C 84006 (Pages : 4) Name..................................

Reg. No..............................

M.Sc. (PREVIOUS) DEGREE EXAMINATION APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics

Paper V—DIFFERENTIAL EQUATIONS

(2000 Admission onwards)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

I. (a) Find the general solution of the differential equation $\left(2x^2+2x\right)y''+\left(1+5x\right)y'+y=0$ near its singular point $\left(x=0\right)$.

(b) Show that $J_{-m}\left(x\right)=\left(-1\right)^m J_m\left(x\right)$ where m is a non-negative integer.

(c) Show that a function of the form $ax^3+bxy^2+cxy^2+dy^3$ cannot be either positive definite or negative definite.

(d) Show that $f\left(x,y\right)=xy^2$ satisfies a Lipschitz condition on any rectangle $a\le x\le b$ and $c\le y\le d$.

(e) Show that the solution to the Dirichlet problem is stable.

(f) Show that the surfaces $x^2+y^2+z^2=cx^{2/3}$ can form an equipotential family of surfaces, and find the general form of the potential function.

(6 × 4 = 24 marks)

Turn over

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 24 marks.

Unit I

II. (a) Find the power series solution of the differential equation $(1+x)\,y' = py,\; y(0) = 1.$

(b) Show that the hypergeometric equation

$x(1-x)\,y'' + [c-(a+b+1)\,x]\,y' - aby = 0$ has three regular singular points 0, 1 and ∞.

(c) Show that $\int_{-1}^{1} p_m(x)\,p_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$, when $p_n(x)$ denotes the n^{th} degree Legendre polynomial.

III. (a) Find a series solution $y_1(x)$ of the equation $y'' + y' - xy = 0$ such that $y_1(0) = 1,\; y_1'(0) = 0.$

(b) Find two independent Frobenius series solutions of Bessel's equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0.$$

IV. (a) Show that $\dfrac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x).$

(b) Find the general solution of the system :

$$\frac{dx}{dt} = 5x + 4y,\; \frac{dy}{dt} = -x + y.$$

Unit II

V. (a) Find the critical points and differential equation of the paths of the non-linear system :

$$\frac{dx}{dt} = y\left(x^2+1\right),\; \frac{dy}{dt} = -x\left(x^2+1\right).$$

(b) Determine the nature and stability properties of the critical point $(0,0)$ for the linear system :

$$\frac{dx}{dt} = -x - 2y, \ \frac{dy}{dt} = 4x - 5y.$$

(c) Verify that $(0,0)$ is a simple critical point for the following system, and determine its nature and stability properties :

$$\frac{dx}{dt} = x + y - 2xy, \frac{dy}{dt} = -2x + y + 3y^2.$$

VI. (a) State and prove Liapunov's theorem.

(b) Show that (0, 0) is an unstable critical point for the system :

$$\frac{dt}{dt} = 2xy + x^3, \frac{dy}{dt} = -x^2 + y^5.$$

VII. (a) Apply Picard's method to calculate $y_1(x), y_2(x), y_3(x)$ of the initial value problem $y^1 = y^2, y(0) = 1.$

(b) Find the general integral of :

$$\left(x^2 + y^2\right) p + 2xyq = (x + y)z.$$

(c) Find the integral of the Ptaffian differential equation $yzdx + 2xydy - 3xydz = 0.$

Unit III

VIII. (a) Determine the region in which the two equations $xp - yq - x = 0, x^2 p + q - xz = 0$ are compatible.

(b) Find a complete integral of the equation $\left(p^2 + q^2\right) y = qz$ by Jacobi's method.

(c) Solve the Cauchy problem for $2z_x + yz_y = z,$ when the initial data curve c is $x_0 = s, y_0 = s^2, z_0 = s, \ 1 \le s \le 2.$

IX. (a) Find an integral surface of $p^2x + qy - z = 0$ containing the initial line $y = 1, x + z = 0$; by Monge's method.

(b) Reduce the equation $u_{xx} - 4x^2 u_{yy} = \frac{1}{x} u_x$ into its canonical form.

X. (a) Solve :

$$\begin{array}{ll} y_{tt} - c^2 y_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ y(0, t) = y(1, t) = 0 & \\ y(x, 0) = 0, & 0 \leq x \leq 1 \\ y_t(x, 0) = x^2, & 0 \leq x \leq 1. \end{array}$$

(b) Solve the Naumann problem for a circle.

(c) State the heat conduction problem in an infinite rod.

(4 × 24 = 96 marks)

C 84005 (Pages : 2) Name.......................................

Reg. No..................................

M.Sc. (PREVIOUS) DEGREE EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

Mathematics

Paper IV—TOPOLOGY

(2000 Admission onwards)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 marks.

I. (a) Prove that the lower limit topology on $\mathfrak{R}$ is strictly finer than the standard topology.

(b) In the set of real numbers with standard topology, prove that closed intervals are closed sets.

(c) Show that interior of a connected set is connected in a topological space.

(d) Prove that every locally compact Hausdorff space is regular.

(e) Show that closed subset of a normal space is normal.

(f) Show that the order topology on the set of real numbers is regular.

($6 \times 4 = 24$ marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 24 marks.

Unit I

II. (a) If X is any set, prove that the collection of all one point subsets of X is a basis for the discrete topology on X.

(b) Let Y be a subspace of X. Then prove that a set A is closed in Y if and only if A equals the intersection of a closed set of X with Y.

III. (a) If X and Y are any two topological spaces, give a basis for the topology on $X \times Y$ in terms of bases of topologies on X and Y. Establish your claim.

(b) Prove that a subset of a topological space is closed if and only if it contains all its limit points.

Turn over

IV. (a) Let f : A —> X × Y be given by the equation $f(a)=(f_1(a),f_2(a))$. Then prove that f is continuous if and only if the functions f_1 : A —> X and f_2 : A —> Y are continuous.

(b) Let $\{X_\alpha\}$ be an indexed family of spaces ; let $A_\alpha \subset X_\alpha$ for each α . If $\prod X_\alpha$ is given either the product or the box topology, then prove that $\prod \overline{A}_\alpha = \overline{\prod A_\alpha}$.

UNIT II

V. (a) Let the sets C and D form a separation of X, and if Y is a connected subspace of X, then prove that Y lies entirely within either C or D.

(b) Prove that every compact subspace of a Hausdorff space is closed.

VI. (a) Prove that the product of finitely many compact spaces is compact.

(b) Suppose that the topological space X has a countable basis. Then prove that there exists a countable subset of X that is dense in X.

VII. (a) Prove that every regular space with a countable basis is normal.

(b) Show that every locally compact Hausdorff space is regular.

UNIT III

VIII. (a) State and prove Tychonoff theorem.

(b) Let X be a space. Let $\mathscr{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Let $D \times \mathscr{D}$. Show that if $A \supset D$, then $A \in \mathscr{D}$.

IX. (a) Let (X, d) be a metric space. Prove that there is an isometric imbedding of X into a complete metric space.

(b) Let X be a topological space and (Y, d) be a metric space. Prove that the set C (X, Y) of all continuous functions is closed in Y^X under the uniform metric.

X. (a) Prove that a sequence $\{f_n\}$ of functions converges to the function f in the topology of pointwise convergence if and only if for each $x \in X$, the sequence $\{f_n(x)\}$ of points of Y converges to the point $f(x)$.

(b) Prove that a metric space (X, d) is compact if and only if it is complete and totally bounded.

(4 × 24 = 96 marks)

C 84004 (Pages : 3) Name..

Reg. No.....................................

M.Sc. (PREVIOUS) DEGREE EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics

Paper III—REAL ANALYSIS

(2000 Admission onwards)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. (a) Prove that every closed subset of a compact set is compact.

(b) Let f be a decreasing function on (a,b). Prove that the set of discontinuities of f on (a,b) is at most countable.

(c) Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval.

(d) Prove that the characteristic function χ_E of a set E is measurable if and only if E is measurable.

(e) Let f be integrable on a measurable set E. Prove that $|f|$ is integrable and $\left|\int_E f\right| \le \int_E |f|$.

(f) Prove that monotonic functions on [0, 1] are of bounded variation on $[a,b]$.

(6 × 4 = 24 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 24 marks.

Unit I

II. (i) Prove that for every real $x > 0$ and every integer $n > 0$, there is one and only one real $y > 0$ such that $y^n = x$.

(ii) Let $\{G_\alpha\}$ be a collection of open sets in a metric space X. Prove the $\bigcup_\alpha G_\alpha$ is open in X.

(iii) Prove that finite point set has no limit points.

Turn over

III. (i) Prove that a set E is open if and only if its complement is open.

(ii) Let X be a metric space and let E be a subset of X. Prove that E is closed if and only if $\overline{E} = E$.

(iii) Prove that non-empty perfect sets in $\mathbb{R}^k$ are uncountable.

IV. (i) Let f be a continuous mapping of a compact metric space X into a metric space Y. Prove that f is uniformly continuous on X.

(ii) If f is a continuous mapping of a connected metric space X into a metric space Y, then prove that $f(X)$ is connected.

(iii) Let f be a differentiable function on (a,b). If $f'(x) \geq 0$ for all $x \in (a,b)$, then prove that f is monotonically increasing.

UNIT II

V. (i) Let α be a monotonically increasing function on $[a,b]$ and let f be a real bounded function on $[a,b]$. Prove that $f \in \mathfrak{R}(\alpha)$ (f is Riemann-Stieltjes integrable with respect α) on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

(ii) Let α be monotonically increasing and α' is Riemann integrable $(\alpha' \in \mathfrak{R})$ on $[a,b]$. Let f be a bounded real function on $[a,b]$. Prove that $f \in \mathfrak{R}(\alpha)$ if and only if $f\alpha' \in \mathfrak{R}$. In that case prove that

$$\int_a^b f d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx.$$

VI. (i) If $\{f_n\}$ is a sequence of continuous functions on E and if $f_n \to f$ uniformly on E, then prove that f is continuous on E.

(ii) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

VII. (i) Prove that the family $\mathcal{M}$ of measurable sets is a σ-algebra.

(ii) Let $\{E_n\}$ be an infinite decreasing sequence ($E_{n+1} \subset E_n$ for each n) of measurable sets. If $m(E_1)$ is finite, then prove that

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{x \to \infty} m(E_n).$$

(iii) Let f be a real valued function and g be a continuous function defined on $(-\infty,\infty)$. Prove that $g \circ f$ is measurable.

UNIT III

VIII. (i) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure and let M be a real number such that $|f_n(x)| \leq M$ for all n and all x. Prove that

$$\int_E f = \lim \int_E f_n.$$

(ii) Let f and g be integrable functions over a measurable set E. Prove that $f + g$ is integrable over E and

$$\int_E (f+g) = \int_E f + \int_E g.$$

(iii) Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Prove that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges to f almost everywhere.

IX. (i) Let f be an increasing real-valued function on the interval $[a,b]$. Prove that f is differentiable almost everywhere. Also prove that the derivative f' is measurable and

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

(ii) If f is absolutely continuous, then prove that f has a derivative almost everywhere.

(iii) If φ is convex on (a,b), then prove that φ is absolutely continuous on each closed subinterval of (a,b).

X. (i) Let υ be a signed measure on the measure space $(X,\mathscr{B})$. Prove that there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \emptyset$.

(ii) Prove that the total variation of a signed measure is a measure.

(iii) State and prove Lebesgue decomposition theorem.

(4 × 24 = 96 marks)

C 84003 (Pages : 3) Name..

Reg. No...................................

M.Sc. (PREVIOUS) DEGREE EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

Mathematics

Paper II—LINEAR ALGEBRA

(2000 Admission onwards)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions in this part.*
Each question carries 4 marks.

1. (a) Find all the units in $\mathbb{Z}_{14}$. Describe all units in $\mathbb{Z}_n$.

(b) Let M, N be R-modules and $f : M \to N$ be a module homomorphism. Define $K = \{x \in M : f(x) = 0\}$. Show that K is a submodule of M.

(c) Find k for which $u = (1, -2, k)$ in $\mathbb{R}^3$ is a linear combination of $v = (3, 0, -2)$ and $w = (2, -1, -5)$.

(d) Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$.

(e) Let $W_1 = (0)$ and $W_2 = \mathbb{R}^2$. Verify whether $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(f) Let $\mathbb{C}$ be the vector space of all complex numbers over $\mathbb{R}$ and $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = \bar{z}$. Find the matrix of T relative to the basis $(1+i, 1+2i)$ and its characteristic values.

(6 × 4 = 24 marks)

Turn over

Part B

Answer any **four** *questions from this part without omitting any unit.*
Each question carries 24 marks.

Unit 1

2. (a) Prove that the cancellation laws hold in a ring R if and only if R has no divisors of zero.

   (b) Show that a division ring contains exactly two idempotent elements.

   (c) Find all solutions of $x^2 + 2x + 2 = 0$ in $\mathbb{Z}_6$.

3. (a) Find the remainder when 3^{47} is divided by 23.

   (b) Let d be the gcd of a and m. Prove that $ax \equiv b \pmod{m}$ has a solution if and only if $d | b$.

   (c) Find all solutions of $15x \equiv 27 \pmod{18}$.

4. (a) If R is a commutative ring with unity and M and N are unitary free R-modules, prove that $\mathrm{Hom}_R(M, N)$ is a free R-module.

   (b) Prove that a unitary module over a division ring is a free module.

Unit 2

5. (a) If W is a subspace of a finite dimensional vector space V then prove that every linearly independent subset of W is finite and is part of a basis for W. Further prove that dim W < dim V if W is a proper subspace of V.

   (b) If W_1 and W_2 are finite dimensional subspaces of a vector space V, then prove that $W_1 + W_2$ is finite dimensional and $\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2)$.

6. (a) Let $T : V \to W$ be a linear transformation from a finite dimensional vector space V into a vector space W. Show that $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$.

   (b) Verify the Rank-Nullity theorem for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(a, b, c) = (a, a + b, a + b + c, c)$.

   (c) Let V be the vector space of all $n \times n$ matrices over the field F. Let B be a fixed $n \times n$ matrix over F. If $T(A) = AB - BA$, verify whether T is a linear transformation from V into V.

7. (a) If T is a linear transformation from the vector space V into the vector space W, prove :

   (i) There exists a unique linear transformation $T^t = W^* \to V^*$ such that $T^t(g)(\alpha) = g(T\alpha)$ for every $g \in W^*$ and $\alpha \in V$.

   (ii) The null space of T^t is the annihilator of the range of T.

   (iii) If V and W are finite dimensional, then the range of T^t is the annihilator of the null space of T.

   (b) Let $f \in (\mathbb{R}^2)^*$ be defined by $f(x_1, x_2) = ax_1 + bx_2$ and $T(x_1, x_2) = (x_1, 0)$ be a linear operator on $\mathbb{R}^2$. If $g = T^t(f)$, find $g(x_1, x_2)$.

UNIT 3

8. (a) Let $c_1, c_2, \ldots, c_k$ be the distinct characteristic values of the linear operator T on the finite dimensional vector space V. Let W_i be the space of characteristic vectors associated with c_i. If $W = W_1 + W_2 + \ldots + W_k$, then prove that $\dim W = \dim W_1 + \dim W_2 + \ldots + \dim W_k$.

   (b) Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$ each of which is a characteristic vector.

9. In the following T is a linear operator on a finite dimensional vector space V :

   (a) If W is an invariant subspace for T, prove that the characteristic polynomial for T_w divides the characteristic polynomial for T.

   (b) Prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1)(x - c_2)\ldots(x - c_k)$ where $c_1, c_2, \ldots, c_k$ are distinct elements of F.

10. (a) Let T be a linear operator on a finite dimensional vector space V over F. Prove that the T-cyclic subspace generated by a $\alpha \in V$ is one dimensional if and only if α is a characteristic vector for T.

   (b) State and prove the primary decomposition theorem.

(4 × 24 = 96 marks)

C 84002 (Pages : 3) Name..............................

Reg. No..............................

M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics

Paper I—ALGEBRA

(2000 Admission onwards)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions.*
Each question is of 4 *marks.*

I. (a) Find the order of (1, 2) in $\mathbb{Z}_4 \times \mathbb{Z}_6$.

(b) Give a refinement of the series (0) < H < G where G = $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ and H is the subgroup generated by (2, 3).

(c) Find the number of Sylow 2 subgroups of the symmetric group S_3.

(d) Find all maximal ideals of the ring Z of integers.

(e) Find the degree $[\mathbb{Q}(\alpha):\mathbb{Q}]$ where α is the real cube root of 2.

(f) Show that squaring a circle is an impossible construction.

(6 × 4 = 24 marks)

Part B

Answer any **four** *questions without omitting any Unit.*
Each question is of 24 *marks.*

Unit I

II. (a) Prove that if m and n are relatively prime then $\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

(b) Let G_1, G_2 be groups and $(a, b) \in G_1 \times G_2$. Let α be of order s in G_1 and b be of order t in G_2. Prove that the order of (a, b) in $G_1 \times G_2$ is the least common multiple of s and t.

Turn over

III. Let X be a G-set. For $g \in G$ let $\sigma_g : X \to X$ be defined by $x \mapsto gx$. Show that :

(a) σ_g is a permutation of X.

(b) The map $\phi : G \to S_x$ is a homomorphism of groups.

(c) Let S_3 act on $X = \{1, 2, 3\}$ by $\sigma \cdot x = \sigma(x)$ for $\sigma \in S_3$ and $x \in X$. Find the image of ϕ where ϕ as given above.

IV. (a) Let X be a G-set. For $x, y \in X$ define $x \sim y$ if there exists $g \in G$ such that $gx = y$. Show that $\sim$ is an equivalence relation on X.

(b) Describe a non-trivial action of the Klein four group on the set X = {1, 2, 3, 4}. Find the number of orbits in this action.

Unit II

V. (a) Define maximal ideal of a ring.

(b) Let R be a commutative ring with unity and M be a maximal ideal of R. Show that R/M is a field.

(c) Give a maximal ideal in the polynomial ring $\mathbb{R}[x]$.

VI. (a) Define unique factorizaion domain.

(b) Show that the polynomial ring F [x] where F is a field is a unique factorization domain.

(c) Show that the ring $\mathbb{Z}$ of integers is a UFD.

VII. (a) Define algebraic extension of a field.

(b) Let F be a field and $p(x)$ be an irreducible polynomial in F [x]. Show that there is an extension E of F and $\alpha \in E$ such that $p(\alpha) = 0$.

(c) Show that every finite extension of F is an algebraic extension.

Unit III

VIII. (a) Let F be a finite field of characteristic p. Show that the number of elements in F is p^n for some positive integer n.

(b) Let E be a finite field of $q = p^n$ elements where p is a prime. Show that all elements of E are zeros of the polynomial $x^q - x \in \mathbb{Z}_p[x]$.

IX. (a) Define splitting field and give an example.

(b) Let $E \leq \overline{F}$ where $\overline{F}$ is an algebraic closure of F, Show that E is a splitting field over F if and only if every automorphism of $\overline{F}$ leaving F fixed maps E onto E.

X. (a) Define n^{th} cyclotomic polynomial $\Phi_n(x)$.

(b) Show that $\Phi_8(x) = x^4 + 1$.

(c) Show that the Galois group of the p^{th} cyclotomic extension of $\mathbb{Q}$ for a prime p is cyclic of order $p - 1$.

(4 × 24 = 96 marks)