper colouring of a graph ?
3. Define isomorphism between two graphs.
4. Define an asymmetric graph with an example.
5. Petersen graph is a Cayley graph : Justify your answer ?
6. Define edge connectivity of a graph.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 *marks.*

7. Prove that the automorphism group of a graph is equal to the automorphism group of its complement.
8. Characterize a line graph in terms of its induced subgraph.
9. State and prove Orbit - Stabilizer Lemma.
10. Let G be a transitive permutation group on V. Show that G is primitive if and only if each nondiagonal orbit is connected.
11. Prove that $|\mathrm{Aut}(C_n)| = 2n$.
12. Show that the Cayley graph X(G, C) is vertex transitive.
13. Let X be vertex and edge transitive, but not are transitive, Show that its valency is even.
14. Prove any two properties of fragments.

(5 × 4 = 20 marks)

Turn over

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Part C

Answer either A **or** *B of each of the following questions.*
Each question carries 16 *marks.*

Unit I

15. A) (i) Prove that the chromatic numbers of a graph X is the least integer r such that there is a homomorphism from X to K_r.

(ii) Show that X and $\overline{X}$ have the same automorphism group, for any graph X.

B) State and prove Euler's polyhedral formula for planar graphs.

Unit II

16. A) (i) Prove that almost all graphs are asymmetric.

(ii) Let D be a directed graph such that the in-valency and out-valency of any vertex are equal. Then, show that D is strongly connected if and only if it is weakly connected.

B) Let G be a transitive permutation group on V and let x be point in V. Prove that G is primitive if and only if G_x is maximal subgroup of G.

Unit III

17. A) (i) Prove that k—cube Q_k is vertex transitive.

(ii) Let u and v be distinct vertex in X and let P be a path from u to v. If no vertex in $V(P)\setminus\{u, v\}$ is critical then show that maximum matching misses both u and v.

B) (i) Show that any two paths of maximum length in a connected graph must have at least one vertex in common.

(ii) Prove that a transitive abelian permutation group is regular.

(3 × 16 = 48 marks)

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C 2710 (Pages : 2) Name.......................................

Reg. No...................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 4E 04—ALGEBRAIC GRAPH THEORY

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1) Find the automorphism group of C_3.

2) Give an example of a maximal planar graph.

3) Prove that stabilizer of an element in a permutation group V is a subgroup of V.

4) Prove that a directed graph is weakly connected if and only if its underlying undirected graph is connected.

5) Draw the Cayley graph $X(\mathbb{Z}_4, \{1,2,3\})$.

6) Is an edge transitive graph regular ? Justify your answer.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 *marks.*

7) Show that any edge in a bipartite graph X is a retract of X.

8) Is $Aut(K_4)$ isomorphic to $Aut(L(K_4))$? Justify your answer.

9) Let X be a maximal planar graph on n vertices. Prove that X has $3n - 6$ edges.

10) Let G_v denote the class of graphs on the vertex set V. For $X, Y \in G_v$, define ~ by $X \sim Y \Leftrightarrow X$ and Y are isomorphic.

Prove that ~ is an equivalence relation on G_v.

11) Prove that Sym(3) is a primitive group.

12) Prove that the k-cube Q_k is vertex transitive.

Turn over

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13) Prove that the Cayley graph X (G, C) is connected if and only if C generates the group G.

14) If A and B are fragments in a graph X, then prove that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

(5 × 4 = 20 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 16 marks.

Unit I

15) A (a) Let x and y be vertices in a graph X and let g be an automorphism of X. Prove that the distance between x and y in X is equal to the distance between x^g and y^g in X.

(b) If the line graph of a connected graph X is regular, then prove that X is regular or bipartite and semiregular.

B (a) Let X be a graph with n vertices. If $d_1, d_2, ..., d_n$ are the valencies of n vertices of X, then prove that the number of edges in the line graph L(X) is

$$\frac{1}{2}\left(\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i\right).$$

(b) Prove that K_5 is not planar.

Unit II

16) A (a) Is K_3 asymmetric ? Justify your answer.

(b) Prove that the number of isomorphism classes of graphs on n vertices is at most

$$(1+o(1))\frac{2^{\binom{n}{2}}}{n!}.$$

B (a) Let G be a permutation group on the set V and let x be a point in V. If $g \in G$, then prove that $g^{-1}G_x g = G_{x^g}$.

(b) Let G be a transitive permutation group on V. Prove that G is primitive if and only if each non-diagonal orbit is connected.

Unit III

17) A (a) Let X is a connected vertex-transitive graph, then prove that its edge connectivity is equal to its valency.

(b) If a graph X is vertex and edge transitive, but not arc transitive, then prove that its valency is even.

B (a) Prove that Petersen graph is not a Cayley graph.

(b) If A is an atom and B is a fragment of X, then prove that A is a subset of exactly one of B, N(B) and $\overline{B}$.

(3 × 16 = 48 marks)

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C 2713 (Pages : 3) Name.....................................

Reg. No.................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

M.Sc. Mathematics

MAT 4E 02—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

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Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Let X be a vector space over $\mathbb{K}\ (= \mathbb{R}\ or\ \mathbb{C})$ and $A, B : X \to X$ be two linear maps. Show that A^{-1} and B^{-1} exist if and only if $(AB)^{-1}$ and $(BA)^{-1}$ exist.
2. Show that a normed space X is separable if the dual space X' is separable.
3. Define compact operators with examples. Show that a compact operator on an infinite dimensional Banach space is never invertible.
4. State True *or* False. Justify your claim : Every Banach space is a Hilbert space.
5. Give example for an infinite orthonormal set. Show that every orthonormal set is linearly independent.
6. Let $A : l^2 \to l^2$ be defined by $A(x(1), x(2), \ldots) = (0, x(1), x(2), \ldots)$; for all $(x(1), x(2), \ldots) \in l^2$. Find the adjoint A*.
7. Define normal, unitary and self-adjoint operators. Give an example for a normal operator that is not unitary.
8. Define Hilbert-Schmidt operators and show that A is Hilbert-Schmidt if and only if A* is Hilbert-Schmidt.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Define the spectral radius $r_\sigma(A)$ of a bounded operator on a normed space X. Show by an example that $r_\sigma(A)$ can be strictly less than $\inf_{n=1,2,\ldots} \|A^n\|^{\frac{1}{n}}$.

Turn over

10. Show that the spectrum of a compact operator on a Banach space is at most countable.

11. Let $\{u_1, u_2, ...\}$ be a countable orthonormal set in an inner product space X and $x \in X$. Show that the sequence $\{\langle x, u_n \rangle ; n \in \mathbb{N}\}$ is in l^2.

12. Show that if A is a normal operator on a Hilbert space, and λ is an eigenvalue of A, then $\bar{\lambda}$ is an eigenvalue of A*. Show by an example that this need not be true if A is not normal.

13. Show that $x_n \xrightarrow{w} x$ if and only if $x_n \to x$ in l^1.

14. Show that sum of two compact operators is compact. Also show that composition of a compact operator and a bounded operator is compact.

(4 × 4 = 16 marks)

Part C

Answer **either** *part (a)* **or** *(b) of each of the following questions.*
Each question carries 12 *marks.*

15. (a) i) Introduce the spectrum $\sigma(A)$, eigenspectrum $\sigma_e(A)$, approximate eigenspectrum $\sigma_a(A)$ of a bounded operator A on a normed space X. Show that $\sigma_e(A) \subset \sigma_a(A) \subset \sigma(A)$.

ii) Show by examples that the above inclusions can be strict. Also give examples of equality also.

iii) Give example for an operator with no eigenvalues but $\sigma(A)$ is non-empty.

Or

(b) For $1 \le p < \infty$, show that the dual $(l^p)'$ of l^p is isometrically isomorphic to l^q, where $\frac{1}{p} + \frac{1}{q} = 1$.

16. (a) Show that the class CL(X) of all compact operators on a Banach space X is a closed two sided ideal in BL(X), the class of all bounded operators on X.

Or

(b) Let X be a normed space and A be a compact operator on X.

i) Show that every non-zero spectral value of A is an eigenvalue of A.

ii) Show that $\sigma_a(A) = \sigma(A)$.

17. (a) i) State and prove Projection theorem.

ii) State and prove Riesz Representation theorem.

Or

(b) Let $\mathbb{H} = L^2(\mathbb{R})$ and $z \in L^\infty(\mathbb{R})$. Define A on $\mathbb{H}$ by $A(x) = zx$, $x \in \mathbb{H}$. Show that A is a bounded linear map on $\mathbb{H}$ and $\|A\| = \|z\|_\infty$. Also find A*.

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18. (a) Let $\mathbb{H}$ be a Hilbert space and $A \in BL(\mathbb{H})$.

i) If A is self-adjoint, then show that $\|A\| = \sup\{|\langle A(x), x\rangle| : x \in \mathbb{H}, \|x\| \leq 1\}$.

ii) Show that A is unitary if and only if $\|A(x)\| = \|x\|$, for every $x \in \mathbb{H}$. Also show that in this case, $\|A^{-1}(x)\| = \|x\|$ for every $x \in \mathbb{H}$ and $\|A\| = 1 = \|A^{-1}\|$.

iii) Show that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$, for every $x \in \mathbb{H}$. Also show that in this case, $\|A^2\| = \|A\|^2 = \|A^*A\|$.

Or

(b) i) Introduce the numerical range $\omega(A)$ of a bounded operator A on a Hilbert space $\mathbb{H}$. Show that neither $\sigma(A)$ nor $\omega(A)$ is contained in the other in general.

ii) Show that $\omega(A)$ need not be closed.

iii) Show that $\sigma(A) \subset \overline{\omega(A)}$.

(4 × 12 = 48 marks)

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C 2709 (Pages : 3) Name.....................................

Reg. No.................................

FOURTH SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 4E 02—ADVANCED FUNCTIONAL ANALYSIS

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Let X be a normed space over K and (K_n) be a sequence of eigen values of $A \in BL(X)$. Show that if $K_n \to K$ in K, then k is an approximate eigen value of A.

2. Show that if X is a finite dimensional normed space, then its dual X' has the same dimension as X.

3. Show that the dual space of a reflexive normed space is reflexive.

4. Let X be an inner product space and $x \in X$. Show that $\langle x, y \rangle = 0$ for all $y \in X$ iff $x = 0$.

5. Let F be a subspace of a normed space X and $x \in X$. Show that if $y \in F$ with $x - y \perp F$, then y is a best approximation from F to x.

6. Let H be a Hilbert space, $\{u_1, u_2, \ldots\}$ a countable orthonormal set and $k_1, k_2, \ldots$ belong to K. Show that if $\sum_n |k_n|^2 < \infty$, then $\sum k_n u_n$ converges in X.

7. Let H be a Hilbert space and $A \in BL(H)$ be self-adjoint. Show that $A^2 \geq 0$ and $A \leq \|A\|.I$.

8. Let H be a Hilbert space and $A \in BL(H)$. Show that if $A(x_1) = k_1 x_1$ and $A^*(x_2) = \bar{k}_2 x_2$ for $k_1 \neq k_2$ in K and $x_1, x_2 \in X$, then $x_1 \perp x_2$

(8 × 2 = 16 marks)

Turn over

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Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Let X be a finite dimensional normed space. Show that $x_n \xrightarrow{w} x$ in X iff $x_n \to x$ in X.

10. Let X be a Banach space and $P \in BL(X)$ be a projection. Show that $P \in CL(X)$ iff P is of finite rank.

11. Show that every inner product space is a normed linear space.

12. Let $X = C([-1,1]), x(t) = 1 - t^2, x_0(t) = 1$ and $x_1(t) = \cos \pi t$, for t in $[0,1]$. Show that the best approximation from span $\{x_0, x_1\}$ is $2/3 + \frac{4x_1}{\pi^2}$ to x.

13. Let H be a Hilbert space and $A \in BL(H)$. Show that $\| A^* \| = \| A \|$ and $\| A^* A \| = \| A \|^2 = \| A A^* \|$.

14. Show that every compact operator is bounded. Is the converse true ? Justify your answer.

(4 × 4 = 16 marks)

Part C

Answer **either** *part A* **or** *B of each of the following questions.*
Each question carries 12 *marks.*

15. A (i) Let X be a Banach space. Show that the set of all invertible operators is open in BL (X) and that the map $A \mapsto A^{-1}$ is continuous on this set.

(ii) Let $X = l^2$ and $A : X \to X$ be defined by :

$$A(x) = \left(0, x(1), \frac{x(2)}{2}, \dots\right) \text{ for } x = (x(1), x(2), \dots) \in X.$$ Show that $\sigma_e(A) = \phi$ and :

$\sigma_a(A) = \{0\} = \sigma(A)$.

B Let $1 \le p \le \alpha$ and $\frac{1}{p} + \frac{1}{q} = 1$. For a fined $y \in l^q$, let $f_y(x) = \sum_{j=1}^{\infty} x(j)\, y(j)$, for $x \in l^p$. Show that :

(i) $f_y \in (l^p)'$ and $\| f_y \| = \| y \|_q$.

(ii) the map $F : l^2 \to (l^p)'$ defined by $F(y) = f_y, y \in l^q$, is a linear isometry from l^q into $(l^p)'$

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16. A (i) Let X and Y be Banach spaces and $F : X \to Y$ be linear. Show that F is a compact map and R (F) is closed in Y iff F is continuous and of finite rank.

(ii) Let $X = l^2$, the Banach sequence space, and let $M = \text{diag}(k_1, k_2, \ldots)$. Show that if $K_n \to 0$ as $n \to \infty$, then M defines a map in CL (X).

B Let X be a normed space and $A \in CL(X)$. Show that every non-zero spectral value of A is an eigen value of A.

17. A (i) Let X be an inner product space and $f \in X'$ and let $\{u_1, u_2, \ldots\}$ be an orthonormal set in X. Show that $\sum_n \left| f(u_n)^2 \right| \leq \| f \|^2$.

(ii) State and prove unique Hahn–Banach extension theorem for Hilbert spaces.

B (i) Prove or disprove : Let X be an inner product space and $A \in BL(X)$. Then there is a $B \in BL(X)$ such that $\langle A(x), y \rangle = \langle x, B(y) \rangle$ for all $x, y \in X$.

(ii) Let H be a Hilbert space and $A \in BL(H)$. Show that R (A) = H iff A^* is bounded below, and that R (A^*) = H iff A is bounded below.

18. A (i) Let H be a Hilbert space and $A \in BL(H)$. Show that A is unitary iff $\| A(x) \| = \| x \|$ for all $x \in H$ and A is surjective.

(ii) Let H be a finite dimensional Hilbert space over $K = C$ and $A \in BL(H)$ be normal. Show that there is an orthonormal basis for H consisting of eigenvectors of A.

B (i) Let H be a Hilbert space and $A \in BL(H)$ be compact show that A* is compact.

(ii) Let A be a compact operator on a non-zero Hilbert space H. Show that every non-zero approximate eigenvalue of A is an eigenvalue of A and the corresponding eigen space is finite dimensional.

(4 × 12 = 48 marks)

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C 2053 (Pages : 4) Name.......................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 13—WAVELET THEORY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries weightage 1.

1. Define conjugate reflection of $w \in l^2(\mathrm{Z_N})$. Prove that $z * \bar{w}(k) = \langle z, \mathrm{R}_k w \rangle$.

2. Compute $\hat{z}$ for $z = (1, i, 2+i, -3) \in l^2(\mathrm{Z}_4)$.

3. Let $w \in l^2(\mathrm{Z_N})$. Then prove that $\{\mathrm{R}_k w\}_{k=0}^{\mathrm{N}-1}$ is an orthonormal basis for $l^2(\mathrm{Z_N})$ if and only if $|\hat{w}(n)| = 1$ for all $n \in \mathrm{Z_N}$.

4. Suppose $z \in l^2(\mathrm{Z})$. Then prove that $\tilde{z}, z^* \in l^2(\mathrm{Z})$, and $\mathrm{R}_k z \in l^2(\mathrm{Z})$, for all $k \in \mathrm{Z}$.

5. Suppose $z, w \in l^2(\mathrm{Z})$. Then prove that :

   (i) $(\tilde{z} * w) = \tilde{z} * \tilde{w}$.

   (ii) $\mathrm{U}(\tilde{z}) = \mathrm{U}(\tilde{z})$.

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6. Prove that the trigonometric system is an orthonormal set in $L^2[-\pi,\pi)$.

7. For $g : \mathscr{R} \to C$ and $t \in \mathscr{R}, t > 0$, define the t-dilation of g. What do you mean by an approximate identity ?

8. What is Multi-Resolution Analysis ?

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each unit.*
Each question carries weightage 2.

Unit 1

9. Let $T : l^2(Z_N) \to l^2(Z_N)$ be a translation-invariant linear transformation. Then prove that T is diagonalizable.

10. Let $b \in l^2(Z_N)$. Then define the convolution operator T_b. Prove that T_b is translation-invariant.

11. State and prove the Folding lemma.

Unit 2

12. Suppose $f \in L^1([-\pi,\pi))$ and $\langle f, e^{in\theta} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta} d\theta = 0$ for all $n \in Z$. Then prove that $f(\theta) = 0$ a.e.

13. Suppose that $u, v \in l^1(Z)$. Then prove that $B = \{R_{2k}v\}_{k \in Z} \cup \{R_{2k}u\}_{k \in Z}$ is a complete orthonormal set in $l^2 Z$ if and only if the system matrix $A(\theta)$ is unitary for all $\theta \in [0,\pi)$.

14. Suppose $u \in l^1(Z)$ and $\{R_{2k}u\}_{k \in Z}$ is orthonormal in $l^2(Z)$. Define a sequence $v \in l^1(Z)$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$. Then prove that $\{R_{2k}v\}_{k \in Z} \cup \{R_{2k}u\}_{k \in Z}$ is a complete orthonormal system in $l^2(Z)$.

Unit 3

15. Prove that $L^2(\mathscr{R})$ is an innerproduct space.

16. If $f \in L^1(\mathscr{R})$ and $\hat{f} \in L^1(\mathscr{R})$, then show that $\frac{1}{2\pi}\int_R \hat{f}(\xi) e^{ix\xi} d\xi = f(x)$ at every Lebesgue point x of f.

17. Explain Haar MRA.

(6 × 2 = 12 weightage)

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Part C

Answer any **two** *questions.*
Each question carries weightage 5.

18. (i) State and prove convolution theorem in $l^2(Z_n)$.

(ii) Verify convolution theorem for $z = (2, i, 1, 0), w = (1, 0, 2, 3) \in l^2(Z_n)$.

19. (i) Suppose $\theta_0 \in (-\pi, \pi)$ and $\alpha > 0$ is sufficiently small that $-\pi < \theta_0 - \alpha < \theta_0 + \alpha < \pi$. Define intervals $I = (\theta_0 - \alpha, \theta_0 + \alpha)$ and $J = \left(\theta_0 - \frac{\alpha}{2}, \theta_0 + \frac{\alpha}{2}\right)$. Then prove that there exists $\delta > 0$ and a sequence of real valued trigonometric polynomials $\{p_n(\theta)\}_{n=1}^{\infty}$ such that :

(a) $p_n(\theta) \geq 1$ for $\theta \in I$.

(b) $p_n(\theta) \geq (1+\delta)^n$ for $\theta \in J$.

(c) $|p_n(\theta)| \leq 1$ for $\theta \in [-\pi, \pi) \setminus I$.

(ii) Suppose N is even, say N = 2M, $z \in l^2(Z_N)$, and $x, y, w \in l^2\left(Z_{\frac{N}{2}}\right)$. Then prove that :

(a) $D(z) * w = D(z * U(w))$.

(b) $U(x * y) = U(x) * U(y)$.

20. (i) Suppose $v, w \in l^1(Z)$ and $z \in l^2(Z)$. Then prove that :

(a) $(z \hat{*} w)(\theta) = \hat{z}(\theta)\hat{w}(\theta)$ *a.e.*

(b) $z * w = w * z$.

(c) $v * (w * z) = (v * w) * z$.

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(ii) Suppose $u_1, v_1 \in l^1(\mathrm{Z})$ for each $l \in \mathcal{N}$, and the system matrix $\mathrm{A}_l(\theta)$ is unitary for all $\theta \in [0,\pi)$. Define $f_1 = v_1, g_1 = u_1$ and inductively, for $l \in \mathcal{N}, l \geq 2$, define $f_l = g_{l-1} * \mathrm{U}^{l-1}(v_l)$ and $g_l = g_{l-1} * \mathrm{U}^{l-1}(u_l)$. For each $l \in \mathcal{N}$, define

$\mathrm{V}_{-l} = \left\{\sum_{k \in Z} z(k) \mathrm{R}_{2^l k} g_l : z = (z(k))_{k \in \mathrm{Z}} \in l^2(\mathrm{Z})\right\}$ .Suppose $\bigcap_{l \in \mathrm{N}} \mathrm{V}_{-l} = \{0\}$.

Then prove that $\mathrm{B} = \left\{\mathrm{R}_{2^l k} f_l : k \in \mathrm{Z}, l \in \mathrm{N}\right\}$ is a homogeneous wavelet system in $l^2(\mathrm{Z})$.

21. (i) Define approximate identity.

(ii) Suppose $f \in \mathrm{L}^1(\mathscr{R})$ and $\{g_t\}_{t>0}$ is an approximate identity. Then prove that for every Lebesgue point x of f, $\lim_{t \to 0} + g_t * f(x) = f(x)$.

(2 × 5 = 10 weightage)

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C 2052 (Pages : 3) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 12—REPRESENTATION THEORY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Verify whether $A(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives a representation of the cyclic group $\{1, a : a^2 = 1\}$.

2. Let A (x) be the permutation representation of the symmetric group S_4. Find the matrix A (x) for $x = (12)$.

3. Let $\mathbb{R}^3$ be the S_3-module given by :

   $(x_1, x_2, x_3)\sigma = (x_{1\sigma}, x_{2\sigma}, x_{3\sigma})$

   for $(x_1, x_2, x_3) \in \mathbb{R}$ and $\sigma \in S_3$. Find a one dimensional submodule of $\mathbb{R}^3$.

4. Let υ be the character of the permutation representation of S_3. Find $\upsilon(a)$ where $a = (123)$.

5. Let $\phi = \chi^{(1)} + 2\chi^{(2)}$ be a character of the cyclic group $\mathbb{Z}_5$ where $\chi^{(1)}$ and $\chi^{(2)}$ are simple characters.

   Find the inner product $\langle \phi, \chi^{(1)} \rangle$.

6. Find the number of simple characters of $\mathbb{Z}_3$.

Turn over

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7. Let G be the subgroup of S_4 generated by (12) (34). Verify whether G is transitive.

8. Let ψ be the trivial character of the subgroup A_4 of the symmetric group $G = S_4$. Find $\psi^G(1)$ where ψ^G is the induced character.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each unit. Each question has weightage* 2.

Unit I

9. Let G be the cyclic group of order 3. Give a G-module action on $\mathbb{R}^3$.

10. Let A and B be 2 × 2 matrices. Show that trace (A B) = trace (B A).

11. Let V, U be G-modules and $\theta : V \to U$ be a module homomorphism. Show that ker θ is a G-submodule of V.

Unit II

12. Let v be the permutation character and ψ be the alternating character of S_3, find the inner product $\langle v, \psi \rangle$.

13. Let ϕ be a character of a group G. Show that $\phi(x) = \phi(y)$ if x and y are conjugates in G.

14. Let $\mathbb{R}^3$ be the S_3-module given by :

$$(x_1, x_2, x_3)\sigma = (x_{1\sigma}, x_{2\sigma}, x_{3\sigma})$$

for $(x_1, x_2, x_3) \in \mathbb{R}$ and $\sigma \in S_3$. Find all $(x, y, z) \in \mathbb{R}^3$ such that $(x, y, z)\sigma = (x, y, z)$ for all $\sigma \in S_3$.

Unit III

15. Define doubly transitive permutation group.

16. Let G be a transitive permutation group on 1, 2, . . . , n. Let H be the stabilizer of 1. Show that the stabilizer of α is $p_\alpha^{-1} H p_\alpha$ where p_α maps 1 to α.

17. Let ϕ be the character of a subgroup H of a group G. Show that :

$$\phi^G(x) = \frac{1}{h} \sum_{y \in G} \phi\left(yxy^{-1}\right)$$

where h is the order of H.

(6 × 2 = 12 weightage)

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Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Define G-module.

(b) Let V be a G-module and $\{v_1, v_2, \ldots, v_3\}$ be a basis of V. Describe the matrix representation of G provided by the G-module V with respect to the above basis.

19. (a) Define complete reducibility of a representation.

(b) Give an example of a non-trivial completely reducible representation of degree 4 of the cyclic group $\mathbb{Z}_4$. Verify the complete reducibility.

20. (a) Let $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(k)}$ be the simple characters of degree $f_1, f_2, \ldots, f_k$ respectively of a finite group G of order g. Show that $f_1^2 + f_2^2 + \ldots + f_k^2 = g$.

(b) Let G be a group of order 8 with five simple characters. Find the degrees of each of the characters.

21. Let H be a subgroup of a group G and B (x) be a representation of H.

(a) Describe the induced representation on G induced by B (x).

(b) Let ϕ be a character of G and ψ be a character of H. Let ψ^G be the induced character of ψ and ϕ_H be the character of H obtained by restricting ϕ to H. Show that :

$$\left\langle \psi^G, \phi \right\rangle = \langle \psi, \phi_H \rangle.$$

(2 × 5 = 10 weightage)

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C 2051 (Pages : 2) Name.......................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 11—GRAPH THEORY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Define a Forest.
2. Define independence number and covering number of a graph.
3. Define matching and perfect matching.
4. Find edge connectivity of a disconnected graph.
5. State Euler's formula.
6. Define critical graph.
7. Define Bridge.
8. Find Chromatic polynomial of a complete graph.

($8 \times 1 = 8$ weightage)

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Part B

Answer any **two** *questions from each unit.*
Each question carries a weightage of 2.

Module I

9. Show that :

   (1) every connected graph contains a Spanning tree.

   (2) if G is connected, then $\varepsilon \geq \upsilon - 1$.

10. Show that C (G) is well defined.

11. Prove that a vertex υ of G is a cut vertex implies that $d(\upsilon) > 1$.

Module II

12. Prove that if G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

13. Prove that in a critical graph no vertex cut is a clique ?

14. Prove that a connected graph has an Euler trail if and only if it has atmost two vertices of odd degree ?

Module III

15. Prove that :

   (a) If G is k-critical then $\delta \geq k - 1$.

   (b) Every k-chromatic graph has atleast k vertices of degree atleast $k - 1$.

16. Prove that if G is a simple planar graph with $\upsilon \geq 3$, then $\varepsilon \leq 3\upsilon - 6$.

17. Prove that for any graph G, $\pi_k(G)$ is a polynomial in k of degree υ with integer co-efficients.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. In a graph G prove that $k \leq k' \leq \delta$.

19. If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

20. Prove that every 3-regular graph without cut edges has a perfect matching.

21. If G is a simple graph with $\upsilon \geq 3$ and $\delta \geq \frac{\upsilon}{2}$, then prove that G is Hamiltonian.

(2 × 5 = 10 weightage)

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C 2050 (Pages : 3) Name..

Reg. No..

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 10—FLUID DYNAMICS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *the questions.*
Each question carries weightage 1.

1. Show that a vortex filament cannot terminate at a point within the fluid.
2. Show that the constancy of circulation in a circuit moving with the fluid in an inviscid fluid in which the density is either constant or is a function of the pressure.
3. Define irrotational motion.
4. Obtain the equation satisfied by the velocity potential.
5. What is Cavitation ?
6. Find the point of minimum pressure on the elliptic cylinder $\xi = \xi_0$.
7. Let there be a source of strength m at $z = f$, where f is real, outside the cylinder of radius a whose center is at the origin. Determine the complex potential.
8. How are air ship forms formed ?

(8 × 1 = 8 weightage)

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Turn over

Part B

Answer any **six** *questions.*
Each question carries weightage 2.

Unit 1

9. Establish the equation of continuity for an incompressible fluid in the from :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

10. State and prove Kelvin's minimum energy theorem.

11. In irrotational motion in two dimensions, prove that $\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=q\nabla^2 q.$

Unit 2

12. State and prove Blasius's theorem.

13. State and prove the theorem of Kutta and Joukowski.

14. Discuss the type of transformation that will convert the flow past a circular cylinder (with or without circulation) to the flow past a body of aerofoil shape in a perfect fluid.

Unit 3

15. Show that if we map the z-plane on the ξ-plane by a conformal transformation $\xi = f(z)$, then a source in the z-plane will transform into a source at the corresponding point of the ξ-plane.

16. Discuss the force exerted on a circular cylinder by a source.

17. Verify that $\psi=\left(\frac{A}{r^2}\cos\theta+Br^2\right)\sin^2\theta$ is a possible form of Stoke's stream function, and find the corresponding velocity potential.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage 5.

18. (a) Explain the method of differentiation following the fluid, and find the condition that the surface F $(x, y, z, t) = 0$ may be boundary surface.

(b) Prove that acyclic irrotational motion is uniquely determined when the boundary velocities are given.

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19. Explain the derivation of Joukowski aerofoil by the transformation $\xi = z + \sum_{r=1}^{n} \frac{a_x}{z_r}$ applied to the circle centre z_0 and radius a. Obtain the lift formula $L = 4\pi\rho U^2 \sin(\alpha + \beta)$ and show that the momentum about the point $\xi = z_0$ is $M = 2\pi\rho b^2 U^2 \sin 2(\alpha + \gamma)$, where α is the angle of attack and b, β, γ constants of transformation.

20. Discuss the streaming and circulation for a circular cylinder.

21. Determine the effect on a wall of a source parallel to the wall.

(2 × 5 = 10 weightage)

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C 2049 (Pages : 3) Name...

Reg. No...................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 09—DIFFERENTIAL GEOMETRY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Find and sketch the gradient field of the function $f(x_1, x_2) = x_1 + x_2$.

2. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t), \forall t \in I$.

3. Define Euclidean parallel and Levi-Civita parallel vector fields.

4. Define (i) Global parametrization ; (ii) Circle of curvature ; and (iii) Radius of curvature of a plain curve.

5. Find the length of the parametrized curve $\alpha : I \to \mathbb{R}^3$ where $\alpha(t) = (\cos 3t, \sin 3t, 4t), I = [-1, 1]$.

6. Compute $\nabla_{\overline{v}} f$ where $f : \mathbb{R}^2 \to \mathbb{R}$ and $\overline{v} \in \mathbb{R}^2_p, p \in \mathbb{R}^2$ given by $f(x_1, x_2) = x_1{}^2 - x_2{}^2$, $\overline{v} = (1, 1, \cos\theta, \sin\theta)$.

Turn over

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7. Define a parametrized n-surface. Write the map which represent the parameterized torus in $\mathbb{R}^4$.

8. State inverse function theorem for n-surface.

(8 × 1 = 8 weightage)

Part B

Answer **two** *questions from each unit in this part.*
Each question has weightage 2.

Unit 1

9. Define (i) Level set ; and (ii) Graph of a function. Sketch the Graph f and level set of the function $f(x_1, x_2) = x_1^2 - x_2^2$.

10. Find the integral curve through $P(1, 0)$ and $P(a, b)$ for the vector field

$\bar{X}(p) = \left(x_1, x_2, -2x_2, \frac{1}{2}x_1\right)$ on $U \subseteq \mathbb{R}^2$.

11. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and $c = f(p)$. Then show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

Unit 2

12. Prove that the Weingarten map L_p is self-adjoint.

13. Let $\bar{X}, \bar{Y}$ be two smooth tangent vector fields on S and $f : U \to \mathbb{R}$ any smooth function and $\alpha : I \to S$ is a parametrized curve with $\alpha(t_0) = p$ and $\dot{\alpha}(t_0) = \bar{v}$. Then prove that :

(i) $D_v - (\bar{X} + \bar{Y}) = D_v - \bar{X} + D_v - \bar{Y}$; and (ii) $\nabla_v - (\bar{X} \cdot \bar{Y}) = D_v - \bar{X} \cdot \bar{Y}(p) + \bar{X}(p) \cdot D_v - \bar{Y}$.

14. Find the global parametrization and curvature K of the circle $(x_1 - a)^2 + (x_1 - b)^2 = r^2$, oriented by the out ward normal $\nabla f / \| \nabla f \|$.

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Unit 3

15. Find the Gaussian curvature for the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 ; a, b, c \neq 0$ oriented by its outward normal.

16. Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$, there exists a point p such the second fundamental form of p is definite.

17. Show that $\varphi : U \to \mathbb{R}^3$ where $U = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \phi < \pi\}$ and $a > b > 0$ given by

$\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$ is a parametrized 2-surface in $\mathbb{R}^3$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$. Then the Gauss map maps S onto the unit sphere S^n.

19. Let C be an a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parameterization of C. Then β is either one to one or periodic. Also show that β is periodic if and only if C is compact.

20. (a) Prove that if V is a finite dimensional vector space with dot product and $L : V \to V$ a self-adjoint linear transformation on V. Then there exists an orthonormal basis for V consisting of eigenvectors of L.

(b) Find the Gaussian curvature of a cylinder over a plain curve.

21. Find the Gaussian curvature of the parametrized 2-surface

$\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$ in R^3.

(2 × 5 = 10 weightage)

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C 2048 (Pages : 3) Name..

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 08—COMMUTATIVE ALGEBRA

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Let A be a ring $\neq 0$, and every homomorphism of A into a non-zero ring B is injective. Prove that A is a field.
2. If $L \supseteq M \supseteq N$ are A-modules, prove that $(L/N)/(M/N) \cong L/M$.
3. Let $f : A \to B$ be a homomorphism of rings and let N be a finitely generated B-module and B is finitely generated A-module. Prove that N is finitely generated as an A-module.
4. If N, P are submodules of an A-module M, prove that $S^{-1}(N \cap P) = S^{-1}(N) \cap S^{-1}(P)$.
5. Let I be a primary ideal in a ring A. Prove that r (I) is the smallest prime ideal containing I.
6. Let $A \subseteq B$ be rings and let C be the integral closure of A in B. Prove that C is integrally closed in B.
7. Let B be an integral domain, K its field of fractions. Prove that B is integrally closed in K.
8. If A $[x]$ is Noetherian, is A necessarily Noetherian ?

(8 × 1 = 8 weightage)

Turn over

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Part B

Answer any **two** *questions from each module.*
Each question carries a weightage of 2.

Module I

9. Prove that the nilradical of a ring A is the intersection of all prime ideals of A

10. Prove that M is a finitely generated A-module if and only if M is isomorphic to a quotient of A^n for some positive integer n.

11. State and prove Nakayama's Lemma.

Module II

12. Let $\phi : M \rightarrow N$ be an A-module homomorphism. Prove that ϕ is injective if and only if $\phi_m : M_m \rightarrow N_m$ is injective for each maximal ideal m.

13. Let A be a ring, S a multiplicatively closed subset of A, prove that the prime ideals of $S^{-1}A$ are in one-to-one correspondence with the prime ideals of A which donot meet S.

14. State and prove second uniqueness theorem for a decomposable ideal.

Module III

15. Let $A \subset B$ be integral domains, B is integral over A. Prove that B is a field if and only if A is a field.

16. Prove that M is a Noetherian A-module if and only if every submodule of M is finitely generated.

17. In an Artin ring prove that every prime ideal is maximal ideal.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) Prove that every ring $A \neq 0$ has atleast one maximal ideal.

    (b) Prove that the set of all nilpotent elements in a ring A is an ideal.

19. (a) Let $g : A \rightarrow B$ be a ring homomorphism such that $g(s)$ is a unit in B for all $s \in S$, where S is a multiplicatively closed subset of A. Prove that there exist a unique ring homomorphism $h : S^{-1}A \rightarrow B$ such that $g = h \circ f$.

    (b) Prove that every ideal in $S^{-1}A$ is an extended ideal.

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20. (a) Let B be a ring and A is a subring of B, prove that $x \in B$ is integral over A if and only if $A[x]$ is a finitely generated A-module.

(b) State and prove the Going-down theorem.

21. (a) State and prove Hilbert's Basis theorem for Noetherian ring.

(b) In an Artin ring prove that the nilradical is nilpotent.

($2 \times 5 = 10$ weightage)

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C 2047 (Pages : 3) Name..

Reg. No.....................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 07—ALGEBRAIC TOPOLOGY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Define a geometric complex K and r-skeleton of K.
2. Define homologous p-cycles and the quotient group $Z_p(K)/B_p(K)$.
3. Define connected simplexes in a complex.
4. Let K be a 2-pseudomanifold with α_0 vertices, α_1 1-simplexes and α_2 2-simplexes. Then show that $\alpha_1 = 3(\alpha_0 - x(k))$.
5. Show that a chain mapping $\{\varphi_p\}_0^{\alpha}$ from a complex K into a complex L maps Z_p (K) into Z_p (L).
6. Prove that the n-sphere is not contractible for $n \geq 0$.
7. Define the equivalence of loops α, β having common base point x_0.
8. Define the degree of a loop α in S^1.

(8 × 1 = 8 weightage)

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Turn over

Part B

Answer **six** *questions choosing* **two** *from each module.*
Each question has weightage 2.

Module 1

9. Let σ^p be an oriented p-simplex of an oriented complex K and σ^{p-2} be a $(p-2)$ face of σ^p.

   Show that $\sum\left[\sigma^p, \sigma^{p-1}\right]\left[\sigma^{p-1}, \sigma^{p-2}\right] = 0, \sigma^{p-1} \in K$.

10. If K is an oriented complex and $p \geq 2$, then show that the composition $\partial\partial : C_p(K) \to C_{p-2}(K)$ is the trivial homomorphism.

11. Show that Z_1 (K) is isomorphic to $\mathbb{Z}$, where K is the closure of a 2-simplex $\langle a_0\ a_1\ a_2 \rangle$ with the orientation induced by the ordering $a_0 < a_1 < a_2$.

Module 2

12. If S is a simple polyhedron with V vertices, E edges and F faces, then show that V – E + F = 2.

13. For K, L complexes, define the simplicial approximation of a continuous function $f : |K| \to |L|$.

    Also, for an arbitrary complex K and the closure L of a p-simplex $\sigma^p = \langle a_0, \ldots, a_p \rangle$ show that any continuous map $f : |K| \to |L|$ has a simplicial approximation the constant map $g : |K| \to |L|$ which collapses all of K to the vertex a_0.

14. Show that S^m and S^n are not homeomorphic, for $m \neq n$.

Module 3

15. Prove that equivalence of loops is an equivalence relation on the set of loops in a space X with base point x_0.

16. Prove that two loops α and β in S^1 with base point 1 are equivalent if and only if they have the same degree.

17. Let X and Y be spaces with x_0 in X and y_0 in Y. Prove that :

    $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$.

(6 × 2 = 12 weightage)

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Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. Show that the homology groups of a complex are independent of the choice of the orientation of the complexes.

19. Prove that an n-pseudomanifold K is orientable if and only if the n^{th} homology group $H_n(K)$ is not the trivial group.

20. Prove that for any complex $K, \lim_{s \to \infty}$ mesh $K^{(s)} = 0$.

21. (a) Let the space X be path connected and x_0, x_1 be points in X. Prove that the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

    (b) Show that $\pi_1(\mathbb{R}^2 \setminus \{p\}) \cong \pi_1(A)$, where A is a circle in $\mathbb{R}^2$ with center p and $\mathbb{R}^2 \setminus \{p\}$ is the punctured plane.

(2 × 5 = 10 weightage)

CHMK LIBRARY UNIVERSITY OF CALICUT

C 2046 (Pages : 3) Name..............................

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4E 06—ALGEBRAIC NUMBER THEORY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Prove that $\mathbb{Z}\times\mathbb{Z}$ is finitely generated.
2. Find all monomorphisms of $\mathbb{Q}(i)\to\mathbb{C}$.
3. Find the degree of $\mathbb{Q}(\pi)$ over $\mathbb{Q}$.
4. Find an integral basis and discriminant for $\mathbb{Q}(\sqrt{5})$.
5. Write all units in $\mathbb{Q}$ and $\mathbb{Z}$.
6. What is the discriminant of $\mathbb{Q}(\zeta)$ where $\zeta = e^{\frac{2\pi i}{p}}, p$ an odd prime.
7. State Minkowski's theorem.
8. Find the ideals contained in the ideal $\langle 120\rangle$.

(8 × 1 = 8 weightage)

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Turn over

Part B

Answer any **two** *questions from each module.*
Each question carries a weightage of 2.

Module I

9. Prove that a subgroup of a finitely generated abelian group is finitely generated.

10. Prove that the set of algebraic numbers is a subfield of the complex field $\mathbb{C}$.

11. Find the minimum polynomial of $\zeta = e^{\frac{2\pi i}{p}}$, p an odd prime over $\mathbb{Q}$. Also find the degree of $\mathbb{Q}(\zeta)$.

Module II

12. Let R be a ring and I be an ideal of R. Prove that I is prime if and only if R/I is an integral domain.

13. Prove that the factorization of elements of $\mathcal{O}$ into irreducibles is unieque if and only if every ideal of $\mathcal{O}$ is principal.

14. Prove that the class group of a number field is a finite abelian group and the class number h is finite.

Module III

15. Prove that the quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group S.

16. Prove that $x^4 + y^4 = z^2$ has no integer solutions with $x, y, z \neq 0$.

17. Prove that the additive group of $\mathbb{R}^n$ is a Lattice if and only if it is discrete.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) If K is a number field then prove that $K = \mathbb{Q}(\theta)$ for some algebraic number 0.

    (b) Find the degree of $\mathbb{Q}\left(\sqrt{2}, \sqrt{6} + \sqrt{10}\right)$ over $\mathbb{Q}\left(\sqrt{3} + \sqrt{5}\right)$.

19. (a) Find the ring of integers of $\mathbb{Q}\left(\sqrt{2}, i\right)$.

    (b) Let $K = \mathbb{Q}\left(\sqrt[4]{2}\right)$. Find all monomorphisms of $\mathbb{Q}\left(\sqrt[4]{2}\right) \to \mathbb{C}$, minimum polynomial over $\mathbb{Q}$ and field polynomial over K.

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20. (a) Derive all solutions for the Fermat's equation $x^n + y^n = z^n$ for $n = 2$.

(b) Show that if π is an irreducible in $\mathbb{Z}[i]$ then $\mathbb{Z}[i]/\langle\pi\rangle$ is a field.

21. (a) Prove that every number field possesses an integral basis and the additive group of $\mathcal{O}$ is free abelian of rank n equal to the degree of K.

(b) Prove that an algebraic number α is an algebraic integer if and only if the minimum polynomial over $\mathbb{Q}$ has coefficients in $\mathbb{Z}$.

(2 × 5 = 10 weightage)

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C 2044 (Pages : 3) Name.....................................

Reg. No.....................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION MARCH 2021

(CBCSS)

Mathematics

MTH 4C 15—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *the questions.*
Each question carries a weightage of 1.

1. Show that every $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$ is a regular point of the operator A.
2. Give an example for a residual spectrum.
3. Show that for every compact operator $T, 0 \in \sigma(T)$, the spectrum of the operator T.
4. Show that every ortho projection satisfies $0 \leq P \leq I$.
5. State the Banach-Steinhaus theorem.
6. Give an example of a set which is convex but is not perfectly convex.
7. Let E_1, E_2 be Banach spaces, $A \in L(E_1, E_2)$ and let $T_1 \subseteq E_1, T_1 \subseteq E_2$ be perfectly convex sets. Prove that if T_1 is bounded, then AT_1 is perfectly convex.
8. Give an example of a Banach algebra without identity.

(8 × 1 = 8 weightage)

Turn over

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Part B

Answer any **six** *questions.*
Each question carries a weightage of 2.

Unit 1

9. Show that $\langle Ax, x\rangle \in \mathbb{R}$ for any $x \in H$ if and only if A is symmetric.

10. Define the operator $K : L_2[0,1] \mapsto L_2[0,1]$ by $(Kf)(t) = \int_0^1 k(t,s) f(s) ds$, where $k(t,s) = \begin{cases} 1, s \le t, \\ 0, s > t. \end{cases}$

    Find the spectrum of K.

11. Let A be a symmetric operator and let $\| A \| = \mu = \sup\{| \langle Ax, x\rangle | : \| x \|\} = 1$. Show that at least one of μ or $-\mu$ is an element of $\sigma(A)$.

Unit 2

12. Prove that for any self-adjoint operator $A \in L(H)$ the residual spectrum is empty.

13. Let $\varphi(t) \in K[a,b]$, the set of piece-wise continuous bounded functions which are monotone decreasing limits of continuous functions. Show that there exists a sequence of polynomials $P_n(t) \searrow \varphi(t)$ as $n \to \infty$ for all $t \in [a,b]$.

14. State the Hilbert theorem on the spectral decomposition of self- adjoint bounded operators.

Unit 3

15. Define closed graph operator and give an example for a closed graph operator.

16. If X^* is separable, then show that X is also separable.

17. Let $A : X \mapsto Y$ be a linear operator such that Im (A) is closed in Y and there exists $m > 0$ such that for any $x \in \text{Dom } A, \| Ax \| \ge m \| x \|$. Prove that A is closed.

(6 × 2 = 12 weightage)

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Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. Show that a sequence of operators $T_n \in L(X,Y)$ converges strongly to an operator $T \in L(X,Y)$ if and only if :

    (i) the sequence $\{T_n(x)\}$ converges for any x from a dense subset of X.

    (ii) there exists $C > 0$ such that $\| T_n \| \leq C$.

19. State and prove the Gelfand's theorem on maximal ideals.

20. Let the operator $K : L_2[-\pi,\pi] \mapsto L_2[0,1]$ be given by $(Kf)(t) = \int_{-\pi}^{\pi} | t-s | f(s) ds.$

    (a) Prove that K is a compact self - adjoint operator.

    (b) Find the spectrum of K.

    (c) Is K a positive operator ? Justify.

21. State and prove the Fredholem's first theorem.

(2 × 5 = 10 weightage)

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