Ph.D. THESIS MATHEMATICS

STUDIES ON THE GROUP OF HOMEOMORPHISMS AND THE GROUP OF L-FUZZY HOMEOMORPHISMS

Thesis submitted to the

University of Calicut

for the award of the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

under the Faculty of Science

by

SINI P.

Department of Mathematics, University of Calicut

Kerala, India 673 635.

JANUARY 2017

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CALICUT

Ramachandran P.T. University of Calicut

Associate Professor 27 January 2017

CERTIFICATE

I hereby certify that the thesis entitled "Studies on the Group of Homeomorphisms and the Group of L-Fuzzy Homeomorphisms" is a bonafide work carried out by **Smt. Sini P.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Ramachandran P. T.

(Research Supervisor)

DECLARATION

I hereby declare that the thesis, entitled "**Studies on the Group of Homeomorphisms and the Group of L-Fuzzy Homeomorphisms**" is based on the original work done by me under the supervision of **Dr. Ramachandran P. T.**, Associate Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

University of Calicut,

27 January 2017. Sini P.

ACKNOWLEDGEMENT

First and foremost, I would like to express my sincere gratitude to my research guide Dr. Ramachandran P. T., Associate Professor, Department of Mathematics, University of Calicut for the continuous support during my Ph.D course, for his patience, encouragement, enthusiasm and immense knowledge.

I would like to thank the teachers Dr. Raji Pilakkat and Dr. Anil Kumar V. for their help and support. I am grateful to my teacher and colleague Dr. Preethi Kuttipulackal for giving me positive energy and moral support. Sincere thanks to Ms. Shifna M. K., my FDP substitute for her cheerful friendship and discussions she had with me.

It gives me great pleasure to thank Mrs. Dhanya P. M. for collaborative research and allowing me to include here a part of our joint work.

I am very much fortunate to have excellent friends who made my research life memorable.

Heartful gratitude to Dr. Abraham Joseph, Professor, Department of Chemistry for giving moral support and inspiration for the completion of this work

I am also happy to express my thanks to all non-teaching staff, Librarian and students of the Department of Mathematics, University of Calicut for their co-operation and support.

Thankful acknowledgement to the University Grants Commission and the University of Calicut for awarding the Teacher Fellowship for doing the research and providing me the research facilities for the entire course period.

I would like to acknowledge all the teachers who taught me since my childhood.

I am also thankful to my dear parents, husband Mr. Anilkumar M., children Nandanamithra and Harithsyam and in-laws who encouraged and helped me at every stage of my personal and academic life, and longed to see this achievement come true.

Last but not at the least, I am thankful to all others who have helped me in this work.

Above all I thank God for giving me the strength and patience to work through all these years.

University of Calicut,
27 January 2017. Sini P.

Contents

Chapter 0

Introduction

The set of all homeomorphisms on a topological space onto itself is a group under function composition and is called the group of homeomorphisms of the topological space. The group of homeomorphisms of a topological space (X, τ) is a subgroup of the group $S(X)$ of all bijections of the set X. Group of homeomorphisms are very important in the theory of topological spaces. Analogous to the group of homeomorphisms in topology, the group of L-fuzzy homeomorphisms is defined in L-fuzzy topology. The objective of the present thesis is to study some problems related to the concept of the group of homeomorphisms of a topological space and the group of L-fuzzy homeomorphisms of an L-fuzzy topological space.

In the present study we use Set theoretical, Topological, Algebraic and Order theoretical methods.

0.1 Motivation and Survey of Literature

Several authors studied the concept of the group of homeomorphisms of a topological space and the relation between the algebraic properties of the group of homeomorphisms of the topological space X and the topological properties of the space X. A topological space is said to be rigid if the group of homeomorphisms of the space is the trivial group. De Groot, V. Kannan, Rajagopalan etc. studied rigid spaces and have done commendable works in this area [9, 10, 24–26].

In 1920, the concept of homogeneity in topological spaces was introduced by W. Sierpinski. A topological space (X, τ) is said to be homogeneous if for any two points $x_1, x_2 \in X$ there exists a homeomorphism h from (X, τ) onto itself such that $h(x_1) = x_2$. In other words a topological space (X, τ) is homogeneous if the group of homeomorphisms of the space is a transitive permutation group on X. Ginsburg [14] characterized the finite homogeneous topological spaces. The various properties related to homogeneity and rigidity were studied in [43].

A topological space X is said to be completely homogeneous if the group of homeomorphisms of the space is the symmetric group $S(X)$. R. E. Larson studied the concept of complete homogeneity and characterized the spaces which are minimum and maximum with respect to a topological property [29]. He also determined the completely homogeneous topological spaces.

In 1959, J. De Groot proved that any group is isomorphic to the group of homeomorphisms of a topological space [10]. If two topological spaces are homeomorphic, then the corresponding groups of homeomorphisms are isomorphic.

But nonhomeomorphic topological spaces can have isomorphic groups of homeomorphisms.

P. T. Ramachandran studied the problem of representing a subgroup of $S(X)$ as the group of homeomorphisms of a topological space (X, τ) for some topology τ on X [34]. He proved that no nontrivial proper normal subgroups of the symmetric group $S(X)$ can be the group of homeomorphisms of (X, τ) for any topology T on X [35]. He also showed that if $X = \{x_1, x_2, \ldots, x_m\}$, $m > 2$, the subgroup of the symmetric group $S(X)$ generated by the cycle $(x_1, x_2, \ldots, x_m)$ cannot be represented as the group of homeomorphisms of (X, τ) for any topology τ on X [34].

In 1965, L. A. Zadeh [46] introduced the theory of fuzzy set describing fuzziness mathematically for the first time. Later J. A Gougen introduced the concept of $L-$fuzzy set where L is a semigroup, a partially ordered set, a lattice or a boolean ring [15]. We can extend most of the mathematical theories using the concept of a fuzzy set since fuzzy set is a generalization of the fundamental mathematical concept of a set. Based on the notion introduced by L. A. Zadeh [46], C. L Chang introduced fuzzy topology and studied its properties [8].

T. P. Johnson [16] and P. T. Ramachandran [36,37] considered the problem of representing a subgroup H of the group $S(X)$ as the group of L- fuzzy homeomorphisms of some L- fuzzy topological space (X, δ). In [16,17,19] T. P. Johnson proved that the subgroups generated by a finite cycle and some proper non-trivial normal subgroups can be represented as group of L- fuzzy homeomorphisms for some L- fuzzy topology δ on X, when $|X| \leq |L|$. P. T. Ramachandran proved that the group of permutations on a set X generated by a finite cycle and the group generated by an arbitrary product of infinite cycles can be represented as the group of L- fuzzy homeomorphisms for some L- fuzzy topology, if the membership lattice $L \neq \{0,\ 1\}$ [36,37]. One of our main aim is to continue this work.

An L- fuzzy topological space $(X,\ \delta)$ is said to be homogeneous if given any two points x_1 and x_2 in X, there is an L-fuzzy homeomorphism h of X onto itself such that $h(x_1) = x_2$ [3]. The order of the group of L-fuzzy homeomorphisms depends on the structure of the L-fuzzy topological space. For example, if the L-fuzzy topological space is homogeneous, the order of the group of L-fuzzy homeomorphisms on X is greater than or equal to the cardinality of X. An L-fuzzy topological space $(X,\ \delta)$ is called completely homogeneous if the group of L-fuzzy homeomorphisms equals $S(X)$, the set of all permutations of the set X.

In [16] T. P. Johnson defined the concept of a completely homogeneous fuzzy topological space analogous to complete homogeneity in topological space. In [18] the author extended complete homogeneity to L-fuzzy topological spaces. He generalized the result of R. E. Larson for L-fuzzy topological spaces. He also considered the lattice of completely homogeneous fuzzy topologies also [18]. In [22] the authors studied complete homogeneity and reversibility in L-fuzzy topological spaces.

0.2 Organisation of the Thesis

In this thesis, we are studying the group of homeomorphisms of topological spaces and the group of L-fuzzy homeomorphisms of L-fuzzy topological spaces. We are also interested in L-fuzzy topological spaces whose group of L-fuzzy homeomorphisms is the symmetric group.

The thesis consists of seven chapters besides the introduction.

The **first chapter** gives a brief account of preliminary definitions, theorems and results which we used in the forthcoming chapters. Here we discuss some basics of algebra, set theory, topology and L-fuzzy topology.

The **second chapter** is concerned with the study of group of homeomorphisms of topological spaces. We start with the definition of a $t-$representable permutation group. A subgroup H of the symmetric group on a nonempty set X is called $t-$representable on X if there exists a topology τ on X such that the group of homeomorphisms of (X, τ) equals H [39]. Some properties of t-representable permutation groups are also discussed. Further we determine $t-$representability of some permutation groups. Here we prove that the direct sum of finite $t-$representable permutation groups is $t-$representable. Also the $t-$representability of transitive permutation groups and some maximal permutation groups are studied.

In the **third Chapter**, we continue the work carried out in chapter two. Here we determine the $t-$representability of some cyclic subgroups of the symmetric group. It can be seen that if σ is a permutation on a set X which is a product of two disjoint cycles, then the permutation group generated by σ, $< \sigma >$ is not t-representable on X and whereas if σ is a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n, then the permutation group generated by σ, $< \sigma >$ is t-representable on X. We also determined the $t-$representability of groups generated by a permutation which is a product of disjoint infinite cycles.

In the **fourth chapter** we study the group of L-fuzzy homeomorphisms of an L-fuzzy topological space. Analogous to $t-$representability in topological space, here we define L_f-representability of permutation groups. A subgroup H of the symmetric group on a nonempty set X is called L_f-representable on X if there exists an $L-$ fuzzy topology δ on X such that the group of $L-$ fuzzy homeomorphisms on X equals H. Some properties of L_f-representable permutation groups are also studied.

We establish the L_f-representability of semiregular permutation groups on X when $|L| \geq |X|$. A necessary and sufficient condition for a dihedral group D_n to be L_f-representable is provided. We also determine the L_f-representability of all subgroups of S_n when $n \leq 4$. In contrast with results in topology, it can be seen that the group generated by a permutation which is a product of two disjoint cycles having equal length is L_f-representable.

Chapter five is a study of completely homogeneous L-fuzzy topological spaces. An investigation is conducted on the relation between the group of L-fuzzy homeomorphisms of an $L-$fuzzy topological space and the group of homeomorphisms of its level topologies. It is also observed that an L-fuzzy topological space is completely homogeneous if and only if its stratification is completely homogeneous. A characterization of completely homogeneous Alexandroff discrete L-fuzzy topological spaces is obtained when L is a complete chain. Also we prove that supersets of a non- zero L-fuzzy open set having the same range are L-fuzzy open in a completely homogeneous L-fuzzy topology.

A stronger notion of completely homogeneity, called principal completely homogeneity in topology and L-fuzzy topology are introduced in the **sixth chapter**. Let $f \in L^X$. Then the smallest completely homogeneous L-fuzzy topology containing f is called the principal completely homogeneous L- fuzzy topology generated by f and is denoted by $CHLFT(f)$ [42]. A completely homogeneous L-fuzzy topological space (X, δ) is called principal completely homogeneous L-fuzzy topological space if $\delta = CHLFT(f)$ for some L- fuzzy set $f \in L^X$ [42]. Some properties of principal completely homogeneous L-fuzzy topological spaces are also discussed in this chapter. Principal completely homogeneous L-fuzzy topological spaces are studied when $L = \{0, \frac{1}{2}, 1\}$ with the usual order.

The last chapter contains the conclusion and some unsolved problems. A bibliography is also given at the end.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we give a brief account of the preliminary concepts which are used in the forthcoming chapters. It includes the basics of Set Theory, Group Theory, especially Permutation Groups, Topology and L- fuzzy topology.

Through out our discussions X denote a nonempty set. If A is a given set, we will use $|A|$ to denote the cardinality of A.

1.2 Partially Ordered Sets

Let us start with the definition of a partial ordering.

Definition 1.2.1. [23]

Let X be a set and $\leq$ be a relation on X. Then $\leq$ is called

1. *reflexive* if $a \leq a$ for all $a \in X$.

2. *anti-symmetric* if for all a, $b \in X$, $a \leq b$ and $b \leq a$ implies $a = b$.

3. *transitive* if for all $a, b, c \in X$, $a \leq b$ and $b \leq c$ implies $a \leq c$.

A relation on X which is reflexive, anti-symmetric and transitive is called a *partial order*.

A *partially ordered set (or a poset)* is an ordered pair $(X, \leq)$ where X is a set and $\leq$ is a partial order on X. The partial order $\leq$ is said to be *linear (or chain)* if for all a, $b \in X$, either $a \leq b$ or $b \leq a$.

Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be two partially ordered sets and $f : X_1 \to X_2$ be a function. Then f is said to be *order-preserving* if for all $a, b \in X$, $a \leq_1 b \Rightarrow f(a) \leq_2 f(b)$. If f is a bijection and f as well as its inverse f^{-1} are both order preserving then f is called an *order isomorphism* from $(X, \leq_1)$ to $(X, \leq_2)$ [23]. That is, a bijection f from $(X_1, \leq_1)$ to $(X_2, \leq_2)$ is an order isomorphism if and only if $a \leq_1 b \iff f(a) \leq_2 f(b)$ for all $a, b \in X_1$. The set of all order isomorphisms of a partially ordered set onto itself forms a group under composition of functions and is called the *group of order isomorphisms*, which is denoted by $G(X, \leq)$.

Now we will discuss some terms related to the partial ordering.

Definition 1.2.2. [30]

If $(X, \leq)$ is a partially ordered set, A is a nonempty subset of X and $a \in X$, then

1. a is an *upper bound* for A if for all $x \in A$, $x \leq a$.
2. A is said to be *bounded above* if A has atleast one upper bound.

Dually we define lower bound and bounded below.

Definition 1.2.3. [23]

Let $(X, \leq)$ be a partially ordered set and $A \subseteq X$

1. An element a of A is said to be *maximal element* of A if it is not strictly less than any element of A.
2. a is said to be *maximum (largest) element* of A if for all $x \in A \setminus \{a\}$, $x < a$.
3. An element $a \in X$ is said to be the *join (or the least upper bound, or the supremum)* of A, denoted by $\vee A$, if a is an *upper bound* of A and if b is an upper bound for A, then $a \leq b$.

If A consists of two elements a and b, write $a \vee b$ for $\vee\{a, b\}$ for convenience.

The concept of a minimal, smallest or minimum element, meet $(\wedge)$ or greatest lower bound are dully defined.

Definition 1.2.4. [23]

A partial order $(X, \leq)$ is said to be *complete* if every nonempty subset of A, which is bounded above has a least upper bound.

Definition 1.2.5. [30]

A partially ordered set $(X, \leq)$ is called a *well-ordered set*, if for every nonempty subset A of X, there exists a_0 in A such that $a_0 \leq a$ for every $a \in A$ and the partial order is called a well-order.

Every element in a well-ordered set other than the last element has a unique immediate successor.

Theorem 1.2.6. *[23] Well-ordering Theorem*

Every set can be well-ordered.

This can be proved using the Axiom of choice and is equivalent to it.

One of the important facts about well-ordered sets is that we can prove things about their elements by a process similar to mathematical induction which is called *Principle of Transfinite Induction.*

Theorem 1.2.7. [1] *Principle of Transfinite Induction*

Let W be a well-ordered set and V a subset of W in which, for every element $x \in W$, satisfies the following condition:

If every predecessor of x belongs to V, then, x belongs to V.

Then $V = W$.

1.3 Lattice Theory

Definition 1.3.1. [23]

Let $(L, \leq)$ be a partially ordered set. Then L is called a *lattice*, if every two element subset of L has both a meet and join in L and complete lattice if every subset of L has a meet and join in L.

The smallest and largest elements of a lattice L are denoted by 0 and 1. A lattice always be nonempty. A lattice is called trivial if it consist only one element. In a trivial lattice the smallest element coincide with the largest element. This kind of lattice will not be considered in our study. Thus we can assume that a lattice contain at least two elements $0 \neq 1$.

Definition 1.3.2. [30]

Let L be a lattice with 0 and 1. An element $x \in L$ called *complement* of an element $y \in L$, if $x \wedge y = 0$ and $x \vee y = 1$.

A mapping $h : L \to L$ is called *order-reversing*, if for all $x, y \in L$, $x \leq y \Rightarrow h(y) \leq h(x)$ and called an *involution* on L, if $h \circ h$ is the identity map on L.

Definition 1.3.3. [30]

Let L be a complete Lattice. L is called *completely distributive*, if L satisfies the following two conditions,

for all $\{\{a_{i,j} : j \in J_i\} \subseteq \mathscr{P}(L) \setminus \{\emptyset\}, I \neq \emptyset$,

1. $\bigwedge\limits_{i\in I} (\bigvee\limits_{j\in J_i} a_{i,j}) = \bigvee\limits_{\emptyset \in \prod\limits_{i\in I} J_i} (\bigwedge\limits_{i\in I} a_{i,\emptyset(i)})$,

2. $\bigvee\limits_{i\in I}(\bigwedge\limits_{j\in J_i} a_{i,j}) = \bigwedge\limits_{\substack{\emptyset\in\Pi J_i\\ i\in I}}(\bigvee\limits_{i\in I} a_{i,\emptyset(i)})$.

Definition 1.3.4. [30]

A completely distributive lattice L is called an *F-lattice*, if L has an order reversing involution $h : L \to L$.

1.4 Permutation Groups

Here are some basic definitions and theorems from Group theory which we will use in this Thesis. For more details see [12, 32]

Let X be a nonempty set. A bijection of X onto itself is called a *permutation* of X. The set of all permutations of X forms a group under composition of mappings, called the *symmetric group* on X. We denote the symmetric group by $S(X)$ and S_n to denote the special group $S(X)$ when n is a positive integer and $X = \{1, 2, \ldots, n\}$. A *permutation group* is a subgroup of the symmetric group.

We know that a *binary operation* $*$ on a set S is a function mapping $S\times S$ into S. More generally, for any sets A, B and C, we can view a map $* : A \times B \longrightarrow C$ as defining a multiplication of an element $a \in A$ with an element $c \in C$. The notion group action is used in the case where X is a set, G is a group and we have a map $* : G \times X \longrightarrow X$.

Definition 1.4.1. [13]

Let $(G, .)$ be a group and X be a set. An *action* of G on X is a map

$* : G \times X \longrightarrow X$ such that

1) $ex = x$ for all $x \in X$.

2) $(g_1g_2)(x) = (g_1)(g_2x)$ for all $x \in X$ and for all $g_1, g_2 \in X$.

Under these conditions, X is called a G-set. Here $*(g, x)$ is denoted by $g(x)$.

When a group G acts on a set X, a typical point x is moved by some elements of G to various other points and fixed by some elements of G.

Definition 1.4.2. [12]

Let $x \in X$. Then the *orbit* of x under G, x^G is defined by

$$x^G \;=\; \{g(x) :\; g \in G\}$$

and the *stabilizer* of x in G, G_x is defined by

$$G_x \;=\; \{g \in G :\; g(x) = x\}.$$

Example 1.4.3. Let X be any set and G be a permutation group on X. Then G acts naturally on X where $g(x)$ is the image of x under the permutation g.

We assume that such a group action implicitly in some contexts.

A group G acting on X is said to be *transitive* on X if it has only one orbit and so $x^G \;=\; X$ for all $x \in X$. That is G is *transitive* if given $x, y \in X$, there exists $g \in G$ such that $g(x) = y$. A group which is not transitive is called *intransitive*.

Definition 1.4.4. Semi Regular Permutation Groups [12]

Let X be any set and H be a permutation group on X. Then H is called *semiregular* if the identity permutation is the only element in H with any fixed points.

Definition 1.4.5. Regular Permutation Groups [12]

Let X be any set and H be a permutation group on X. Then H is called *regular* if H is transitive and semiregular.

Lemma 1.4.6. *[32] Any transitive abelian permutation group is regular.*

Let G be a group acting on a set X, k be an integer with $1 \leq k \leq |X|$ and $X^{(k)}$ denote the set of all k-tuples of distinct points. Then G acts on $X^{(k)}$ in a natural way, namely $g(x_1, x_1, \ldots x_k) = (g(x_1), g(x_2), \ldots, g(x_k))$ where $g \in G$ and $(x_1, x_2, \ldots x_k) \in X^{(k)}$. If G is transitive on $X^{(k)}$, then we say G is k-*transitive*. We say that G is *highly transitive* if X is infinite and G is k-transitive for all integers $k \geq 1$.

Definition 1.4.7. [12]

Let G be a group acting transitively on a set X. A nonempty subset A of X is called a *block* for G if for each $g \in G$, either $g(A) = A$ or $g(A) \cap A = \emptyset$.

Every group acting transitively on X has X and the singletons $\{x\}$ $(x \in X)$

as blocks and these blocks are called *trivial blocks*. Any other block is called *non-trivial block*.

Definition 1.4.8. [12]

Let G be a group acting transitively on a set X. We say that the group G is *primitive* if G has no nontrivial blocks on X. Otherwise G is called *imprimitive*.

Example 1.4.9.

- S_n is primitive for $n \geq 2$ and A_n is primitive for $n \geq 3$.
- The cyclic group generated by the cycle (1, 2, 3, 4, 5, 6) has five nontrivial blocks, $\{1, 3, 5\}$, $\{2, 4, 6\}$, $\{1, 4\}$, $\{2, 5\}$, $\{3, 6\}$. So this group is an imprimitive permutation group.

Definition 1.4.10. [12]

Let G be a group acting on a set X and A be a subset of X. Then the *setwise stabilizer* of A in G is

$$G_{\{A\}} = \{g \in G : g(A) = A\}.$$

Note that $G_{\{A\}}$ is a subgroup of G.

1.4.1 Normal Subgroups of Symmetric Groups

Now we consider the normal subgroups of the symmetric groups.

If X is a finite set then the *alternating group* A_n of degree n is the collection of all even permutations in S_n. It is also a fact that A_n is the only non-trivial proper normal subgroups of S_n except when $n = 4$. If $n = 4$, S_n has another non-trivial normal subgroup $\{I, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$.

Let X be an infinite set, in order to define the alternating group $A(X)$, we have to define what we mean by odd and even permutations.

Definition 1.4.11. [32]

Let G be a group acting on a set X and $g \in G$. Then *support* of g is defined by

$$supp(g) = \{x \in X : g(x) \neq x\}.$$

Thus $supp(g)$ is the set of all points moved by g and $|supp(g)|$ is called the *degree* of the permutation g. A permutation having finite degree is called a *finitary permutation*. Let $FS(X)$ be the set of all elements in $S(X)$ which have finite support. Then $FS(X)$ is a primitive normal subgroup of $S(X)$ and is a proper subgroup when X is infinite. This group is called *finitary symmetric group* on X. Let g be an element of a finitary symmetric group $FS(X)$. Then g can be written as a product of finite number of transpositions. A finitary permutation g is called even if it can be written as a product of even number of transpositions and odd if it can be written as a product of odd number of transpositions. Then the *alternating group* $A(X)$ is defined as $\{g \in FS(X) : g \text{ is even }\}$. Let m be an infinite cardinal such that $\aleph_0 \leq m \leq |X|$. Then we define the bounded symmetric group $BS(X, m)$ as follows,

$$BS(X, m) = \{g \in S(X) : |supp(g)| < m\}.$$

We end this section by stating a theorem which gives a list of all the normal subgroups of $S(X)$.

Theorem 1.4.12. *[6] Let X be any set with $|X| > 4$. Then the normal subgroups of $S(X)$ are precisely : $\{I\}$, $A(X)$, $S(X)$ and the subgroups of the form $BS(X, m)$ with $\aleph_0 \leq m \leq |X|$.*

1.5 Topology

Now let us go through some basics of topology which will make the reading of this Thesis easier.

Definition 1.5.1. [45]

If X and Y are topological spaces, a function f from X to Y is a *homeomorphism* if f is one-one, onto, continuous and f^{-1} is also continuous. When such a homeomorphism exists, X is said to be *homeomorphic* to Y.

If we denote X is homeomorphic with Y by $X \sim Y$, then $\sim$ is an eqivalence relation on any set of topological spaces.

Consider the collection of all homeomorphisms of a topological space onto itself, which is denoted by $H(X, \tau)$. Let $f, g \in H(X, \tau)$. Then the composition of f and g, $f \circ g$ is also a homeomorphism on X. By definition, the identity map on X which is denoted by I_X is a homeomorphism. It is easy to prove that if $h \in H(X, \tau)$, then $h^{-1} \in H(X, \tau)$. So we can define the group of homeomorphisms of a topological space as given below.

Definition 1.5.2. [45] Group of Homeomorphisms

Let (X, τ) be any topological space and let $H(X,\ \tau)$ be the set of all homeomorphisms of $(X,\ \tau)$ onto itself. Then $H(X, \tau)$ is a group under function composition and it is called the *group of homeomorphisms* of X onto itself.

Note that the group $H(X, \tau)$ of all homeomorphisms of a topological space (X, τ) onto itself is a subgroup of the symmetric group $S(X)$.

We define several topological properties using the group of homeomorphisms. Following are some of them.

Definition 1.5.3. [23] Homogeneous Space

A topological space (X, τ) is said to be *homogeneous* if for any $x, y \in X$, there exists a homeomorphism h from X onto itself such that $h(x) = y$.

Definition 1.5.4. A topological space $(X,\ \tau)$ is said to be *rigid* if the group of homeomorphisms, $H(X,\ \tau)\ =\ \{I\}$.

There is a situation where the group of homeomorphisms equals the set of all permutations on the underlying set.

Definition 1.5.5. [29] Completely Homogeneous Topological Space

A topological space (X, τ) is called *completely homogeneous* if the group of homeomorphisms of X onto itself is equal to the group of all permutations of X i.e., $H(X, \tau) = S(X)$.

The discrete topological spaces and indiscrete topological spaces are completely homogeneous space.

Definition 1.5.6. [2] Alexandroff Discrete Space

A topological space (X, τ) is called Alexandroff discrete space if arbitrary intersections of open sets are open in X.

Any finite topological space and discrete spaces are Alexandroff discrete space.
A topological space (X, τ) is Alexandroff discrete if and only if it has a minimal open neighbourhood at every point in X.

1.6 L-fuzzy Topology

Here we give some preliminary results on L-fuzzy topology.

Let X be a nonempty set, L a complete lattice, then an L-fuzzy subset f of X is a mapping from X to L. The set of all L-fuzzy subsets of X is denoted by L^X, which is called an *L-fuzzy space.* An L- fuzzy set with constant membership value $\alpha \in L$ is denoted by $\underline{\alpha}$.

Definition 1.6.1. [30]

The L-fuzzy subset x_l with $x \in X$ and $l \in L, l \neq 0$ defined by

$$x_l(y) = \begin{cases} l & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

is called an *L-fuzzy point* in X with support x and value l.

The L-fuzzy subset x^l, with $x \in X$ and $l \in L, l \neq 1$ is defined by

$$x^l(y) = \begin{cases} l & \text{if } y = x \\ 1 & \text{otherwise.} \end{cases}$$

Definition 1.6.2. [30]

Let X and Y be two sets and $h : X \to Y$ be a function. Then for any L-fuzzy set f in X, $h(f)$ is an L-fuzzy set in Y defined by

$$h(f)(y) = \begin{cases} \bigvee\{f(x) : \ x \in X \ \ h(x) = y\}; & h^{-1}(y) \neq \emptyset \\ 0; & h^{-1}(y) = \emptyset. \end{cases}$$

For an L-fuzzy set g in Y, we define $h^{-1}(g)(x) = g(h(x))$ for all $x \in X$.

Definition 1.6.3. $L-$ fuzzy Topological Space [30]

Let X be a nonempty set, L an F-lattice, $\delta \subseteq L^X$. Then δ is called an *L-fuzzy topology* on X, and (X, δ) is called an *L-fuzzy topological space* if δ satisfies the following three conditions.

1. $\underline{0}, \underline{1} \in \delta$;

2. $f \wedge g \in \delta$ for all $f, g \in \delta$;

3. $\vee A \in \delta$ for all $A \subseteq \delta$.

Every element in δ is called an *L-fuzzy open* subset of X.

Examples 1.6.4. Let X be any set and L an F-lattice

1. Let $\delta = L^X$. Then δ is an L-fuzzy topology on X and is called *discrete L-fuzzy topology* on X.

2. Let $\delta = \{\underline{0}, \underline{1}\} \subseteq L^X$, then δ is an L-fuzzy topology on X and which is called the *trivial L-fuzzy topology* on X.

Definition 1.6.5. [30]

Let (X, δ) be an L-fuzzy topological space, $\delta_0 \subseteq \delta$. δ_0 is called a *base* of δ, if

$$\delta = \{\vee\mathscr{A} : \mathscr{A} \subseteq \delta_0\}.$$

δ_0 is called a *subbase* of δ, if the family

$$\{\wedge\mathscr{B} : \mathscr{B} \in [\delta_0]^{<w} \setminus \{\emptyset\}\}$$

is a base, where $[\delta_0]^{<w}$ denote the family of all the finite subsets of δ_0.

1.6.1 $L-$ fuzzy Homeomorphism

Let (X, δ) and (Y, δ') be any two $L-$fuzzy topological spaces and h be a mapping from (X, δ) to (Y, δ').

Definition 1.6.6. [30]

1. h is said to be an L-*fuzzy continuous* map from X to Y, if $h^{-1}(f') \in \delta$ for every f' in δ' where $h^{-1}(f')$ means $f' \circ h$.
2. h is said to be L-*fuzzy open* if it maps every L-fuzzy open subset of X as an $L-$fuzzy open one in Y.
3. h is said to be an $L-$ *fuzzy homeomorphism* if it is bijective, continuous and open.

So a necessary and sufficient condition for a permutation h of a set X to be an L-fuzzy homeomorphism of (X, δ) on to itself is that $f \in \delta$ if and only if $f \circ h \in \delta$.

Definition 1.6.7. Group of $L-$ fuzzy homeomorphisms

The set of all L- fuzzy homeomorphisms of an L- fuzzy topological space (X, δ) onto itself is a group under composition, which is a subgroup of the group of all permutations on the set X. It is called the *group of L- fuzzy homeomorphisms* of (X, δ) and is denoted by $LFH(X, \delta)$.

Chapter 2

Group of Homeomorphisms

Here we study the group of homeomorphisms of topological spaces. We define the t-representability of permutation groups and study some properties of the t-representable permutation groups. We determine the t-representability of all finite transitive permutation groups. Also we study the t-representability of some maximal subgroups of the symmetric group.

2.1 $t-$representability of Permutation Groups

We start this section by defining the $t-$representability of permutation groups. Some examples and non-examples of $t-$ representable permutations groups are given.

Definition 2.1.1. *t-representable Permutation Group*

A subgroup K of $S(X)$ is called t-representable on X if there exists a topology

τ on X such that $H(X,\ \tau) = K$.

Examples and non-examples of $t-$representable permutation groups

1. We have the group of homeomorphisms of the discrete space on X is the symmetric group $S(X)$ and hence the symmetric group $S(X)$ is a $t-$representable permutation group.

2. In [34, 35], P. T. Ramachandran proved that no non-trivial proper normal subgroups of the group of all permutations of a set X can be the group of homeomorphisms of $(X,\ \tau)$ for any topology τ on X. So no non-trivial proper normal subgroup of the Symmetric group is $t-$representable on X.

3. Let $X = \{a_1,\ a_2,\ \ldots,\ a_n\}$, n ≥ 3, the group of permutations of X generated by the cycle $(a_1,\ a_2,\ \ldots,\ a_n)$ cannot be represented as the group of homeomorphisms of (X, τ) for any topology τ on X [34]. It follows that the cyclic group generated by the cycle $(a_1,\ a_2,\ \ldots, a_n)$ is not $t-$representable on X.

2.2 Properties of $t-$representable Permutation Groups

Here we derive two important properties of $t-$representable permutation groups.

Definition 2.2.1. [13] Conjugate of a subgroup

A subgroup H is *conjugate* to a subgroup K of a group G if there exists an element $g \in G$ such that $gHg^{-1} = K$.

Theorem 2.2.2. *Let X be any set and H be a permutation group on X. Then H is $t-$representable on X if and only if its conjugate is also t-representable on X.*

Proof. Let H be a t-representable permutation group on X. Then there exists a topology τ on X such that $H(X, \tau) = H$. It follows that

$$H = \{h \in S(X) : h(\tau) = \tau\}.$$

Let $g \in S(X)$. Then $g(\tau) = \{g(U) : U \in \tau\}$ is a topology on X. Now we claim that $H(X, g(\tau)) = gHg^{-1}$.

Let $h \in H(X, \tau)$

$$\begin{aligned}
&\Rightarrow h(\tau) = \tau \\
&\Rightarrow hg^{-1}g(\tau) = \tau \\
&\Rightarrow hg^{-1}(g(\tau)) = \tau \\
&\Rightarrow ghg^{-1}(g(\tau)) = g(\tau) \\
&\Rightarrow ghg^{-1} \in H(X, g(\tau)).
\end{aligned}$$

This implies that

$$gHg^{-1} \subseteq H(X, g(\tau)) \tag{2.1}$$

For the other way inclusion, let $h \in H(X, g(\tau))$

$$\begin{aligned}
&\Rightarrow h(g(\tau)) \;=\; g(\tau)\\
&\Rightarrow hg(\tau) \;=\; g(\tau)\\
&\Rightarrow g^{-1}(hg(\tau)) \;=\; g^{-1}g(\tau)\\
&\Rightarrow g^{-1}hg(\tau) \;=\; \tau\\
&\Rightarrow g^{-1}hg \in H(X,\, \tau) = H\\
&\Rightarrow h \in gHg^{-1}.
\end{aligned}$$

This implies that

$$H(X, g(\tau)) \subseteq gHg^{-1} \tag{2.2}$$

From equations 4.1 and 4.2, we get $H(X, g(\tau)) = gHg^{-1}$. Thus the conjugate of a t-representable permutation group is t-representable on X.

Conversely assume that gHg^{-1} is a t-representable permutation group on X. Then by what we have proved above, the conjugate of gHg^{-1} is a t-representable permutation group on X. So H is t-representable on X. This completes the proof. $\square$

Note that conjugacy is an equivalence relation on the set of all subgroups of $S(X)$. Thus in order to determine $t-$representability of permutation groups, it suffices to consider the t-representability of conjugacy classes of subgroups of $S(X)$.

Definition 2.2.3. *Direct Sum*

Let $\{X_i : i \in I\}$ *be an arbitrary family of mutually disjoint sets and* K_i *be a subgroup of* $S(X_i)$ *for every* $i \in I$*. Then the direct sum of permutation groups* $\{K_i : i \in I\}$ *is the permutation group* $\bigoplus_{\imath \in I} K_i$ *on* $X = \bigcup_{i \in I} X_i$ *whose elements are* $\bigoplus_{i \in I} k_i$ *where* $k_i \in K_i$ *and the action of* $\bigoplus_{i \in I} k_i$ *is given by* $\bigoplus_{i \in I} k_i(x) = k_i(x)$ *if* $x \in X_i$*,* $i \in I$*.*

Theorem 2.2.4. *Let* X *be any set and* Y *be a nonempty subset of* X*. If* H *is a t-representable permutation group on* Y*, then the permutation group* $\{I_{X \setminus Y}\} \oplus H$ *is t-representable on* X*.*

Proof. Let τ_1 be the topology on Y such that $H(Y, \tau_1) = H$. Define

$$\tau' = \{(X \setminus Y) \cup U : U \in \tau_1\}.$$

If $X \setminus Y = \emptyset$, there is nothing to prove. Otherwise by using the well-ordering Theorem, well-order the set $X \setminus Y$ by the order relation $<$. Define a topology τ_2 on $X \setminus Y$ as

$$\tau_2 = \{X \setminus Y\} \cup \{\{y \in X \setminus Y : y < x\} : x \in X \setminus Y\}.$$

Let

$$\tau = \tau_2 \cup \tau'.$$

It is easy to see that τ is a topology on X.

Claim: $H(X, \tau) = \{I_{X\setminus Y}\} \oplus H$.

Let $h \in \{I_{X\setminus Y}\} \oplus H$. This gives that $h = I_{X\setminus Y} \oplus h_1$ for some $h_1 \in H$. Let $U \in \tau$. If $U \in \tau_2$, then $h(U) = U$ and so $h(U) \in \tau$. If $U \in \tau'$, then $U = (X \setminus Y) \cup U_1$ for some U_1 in τ_1 . Since h_1 is a homeomorphism on (Y, τ_1), $h_1(U_1) \in \tau_1$ and hence $h(U) = (X \setminus Y) \cup h_1(U_1) \in \tau$. Since U is arbitrary, h is a homeomorphism on (X, τ). So

$$\{I_{X\setminus Y}\} \oplus H \subseteq H(X, \tau). \tag{2.3}$$

Conversely assume that $h \in H(X, \tau)$. First we prove that $h(x) = x$ for all $x \in X \setminus Y$. Let x_0 and x_1 be the first and the second elements of the set $X \setminus Y$ and $U = \{y \in X \setminus Y : y < x_1\}$. Then $U = \{x_0\}$ and $U \in \tau$. Since h is a homeomorphism, $h(U) \in \tau$ and hence $h(x_0) = x_0$. Let x_α be any element of $X \setminus Y$ such that $h(x) = x$ for all x in $X \setminus Y$ such that $x < x_\alpha$.

If x_α has an immediate successor x_β in $X \setminus Y$, consider $U = \{x \in X \setminus Y : x < x_\beta\}$, which is an open set and hence $h(U)$ is open in τ. Now

$$h(U) = \{x \in X \setminus Y : x < x_\alpha\} \cup \{h(x_\alpha)\}.$$

By the definition of topology, this gives that $h(U) = U$ and hence $h(x_\alpha) = x_\alpha$.

If x_α has no immediate successor, then x_α is the last element of the set $X \setminus Y$.

Since $X \setminus Y \in \tau_2$, $X \setminus Y \in \tau$. Therefore $h(X \setminus Y) \in \tau$ and $h(X \setminus Y) = (X \setminus Y) \setminus \{x_\alpha\} \cup \{h(x_\alpha)\}$. This implies that $h(x_\alpha) = x_\alpha$ and hence $h(x) = x$ for all $x \in X \setminus Y$. Thus $h_{|X\setminus Y} = I_{X\setminus Y}$ and $h_{|Y}$ will be a homeomorphism of (Y, τ_1). Thus $h_{|Y} \in H$ and we get $h = I_{X\setminus Y} \oplus h_1$ where $h_1 = h_{|X\setminus Y} \in H$. So $h \in \{I_{X\setminus Y}\} \oplus H$. Thus we get

$$H(X, T) \subseteq \{I_{X\setminus Y}\} \oplus H. \tag{2.4}$$

From equations 2.3 and 2.4, we have $H(X, \tau) = \{I_{X\setminus Y}\} \oplus H$. This completes the proof. $\square$

Remark 2.2.5. Let H be a non-trivial permutation group on a set X. Let $Y = X \setminus \{x \in X : h(x) = x \text{ for all } h \in H\}$. Define $H' = \{h_{|Y} : h \in H\}$, which is a permutation group on Y. Note that H' moves all the elements of Y and $H = H' \oplus \{I_{X\setminus Y}\}$. By Theorem 2.2.4, it follows that, if H' is a t-representable permutation group on Y, then H is t-representable on X. So if $(X,\, \tau)$ is a topological space which is not rigid and $H = H(X,\, \tau)$ then without loss of generality we can assume that H moves all the elements of X.

Remark 2.2.6. The intersection of two $t-$representable permutation groups on X need not be $t-$representable on X. The following example illustrates this.

Example 2.2.7. Let $X = \{a,\, b,\, c,\, d\}$ and H_1, H_2 be two permutation groups on X defined by

$H_1 = \{I,\, (a, b),\, (c, d),\, (a, b)(c, d),\, (a, c, b, d),\, (a, d, b, c),\, (a, c)(b, d),\, (a, d)(b, c)\}$

and

$H_2 \;=\; \{I,\,(a,c),\,(b,d),\,(a,b)(c,d),\,(a,b,c,d),\,(a,d,c,b),\,(a,c)(b,d),\,(a,d)(b,c)\}$

respectively. Let τ_1 and τ_2 be two topologies on X defined by

$$\tau_1 \;=\; \{\emptyset,\, X,\, \{a,b\},\, \{c,d\}\}$$

and

$$\tau_2 \;=\; \{\emptyset,\, X,\, \{a,c\},\, \{b,d\}\}.$$

Then we have $H(X,\,\tau_1) \;=\; H_1$ and $H(X,\,\tau_2) =\; H_2$.

So H_1 and H_2 are $t-$representable permutation groups.

Here $H_1 \cap H_2 \;=\; \{I,\,(a,b)(c,d),\,(a,c)(b,d),\,(a,d)(b,c)\}$ which is a normal subgroup of S_4 and hence not $t-$representable on X.

Remark 2.2.8. The permutation group generated by the union of two t-representable permutation groups on X need not be t-representable on X.

Example 2.2.9. Let $X = \{a,b,c,d\}$ and H_1, H_2 be two permutation groups on X defined by $H_1 \;=\; \{I,\,(a,b)(c,d)\}$ and $H_2 = \{I,(a,c)(b,d)\}$.

Let τ_1 and τ_2 be two topologies on X defined by

$$\tau_1 = \{X,\emptyset,\{a\},\{b\},\{a,c\},\{b,d\}\}$$

and

$$\tau_2 = \{X,\emptyset,\{a\},\{c\},\{a,b\},\{c,d\}\}.$$

Then $H(X, \tau_1) = H_1$ and $H(X, \tau_2) = H_2$. Now the permutation group generated by the union of H_1 and H_2 is $\{I, (a, b)(c, d), (a, c)(b, d), (a, d)(b, c)\}$, which is not a t-representable permutation group on X.

2.2.1 $t-$representability of Direct Sum of Permutation Groups

Now we investigate the t-representability of a direct sum of t-representable permutation groups.

Theorem 2.2.10. *Let $\{X_i\}_{i\in I}$ be an arbitrary family of mutually disjoint finite sets and K_i be a t-representable subgroup of $S(X_i)$ for $i \in I$. Then $\bigoplus\limits_{i\in I} K_i$ is t-representable on $X = \underset{i\in I}{\oplus} X_i$.*

Proof. For all $i \in I$, there exists a topology τ_i on X_i such that $H(X, \tau_i) = K_i$. By the well-ordering Theorem, we can choose a well order $<$ on I. Let $\tau_i' = \{(\underset{j<i}{\cup} X_j) \cup U : \quad U \in \tau_i\}$ for all $i \in I$. Now define

$$\tau = \bigcup_{i\in I} \tau_i' \cup \{X\}.$$

Claim-I: τ is a topology on X.

Clearly $\emptyset$ and X are in τ.

Let U and V be in τ. If $U = X$ or $V = X$, then $U \cap V \in \tau$. If $U \neq X$, $V \neq X$, then $U = \bigcup\limits_{j<i} X_j \cup U_i$ where $U_i \in \tau_i$ and $V = \bigcup\limits_{j<k} X_j \cup V_k$ where $V_k \in \tau_k$

for some i and k. If $i = k$, then $U \cap V = (\bigcup_{j<i} X_j) \cup (U_i \cap V_i) \in \tau$. If $i < k$, $U \cap V = (\bigcup_{j<i} X_j) \cup U_i \in \tau$. If $k < i$, then $U \cap V = (\bigcup_{j<k} X_i) \cup V_k) \in \tau$. Thus τ is closed under finite intersections.

Let $\{U_\alpha : \alpha \in J\}$ be an arbitrary family of sets in τ. If $U_\alpha = X$ for some $\alpha \in J$, then $\bigcup_{\alpha \in J} U_\alpha = X \in \tau$. Otherwise for all $\alpha \in J$, $U_\alpha = \bigcup_{j<i_\alpha} X_j \cup U^\alpha_{i_\alpha}$ where $U^\alpha_{i_\alpha}$ is an open subset of X_{i_α}. Consider the set $A = \{i_\alpha : \alpha \in J\}$. If A is an unbounded subset of I, then for any $i \in I$, there exists some $\alpha \in J$ such that $i < i_\alpha$. Therefore, for all $i \in I$, $X_i \subseteq U_\alpha$ for some α. Therefore $\bigcup_{\alpha \in J} U_\alpha = X \in T$. Now assume that A is a bounded subset of I. Let m be the least upper bound of A in I.

Case(1): $m \in A$.

Let $B = \{\alpha \in J : i_\alpha = m\}$. Then $U^\alpha_{i_\alpha}$ is an open subset of X_m for all α . Let $W = \bigcup_{\alpha \in B} U^\alpha_{i_\alpha}$. Then W is open in X_m and $\bigcup_{\alpha \in J} U_\alpha = (\bigcup_{j<m} X_j) \cup W \in \tau$.

Case(2): $m \notin A$.

For all α, we have $i_\alpha < m$ and consequently, $\bigcup_{j<i_\alpha} X_j \subseteq \bigcup_{j<m} X_j$. Also $U^\alpha_{i_\alpha} \subseteq X_{i_\alpha} \subseteq \bigcup_{j<m} X_j$. Therefore $U_\alpha \subseteq \bigcup_{j<m} X_j$, for all $\alpha \in J$. Thus $\bigcup_{\alpha \in J} U_\alpha \subseteq \bigcup_{j<m} X_j$. Since m is the least upper bound of A, for all $j < m$, there exists $\alpha_j \in J$ such that $i_{\alpha_j} > j$. Therefore $X_j \subseteq \bigcup_{j<i_{\alpha_j}} X_j \subseteq U_{\alpha_j}$. Thus $\bigcup_{j<m} X_j \subseteq \bigcup_{\alpha \in J} U_\alpha$. Hence $\bigcup_{\alpha \in J} U_\alpha = \bigcup_{j<m} X_j \in \tau$.

From the above two cases, it follows that τ is closed under arbitrary union. Hence τ is a topology on X.

Claim-II: $\bigoplus\limits_{i\in I} K_i = H(X, \tau)$.

Let $K = H(X, \tau)$. Let $k_i \in K_i$ for all $i \in I$ and $k = \bigoplus\limits_{i\in I} k_i$. Clearly k is a bijection of X onto itself. Let $U \in \tau$. If $U = X$, $k(X) = X$ and therefore $k(X)$ is open in X. If $U \neq X$, $U = (\bigcup\limits_{j<i} X_j) \cup U_i$ where U_i is open in X_i and $k(U) = k(\bigcup\limits_{j<i} X_j) \cup k(U_i) = (\bigcup\limits_{j<i} k(X_j)) \cup k(U_i) = (\bigcup\limits_{j<i} X_j) \cup k(U_i)$. Since $U_i \subseteq X_i$, $k(U_i) = k_i(U_i)$, which is open in X_i. Thus $k(U)$ is open in X. Similarly we can prove that $k^{-1}(U)$ is open in X for every open set U in X. Hence $k \in K$. Therefore

$$\bigoplus_{i\in I} K_i \subseteq K. \tag{2.5}$$

Conversely suppose that $k \in K$. Let i_0 be the smallest element of I. Assume that $k(X_{i_0}) \neq X_{i_0}$ Then there exists $x \in X_{i_0}$ such that $k(x) \notin X_{i_0}$ or there exists $x \notin X_{i_0}$ such that $k(x) \in X_{i_0}$. In the second case also we can see that there exists $x \in X_{i_0}$ such that $k(x) \notin X_{i_0}$, since X_{i_0} is finite. Thus without loss of generality we can assume that there exists $x \in X_{i_0}$ such that $k(x) \notin X_{i_0}$. Since X_{i_0} is open in (X, T), $k(X_{i_0})$ is an open in X. Also we have that $k(x) \in k(X_{i_0})$ and $k(x) \notin X_{i_0}$. Then by the definition of the topology τ, it follows that $X_{i_0} \subsetneqq k(X_{i_0})$. Since X_{i_0} is finite, we get $|X_{i_0}| < |k(X_{i_0})|$. This is a contradiction, since k is an injection. Thus $k(X_{i_0}) = X_{i_0}$.

Now assume that $j \in I$ and $K(X_i) = X_i$, for all $i \in I$ and $i < j$. To prove that $k(X_j) = X_j$. Assume that $k(X_j) \neq X_j$. Then arguing as above there exists $x \in X_j$ such that $k(x) \notin X_j$. Then $k(x) \in X_k$ for some $k > j$.

Consider $\bigcup\limits_{i\leq j} X_i$ which is open in X. Therefore $k(\bigcup\limits_{i\leq j} X_i)$ is also open in X. Now $k(\bigcup\limits_{i\leq j} X_i) = \bigcup\limits_{i<j} k(X_i) \cup k(X_j) = (\bigcup\limits_{i<j} X_i) \cup k(X_j)$. Since $x \in X_j$, $k(x) \in k(\bigcup\limits_{i\leq j} X_i)$ and $X_j \subseteq \bigcup\limits_{i<j} X_i \cup k(X_j)$ and hence $X_j \subseteq k(X_j)$. But $k(x) \in k(X_j)$ and $x \notin k(X_j)$. Thus $|k(X_j)| > |X_j|$, since X_j is finite. It is also a contradiction, since k is an injection. Thus $k(X_i) = X_i,$ for all $i \in I$.

Then $k|_{X_i} = k_i$ is a homeomorphism of $X_i,$ for all $i \in I$. Therefore $k = \bigoplus\limits_{i\in I} k_i$, so that $k \in \bigoplus\limits_{i\in I} K_i$ and hence

$$K \subseteq \bigoplus_{i\in I} K_i. \tag{2.6}$$

From the above two inequalities 2.5 and 2.6, we have

$$K = \bigoplus_{i\in I} K_i.$$

So $\bigoplus\limits_{i\in I} K_i$ is a t-representable permutation group on $X = \underset{i\in I}{\cup} X_i$. $\square$

The following example ensures that finiteness of X_i cannot be dropped in the proof of Theorem 2.2.10 even when I is finite.

Example 2.2.11. Let X_1 be the set of all negative integers and τ_1 be the topology on X_1 defined by

$$\tau_1 = \{X_1, \emptyset\} \cup \{\{a \in X_1 : a \leq m\} : m \in X_1\}.$$

Let X_2 be the set of all non negative integers and τ_2 be the topology on X_2

defined by

$$\tau_2 = \{X_2, \emptyset\} \cup \{\{a \in X_2 : a \leq m\} : m \in X_2\}.$$

Here $H(X_1, \tau_1) = \{I_{X_1}\}$ and $H(X_2, \tau_2) = \{I_{X_2}\}$. Let $X = X_1 \cup X_2$, the set of integers. Now define the topology τ on X as in the proof of Theorem 2.2.10. That is

$$\tau = \{X, \emptyset\} \cup \{\{a \in X : a \leq m\} : m \in X\}.$$

Now we can easily prove that $H(X, \tau)$ is the group generated by the infinite cycle

$$(\ldots, -2, -1, 0, 1, 2, \ldots)$$

which is not equal to the direct sum of $H(X_1, \tau_1)$ and $H(X_2, \tau_2)$.

2.3 $t-$representability of Transitive Permutation Groups

In this section we consider the $t-$representability of transitive permutation groups.

Recall that a topological space (X, τ) is said to be *homogeneous* if for any $x, y \in X$, there exists a homeomorphism h from (X, τ) onto itself such that $h(x) = y$. From the definition of a homogeneous space, it follows that a topological space (X, τ) is homogeneous if and only if $H(X, \tau)$ is a transitive permutation group on X.

If a topological space (X, τ) is homogeneous, then the order of the group $H(X, \tau)$ is greater than or equal to the cardinality of X. As the homogeneity of a topological space increases, the number of homeomorphisms are also increase. But the converse is not true. The following example illustrates this.

Example 2.3.1. Let X be any infinite set and $x_0 \in X$. Let τ be a topology on X defined by

$$\tau \;=\; \{U \subseteq X : \, x_0 \in U\} \cup \{\emptyset\}.$$

Then it is easy to prove that

$$\begin{aligned} H(X,\, \tau) \;&=\; \{h \in S(X) \;:\; h(x_0) \;=\; x_0\} \\ &=\; S_{\{x_0\}}, \end{aligned}$$

which is the symmetric group on $X \setminus \{x_0\}$. Here $|H(X,\, \tau)| \;=\; |S(X)|$ and $(X,\, \tau)$ is not a homogeneous space..

Here we need the following result due to J. Ginsburg [14] in which he characterized the finite homogeneous topological spaces.

Theorem 2.3.2. *[14] A finite nonempty topological space X is homogeneous if and only if there exist positive integers m and n such that X is homeomorphic to $D(m) \times I(n)$ where $D(k)$ and $I(k)$ denote the set $\{1,\, 2,\, 3,\, \ldots,\, k\}$ with the discrete topology and indiscrete topology respectively.*

Using Ginsburg characterization theorem on finite homogeneous space, we observe the following results.

Remark 2.3.3. [39] Let (X, τ) be a finite homogeneous space, then

(a) If $|X|$ is a prime, then the topology τ on X is either discrete or indiscrete.

(b) When $|X| \geq 2$, there exist at least one transposition which is a homeomorphism of (X, τ).

(c) There exists a partition of X using sets with equal number of elements, which forms a base for the topology on X.

Moreover, the symmetric group is the only t-representable transitive permutation group of prime degree.

Using the above remarks, one can prove the following proposition.

Proposition 2.3.4. *A regular permutation group K on a finite set X is not t-representable on X when $|X| \geq 3$.*

Proof. Suppose that K is a t-representable permutation group on X. Then there exists a topology τ on X such that $H(X, \tau) = K$. Since K is transitive, the topology τ on X is homogeneous. Since $|X| \geq 3$, then by Remark 2.3.3(b), there exists a transposition which is a homeomorphism of (X, τ) onto itself. This implies that K is not a regular permutation group. This is a contradiction. So a finite regular permutation group K is not t-representable on X when $|X| \geq 3$. $\square$

Remark 2.3.5. A regular permutation group K on a finite set X is $t-$representable if and only if $|X| < 3$.

Example 2.3.6. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $H = \{(1, 2)(3, 4)(5, 6)$, $(1, 6)(2, 3)(4, 5)$, $(1, 4)(2, 5)(3, 6)$, $(1, 3, 5)(2, 6, 4)$, $(1, 5, 3)(2, 4, 6), I\}$. Here H is a regular permutation group on X and $|X| = 6$. Then by Proposition 2.3.4, H is not $t-$representable on X.

Observe that, in Proposition 2.3.4, the condition of finiteness of X cannot be relaxed.

Example 2.3.7. Consider $\mathbb{Z}$, the set of all integers with the topology τ consisting precisely of $\mathbb{Z}$, $\emptyset$, and all the subset of $\mathbb{Z}$ of the form $\{x \in \mathbb{Z} : x \leq m\}$ for some $m \in \mathbb{Z}$. Then $H(\mathbb{Z}, \tau)$ is the group generated by the infinite cycle $(\ldots, -2, -1, 0, 1, 2, \ldots)$. Obviously, $H(\mathbb{Z}, \tau)$ is a regular permutation group on $\mathbb{Z}$. So there exists t-representable regular permutation group on an infinite set X.

Remark 2.3.8. Any transitive abelian permutation group on a finite set X is not t-representable when $|X| \geq 3$.

Proof. We have that any transitive abelian permutation group is regular by Lemma 1.4.6. Proof follows from Proposition 2.3.4. □

Next we consider the t-representability of the dihedral group D_n. Let us start with the following Lemma.

Lemma 2.3.9. *Let X be a finite set with a partition topology τ having a partition $\mathscr{B}$ consisting of m sets each with n elements as a base. Then the order of the* *group of homeomorphisms of* $(X,\,\tau)$ *is* $(m!)(n!)^m$

Proof. Let $\mathscr{B} = \{A_1,\; A_2,\; A_3,\; \ldots,\; A_m\}$ where $|A_i| = n$ for all $i = 1, 2, \ldots m$. Then a homeomorphism h of $(X,\,\tau)$ onto itself maps each A_i onto some A_j where $i,\; j \in \{\;1,\; 2,\; \ldots,\; m\}$. Then the order of $H(X, \tau)$ is the number of bijections on X such that, for every i, $h(A_i) = A_j$ for some j. Now elements of A_i can be permuted in $n!$ ways for every $i = 1,\, 2,\, .\; .\; .\; ,\, m$ and elements of $\mathscr{B}$ can be permuted in $m!$ ways. Thus the order of the group of homeomorphisms of $(X,\,\tau)$ is $(m!)(n!)^m$ $\square$

For $n \geq 3$, the *dihedral group* D_n is defined as the rigid motions of the plane preserving a regular n-gon with the operation being composition [13]. The order of the dihedral group D_n is $2n$.

Theorem 2.3.10. *The dihedral group* D_n *is not t-representable for* $n \geq 5$

Proof. The generators of the dihedral group D_n on $X = \{1,\, 2,\, 3,\, \ldots,\, n\}$ are the cycle $(1,\; 2,\; 3,\; .\; .\; .\; ,\, n)$ and
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & i & \ldots & n-1 & n \\ 1 & n & n-1 & n-2 & n-3 & \ldots & n+2-i & \ldots & 3 & 2 \end{pmatrix}$. Hence D_n is a transitive permutation group on X. If D_n is t-representable on X, there exists a topology τ on X such that $H(X,\,\tau) = D_n$. Since D_n is transitive, $(X,\,\tau)$ is a homogeneous topological space. Consider the following cases.

Case(1): n is a prime number.

In this case, by Remark 2.3.3(a), $(X,\,\tau)$ is discrete or indiscrete and the group of homeomorphisms of (X, τ) is $S(X)$, which is not possible. Thus D_n is not t-representable on X.

Case(2): n is not a prime number.

Then by Lemma 2.3.9, the order of $H(X, \tau)$ is of the form $k!(m!)^k$ for some positive integers k and m such that $n = km$. Clearly $k \geq 2$ and $m \geq 2$, since (X, τ) is neither discrete nor indiscrete. But when $n \geq 5$, $k > 2$ or $m > 2$. Then the order of D_n,

$$\begin{aligned} o(D_n) &= k!(m!)^k \\ &= mk(k-1)!m^{k-1}((m-1)!)^k. \end{aligned}$$

This implies that the order of (D_n) is greater than $2n$. This is a contradiction, since order of D_n is $2n$. Hence D_n is not t-representable for $n \geq 5$.

□

Remark 2.3.11. The Theorem 2.3.10 is not true for $n < 5$.

Proof. Consider $X = \{1, 2, 3, 4\}$ and $\tau = \{X, \emptyset, \{1, 3\}, \{2, 4\}\}$. Then $H(X, \tau) = \{I, (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2), (1, 2)(3, 4), (1, 4)(2, 3), (1, 3), (2, 4)\} = D_4$. Hence D_4 is t-representable on X. Since $D_3 = S_3$, D_3 is also t-representable on a set with 3 elements.

Thus the dihedral group D_n is t-representable for $n < 5$. □

As an immediate consequence, we have the following remark.

Remark 2.3.12. The dihedral group D_n is $t-$representable on X if and only if $n \leq 4$.

Now we consider the $t-$representability of finite primitive permutation groups. First we recall the definition of minimal open sets.

Definition 2.3.13. [33] Minimal open set
Let (X, τ) be a topological space. Then a minimal open set in (X, τ) is a nonempty open set having no proper nonempty open subset.

Theorem 2.3.14. *Let X be a finite set and H be a primitive permutation group on X. Then H is $t-$representable on X if and only if $H = S(X)$.*

Proof. If $H = S(X)$, then clearly H is t-representable on X. Conversely assume that H is $t-$representable on X. Then there exist a topology τ on X such that $H(X, \tau) = H$. Let U be a minimal open set in X. Now any homeomorphism on X maps minimal open set to minimal open set. So $h(U)$ is also a minimal open set for all $h \in H$. Then either $h(U) = U$ or $h(U) \cap U = \emptyset$. Otherwise U is not a minimal open set. Thus U is a block for H. But H is a primitive group. So H has no proper non-trivial block on X and hence either U is a singleton or $U = X$. This implies that either τ is discrete or indiscrete topology and hence $H = S(X)$. $\square$

Now we characterize $t-$representable imprimitive permutation group on a finite set X.

Theorem 2.3.15. *Let H be an imprimitive permutation group on a finite set X. Then H is $t-$representable on X if and only if H is the stabilizer of some partition of X into parts of equal size k where $1 < k < |X|$.*

Proof. Let H be a $t-$representable permutation group on X. Since H is transitive, the corresponding topology τ on X is homogeneous. Hence by Remark 2.3.3 (c), $\mathscr{B} = \{A_1,\ A_2,\ A_3,\ \ldots,\ A_m\}$ form a base for τ where $|A_i| = k$, $1 \leq k \leq |X|$ for all $i = 1, 2, \ldots, m$ and $\bigcup\limits_{i=1}^{m} A_i = X$. So

$$H(X,\, \tau) = \{h \in S(X) : h(A_i) = A_j, i, j = 1, 2, \ldots m\}.$$

Since H is an imprimitive permutation group, $|A_i| = k$, $1 < k < |X|$ for all $i = 1, 2, \ldots, m$. Hence H is the stabilizer of some partition of X into parts of equal size k where $1 < k < |X|$.

Conversely assume that H is the stabilizer of some partition of X into equal parts of size k. Then clearly H is $t-$representable on X. $\square$

Example 2.3.16. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $H = \{I, (1,\ 2,\ 3,\ 4,\ 5,\ 6), (1,\ 3,\ 5)(2,\ 4,\ 6), (1,\ 4)(2,\ 5)(3,\ 6), (1,\ 5,\ 3)(2,\ 6,\ 4), (1,\ 6,\ 5,\ 4,\ 3,\ 2)\}$. Here H is an imprimitive group, but not $t-$representable on X

From Theorems 2.3.14 and 2.3.15, we conclude the following Remark.

Remark 2.3.17. Let H be a transitive subgroup of S_n. Then H is t-

representable on X if and only if H is one of the following.

1. $H = S_n$

2. H is the stabilizer of some partition of X into equal parts of size k where $1 < k < n$.

This follows that a transitive subgroup of S_n is t-representable on X if and only if H is the stabilizer of some partition of X into equal parts of size k where $1 \leq k \leq n$.

2.4 $t-$representability of Maximal Subgroups of the Symmetric Groups

By a maximal subgroup, we mean a maximal element of the collection of all proper subgroups of the symmetric group $S(X)$. In this section our main aim is to determine the $t-$representability of maximal subgroups of the symmetric group.

First we consider the $t-$representability of maximal subgroups of $S(X)$ when X is a finite set.

2.4.1 $t-$representability of Maximal Subgroups of the Finite Symmetric Groups

The maximal subgroups M of S_n are the following [12].

1. M is the set stabilizer of some set V with $|V| = m$, $1 \leq m < \frac{n}{2}$, which is an intransitive permutation group.

2. M is the stabilizer of some partition of X into m equal parts of size k with $1 < m < n$.

3. $M = A_n$ or any proper primitive group.

Now we characterize $t-$representable maximal permutation groups on a finite set.

Theorem 2.4.1. *Let X be any finite set and H be a maximal subgroup of $S(X)$. Then H is not $t-$representable on X if and only if H is a proper primitive permutation group on X.*

Proof. Let H be a maximal subgroup of $S(X)$. By Theorem 2.3.14, the maximal subgroups of the form A_n or any proper primitive group are not $t-$representable on X.

Now assume that H is maximal and not a primitive permutation group on X.

Case 1: H is the stabilizer of a subset A of X.

Here we have $H = \{h \in S(X) : h(A) = A\}$.

Now define τ on X as

$$\tau = \{\emptyset\} \cup \{U \subseteq X : A \subseteq U\}.$$

Then clearly τ is a topology on X and it follows that $H(X, T) = H$. So H is a $t-$representable permutation group on X.

Case 2: H is the stabilizer of some partition of X into m equal parts of size k with $1 < k < |X|$.

Let $\mathscr{B} = \{A_1, A_2, \ldots, A_m\}$ be the partition of X and τ be the topology having base $\mathscr{B}$ where $|A_i| = k$ for all $i = 1, 2, \ldots m$ and $mk = |X|$. Then $H(X, \tau)$ consisting of all permutations which preserve the partition. So H is a $t-$representable permutation group on X.

□

So the only finite maximal subgroup of $S(X)$ which are not $t-$representable are of the form A_n or any proper primitive subgroup of $S(X)$.

2.4.2 $t-$representability of Maximal Subgroups of the Infinite Symmetric Groups

If X is an infinite set, we cannot list all maximal subgroup of $S(X)$ as in the case of finite symmetric groups. But Ball, Richman, Neumann, Macpherson etc. found some classes of maximal subgroups of infinite symmetric group [5,7,31,38]. Now we study the $t-$representability of these maximal subgroups.

Theorem 2.4.2. *[31] If A is a nonempty finite subset of X then $S_{\{A\}}$ is a maximal subgroup of $S(X)$. Any maximal subgroup of $S(X)$ which is not of the form $S_{\{A\}}$ for some nonempty subset A of X contains $FS(X)$ and therefore highly transitive.*

Obviously any maximal subgroup of $S(X)$ of the form $S_{\{A\}}$ is $t-$representable on X by defining a topology τ on X as $\tau = \{U \subseteq X : A \subseteq U\} \cup \{\emptyset\}$.

Remark 2.4.3. [12]

1. If X is an infinite set, then the symmetric group $S(X)$ has no imprimitive maximal subgroups.
2. If A and $X \setminus A$ are infinite sets, then the stabilizer of A is not a maximal subgroup of $S(X)$. But the stabilizer of A is a $t-$representable permutation group on X.

If $\mathscr{F}$ is a filter on a nonempty set X, then we define the stabilizer [31] of $\mathscr{F}$

by

$$S_{\{\mathscr{F}\}} = \{h \in S(X) : h(A) \in \mathscr{F} \Leftrightarrow A \in \mathscr{F} \text{ for every } A \subseteq X\}.$$

We also define

$$S_{(\mathscr{F})} = \{h \in S(X) : \{x \in X : h(x) = x\} \in \mathscr{F}\}.$$

Obviously $S_{(\mathscr{F})}$ is a subgroup of $S_{\{\mathscr{F}\}}$ [31].

Proposition 2.4.4. *The stabilizer of a filter on a set X is a $t-$representable permutation group on X.*

Proof. If $\mathscr{F}$ is a filter, then clearly $\mathscr{F} \cup \{\emptyset\}$ is a topology τ on X. It is easy to prove that $H(X, \tau) = S_{\{\mathscr{F}\}}$. This completes the proof. $\square$

Definition 2.4.5. [45] Ultrafilter

A filter $\mathscr{F}$ is an ultrafilter if there is no filter $\mathscr{G}$ strictly finer than $\mathscr{F}$.

In [31], H. D. Macpherson and P. M. Neumann proved the following Theorem.

Theorem 2.4.6. *[31] Let $\mathscr{F}$ is an ultrafilter on an infinite set X. Then $S_{\{\mathscr{F}\}}$ is a maximal subgroup of $S(X)$. Furthermore $S_{\{\mathscr{F}\}} = S_{(\mathscr{F})}$.*

Using the above Theorem we can easily prove the following Theorem.

Theorem 2.4.7. *Let X be an infinite set with $|X| = k$. Then there are 2^{2^k} maximal permutation groups on X which are t-representable.*

Proof. We have the cardinality of the set of all ultrafilters on an infinite set X is 2^{2^k}. By Theorem 2.4.6, the stabilizer of an ultrafilter $\mathscr{F}$ on an infinite set is a maximal subgroup of $S(X)$. We also have $S_{\{\mathscr{F}\}} = S_{(\mathscr{F})}$ by Theorem 2.4.6. It follows that distinct ultrafilters have distinct stabilizers. Thus there are 2^{2^k} maximal subgroups in $S(X)$ which are t-representable on X. $\square$

We know that the cardinality of the set of all topologies on an infinite set X is $2^{2^{|X|}}$. So from Theorem 2.4.7, it follows that the number of t-representable maximal subgroups of $S(X)$ are as many as the number of topologies on X when X is an infinite set.

Definition 2.4.8. [12] Almost Stabilizer

Let X be an infinite set and Y be a nonempty subset of X with $|Y| = \lambda < |X|$. Then the almost stabilizer of Y is defined by

$$AStab(Y) = \{g \in S(X) : \; |Y \Delta g(Y)| < \lambda\}$$

where Δ denotes the symmetric difference of Y and $g(Y)$.

Theorem 2.4.9. *[5] If Y is a nonempty subset of an infinite set X with $|Y| = \lambda < |X|$. Then the almost stabilizer of Y, $AStab(Y) = \{g \in S(X) : \; |Y \Delta g(Y)| < \lambda\}$ is a maximal subgroup of $S(X)$.*

Now we investigate the $t-$representability of the almost stabilizer of a subset of X.

Proposition 2.4.10. *Let Y be a nonempty subset of an infinite set X with $|Y| = \lambda < |X|$. Then $AStab(Y)$ is a $t-$representable permutation group on X.*

Proof. Let Y be a finite set, then $AStab(Y) = S_{\{Y\}}$, which is a $t-$representable permutation group on X by defining a topology τ on X as

$$\tau = \{\emptyset\} \cup \{U \subseteq X : Y \subseteq U\}.$$

Let Y be an infinite subset of X with $|Y| < |X|$. Define

$$\tau = \{\emptyset\} \cup \{U \subseteq X : |U' \cap Y| < \lambda\}$$

where U' denotes the complement of U in X.

Claim I: τ is a topology on X.

Clearly $\emptyset$ and X are in τ.

Let $U_1, U_2, \ldots U_n$ are in τ and $U = U_1 \cap U_2 \ldots \cap U_n$. We have

$$\begin{aligned} U' \cap Y &= (U_1' \cup U_2' \cup \ldots \cup U_n') \cap Y \\ &= (U_1' \cap Y) \cup (U_2' \cap Y) \cup \ldots \cup (U_n' \cap Y). \end{aligned}$$

Since $|U_i' \cap Y| < \lambda$ for all $i = 1, 2, \ldots n$, it follows that $|U' \cap Y| < \lambda$.

Let $\{U_i : i \in I\}$ be an arbitrary collection in τ and $U = \underset{i \in I}{\cup} U_i$. Then

$$U' \cap Y = (\underset{i \in I}{\cup} U_i)' \cap Y$$

$$= (\bigcap_{i \in I} U_i') \cap Y$$

$$\subseteq U_i' \cap Y \text{ for all } i \in I.$$

This implies that $|U' \cap Y| < \lambda$. So τ is a topology on X.

Claim II: $H(X, \tau) = AStab(Y)$.

Let $h \in H(X, \tau)$. Since $Y' \cap Y = \emptyset$, we have $Y \in \tau$ and hence $h(Y)$ is open in (X, τ). This implies that $|h(Y)' \cap Y| < \lambda$. So

$$|Y \setminus h(Y)| < \lambda. \tag{2.7}$$

Similarly we have $h^{-1}(Y)$ is in τ. This implies that $|h^{-1}(Y)' \cap Y| < \lambda$ and hence $|Y \setminus h^{-1}(Y)| < \lambda$. Now we have $[h(Y) \setminus h(h^{-1}(Y)] \subseteq h(Y \setminus h^{-1}(Y))$ and so $|h(Y) \setminus Y| < |h(Y \setminus h^{-1}(Y))| = |Y \setminus h^{-1}(Y)| < \lambda$. This follows that

$$|h(Y) \setminus Y| < \lambda \tag{2.8}$$

From inequalities 2.7 and 2.8, it follows that $|Y \Delta h(Y)| < \lambda$ and hence $h \in AStab(Y)$.

Conversely assume that $h \in AStab(Y)$. Now we claim that h is a homeomorphism on (X, τ). Let $U \in \tau$. Then $|Y \setminus U| = |U' \cap Y| < \lambda$ and hence $|h(Y) \setminus h(U)| \leq |h(Y \setminus U)| = |Y \setminus U| < \lambda$. Since $h \in AStab(Y)$, we have

$|Y \setminus h(Y)| < \lambda$. Now

$$Y \setminus h(U) = [(Y \setminus h(Y) \setminus h(U)] \cup [(Y \setminus h(Y)) \cap (h(Y) \setminus h(U))].$$

Since λ is an infinite cardinal, $|Y \setminus h(U)| < \lambda$ and hence $h(U) \in \tau$. Similarly we get $|Y \setminus h^{-1}(U)| < \lambda$. So h is a homeomorphism on (X, τ). Hence the claim. $\square$

A topological space (X, τ) is said to be hereditarily homogeneous [27] if every subspace of (X, τ) is homogeneous. In [27], V. Kannan and P. T. Ramachandran proved that (X, τ) is hereditary homogeneous space if and only if every permutation of X which moves only a finite number of elements of X is a homeomorphism of (X, τ) onto itself. That is, (X, τ) is a hereditarily homogeneous space if and only if the finitary symmetric group $FS(X)$ is contained in $H(X, \tau)$.

Remark 2.4.11. Let M be a highly transitive maximal subgroup of $S(X)$. Then by Theorem 2.4.2, the finitary symmetric group $FS(X)$ is contained in M and $M \neq S(X)$. If M is $t-$representable on X, then by Lemma 1.2.3 in [34], the corresponding topology τ on X satisfy the following.

(a) Supersets of nonempty open sets of (X, τ) are open.

(b) Every finite subset of (X, τ) is closed.

(c) Intersection of nonempty open sets of (X, τ) is non- empty.

(d) There exist no non- empty finite open set.

(e) (X, τ) is a T_1 hereditarily homogeneous space.

Now we can show that there exists topological space $(X,\, \tau)$ which satisfies all the above conditions but the group of homeomorphisms $H(X,\, \tau)$ is not maximal in $S(X)$.

The following example illustrates this

Example 2.4.12. Let X be any infinite set and $X \;=\; A_1 \cup A_2$ where A_1 and A_2 are disjoint subsets of X such that $|A_1| \;=\; |X| \;=\; |A_2|$. Define

$$\tau \;=\; \{U \subseteq X :\; |(X \setminus U) \cap A_1| < |X|\} \cup \{\emptyset\}.$$

By using similar arguments as in the Proposition 2.4.10, one can easily verify that τ is a topology on X and $H(X,\, \tau) \;=\; \{h \in S(X) :\; |h(A)\Delta A| < |X|\}$.

Note that $H(X, \tau)$ satisfies all the properties in Remark 2.4.11. Here $H(X,\, \tau)$ is not a maximal subgroup of $S(X)$ since $H(X,\, \tau) \subsetneq AStab\mathscr{P}$ where

$$AStab\mathscr{P} \;=\; \{g \in S(X) :\; \text{for all } i \text{ there exist } j \text{ such that } |g(A_i)\Delta A_j| < |X|\}$$

and $\mathscr{P} = \{A_1,\, A_2\}$.

Chapter 3

$t-$representability of Cyclic Group of Permutations

In this Chapter, we continue the work carried out in second chapter. Here we determine the $t-$representability of some cyclic subgroups of the symmetric group $S(X)$ using an order theoretic method.

If σ is a permutation on a set X, then the subgroup of $S(X)$ generated by σ is denoted by $< \sigma >$.

3.1 Preliminaries

Definition 3.1.1. [23] Preorder

A reflexive transitive relation $\leq$ on a set X is called a *preorder* on X. The ordered pair $(X, \leq)$ is called a *preordered set*.

S. J. Andima and W. J. Thron [2] associated with each topology T on a set X with a preorder relation '$\leq$' on X defined by $a \leq b$ if and only if every open set containing b contains a. Then any homeomorphisms of (X, τ) onto itself is also an order isomorphism of $(X, \leq)$. Also we have that the group of homeomorphisms of the topological space (X, τ) is equal to the group of order isomorphisms of the preordered set $(X, \leq)$ if X is finite.

Definition 3.1.2. [45] T_0 Space

A topological space (X, τ) is said to be a T_0 *space* if given any two distinct points in X, there exists an open set which contains one of them but not the other.

From the definition of a T_0 space, it follows that a topological space (X, τ) is a T_0 space if and only if $(X, \leq)$ is a partially ordered set.

If X is a finite nonempty set, then the partially ordered set $(X, \leq)$ has both maximal and minimal elements. Also an order isomorphism of $(X, \leq)$ maps maximal elements to maximal elements and minimal elements to minimal elements.

Remark 3.1.3. Let X be a finite set. Then there is a one-to-one correspondence between the set of all T_0 topologies on X and the set of all partial orders on X [2].

3.2 Groups Generated by a Product of Disjoint Cycles Having Equal Length

We investigate the $t-$representability of cyclic subgroups of $S(X)$ generated by a product of disjoint cycles having equal length.

Theorem 3.2.1. *[39] Let X be any set. Then every subgroup K of $S(X)$ of order two is t-representable on X.*

So cyclic subgroups of $S(X)$ generated by a product of disjoint cycles having length 2 (transpositions) are t-representable on X.

Now we determine the $t-$representability of permutation group generated by a permutation which is a product of two disjoint cycles that have equal length n where $n \geq 3$.

Theorem 3.2.2. *Let X be any set such that $|X| = 2n$, where $n \geq 3$ and σ be a permutation on X which is a product of two disjoint cycles having equal length n, then the group generated by σ is not $t-$representable on X.*

Proof. Let $X = \{a_1, a_2, a_3, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ and

$$\sigma = (a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n).$$

Let K be the group generated by σ. Decompose X as $X_1 \cup X_2$ where

$$X_1 = \{a_1, a_2, \ldots a_n\} \text{ and } X_2 = \{b_1, b_2, \ldots b_n\}.$$

Assume that K is $t-$representable permutation group on X. Then there exists a topology τ on X such that the group of homeomorphisms on X, $H(X,\,\tau) = K$. Since there exist no transposition on X which is a homeomorphism of $(X,\,\tau)$, the space (X, τ) must be T_0. Let $(X,\,\leq)$ be the corresponding partially ordered set. Since X is a finite set, $(X,\,\leq)$ has both maximal and minimal elements. If an element is both minimal and maximal, then all the elements are minimal and maximal and hence $G(X,\,\leq) = S(X)$. This implies that the topology is discrete and $H(X,\,\tau) = S(X)$, which is not possible. So a minimal element can not be a maximal element and conversely. Then either X_1 or X_2 is the set of all minimal elements.

Assume that X_1 is the set of all minimal elements. Then X_2 is the set of all maximal elements. Thus the elements of X_i are incomparable to each other for $i = 1,\,2$. So there exists at least one $a_j \in X_1$ and $b_i \in X_2$ such that b_i succeeds a_j. Otherwise all elements of X are minimal and maximal, which is a contradiction. Since $G(X,\,\leq) = K$, if b_i succeeds a_j, then $\sigma^h(b_i)$ succeeds $\sigma^h(a_j)$ for $h = 1,\,2,\,\ldots, n-1$ and this gives $a_{j\oplus(h-1)} < b_{i\oplus(h-1)}$ where $\oplus$ denotes addition modulo n. So without loss of generality we can assume that b_1 at least succeeds a_1. If b_1 succeeds only a_1, then each b_j succeeds only a_j for $j = 1, 2, \ldots, n$. In this case $(a_1,\,a_i)(b_1,\,b_i)$, where $i = 2,\,3,\,\ldots\,n$ are order isomorphisms, which is not possible.

Now assume that b_1 succeeds k elements in X_1 namely $a_1,\,a_2,\,\ldots,\,a_k$ where $1 < k \leq n$. Then each $b_i \in X_2$ exactly succeeds k elements in X_1 namely

$a_i,\ a_{i\oplus 1},\ \ldots,\ a_{i\oplus(k-1)}$. Then

$$\mathscr{B} = \{\{a_i\}, \{a_i, a_{i\oplus 1}, \ldots, a_{i\oplus k-1}, b_i\},\ i = 1, 2, \ldots, n\}$$

form a base for T.

If $k = n$, then $H(X, T) = S(X_1) \oplus S(X_2)$, which is not possible. So assume that $1 < k < n$

For $j = 1, 2, \ldots n$, define $h : X \to X$ as

$$h(x) = \begin{cases} a_{j\oplus n-(i-1)} & \text{if } x = a_i,\ 1 \leq i \leq n \\ b_{j\oplus n-(k+i-2)} & \text{if } x = b_i,\ 1 \leq i \leq n. \end{cases}$$

Now we claim that h is a is a homeomorphism on $(X,\ \tau)$ onto itself. For let $U \in \mathscr{B}$. Then either U is a singleton or $U = \{a_l, a_{l\oplus 1}, \ldots, a_{l\oplus(k-1)}, b_l\}$, where $1 \leq l \leq n$. If U is a singleton, then $h(U) \in \mathscr{B}$. Now let

$$U = \{a_l, a_{l\oplus 1}, \ldots, a_{l\oplus(k-1)}, b_l\}.$$

Then

$$\begin{aligned} h(U) &= \{a_{j\oplus n-(l-1)}, a_{j\oplus n-l}, a_{j\oplus n-(l+1)}, \ldots, \\ &\qquad a_{j\oplus n-(l+k-2)}, b_{j\oplus n-(l+k-2)}\} \\ &= \{a_{j\oplus(n-l)+1)}, a_{j\oplus n-l}, a_{j\oplus(n-l)-1)}, \ldots, \end{aligned}$$

$$
\begin{aligned}
& a_{j\oplus(n-(l+k)+2)}, b_{j\oplus(n-(l+k)+2)}\} \\
= \; & \{a_{j\oplus n-(l+k)+2}, a_{j\oplus n-(l+k)+3}, \cdots, \\
& a_{j\oplus n-(l+k)+k}, a_{j\oplus n-(l+k)+(k+1)}, b_{j\oplus n-(l+k)+2)}\} \\
= \; & \{a_p, a_{p\oplus 1}, a_{p\oplus 2}, \ldots a_{p\oplus(k-1)}, b_p\}, \\
& \text{ where } p = j \oplus n - (l+k) + 2.
\end{aligned}
$$

Thus $h(U) \in \mathscr{B}$. Observe that h is the disjoint product of transpositions and hence $h^{-1} = h$. Consequently, we have $h^{-1}(U) \in \mathscr{B}$. This is true for all $U \in \mathscr{B}$. So h is a homeomorphism on (X, τ), which is a contradiction to the fact that $H(X, \tau) = K$. This completes the proof. □

The following example illustrates the Theorem 3.2.2.

Example 3.2.3. Let $X = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and

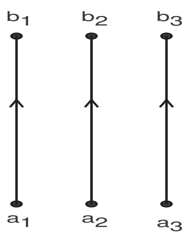
$$\sigma = (a_1, a_2, a_3)(b_1, b_2, b_3).$$

Then the cyclic group generated by σ is,

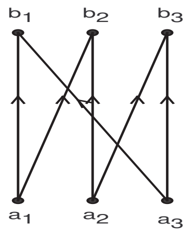
$$< \sigma >= \{I, (a_1, a_2, a_3)(b_1, b_2, b_3), (a_1, a_3, a_2)(b_1, b_3, b_2)\}.$$

Suppose $< \sigma >$ is a $t-$representable permutation group on X, then there exists a topology τ on X such that $H(X, \tau) =< \sigma >$. Consider $(X, \leq)$, then arguing as in the proof of Theorem 3.2.2 either $X_1 = \{a_1, a_2, a_3\}$ or $X_2 = \{b_1, b_2, b_3\}$

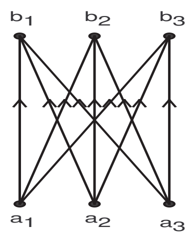
is the set of all minimal elements. Without loss of generality we can assume that X_1 is the set of all minimal elements. So the possible partial orders on X are essentially as in the following figure.



$(X, \leq_1)$



$(X, \leq_2)$



$(X, \leq_3)$

Then

$$\begin{aligned}
G(X, \leq_1) &= \{I,\ (a_1, a_2)(b_1, b_2),\ (a_1, a_3)(b_1, b_3),\ (a_2, a_3)(b_2, b_3), \\
&\quad (a_1, a_2, a_3)(b_1, b_2, b_3),\ (a_1, a_3, a_2)(b_1, b_3, b_2)\}, \\
G(X, \leq_2) &= \{I,\ (a_1, a_2)(b_1, b_3),\ (a_1, a_3)(b_2, b_3),\ (a_2, a_3)(b_1, b_2), \\
&\quad (a_1, a_2, a_3)(b_1, b_2, b_3),\ (a_1, a_3, a_2)(b_1, b_3, b_2)\}, \\
\text{and } G(X, \leq_3) &= S(X_1) \oplus S(X_2).
\end{aligned}$$

So there exists no partial order on X such that the group of order isomorphisms is the group generated σ. Then by Remark 3.1.3, there exists no topology τ on X such that $H(X, \tau) = <\sigma>$. Hence $<\sigma>$ is not $t-$representable on X.

Now we consider the $t-$representability of the cyclic group generated by a permutation on X which is an arbitrary product of more than two disjoint cycles having equal lengths n where $n > 2$. Here we prove that such cyclic subgroups of the symmetric group $S(X)$ are $t-$representable on X.

Theorem 3.2.4. *Let X be any set and σ be a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n where $n > 2$, then the group generated by σ, $< \sigma >$ is $t-$representable on X.*

Proof. Let

$$\sigma = \prod_{i \in I} C_i$$

where $\{C_i : i \in I\}$ be an indexed family of disjoint cycles having equal length n where $n > 2$ and $|I| > 2$. Let $C_i = (a_{i\,1}, a_{i\,2}, \ldots, a_{i\,n})$ for every $i \in I$. By Theorem 2.2.4 without loss of generality we can assume that $X = \bigcup_{i \in I} X_i$ where $X_i = \{a_{i\,1}, a_{i\,2}, \ldots, a_{i\,n}\}$.

By the well-ordering Theorem, well-order the set I by the order relation $<$. We can use the ordinals to index the members of I. Let i_0 be the first element of I and i_1 denote the first element of the set $I \setminus \{i_0\}$. In general i_γ denote the first element of the set $I \setminus \{i_\alpha : \alpha < \gamma\}$ provided $I \setminus \{i_\alpha : \alpha < \gamma\} \neq \emptyset$.

Define a base $\mathscr{B}$ by

$$\mathscr{B} = \{A_{i_\alpha\, j} : i_\alpha \in I, j = 1, 2, \ldots, n\} \cup \{A_j : j = 1, 2, \ldots, n\}$$

where

$$A_{i_\alpha j} = \begin{cases} \{a_{i_k j\,:\,i_k<i_\alpha}\} & \text{if } i_\alpha \leq i_2 \\ \{a_{k j\,:\,i_k<i_\alpha}\} \cup \{a_{i_0\, j\oplus(n-1)}\} & \text{if } i_2 < i_\alpha. \end{cases}$$

and

$$A_j = \{a_{i\,j} \,:\, i \in I\} \cup \{a_{i_0\, j\oplus(n-1)}\}.$$

Here $\oplus$ denotes addition modulo n. Now τ be the topology on X having the base $\mathscr{B}$. Since $\mathscr{B}$ is a collection of subsets of X such that for each $x \in X$ there is a minimal set $M(x) \in \mathscr{B}$ containing x, τ is an Alexandroff discrete topology on X.

It can be verify that σ is a homeomorphism of (X, τ). Then all powers of σ are homeomorphisms of (X, T). Hence $<\sigma>\subseteq H(X,\, \tau)$

Conversely let $h \in H(X,\, \tau)$. We have to prove that h is of the form σ^m for some m such that $1 \leq m \leq n$. Since h is a homeomorphism on $(X,\, \tau)$, $h(A_{i_1\,1}) = A_{i_1\,m}$ for some m, $1 \leq m \leq n$. Thus $h(a_{i_0\,1}) = a_{i_0\,m}$. Now consider the minimal open set $A_{i_2\,1}$ containing $a_{i_1\,1}$. Then $h(A_{i_2\,1})$ is a minimal open set containing $h(a_{i_1\,1})$ and which contains $a_{i_0\,m}$. It follows that $h(a_{i_1\,1}) = a_{i_1\,m}$. Next we consider the minimal open set $A_{i_3\,1} = \{a_{i_0\,1},\, a_{i_1\,1},\, a_{i_2\,1},\, a_{i_0\,n}\}$. Now $h(A_{i_3\,1})$ is a minimal open set containing $h(a_{i_2\,1})$ and which contains $a_{i_0\,m}$ and $a_{i_1\,m}$. This implies that $h(A_{i_3\,1}) = A_{i_3\,m}$ and hence $h(a_{i_0\,n}) = a_{i_0\,m\oplus(n-1)}$.

Assume that $h(a_{i_\alpha\,1}) = a_{i_\alpha\,m}$ for all $\alpha < \gamma$. If i_γ has an immediate successor i_δ in I, consider $A_{i_\delta\,1} = \{a_{i_0\,1},\, a_{i_1\,1},\, \ldots,\, a_{i_\gamma\,1}\} \cup \{a_{i_0\,n}\}$. Then $h(A_{i_\delta\,1}) =$

$\{a_{i_0\,m},\, a_{i_1\,m},\, \ldots,\, h(a_{i_\gamma\,m})\} \cup \{a_{i_0\,m\oplus(n-1)}\}$. Here $h(A_{i_\delta\,1})$ is a minimal open set containing $h(a_{i_\gamma\,1})$ and $A_{i_\gamma\,m} \subseteq h(A_{i_\delta\,1})$. So $h(A_{i_\delta\,1}) = A_{i_\delta\,m}$ and $h(a_{i_\gamma\,1}) = a_{i_\gamma\,m}$.

If i_γ has no immediate successor, then i_γ is the last element of I. Now A_1 is the minimal open set containing $a_{i_\gamma\,1}$ and $h(A_1) = A_{i_\gamma\,1} \cup h(a_{i_\gamma\,1})$. This implies that $h(a_{i_\gamma\,1}) = a_{i_\gamma\,m}$ and hence $h(a_{i\,1}) = a_{i\,m}$ for all $i \in I$.

Now let $h(a_{i_0\,2}) = a_{i_0\,p}$. Then arguing as above, we get $h(a_{i\,2}) = a_{i\,p}$ for all $i \in I$ and $h(a_{i_0\,2\oplus(n-1)}) = a_{i_0\,p\oplus(n-1)}$. That is $h(a_{i_0\,1}) = a_{i_0\,p\oplus(n-1)}$. By the fact that $h(a_{i_0\,1}) = a_{i_0\,m}$ we have that $m = p \oplus (n-1)$. This implies that $p = m \oplus 1$. So $h(a_{i\,2}) = a_{i\,m\oplus 1}$ for all $i \in I$.

Now assume that $h(a_{i_0\,q}) = a_{i_0\,r}$ for some q and r. Then $h(a_{i\,q}) = a_{i\,r}$ for all $i \in I$ and $h(a_{i_0\,q\oplus(n-1)}) = a_{i_0\,r\oplus(n-1)}$. Now let $h(a_{i_0\,q\oplus 1}) = a_{i_0\,s}$. Then $h(a_{i_0\,q\oplus 1\oplus(n-1)}) = a_{i_0\,s\oplus(n-1)}$, which implies that $h(a_{i_0\,q}) = a_{i_0\,s\oplus(n-1)}$. Thus we obtain that $r = s \oplus (n-1)$ and so $s = r \oplus 1$. Consequently we get $h(a_{i\,q\oplus 1}) = a_{i\,r\oplus 1}$ for all $i \in I$ and so $h(a_{i\,k}) = a_{i\,k\oplus(m-1)}$ for all $i \in I$ and $1 \leq k \leq n$. Hence $h(a_{i\,k}) = \sigma^m(a_{i\,k})$ for all $i \in I$ and $1 \leq k \leq n$. This follows that $h = \sigma^m$. Thus $h \in <\sigma>$. So $H(X,\, \tau) \subseteq <\sigma>$. This completes the proof. $\square$

The following Theorem is used to prove our next result.

Theorem 3.2.5. *[35] Let $X = \{a_1, a_2, \ldots a_n\}$, $n \geq 3$ and H be the group of permutations on X generated by the cycle $(a_1, a_2, \ldots a_n)$. Then H is not $t-$representable on X.*

Here we characterize t-representable cyclic groups generated by a permutation which is a product of disjoint cycles having equal length.

Theorem 3.2.6. *Let X be any set and H be the group on X generated by $\sigma = \prod_{i \in I} C_i$ where where $\{C_i, i \in I\}$ be an indexed family of disjoint cycles having equal length n. Then*

1. *H is $t-$representable if $|I| > 2$ or $n < 3$.*
2. *H is not $t-$representable if $|I| \leq 2$ and $n \geq 3$.*

Proof. In Theorem 3.2.1, it is proved that every permutation group of order two is $t-$representable on X. So if the length of each cycle of the permutation σ is less than three, then $< \sigma >$ is $t-$representable on X. Thus H is $t-$representable if $n < 3$. Now H is $t-$representable if $I > 2$ by Theorem 3.2.4 and H is not $t-$representable if $|I| \leq 2$ and $n \geq 3$ by Theorems 3.2.1 and 3.2.2. Hence the theorem. $\square$

Remark 3.2.7. Let

$$H_1 = \{I,\, (a_1,\, a_2,\, a_3)(a_4,\, a_5,\, a_6),\, (a_1,\, a_3,\, a_2)(a_4,\, a_6,\, a_5)\} \text{ and}$$

$$H_2 = \{I,\, (a_1,\, a_2,\, a_3)(a_4,\, a_5,\, a_6)(a_7,\, a_8,\, a_9),\, (a_1,\, a_3,\, a_2)(a_4,\, a_6,\, a_5)(a_7,\, a_9,\, a_8)\}.$$

Observe that H_1 and H_2 are isomorphic permutation groups. Here H_1 is not a $t-$representable permutation group by Theorem 3.2.2. In the case of H_2, by using the proof of Theorem 3.2.4, we can define a topology τ on $X =$

$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ having the base

$$\mathscr{B} = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_4\}, \{a_2, a_5\}, \{a_3, a_6\}, \{a_1, a_4, a_7, a_3\},$$
$$\{a_2, a_5, a_8, a_1\}, \{a_3, a_6, a_9, a_2\}\}.$$

So H_2 is a t-representable permutation group on X.

3.3 Groups Generated by a Product of Two Disjoint Cycles Having Finite Length

In this section we investigate the $t-$representability of cyclic groups generated by a permutation which is a product of two disjoint cycles. The main result in this section is, if σ is a permutation on a set X which is a product of two disjoint cycles having finite length, then the cyclic group generated by σ, $< \sigma >$ is not $t-$representable on X provided the length of at least one of them is greater than two.

Theorem 3.3.1. *Let X be any set such that $|X| = m_1 + m_2$ and σ be a permutation on X which is a product of two disjoint cycles having lengths m_1 and m_2 respectively where $(m_1, m_2) = 1$, then the cyclic group generated by σ is not $t-$representable on X.*

Proof. Let $\sigma_1 = (a_1, a_2, \ldots, a_{m_1})$ and $\sigma_2 = (b_1, b_2, \ldots, b_{m_2})$ and $\sigma = \sigma_1\sigma_2$.

Since $(m_1,\, m_2) \;=\; 1$, we have

$$< \sigma > = < \sigma_1 > \oplus < \sigma_2 >$$

Let $X \;=\; X_1 \cup X_2$ where X_i is the set of all elements in the cycle σ_i for $i \;=\; 1,\, 2$. Assume that $< \sigma >$ is a t-representable permutation group on X and τ be the corresponding topology.

Now we have two possible cases.

Case 1: $m_1,\, m_2 \;>\; 2$

If $m_1,\; m_2 \;>\; 2$, the topology on X is T_0 and so consider the partially ordered set $(X,\, \leq)$. If an element in X is both minimal and maximal, then all the elements of X are minimal and maximal and hence $H(X, \tau) = S(X)$, which is not possible. So a minimal element can not be a maximal element. Then either X_1 or X_2 is the set of all minimal elements. Assume that X_1 is the set of all minimal elements. Then there exists at least one $a_i \in X_1$ and $b_j \in X_2$ such that a_i precedes b_j. Now $a_i < b_j$ gives $p(a_i) \;<\; p(b_j)$ for all $p \in < \sigma >$. Since $< \sigma > = < \sigma_1 > \oplus < \sigma_2 >$, any $p \in < \sigma >$ is of the form $p \;=\; p_1 \oplus p_2$ where $p_1 \in < \sigma_1 >$ and $p_2 \in < \sigma_2 >$. Therefore

$$\begin{aligned} a_i < b_j &\Longrightarrow (I_{X_1} \oplus p_2)(a_i) \;<\; (I_{X_1} \oplus p_2)(b_j) \text{ for all } p_2 \in < \sigma_2 > \\ &\Longrightarrow a_i < p_2(b_j) \text{ for all } p_2 \in < \sigma_2 > \\ &\Longrightarrow a_i < b_k \text{ for } k = 1, 2, \ldots m_2 \end{aligned}$$

So a_i precedes all the elements of X_2 and hence every element in X_1 precedes all the elements of X_2. Thus we get $G(X, \leq) = S(X_1) \oplus S(X_2)$, this is not possible.

Case 2: m_1 or $m_2 = 2$

Let $m_1 = 2$. Assume that $X_1 = \{a_1, a_2\}$ and $\sigma_1 = (a_1, a_2)$. Since $< \sigma > = < \sigma_1 > \oplus < \sigma_2 >$, σ_1 is a homeomorphism on X and so the space (X, T) need not be T_0. If the space is T_0, then arguing as above, we can prove that $< \sigma >$ is not t-representable on X. Assume that the space is not T_0. Since the transposition (a_1, a_2) is a homeomorphism on the space (X, τ), the subspace $(X_1, \tau_{/X_1})$ has either the discrete topology or indiscrete topology.

Now if the subspace $(X_1, \tau_{/X_1})$ has the discrete topology, then there exist open sets of the form $U = U_1 \cup \{a_1\}$ and $V = V_1 \cup \{a_2\}$ where U_1 and V_1 are subsets of X_2. Since (X, τ) is not T_0, there exist two distinct points x and y such that every open set in (X, τ) contains both x and y or else contain neither of them. Since the topology on $(X_1, \tau_{/X_1})$ is discrete, we have at least one of x, y does not belongs to X_1. Hence we get a transposition (x, y) other than (a_1, a_2), which is a homeomorphism on X. This is not possible since $H(X, \tau) = < \sigma >$.

If the subspace $(X_1, \tau_{/X_1})$ has the indiscrete topology, then every open set in (X, τ) contains both a_1 and a_2 or else contain neither of them. Now consider the subspace $(X_2, \tau_{/X_2})$. Since $< \sigma_2 > \subseteq H(X_2, \tau_{/X_2})$, $(X_2, \tau_{/X_2})$

is a homogeneous space and hence there exists a partition $\mathscr{B}$ of X_2 consisting sets with equal number of elements, which forms a base for $(X_2, \tau_{/X_2})$. Also there exists a transposition p which is a homeomorphism on $(X_2, \tau_{/X_2})$. Now any base element of (X, τ) is of the form B or $B \cup X_1$ where $B \in \mathscr{B}$. Suppose there exist a B in $\mathscr{B}$ such that B is a base element for (X, τ), then since $H(X, \tau) = < \sigma >$, each $B \in \mathscr{B}$ is also a base element for (X, τ). Similarly if $B \cup X_1$ is a base element for (X, τ), then for each $B \in \mathscr{B}$, $B \cup X_1$ is also a base element for (X, τ). Hence p is also a homeomorphism on (X, τ), which is a contradiction. Thus in both cases we get $< \sigma >$ is not a t-representable permutation group on X.

□

Theorem 3.3.1 can be extended to a cyclic group generated by a permutation, which is a product of more than two cycles by similar arguments.

Corollary 3.3.2. *Let X be any set such that $|X| = m_1 + m_2 + \ldots m_n$ and σ be a permutation on X which is a product of disjoint cycles having lengths $m_1, m_2, \ldots, m_n$ such that gcd of any two of them is one, then the cyclic group generated by σ is not $t-$representable on X.*

Proof. Proof is obvious since we have

$$< \sigma > = < \sigma_1 > \oplus < \sigma_2 > \oplus \ldots \oplus < \sigma_n > .$$

□

Theorem 3.3.3. *Let X be a set such that $|X| = m_1 + m_2$ and σ be a permutation on X which is a product of two disjoint cycles having different lengths m_1 and m_2 respectively where $(m_1, m_2) = d > 1$, then the cyclic group generated by σ is not $t-$representable on X.*

Proof. Let $m_1 < m_2$. We have $(m_1, m_2) = d > 1$ and hence $m_1 = ld$ and $m_2 = kd$, where l and k are positive integers. Assume that

$$\sigma = (a_1, a_2, \ldots, a_{m_1})(b_1, b_2, \ldots, b_{m_1}, b_{m_1+1}, \ldots, b_{m_2})$$

and $< \sigma >$ be the cyclic group generated by σ. Suppose $< \sigma >$ is a t-representable permutation group on X. Then the corresponding topology τ on X is T_0 and by a similar argument as in Theorem 3.3.1, we get $(a_i, a_{i\oplus d}, \ldots, a_{i\oplus(l-1)d})$, $i = 1, 2 \ldots m_1$ and $(b_j, b_{j\oplus d}, \ldots, b_{j\oplus(k-1)d})$, $j = 1, 2, \ldots m_2$ are homeomorphisms on X, which is a contradiction to the fact that $< \sigma > = H(X, \tau)$. □

Combining previous results, we get the following theorem.

Theorem 3.3.4. *Let X be any set such that $|X| = m_1 + m_2$, σ be a permutation on X which is a product of two disjoint cycles having lengths m_1 and m_2 respectively and H be the cyclic group generated by σ. Then the group H is t-representable on X if and only if order H is less than three.*

Proof. This follows directly from the Theorems 3.2.1, 3.2.5, 3.3.1 and 3.3.3. □

3.4 Groups Generated by a Product of Disjoint Infinite Cycles

We now turn our attention to the $t-$representability of infinite cyclic subgroups of symmetric groups. Here we prove that if X is an infinite set and σ is a permutation on X which can be written as an arbitrary product of disjoint infinite cycles, then the cyclic group generated by σ, $<\sigma>$ is $t-$representable on X.

Let X be the infinite set $\{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$ and σ be the infinite cycle $(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ on X. Then the group generated by σ is t-representable on X by defining a topology $\tau = \{\emptyset, X\} \cup \{\{a_j : j \leq i\} : i \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integers [37].

First we consider the $t-$representability of cyclic group generated by a permutation which is a product of two disjoint infinite cycles. In contrast with result in finite case, here we can prove that the subgroup of $S(X)$ generated by a permutation which is a product of two disjoint infinite cycles is $t-$representable on X.

Theorem 3.4.1. *Let X be an infinite set and σ be a permutation on X which can be written as a product of two disjoint infinite cycles. Then the cyclic group generated by σ, $<\sigma>$ is $t-$representable on X.*

Proof. Let $\sigma = \sigma_1\sigma_2$ where

$$\sigma_1 = (\ldots, a_{-1}, a_0, a_1, \ldots) \text{ and } \sigma_2 = (\ldots, b_{-1}, b_0, b_1, \ldots).$$

By Theorem 2.2.4, without loss of generality we can assume that $X = X_1 \cup X_2$, where $X_1 = \{a_i : i \in \mathbb{Z}\}$ and $X_2 = \{b_i : i \in \mathbb{Z}\}$. Now define a base $\mathscr{B}$ by

$$\mathscr{B} = \{U \subseteq X : U \cap X_1 = A_i \text{ and } (U \cap X_2 = \emptyset \text{ or } U \cap X_2 = B_i) \text{ for some } i \in \mathbb{Z}\}$$

where

$$A_i = \{a_j \in X_1 : j \leq i\} \text{ and } B_i = \{b_j \in X_2 : j \leq i\}$$

and τ be the topology having base $\mathscr{B}$. Here $\mathscr{B}$ is a collection of its subsets such that for each $x \in X$, there is a minimal set $M(x)$ in $\mathscr{B}$ containing x. So (X, τ) is an Alexandroff discrete space. We prove that $H(X, \tau) =< \sigma >$.

Let V be an open set. Then V can be written as the union of some members of $\mathscr{B}$, say, $V = \bigcup\limits_{i \in I} B_i$ where I is an index set and $B_i \in \mathscr{B}$ for all $i \in I$. We have $\sigma(V) = \bigcup\limits_{i \in I} \sigma(B_i)$. So if we show that $\sigma(\mathscr{B}) = \mathscr{B}$ and $\sigma^{-1}(\mathscr{B}) = \mathscr{B}$, then σ is a homeomorphism on (X, τ).

Let $U \in \mathscr{B}$. Recall that $U \cap X_1 = A_i$ and $U \cap X_2 = \emptyset$ or B_i. If $U \cap X_1 = A_i$ and $U \cap X_2 = \emptyset$, then $\sigma(U) = A_{i+1}$ and $\sigma^{-1}(U) = A_{i-1}$. So $\sigma(U)$ and $\sigma^{-1}(U)$ belongs to $\mathscr{B}$. Now consider the next case, $U \cap X_1 = A_i$ and $U \cap X_2 = B_i \neq \emptyset$. We have $\sigma(A_i) = A_{i+1}$ and $\sigma(B_i) = B_{i+1}$ and hence $\sigma(U) = A_{i+1} \cup B_{i+1}$, which implies that $\sigma(U) \in \mathscr{B}$. Similarly $\sigma^{-1}(U) = A_{i-1} \cup B_{i-1} \in \mathscr{B}$. So in both cases $\sigma \in H(X, \tau)$. Hence

$$< \sigma > \subseteq H(X, \tau). \tag{3.1}$$

Let $h \in H(X, \tau)$. First we prove that h maps X_i to X_i for $i = 1, 2$. Suppose that $h(X_1) \neq X_1$. Then either $X_1 \subsetneqq h(X_1)$ or $h(X_1) \subsetneqq X_1$.

If $X_1 \subsetneqq h(X_1)$, then there exists an a_i in X_1 such that $h(a_i) \in X_2$. Let $h(a_i) = b_k$ for some k and $U = A_i$. Then U is a minimal open set containing a_i. This implies that $h(U)$ is a minimal open set containing b_k. This follows that $h(U) = A_k \cup B_k$. Let $V = A_{i+1}$. Since V is a minimal open set containing a_{i+1}, $h(V) = A_k \cup B_k \cup h(a_{i+1})$ is also a minimal open set. This implies that $B_k = \emptyset$, which leads to a contradiction.

Now consider the case $h(X_1) \subsetneqq X_1$. Then there exists $a_j \in X_1$ such that $h^{-1}(a_j) \in X_2$. Since h is a homeomorphism on X, h^{-1} is also a homeomorphism on X. So in this case also we get a contradiction. Thus $h(X_i) = X_i$ for $i = 1, 2$.

Let $h(a_0) = a_j$ for some j. Now consider the minimal open set A_0 containing a_0. Then $h(A_0) = A_j$. This implies that $h(A_1) = A_{j+1}$ and hence $h(a_1) = a_{j+1}$. Now assume that $h(a_n) = a_m$ for some n and m. Then we can easily show that $h(a_{n+1}) = a_{m+1}$. Thus get $h(a_i) = a_{i+j}$ for all $i \in \mathbb{Z}$.

Consider $b_0 \in X_2$. Now $h(A_0 \cup B_0) = h(A_0) \cup h(B_0) = A_j \cup h(B_0)$, which belongs to $\mathscr{B}$. This implies that $h(B_0) = B_j$. Also $h(b_0) = b_j$. Since $h(a_1) = a_{j+1}$, we get $h(b_1) = b_{j+1}$ and so on. Hence $h = \sigma^j$ for some $j \in \mathbb{Z}$. So

$$H(X, \tau) \subseteq < \sigma > . \tag{3.2}$$

From equations 3.1 and 3.2, we get $H(X, \tau) =< \sigma >$. This completes the proof. ☐

If σ is a permutation on X which is a product of more than two disjoint infinite cycles, we can define a topology τ on X such that $H(X, \tau)$ is the group generated by σ.

Theorem 3.4.2. *If σ is a permutation on X which is a product of more than two disjoint infinite cycles, then the cyclic group generated by σ, $< \sigma >$ is $t-$representable on X.*

Proof. Let $\sigma = \prod_{\imath \in I} C_i$ where I be an indexed set, $|I| > 2$ and

$$C_i = (\ldots, a_{i\,-2},\, a_{i\,-1},\, a_{i\,0},\, a_{i\,1},\, a_{i\,2}, \ldots)$$

which is an infinite cycle. Let X_i be the set which consists of all the elements of the cycle C_i. In view of Theorem 2.2.4 without loss of generality we can assume that $X = \bigcup_{i \in I} X_i$. By the well-ordering Theorem, well-order the set I by an order relation $<$. We can use the ordinals to index the members of I. Let i_0 be the first element of I and i_1 denote the first element of the set $I \setminus \{i_0\}$. In general i_β denote the first element of the set $I \setminus \{i_\alpha : \alpha < \beta\}$ provided $I \setminus \{i_\alpha : \alpha < \beta\} \neq \emptyset$.

Define a base $\mathscr{B}$ by

$$\mathscr{B} = \{B_{i_\gamma\, j} : i_\gamma \in I, j \in \mathbb{Z}\} \cup \{B_j : j \in \mathbb{Z}\}$$

where

$$B_{i_\gamma\, j} = \begin{cases} \{a_{i_k\, j\ : i_k < i_\gamma}\} & \text{if } i_\gamma \leq i_2 \\ \{a_{i_k\, j : i_k < i_\gamma}\} \cup \{a_{i_0\, j-1)}\} & \text{if } i_2 < i_\gamma \end{cases}$$

and

$$B_j \;=\; \{a_{i\, j} \,:\, i \in I\} \cup \{a_{i_0\, j-1)}\}.$$

Let τ be the topology having base $\mathscr{B}$. It is straight forward to check that σ is a homeomorphism on (X, τ) and hence all powers of σ are also homeomorphisms. So

$$< \sigma > \subseteq H(X, \tau). \tag{3.3}$$

For the other way inclusion, let $h \in H(X, \tau)$. Since h is a homeomorphism on $(X,\, \tau)$, $h(B_{i_1\, 1}) = B_{i_1\, m}$ for some $m \in \mathbb{Z}$. Thus $h(a_{i_0\, 1}) = a_{i_0\, m}$. Now consider the minimal open set $B_{i_2\, 1}$ containing $a_{i_1\, 1}$. Then $h(B_{i_2\, 1})$ is a minimal open set containing $h(a_{i_1\, 1})$ and which contains $a_{i_0\, m}$. It follows that $h(a_{i_1\, 1}) \;=\; a_{i_1\, m}$. Next we consider the minimal open set $B_{i_3\, 1} \;=\; \{a_{i_0\, 1},\, a_{i_1\, 1},\, a_{i_2\, 1},\, a_{i_0\, 0}\}$. Now $h(B_{i_3\, 1})$ is a minimal open set containing $h(a_{i_2\, 1})$ and which contains $a_{i_0\, m}$ and $a_{i_1\, m}$. This implies that $h(B_{i_3\, 1}) \;=\; B_{i_3\, m}$ and hence $h(a_{i_0\, 0}) \;=\; a_{i_0\, m-1)}$.

Assume that $h(a_{i_\alpha\, 1}) \;=\; a_{i_\alpha\, m}$ for all $\alpha < \gamma$. If i_γ has an immediate successor i_δ

in I, consider $B_{i_\delta 1} = \{a_{i_0 1}, a_{i_1 1}, \ldots, a_{i_\gamma 1}\} \cup \{a_{i_0 0}\}$. Here $h(B_{i_\delta 1})$ is a minimal open set containing $h(a_{i_\gamma 1})$ and $B_{i_\gamma m} \subseteq h(B_{i_\delta 1})$. So $h(B_{i_\delta 1}) = B_{i_\delta m}$ and $h(a_{i_\gamma 1}) = a_{i_\gamma m}$.

If i_γ has no immediate successor, then i_γ is the last element of I. Then $h(B_1) = h(\{a_{i 1} : i \in I\} \cup \{a_{i_0 0}\}) = B_{i_\gamma m} \cup h(a_{i_\gamma 1})$. So the only possibility is $h(B_1) = B_m$ and hence $h(a_{i_\gamma 1}) = a_{i_\gamma m}$. Thus we get $h(a_{i 1}) = a_{i m}$ for all $i \in I$.

Now let $h(a_{i_0 2}) = a_{i_0 p}$. Then arguing as above, we get $h(a_{i 2}) = a_{i p}$ for all $i \in I$ and $h(a_{i_0 1)}) = a_{i_0 p-1}$. By the fact that $h(a_{i_0 1}) = a_{i_0 m}$ we have that $m = p - 1$. This implies that $p = m + 1$. So $h(a_{i 2}) = a_{i m+1}$ for all $i \in I$.

Now assume that $h(a_{i_0 q}) = a_{i_0 r}$ for some q and r. Then $h(a_{i q}) = a_{i r}$ for all $i \in I$ and $h(a_{i_0 q-1}) = a_{i_0 r-1}$. Now let $h(a_{i_0 q+1}) = a_{i_0 s}$. Then $h(a_{i_0 q+1-1}) = a_{i_0 s-1}$, which implies that $h(a_{i_0 q}) = a_{i_0 s-1}$ and we obtain that $r = s - 1$ and so $s = r + 1$. Consequently we get $h(a_{i q+1}) = a_{i r+1}$ for all $i \in I$ and so $h(a_{i k}) = a_{i k+m}$ for all $i \in I$ and $k \in \mathbb{Z}$. Hence $h(a_{i k}) = \sigma^m(a_{i k})$ for all $i \in I$ and $k \in \mathbb{Z}$. This follows that $h = \sigma^{m-1}$ for some $m \in \mathbb{Z}$ and hence $h \in < \sigma >$. Thus

$$< \sigma > \subseteq H(X, \tau). \tag{3.4}$$

From equations 3.3 and 3.4, we get $< \sigma > = H(X, \tau)$. Hence $< \sigma >$ is a $t-$representable permutation group on X. □

We conclude this section by the following theorem.

Theorem 3.4.3. *Let X be an infinite set and σ be a permutation on X which can be written as an arbitrary product of disjoint infinite cycles. Then the cyclic group generated by σ, $< \sigma >$ is $t-$representable on X.*

Proof. Proof follows from Theorems 3.4.1 and 3.4.2 $\square$

Chapter 4

Group of L-fuzzy Homeomorphisms

In this chapter we study the problem of representing a subgroup of the symmetric group $S(X)$ as the group of homeomorphisms of an L-fuzzy topological space (X, δ).

Similar to the definition of a t-representable permutation group, here we define an L_f-representable permutation group. Properties of L_f-representable permutation groups are discussed. Further we investigate the L_f-representability of semiregular permutation group, dihedral group, alternating group etc. L_f-representability of some cyclic subgroup of the symmetric group.

The simplest F-lattice other than $L = \{0, 1\}$ is $L = \{0, a, 1\}$ with the order $0 < a < 1$. Also note that any F-lattice other than $\{0, 1\}$ contains a sublattice isomorphic to $L = \{0, a, 1\}$.

4.1 L_f-representability of Permutation Groups

Analogous to $t-$representability in topology, here we define L_f- representability of permutation groups.

Definition 4.1.1. L_f-representable permutation Groups

A subgroup K of the group $S(X)$ of all permutations of a set X is called L_f-representable on X if there exists an L-fuzzy topology δ on X such that the group of L-fuzzy homeomorphisms of $(X, \delta) = K$.

If we take L as the lattice containing only two elements 0 and 1, then in view point of lattice theory L^X is isomorphic to the power set of X and hence topologies and topological spaces become special cases of L-fuzzy topologies and L-fuzzy topological spaces. So every $t-$representable permutation group on a set X is also L_f- representable on X. But an L_f-representable permutation group need not be t-representable on X.

4.2 Properties of L_f-representable Permutation Groups

In this section, some properties of L_f- representable permutation groups are discussed.

The following theorem plays a major role in proving results related to L_f-

representability of permutation groups.

Theorem 4.2.1. *[36] Let L and L' be complete and distributive lattices such that L is isomorphic to a sublattice of L'. Then if H is a subgroup of $S(X)$ which can be represented as the group of homeomorphisms of an L-fuzzy topological space (X, δ) for some L-fuzzy topology δ on X, then H can also be represented as the group of L'-fuzzy homeomorphisms of the L'-fuzzy topological space $(X,\ \delta')$ for some L'-fuzzy topology δ' on X.*

Using the above Theorem we have the following Remark.

Remark 4.2.2. Let L and L' be two F-lattices such that L is isomorphic to a sub lattice of L'. Then if a permutation group H is L_f-representable on an arbitrary set X, then H is also L'_f-representable on X.

So if we prove a permutation group H is L_f-representable on a set X by taking $L = \{0,\ a,\ 1\}$ with the usual order, then H is L_f-representable on X for any F-lattice $L \neq \{0, 1\}$.

As in the case of t-representable permutation groups, here we prove that conjugate of an L_f- representable permutation group is L_f- representable.

Theorem 4.2.3. *Let X be any set and H be a subgroup of the symmetric group $S(X)$. Then H is L_f-representable on X if and only if its conjugate is also L_f-representable on X.*

Proof. Let H be a L_f-representable permutation group on X. Then there exists an L-fuzzy topology δ on X such that $LFH(X, \delta) = H$. It follows that

$$H = \{h \in S(X) : \delta \circ h = \delta\}.$$

Let $g \in S(X)$. Then $\delta \circ g = \{f \circ g : f \in \delta\}$ is an L-fuzzy topology on X. Now we claim that $LFH(X, \delta \circ g) = gHg^{-1}$.

Let $h \in LFH(X, \delta)$

$$\Rightarrow \delta \circ h = \delta$$

$$\Rightarrow \delta \circ gg^{-1}h = \delta$$

$$\Rightarrow (\delta \circ g) \circ g^{-1}h = \delta$$

$$\Rightarrow (\delta \circ g) \circ g^{-1}hg = \delta \circ g$$

$$\Rightarrow g^{-1}hg \in LFH(X, \delta \circ g).$$

This implies that

$$g^{-1}Hg \subseteq LFH(X, \delta \circ g) \tag{4.1}$$

For the other way inclusion,

let $h \in LFH(X, \delta \circ g)$

$$\Rightarrow \delta \circ g \circ h = \delta \circ g$$

$$\Rightarrow \delta \circ ghg^{-1} = \delta$$

$$\Rightarrow ghg^{-1} \in LFH(X, \delta) = H$$

$$\Rightarrow h \in g^{-1}Hg.$$

This implies that

$$LHH(X,\, \delta \circ g) \subseteq g^{-1}Hg \tag{4.2}$$

From equations 4.1 and 4.2, we get $LFH(X,\, \delta \circ g) = g^{-1}Hg$. Thus the conjugate of an L_f-representable permutation group is L_f-representable on X.

Conversely assume that $g^{-1}Hg$ is L_f-representable on X. Then by what we have proved above, the conjugate of $g^{-1}Hg$ is L_f-representable on X. So H is L_f-representable on X. This completes the proof.

□

Remark 4.2.4. So it suffices to determine the conjugacy classes of subgroups of S(X) which are L_f-representable on X.

Our next Theorem is a generalization Theorem 2 of [36].

Theorem 4.2.5. *Let X be any set and $Y \subseteq X$. Let H be a permutation group on Y. If H is L_f-representable on Y, then the permutation group $H \oplus \{I_{X\setminus Y}\}$ is L_f-representable on X where $I_{X\setminus Y}$ is the identity permutation on $X \setminus Y$.*

Proof. Let δ be the L-fuzzy topology on Y such that $LFH(Y,\, \delta) = H$. Now for any $f \in \delta$, let $f' : X \to L$ defined by

$$f'(x) = \begin{cases} 1 & \text{if } x \in X \setminus Y \\ f(x) & \text{if } x \in Y. \end{cases}$$

By the well-ordering Theorem, well-order the set $X \setminus Y$ with the order relation '$<$'. Now for $a \in X \setminus Y$, define $f_a : X \to L$ as

$$f_a(x) = \begin{cases} 1 \text{ if } x \in X \setminus Y \text{ and } x < a \\ 0 \text{ otherwise.} \end{cases}$$

Let

$$\delta_1 = \{f' \,:\, f \in \delta\} \text{ and } \delta_2 = \{f_a : \, a \in X \setminus Y\}.$$

Using δ_1 and δ_2 we define an L- fuzzy topology δ' on X as follows.

$$\delta' \;=\; \delta_1 \cup \delta_2$$

Now we claim that $LFH(X, \delta') = H \oplus \{I_{X\setminus Y}\}$.

Let $h \oplus I_{X\setminus Y} \in H \oplus \{I_{X\setminus Y}\}$ and $g \in \delta'$. If $g = f_a$ for some $a \in X \setminus Y$, then $(h \oplus I_{X\setminus Y})^{-1}(f_a) = f_a \circ (h \oplus I_{X\setminus Y})$. Now

$$(h \oplus I_{X\setminus Y})^{-1}(f_a)(x) \;=\; \begin{cases} f_a(x) & \text{if } x \in X \setminus Y \\ f_a(h(x)) & \text{if } x \in Y. \end{cases}$$

$$
\begin{aligned}
&= \begin{cases} f_a(x) & \text{if } x \in X \setminus Y \\ 0 & \text{if } x \in Y. \end{cases} \\
&= \begin{cases} 1 & \text{if } x \in X \setminus Y \text{ and } x < a \\ 0 & \text{otherwise} \end{cases} \\
&= f_a(x).
\end{aligned}
$$

Thus if $g = f_a$, then $(h \oplus I_{X\setminus Y})^{-1}(g) = g$, which belongs to δ'.

If $g = f'$, then $(h \oplus I_{X\setminus Y})^{-1}(f') = f' \circ (h \oplus I_{X\setminus Y})$ and

$$
\begin{aligned}
(h \oplus I_{X\setminus Y})^{-1} f'(x) &= \begin{cases} (f \circ h)(x) \text{ if } x \in Y \\ 1 \text{ if } x \in X \setminus Y. \end{cases} \\
&= (f \circ h)'(x)
\end{aligned}
$$

Observe that $f \in \delta$ and $h \in H$. So $f \circ h \in \delta$ and hence $(h \oplus I_{X\setminus Y})^{-1}(f') = (f \circ h)' \in \delta'$. So $(h \oplus I_{X\setminus Y})^{-1}(g) \in \delta'$ for all $g \in \delta'$. Thus $h \oplus I_{X\setminus Y}$ is an L-fuzzy continuous map on X onto itself. Similarly we can prove that $(h \oplus I_{X\setminus Y})(f') \in \delta'$ for all $f' \in \delta'$ and hence $(h \oplus I_{X\setminus Y})^{-1}$ is also an L-fuzzy continuous map on X. Thus $h \oplus I_{X\setminus Y}$ is an L-fuzzy homeomorphism on (X, δ') for all $h \in H$. So

$$
H \oplus \{I_{X\setminus Y}\} \subseteq LFHT(X, \delta'). \tag{4.3}
$$

Conversely let h be an L-fuzzy homeomorphism on (X, δ') onto itself. Let x_0 be the first element of the set $X \setminus Y$ and x_1 be the first element of the set

$(X \setminus Y) \setminus \{x_0\}$. Now consider $h^{-1}(f_{x_1})$, which takes the value 1 at exactly one point $h^{-1}(x_0)$ and 0 elsewhere. Since $f_{x_1} \in \delta'$, $h^{-1}(f_{x_1}) \in \delta'$. This implies that $h^{-1}(f_{x_1}) = f_{x_1}$ and so $h(x_0) = x_0$. Let x_α be any element of $X \setminus Y$ such that $h(x) = x$ for all x in $X \setminus Y$, $x < x_\alpha$. Now we claim that $h(x_\alpha) = x_\alpha$. If x_α has no immediate successor in $X \setminus Y$, x_α be the last element of the set $X \setminus Y$. Since $f = \underline{0} \in \delta$, $f' : X \to L$ defined by

$$f'(x) = \begin{cases} 0 \text{ if } x \in Y \\ 1 \text{ if } x \in X \setminus Y \end{cases}$$

belongs to δ'. So

$$\begin{aligned} h^{-1}(f')(x) &= (f' \circ h)(x) \\ &= \begin{cases} 1 \text{ for all } x \in X \setminus Y \text{ such that } x < x_\alpha \text{ and } h^{-1}(x_\alpha) \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

and $h^{-1}(f') \in \delta'$. This implies that $h^{-1}(f') = f'$ and hence $h(x_\alpha) = x_\alpha$.

If x_α has an immediate successor x_β in $X \setminus Y$, then consider f_β in δ'. We have that $h^{-1}(f_\beta) \in \delta'$ and

$$\begin{aligned} h^{-1}(f_\beta)(x) &= (f_\beta \circ h)(x) \\ &= \begin{cases} 1 \text{ for all } x < x_\alpha \text{ and } h^{-1}(x_\alpha) \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

This gives that $h^{-1}(f_\beta) = f_\beta$ and hence $h(x_\alpha) = x_\alpha$. It follows that $h(x) = x$ for all $x \in X \setminus Y$ and $h(Y) = Y$. So $h_{|Y}$ is a homeomorphism on (Y, δ). Hence $h \in H \oplus I_{X\setminus Y}$. Since h is arbitrary, we have

$$LFH(X, \delta') \subseteq H \oplus \{I_{X\setminus Y}\}. \tag{4.4}$$

From equations 4.3 and 4.4, we get $LFH(X, \delta') = H \oplus \{I_{X\setminus Y}\}$ □

Remark 4.2.6. Let H be a non-trivial permutation group on a set X. Let $Y = X \setminus \{x \in X : h(x) = x \text{ for all } h \in H\}$. Define $H' = \{h_{|Y} : h \in H\}$, which is a permutation group on Y. Note that H' moves all the elements of Y and $H = H' \oplus \{I_{X\setminus Y}\}$. By Theorem 4.2.5, it follows that, if H' is L_f-representable on Y, then H is L_f-representable on X. So if (X, δ) is an L-fuzzy topological space which is not rigid and $H = H(X, \delta)$ then without loss of generality we can assume that H moves all the elements of X.

4.2.1 L_f-representability of Direct Sum of L_f-representable Permutation Groups

Now we turn our attention to the L_f-representability of direct sum of finite L_f-representable subgroups of symmetric groups. In chapter two it is proved that the direct sum of finite $t-$representable permutation groups is $t-$ representable. Here we prove analogues result in the case of L- fuzzy topological spaces.

Theorem 4.2.7. *Let $\{X_i\}_{i\in I}$ be an arbitrary family of mutually disjoint finite sets and K_i be a L_f-representable subgroup of $S(X_i)$ for $i \in I$. Then $\bigoplus\limits_{i\in I} K_i$ is L_f-representable on $X = \bigcup\limits_{i\in I} X_i$.*

Proof. Since K_i is L_f- representable on X_i for all $i \in I$, there exists an L-fuzzy topology δ_i on X_i such that the group of L- fuzzy homeomorphisms, $LFH(X_i, \delta_i) = K_i$. By the well-ordering Theorem, we can choose a well-order $<$ on I. For each $f \in \delta_i$, $i \in I$ define $f' : X \to L$ as follows

$$f'(x) = \begin{cases} 1 & \text{if } x \in \underset{j<i}{\cup} X_j \\ f(x) & \text{if } x \in X_i \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\delta_i' = \{f_i' : f_i \in \delta_i\}.$$

Using this δ', we can define δ on X as follows.

$$\delta = \{\underline{1}\} \cup \underset{i\in I}{\cup} \delta_i'.$$

Then δ is an L- fuzzy topology on X. We claim that $LFH(X, \delta) = K$ where $K = \bigoplus\limits_{i\in I} K_i$.

Let $k_i \in K_i$ for all $i \in I$ and $k = \underset{i\in I}{\oplus} k_i$. Clearly k is a bijection of X onto itself. Let $f \in \delta$. If $f = \underline{1}$, then $k^{-1}(f) = f \circ k = f$. Suppose $f \neq \underline{1}$, then $f = f_i'$

for some $i \in I$. consider $k^{-1}(f_i')$.

Now

$$\begin{aligned}
k^{-1}(f_i')(x) &= (f_i' \circ k)(x) \\
&= f_i'(k(x)) \\
&= \begin{cases} 1 & \text{if } k(x) \in \bigcup\limits_{j<i} X_j \\ f_i(k(x)) & \text{if } k(x) \in X_i \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } x \in \bigcup\limits_{j<i} X_j \\ (f_i \circ k_i)(x)) & \text{if } x \in X_i \\ 0 & \text{otherwise} \end{cases} \\
&= (f_i \circ k_i)'(x).
\end{aligned}$$

Since $f_i \in \delta_i$ and $k_i \in K_i$, we have that $f_i \circ k_i \in \delta_i$ and hence $(f_i \circ k_i)' \in \delta$. This implies that $k^{-1}(f_i')$ belongs to δ. Similarly we can prove that $f_i' \circ k^{-1} \in \delta$. So $K \subseteq LFH(X, \delta)$

Conversely suppose that $k \in LFH(X, \delta)$. Let i_0 be the smallest element of I. Since $\underline{1} \in \delta_{i_0}$, Consider f_{i_0}' defined by

$$f_{i_0}'(x) = \begin{cases} 1 & \text{if } x \in X_{i_0} \\ 0 & \text{otherwise} \end{cases}$$

which takes the value 1 at exactly $|X_{i_0}|$ points and belongs to δ. Since $k \in LFH(X, \delta)$, $k(f'_{i_0}) \in \delta$. Also we have that

$$\begin{aligned} k(f'_{i_0})(x) &= (f'_{i_0} \circ k^{-1})(x) \\ &= \begin{cases} 1 \text{ if } k^{-1}(x) \in X_{i_0} \\ 0 \text{ otherwise} \end{cases} \\ &= \begin{cases} 1 \text{ if } x \in k(X_{i_0}) \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

Note that $k(f')$ takes the value 1 at $|X_{i_0}|$ points and hence the only possibility is $k(f'_{i_0}) = f'_{i_0}$. Thus we get $k(X_{i_0}) = X_{i_0}$.

Now assume that $j \in I$ and $k(X_j) = X_j$ for all $j \in I$ and $j < i$. We prove that $k(X_i) = X_i$. Suppose $k(X_i) \neq X_i$, then there exists x in X_i such that $k(x) \notin X_i$ or there exists $x \notin X_i$ such that $k(x) \in X_i$. In the second case also we can see that, there exists $x \in X_i$ such that $k(x) \notin X_i$, since X_i is finite. Thus without loss of generality, we can assume that there exists x in X_i such that $k(x) \notin X_i$. Then $k(x) \in X_k$ for some $k > i$. Consider $f = f'_i$ where

$$f'_i(x) = \begin{cases} 1 & \text{if } x \in \bigcup\limits_{j \leq i} X_j \\ 0 & \text{otherwise} \end{cases}$$

which is an L-fuzzy open set in δ. Therefore $k(f)$ is also an L-fuzzy open set in X.

Let $x \in X$. Then

$$\begin{aligned} k(f)(x) &= (f \circ k^{-1})(x) \\ &= f(k^{-1}(x)) \\ &= \begin{cases} 1 & \text{if } k^{-1}(x) \in \underset{j \le i}{\cup} X_j \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } x \in k(\underset{j \le i}{\cup} X_j) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } x \in \underset{j < i}{\cup} X_j \cup k(X_i) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is true for all $x \in X$. Since $k(x) \in X_k$ for some $x \in X_i$ and $k(f) \in \delta$, we get $|k(X_i)| > |X_i|$, which is a contradiction. Hence $k(X_j) = X_j$ for all $j \in I$. Thus $k|_{X_j} = k_j$ is an L-fuzzy homeomorphism of X_j for all $j \in I$ and $k = \underset{i \in I}{\oplus} k_i$. So $k \in K = \bigoplus_{i \in I} K_i$. Thus $LFH(X, \delta) = K$ and hence $\bigoplus_{i \in I} K_i$ is an L_f-representable permutation group on X. $\square$

4.3 L_f-representability of Semiregular Permutation Groups

In this section we prove that semiregular permutation groups on a nonempty set X are L_f-representable on X provided $|X| \le L$.

Recall that a permutation group H on a nonempty set X is called semiregular if the identity permutation is the only element in H with any fixed points.

Theorem 4.3.1. *Let X be any set and L be an $F-$lattice such that $|X| \leq |L|$. Let H be a semiregular permutation group on X. Then H is L_f- representable on X.*

Proof. Let $f : X \to L$ be an L- fuzzy set such that f is one-one and f takes the value 0 and 1. We can define such a one-one function since $|X| \leq |L|$. By the well-ordering Theorem, well-order the group H with order relation $<$. Define $\mathcal{S} = \{f_i = f \circ h_i \, : \, h_i \in H\}$. Let δ be the L-fuzzy topology generated by $\mathcal{S}$. Then any element of δ is of the form $\underset{i \in I}{\vee}(\underset{j \in J_i}{\wedge} f_j)$ where I is an index set and J_i is finite for each $i \in I$.

Claim: $LFH(X, \delta) \; = \; H$

Let $h \in H$ and $f_i \in \mathcal{S}$. Then $h^{-1}(f_i) = f_i \circ h \; = \; (f \circ h_i) \circ h = f \circ (h_i \circ h)$. Since H is a group and h, $h_1 \in H$, $h_i \circ h$ is in H. Let $h_i \circ h \; = \; h_k$ for some k. Then it follows that $h^{-1}(f_i) \; = \; f \circ h_k = f_k$. Hence $h^{-1}(f_i)$ is in δ. By similar arguments we can prove that $h(f_i) \in \delta$. So h is an L-fuzzy homeomorphism on X.

$$H \subseteq LFH(X, \, \delta) \tag{4.5}$$

Conversely assume that $h \in LFH(X, \, \delta)$. Then $h^{-1}(f) \in \delta$. Now $h^{-1}(f) \; = \; f \circ h \; = \; \underset{i \in I}{\vee}(\underset{j \in J_i}{\wedge} f_j)$. Note that f takes the value 0 and hence $f \circ h$ also takes the value 0 at some $x \in X$. It follows that for all $i \in I$, there exists a $j_i \in J_i$ such that $f_{j_i}(x) = 0$. Now f takes the value 1 implies that there exists some $y \in X$ such that $f \circ h(y) = 1$. This implies that there exists some $i_0 \in I$ such that $\underset{j \in J_{i_0}}{\wedge} f_j(y) = 1$. It follows that $|J_{i_0}| = 1$ and hence $j_i = k$ for all $i \in I$.

$$
\begin{aligned}
f \circ h &= \underset{i \in I}{\vee} (\underset{j \in J_i}{\wedge} f_j) \\
&= [\underset{i \in I \setminus \{i_0\}}{\vee} (\underset{j \in J_i}{\wedge} f_j)] \vee (\underset{j \in J_{i_0}}{\wedge} f_j) \\
&= \underset{i \in I \setminus \{i_0\}}{\vee} (\underset{j \in J_i}{\wedge} f_j) \vee f_k \\
&= \underset{i \in I \setminus \{i_0\}}{\vee} ((\underset{j \in J_i \setminus \{k\}}{\wedge} f_j) \wedge f_k) \vee f_k \\
&= f_k \\
&= f \circ h_k.
\end{aligned}
$$

Thus we get $h = h_k$ and hence $h \in H$. So

$$LFH(X, \delta) \subseteq H \tag{4.6}$$

From Equations 4.5 and 4.6, it follows that H is L_f-representable on X. □

A permutation group is regular if and only if it is both semiregular and transitive.

Corollary 4.3.2. *Let X be any set and L-be any F-lattice such that $|X| \leq |L|$. Then every regular permutation group on X is L_f-representable.*

Proof. We have a regular permutation group is semiregular. Proof follows from

Theorem 4.3.1. □

Let X be any set on X such that $|X| \leq |L|$. Then any transitive abelian permutation group on a finite set X is regular and hence L_f-representable on X.

4.4 L_f-representability of Cyclic Group of Permutations

Now we investigate the L_f-representability of the cyclic permutation groups. We need the following theorem taken from [36]

Theorem 4.4.1. *[36] Let X be any set and L be any complete distributive lattice containing more than two elements. Then the group of permutations of X generated by any finite cycle on X can be represented as the group of homeomorphisms of the L-fuzzy topological space (X, δ) for some L-fuzzy topology δ on X.*

Now using Theorems 4.2.5, 4.2.7 and 4.4.1, we can easily prove the following Corollary.

Corollary 4.4.2. *Let X be any set and L be an F-lattice containing more than two elements. If σ is any permutation on X, which is a product of n disjoint cycles having lengths $m_1, m_2, \ldots, m_n$ where $(m_i, m_j) = 1$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$, then the permutation group generated by σ is L_f-representable on X .*

Proof. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_{m_n}$ where σ_i is a cycle of length m_i where $1 \leq i \leq n$. Since $(m_i, m_j) = 1$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$, we have that

$$< \sigma > = < \sigma_1 > \oplus < \sigma_2 > \oplus \ldots \oplus < \sigma_n > .$$

By Remark 4.2.6, without loss of generality we can assume that $X = X_1 \cup X_2 \cup \ldots \cup X_n$ where X_i is the set of all elements of the cycle σ_i for $1 \leq i \leq n$. Then by theorem 4.4.1, $< \sigma_i >$ is L_f-representable on X_i for all i, $1 \leq i \leq n$ when $L \neq \{0, 1\}$. So from Theorem 4.2.7, it follows that $< \sigma >$ is L_f-represenable on X. $\square$

In Chapter 2, it is proved that the above group is not $t-$representable on X.

Corollary 4.4.3. *If X is any set and σ is any permutation on X, which is a product of n disjoint cycles having lengths $m_1, m_2, \ldots, m_n$ where $(m_i, m_j) = 1$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. Then the permutation group generated by σ is L_f-representable on X if and only if $L \neq \{0, 1\}$.*

Proof. Proof follows from Corollaries 3.3.2 and 4.4.2. $\square$

In Theorem 3.2.2, we proved that the permutation group generated by a permutation which is a product of two disjoint cycles having equal length n, $n \geq 3$ is not $t-$representable. Here we prove such groups are L_f-representable on X when $|L| \neq 2$.

Theorem 4.4.4. *Let X be any set and $L = \{0, \frac{1}{2}, 1\}$. Let σ be a permutation on X such that σ can be written as a product of two disjoint cycles having equal length n. Then the permutation group generated by σ is L_f- representable on X.*

Proof. Let $\sigma = \sigma_1\sigma_2$ where

$$\sigma_1 = (x_{1\,1}, x_{1\,2}, \ldots, x_{1\,n}),\ \sigma_2 = (x_{2\,1}, x_{2\,2}, \ldots, x_{2\,n})$$

and $L = \{1, .5, 0\}$ with the usual order. By Remark 4.2.6, without loss of generality we assume that $X = X_1 \cup X_2$ where X_i is the elements in the cycle σ_i for $i = 1, 2$.

Define $\delta = \{f \in L^X : f(x_{ij\oplus 1}) \geq f(x_{ij}) - .5$ for every $j = 1, 2, \ldots, n$ and $i = 1, 2$ and if $f(x_{2j}) = 1$, then $f(x_{1j}) = 1$ for all $j = 1, 2, \ldots n\}$, where $\oplus$ denote addition modulo n. Then δ is an L-fuzzy topology on X.

The constant functions $\underline{0}$ and $\underline{1}$ belongs to δ. Let $f_1,\ f_2 \in \delta$. Then

$$\begin{aligned}(f_1 \vee f_2)(x_{ij\oplus 1}) = &\ f_1(x_{ij\oplus 1}) \vee f_2(x_{ij\oplus 1}) \\ &\geq (f_1(x_{ij}) - .5) \vee (f_2(x_{ij}) - .5) \\ &\geq (f_1 \vee f_2)(x_{ij}) - .5\end{aligned}$$

and $(f_1 \vee f_2)(x_{2j}) = 1$ gives $f_1(x_{2j}) = 1$ or $f_2(x_{2j}) = 1$. If $f_1(x_{2j}) = 1$, then $f_1(x_{1j}) = 1$. Similarly if $f_2(x_{2j}) = 1$, then $f_2(x_{1j}) = 1$. Thus if $(f_1 \vee f_2)(x_{2j}) = 1$

then $(f_1 \vee f_2)(x_{1j}) = 1$. Now

$$
\begin{aligned}
(f_1 \wedge f_2)(x_{ij\oplus 1}) &= f_1(x_{ij\oplus 1}) \wedge f_2(x_{ij\oplus 1}) \\
&\geq (f_1(x_{ij}) - .5) \wedge (f_2(x_{ij}) - .5) \\
&\geq (f_1 \wedge f_2)(x_{ij}) - .5
\end{aligned}
$$

Now $(f_1 \wedge f_2)(x_{2j}) = 1$ gives $f_1(x_{2j}) = 1$ and $f_2(x_{2j}) = 1$. Thus $(f_1 \wedge f_2)(x_{1j}) = 1$. So δ is an L-fuzzy topology on X.

Now we prove that σ is an L-fuzzy homeomorphism on X. Let $f \in \delta$. Then $\sigma^{-1}(f) = f \circ \sigma$ and for every $i = 1, 2$

$$
\begin{aligned}
f \circ \sigma(x_{i,j\oplus 1}) &= f(x_{i,j\oplus 1\oplus 1}) \\
&\geq f(x_{ij\oplus 1}) - .5 \\
&= f \circ \sigma(x_{i,j}) - .5
\end{aligned}
$$

and

$$
\begin{aligned}
f \circ \sigma(x_{2,j}) = 1 & \\
&\Rightarrow f(x_{2,j\oplus 1}) = 1 \\
&\Rightarrow f(x_{1j\oplus 1}) = 1 \\
&\Rightarrow f \circ \sigma(x_{1,j}) = 1.
\end{aligned}
$$

Hence $f \circ \sigma \in \delta$. Similarly $\sigma(f) = f \circ \sigma^{-1} \in \delta$. Hence σ is an L- fuzzy homeomorphism on (X, δ) and consequently all the powers of σ are also L-fuzzy homeomorphisms on X. Thus

$$K \subseteq LFH(X, \delta). \tag{4.7}$$

Conversely let $h \in LFH(X, \delta)$. For $i = 1, 2$ and $j = 1, 2, \ldots n$,

Define $f_{ij} : X \to L$ as follows

$$f_{ij}(x) = \begin{cases} 1 & \text{if } x = x_{k_j},\ k \leq i \\ .5 & \text{if } x = x_{k_j \oplus 1},\ k \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_{ij} \in \delta$ for all $i = 1,\ 2$ and $j = 1,\ 2,\ \ldots,\ n$. Since h is an L-fuzzy homeomorphism, $h(f_{ij}) = f_{ij} \circ h^{-1} \in \delta$ for all $i = 1,\ 2$ and $j = 1,\ 2,\ \ldots,\ n$. Consider $h(f_{11})$. We have $f_{11} \circ h^{-1}(x) = 1$ for an unique element of X namely $h(x_{11})$. Then $h(x_{11}) = x_{1k}$ for some k. Now $f_{11} \circ h^{-1}(x_{1k}) = 1$ gives $f_{11} \circ h^{-1}(x_{1k\oplus 1}) = .5$ and $f_{11} \circ h^{-1}(x) = 0$ for all other values of X. Thus $f_{11} \circ h^{-1} = f_{1k}$. Also $f_{11} \circ h^{-1}(x_{1k\oplus 1}) = f_{1k}(x_{1k\oplus 1}) = .5$ and hence $h^{-1}(x_{1k\oplus 1}) = x_{12}$ or $h(x_{12}) = x_{1k\oplus 1}$.

Now we prove that if $h(x_{1\alpha}) = x_{1\beta}$ for some α and β, then $h(x_{1\alpha\oplus 1}) = x_{1\beta\oplus 1}$. Note that $h(f_{1\alpha}) = f_{1\alpha} \circ h^{-1} \in \delta$. Now $f_{1\alpha} \circ h^{-1}(x_{1\beta}) = 1$ and this gives $f_{1\alpha} \circ h^{-1}(x_{1\beta\oplus 1}) = 0.5$. So $h^{-1}(x_{1\beta\oplus 1}) = x_{1\alpha\oplus 1}$ or $h(x_{1\alpha\oplus 1}) = x_{1\beta\oplus 1}$. Thus $h(x_{1j}) = x_{1j\oplus(k-1)}$ for all $j = 1, 2, ..., n$. So $h(x_{1j}) = \sigma^{k-1}(x_{1j})$ and h maps

$(x_{11}, x_{12}, \ldots, x_{1n})$ on to itself cyclically.

Let $h(x_{2i}) = x_{2\rho}$. We claim that $\rho = i \oplus (k-1)$. Suppose $\rho \neq i \oplus (k-1)$. Now $f_{2i} \circ h^{-1}(x_{2\rho}) = f_{2i}(x_{2i}) = 1$ and this gives $f_{2i} \circ h^{-1}(x_{1\rho}) = 1$. Now $h^{-1}(x_{1\rho}) = x_{1\rho \oplus (n-k) \oplus 1}$ and so, $f_{2i}(x_{1\rho \oplus n-k \oplus 1}) = 1$, which is a contradiction to the fact that $f_{2i} \in \delta$. So $\rho = i \oplus (k-1)$ and hence $h(x_{2i}) = x_{2i \oplus (k-1)} = \sigma^{k-1}(x_{2i})$ and it follows that $h = \sigma^{k-1}$. Thus

$$LFH(X, \delta) \subseteq K. \tag{4.8}$$

From equations 4.7 and 4.8 we get $LFH(X, \delta) = K$. This completes the proof. □

So if X is any set and L be any F-lattice containing more than two elements, then the group generated by a permutation which is a product of two disjoint cycles having equal length is L_f-representable on X.

Theorem 4.4.5. *Let X be any set and σ be a permutation on X such that σ can be written as a product of two disjoint cycles having equal length n where $n \geq 3$. Then the group generated by σ is L_f- representable on X if and only if $L \neq \{0, 1\}$.*

Proof. Proof follows from Theorems 4.2.2, 3.2.2 and 4.4.4. □

If σ is a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n, then the cyclic group generated by σ, $< \sigma >$

is $t-$representable on X by the Theorem 3.2.4. So by using Theorem 4.4.4, we conclude the following Theorem.

Theorem 4.4.6. *If X is any infinite set and if σ is any permutation on X, which is an arbitrary product of disjoint cycles having equal lengths. Then the permutation group generated by $< \sigma >$ is L_f-representable on X.*

Proof. Proof follows from Theorems 3.2.4 and 4.4.4. □

Corollary 4.4.7. *Let X be any set. Then every permutation group of prime order is L_f-representable on X.*

Proof. Let H be a permutation group on X having order p where p is a prime number. By Remark 4.2.6, without loss of generality we can assume that H moves all the elements of X. Since H is of order p, H is a cyclic group generated by a permutation σ which is of order p. This implies that σ is a product of disjoint cycles having equal length p. So by Theorem 4.4.6, H is L_f-representable on X. □

4.5 L_f-representability of Dihedral Groups

In chapter two, we determined the t-representability of dihedral group D_n. We proved that D_n is not t-representable if and only if $n > 5$.

We investigate the L_f-representability of dihedral groups when $|L| \geq 3$.

Lemma 4.5.1. *Let $X = \{1, 2, \ldots n\}$ and $L = \{0, \frac{1}{2}, 1\}$ with the usual order. Then the dihedral group D_n is L_f-representable for all n.*

Proof. Recall that the group D_n has $2n$ elements generated by the cycle $r = (1, 2, \ldots n)$ and the permutation

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & i & \ldots & n-1 & n \\ n & n-1 & n-2 & n-3 & \ldots & n+1-i & \ldots & 2 & 1 \end{pmatrix}.$$

Let δ be the L-fuzzy topology having base $B = \{g_i : i = 1, 2, \ldots n\} \cup \{f_i : i = 1, 2, \ldots n\}$ where g_i and f_i are functions from X to L defined by

$$g_i(j) = \begin{cases} .5 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_i(j) = \begin{cases} 1 & \text{if } i = j \\ .5 & \text{if } j = i \oplus 1, i \oplus (n-1) \\ 0 & \text{otherwise.} \end{cases}$$

where $\oplus$ denotes addition modulo n.

Now we prove that $LFH(X, \delta) = D_n$. First we claim that the rotation r and reflection s are homeomorphisms.

Consider $f_i \circ r$ and $g_i \circ r$. Now

$$(f_i \circ r)(x) = \begin{cases} 1 & \text{if } r(x) = i \\ .5 & \text{if } r(x) = i \oplus 1, i \oplus (n-1) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x = i \oplus (n-1) \\ .5 & \text{if } x = i, i \oplus (n-2) \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_i \circ r(x) = \begin{cases} .5 & \text{if } r(x) = i \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} .5 & \text{if } x = i \oplus (n-1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f_i \circ r = f_{i \oplus (n-1)}$ and $g_i \circ r = g_{i \oplus (n-1)}$, which belongs to δ. Now consider $f_i \circ s$ and $g_i \circ s$

$$(f_i \circ s)(x) = \begin{cases} 1 & \text{if } s(x) = i \\ .5 & \text{if } s(x) = i \oplus 1, i \oplus (n-1) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x = (n-i) \oplus 1 \\ .5 & \text{if } x = (n-i), (n-i) \oplus 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_i \circ s(x) = \begin{cases} .5 & \text{if } s(x) = i \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} .5 & \text{if } x = (n-i) \oplus 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f_i \circ s = f_{i \oplus (n-1)}$ and $g_i \circ s = g_{i \oplus (n-1)}$, which belongs to δ for all $i = 1, 2, \ldots n$. So r and s belongs to $LFH(X, \delta)$ and hence

$$D_n \subseteq LFH(X, \delta). \tag{4.9}$$

Let $h \in LFH(X, \delta)$. Consider $f_1 \circ h^{-1}$ which takes the value 1 at $h(1)$ and .5 at $h(n)$ and $h(2)$. Let $h(1) = k$. Then either $h(2) = k \oplus 1$ and $h(n) = k \oplus (n-1)$ or $h(2) = k \oplus (n-1)$ and $h(n) = k \oplus 1$.

Case(1): $h(2) = k \oplus 1$ and $h(n) = k \oplus (n-1)$.

Consider $f_2 \circ h^{-1}$. Now $f_2 \circ h^{-1}$ takes the value 1 at $h(2)$ and .5 at $h(1)$ and $h(3)$, which gives $f_2 \circ h^{-1} = f_{k \oplus 1}$ and hence $h(3) = k \oplus 2$. Similarly by considering $f_3, f_4, \ldots f_n$, we get $h(i) = k \oplus (i-1)$. That is $h(i) = r^{k-1}(i)$

for all $i = 1, 2, \dots n$.

Case(2): $h(2) = k \oplus (n-1)$ and $h(n) = k \oplus 1$.

Then $f_2 \circ h^{-1}$ takes the value 1 at $h(2) = k \oplus (n-1)$ and .5 at $h(1) = k$ and $h(3)$ which implies that $f_2 \circ h^{-1} = f_{k \oplus (n-1)}$ and $h(3) = k \oplus (n-2)$. Similarly by considering $f_3, f_4, \dots f_n$ we get $h(i) = k \oplus (n-(i-1))$. That is $h = (1,k)(2,k-1)\dots(k\oplus 1, n)(k \oplus 2, n-1)\dots = r^{n-k}s \in D_n$.

So in both cases $h \in D_n$ and hence

$$LFH(X, \delta) \subseteq D_n. \tag{4.10}$$

From equations 4.9 and 4.10, we get $LFH(X, \delta) = D_n$. This completes the proof. ☐

So if X is any set and L be any F-lattice containing more than two elements, then the dihedral group D_n is L_f-representable on X, by Remark 4.2.2. An immediate consequence of Theorem 2.3.10 and Lemma 4.5.1 is the following.

Theorem 4.5.2. *Let X be any set such that $|X| = n > 4$. Then the dihedral group D_n on X is L_f-representable on X if and only if $L \neq \{0, 1\}$.*

Proof. If $L \neq \{0,1\}$, then D_n is L_f-representable by Lemma 4.5.1 and Remark 4.2.2. Now assume that $L = \{0, 1\}$, which means the crisp case. In this case D_n is not $t-$representable when $n > 4$ by Theorem 2.3.10. So the dihedral group D_n on X is L_f-representable on X if and only if $L \neq \{0, 1\}$. ☐

4.6 L_f- representability of Alternating Groups

Here we enquire the L_f-representability of $A(X)$ when $|X| > |L|$.

Theorem 4.6.1. *If X is any set such that $|X| \geq 4$ and $|L| < |X|$, then the alternating group $A(X)$ is not L_f-representable on X.*

Proof. Suppose that $A(X)$ is L_f-representable on X. Then $A(X) = LFH(X, \delta)$ for some L- fuzzy topology δ on X.

Now we claim that if $(x, y) \circ h \in LFH(X, \delta)$ for every transposition (x, y) in X, then $h \in LFH(X, \delta)$. Let $h \notin A(X)$. It follows that h is not an L-fuzzy homeomorphism on (X, δ). Then there exists at least one $f \in \delta$ such that $f \circ h \notin \delta$ or $f \circ h^{-1} \notin \delta$. Now since $|L| < |X|$ and $f \in \delta$ implies that there exist at least two points $x_0, y_0 \in X$ such that $f(x_0) = f(y_0)$. Suppose $f \circ h \notin \delta$. Since h is a permutation on X and $x_0,\, y_0 \in X$ gives that there exist $x_1,\, y_1 \in X$ such that $h(x_1) = x_0$ and $h(y_1) = y_0$ and hence $(f \circ h)(x_1) = (f \circ h)(y_1)$.

Consider $(x_0, y_0) \circ h$. Here we prove that $f \circ ((x_0,\, y_0) \circ h) = f \circ h$.

For, if $x \in X$, $x \neq x_1,\, y_1$,

$$f \circ ((x_0, y_0) \circ h)(x) = f \circ (x_0,\, y_0)(h(x)) = f \circ h(x).$$

If $x = x_1$, then

$$f \circ ((x_1,\, y_1) \circ h)(x) = f \circ ((x_0,\, y_0) \circ h)(x_1)$$

$$= f(y_0)$$

$$= f(x_0)$$

$$= f \circ h(x_1)$$

$$= f \circ h(x).$$

Similarly if $x = y_1$ we get $f \circ ((x_0, y_0) \circ h)(y_1) = f \circ h(y_1)$. Thus

$$f \circ (x_0, y_0) \circ h(x) = f \circ h(x) \text{ for all } x \in X.$$

It follows that there is an $f \in \delta$ such that $f \circ (x_0, y_0) \circ h \notin \delta$. So there is a transposition (x_0, y_0) such that $(x_0, y_0) \circ h \notin LFH(X, \delta)$.

Similarly if $f \circ h^{-1} \notin \delta$, then we can prove that $f \circ h^{-1} \circ (x_2, y_2) = f \circ h^{-1}$ where $x_0 = h^{-1}(x_2)$ and $y_0 = h^{-1}(y_2)$. Thus in this case also there exists $f \in \delta$ such that $f \circ (h^{-1} \circ (x_2, y_2)) \notin \delta$. Hence $((x_2, y_2) \circ \sigma))^{-1} \notin LFH(X, \delta)$. Here also there is a transposition (x_2, y_2) such that $((x_2, y_2) \circ \sigma)^{-1} \notin LFH(X, \delta)$. So if $(x, y) \circ h \in LFH(X, \delta)$ for every transposition (x, y) in X, then $h \in LFH(X, \delta)$, which is a contradiction since $LFH(X, \delta) = A(X)$ and the alternating group $A(X)$ does not satisfy this if $|X| > 3$. So there exist no L- fuzzy topology δ on X such that the group of L- fuzzy homeomorphisms of (X, δ) is the Alternating group. $\square$

4.6.1 L_f-representability of Normal Subgroups of S_n

Now we investigate the L_f-representability of normal subgroups of S_n when $|L| = 3$. If $n \geq 3$ and $n \neq 4$, the only proper non-trivial normal subgroup of S_n is the alternating group A_n and by Theorem 4.6.1 we determined L_f-representability of A_n. If $n = 4$, then S_n has another normal subgroup and we can determine the L_f-representability of that subgroup.

Theorem 4.6.2. *Let H be the normal subgroup of S_4 given by $H = \{I, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Then H is L_f-representable if and only if the membership lattice $L \neq \{0,1\}$.*

Proof. Assume that $L \neq \{0,1\}$. Let $L = \{1, .5, 0\}$ and δ be the L-fuzzy topology having base

$$\mathbb{B} = \{f_1, f_2, f_3, f_4\}$$

where

$$f_1(1) = .5, f_1(2) = 0, f_1(3) = .5, f_1(4) = 0$$

$$f_2(1) = 0, f_2(2) = .5, f_2(3) = 0, f_2(4) = .5$$

$$f_3(1) = 1, f_3(2) = .5, f_3(3) = .5, f_3(4) = 1$$

$$f_4(1) = .5, f_4(2) = 1, f_4(3) = 1, f_4(4) = .5$$

Now we claim that $LFH(X, \delta) = H$. Reader can easily verify that every element of H is an L-fuzzy homeomorphism of (X, δ) onto itself. Hence $H \subseteq LFH(X, \delta)$.

Let h be an L-fuzzy homeomorphism on X. Then $h^{-1}(f)$ and $h(f)$ are in δ for all $f \in \delta$. Now consider $h^{-1}(f_1)$ and $h^{-1}(f_2)$. Then either $h^{-1}(f_1) = f_1$ and $h^{-1}(f_2) = f_2$ or $h^{-1}(f_1) = f_2$ and $h^{-1}(f_2) = f_1$.

Case 1: $h^{-1}(f_1) = f_1$ and $h^{-1}(f_2) = f_2$

That is $f_1 \circ h = f_1$ and $f_2 \circ h = f_2$, which implies that $h(1) = 1$ or 3 and $h(2) = 2$ or 4. If $h(1) = 1$, then $h(3) = 3$. Suppose $h(2) = 4$. So $h(4) = 2$. Hence $h = (2, 4)$. Then $f_3 \circ h \neq f_3$ or f_4. Hence h is not an L-fuzzy homeomorphism on X, which is a contradiction. So if $h(1) = 1$, then h is the identity permutation.

Now suppose $h(1) = 3$, then $h(3) = 1$. Suppose $h(2) = 2$. So $h(4) = 4$. Hence $h = (1, 3)$. Then $f_3 \circ h \neq f_3$ or f_4. Hence h is not an L-fuzzy homeomorphism, which is also a contradiction. So if $h(1) = 3$, then $h = (1, 3)(2, 4)$.

Case 2 : $h^{-1}(f_1) = f_2$ and $h^{-1}(f_2) = f_1$

In this case $h(1) = 2$ or 4 and $h(2) = 3$ or 1. If $h(1) = 2$, then $h(3) = 4$. Suppose $h(2) = 3$. So $h(4) = 1$. Hence $h = (1, 2, 3, 4)$. Then $f_3 \circ h \neq f_3$ or f_4. Hence h is not an L-fuzzy homeomorphism, which is a contradiction. So if $h(1) = 2$, then $h = (1, 2)(3, 4)$.

Now suppose $h(1) = 4$, then $h(3) = 2$. Suppose $h(2) = 1$. So $h(4) = 3$. Hence $h = (1, 4, 3, 2)$. Then $f_3 \circ h \neq f_3$ or f_4. Hence h is not an L-fuzzy homeomorphism, which is a contradiction. So if $h(1) = 4$, then

$h = (1,\, 4)(2,\, 3)$.

So if h is an L-fuzzy homeomorphism on X, then $h \in H$. Thus $LFH(X, \delta) \subseteq H$ Then $LFH(X, \delta) = H$.

If $L = \{0, 1\}$, H cannot be represented as the group of homeomorphisms of any topological space (X, T) for any topology T on X [34]. So if H is L_f-representable, then $L \neq \{0, 1\}$. This completes the proof. □

Remark 4.6.3. T. P. Johnson [16] proved that the above group H is L_f-representable if $|L| \geq 4$. Here we get H is L_f-representable on X if $L \neq \{0, 1\}$

4.7 t-representability and L_f-representability of Subgroups of S_n when $n \leq 4$

Here we determine the L_f- representability of all subgroups S_n when $n \leq 4$ and $|L| < 4$.

If $n < 3$, all subgroups on S_n are $t-$ representable and hence L_f-representable. If $n = 3$, then the only subgroup which is not $t-$representable on X is A_3, which is L_f-representable by Theorem 4.4.1. So all subgroups of S_3 are also L_f-representable.

Now we consider the case of S_4. The symmetric group S_4 has exactly 30 subgroups and there are exactly 11 conjucacy types of subgroups of S_4.

1. $H_1 = \{I\}$
2. $H_2 = \{I, (1,2)\}$
3. $H_3 = \{I, (1,2)(3,4)\}$
4. $H_4 = \{I, (1,2,3), (1,3,2)\}$
5. $H_5 = \{I, (1,2), (3,4), (1,2)(3,4)\}$
6. $H_6 = \{I, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$
7. $H_7 = \{I, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$
8. $H_8 = \{I, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$
9. $H_9 = \{I, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2), (3,4), (1,4,2,3), (1,3,2,4)\}$
10. $H_{10} = A_4$
11. $H_{11} = S_4$

First we investigate the t-representability of subgroups of S_4.

Here H_4 and H_7 are cyclic groups generated by cycles having length 3 and 4 respectively. So they are not t-representable by Theorem 3.2.5. Now H_6 and H_{10} are normal subgroups of S_4 and hence not t-representable on X. Thus there are four subgroups of S_4 that do not arise as the group of homeomorphisms of any topology.

Clearly H_1 is the group of homeomorphisms of the space

$$\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, X\}.$$

Let τ be the topology defined by $\{X, \emptyset, \{3\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$. Then $H(X,\tau) = H_2$. Consider the topology τ defined by $\{\emptyset, \{1\}, \{2\}, \{1,3\}, \{2,4\}, \{1,2,4\}, \{1,2,3\}, X\}$. Then $H(X,\, \tau) \;=\; H_3$. Let τ be the topology defined by $\{\emptyset, X, \{1\}, \{2\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}$.
Then $H(X,\, \tau) \;=\; H_5$.
Let τ be the topology $\{\emptyset, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}, X\}$
$H(X,\tau) \;=\; H_8$.
If we define $\tau = \{\emptyset, X, \{1,2\}, \{3,4\}\}$, then $H(X,\tau) \;=\; H_9$. Now H_{11} is the group of homeomorphisms of discrete and indiscrete space. So exactly 7 of them are the group of homeomorphism for some topology on X.

Now we get all but one of thirty subgroups arise as the group of fuzzy homeomorphisms for some L-fuzzy topology where $|L| = 3$.

In the above 11 groups, H_4 and H_7 are cyclic. So we can apply Theorem 4.4.1 and hence H_4 and H_7 are L_f-representable. H_6 is L_f-representable on X by Theorem 4.6.2. Our Theorem 4.6.1 applies to exclude H_{10}. So the only subgroup of S_4 which is not L_f- representable is A_4 when $|L| = 3$.

Table 4.1: t-representability and L_f-representability of Subgroups of S_4

SlNo.	Permutation group	t-repre sentability	L_f-repre sentability
1	$\{I\}$	Yes	Yes
2	$\{I, (1,2)\}$	Yes	Yes
3	$\{I, (1,2)(3,4)\}$	Yes	Yes
4	$\{I, (1,2,3), (1,3,2)\}$	No	Yes
5	$\{I, (1,2), (3,4), (1,2)(3,4)\}$	Yes	Yes
6	$\{I, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$	No	Yes
7	$\{I, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$	No	Yes
8	$\{I, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$	Yes	Yes
9	$\{I, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3),$ $(1,2), (3,4), (1,4,2,3), (1,3,2,4)\}$	Yes	Yes
10	A_4	No	No
11	S_4	Yes	Yes

Chapter 5

Completely Homogeneous L-fuzzy Topological Spaces

Larson studied the concept of complete homogeneity in topological spaces [29]. In [16] T. P. Johnson defined the concept of a completely homogeneous fuzzy topological space in an analogous way. In [18] the author extended complete homogeneity to L-fuzzy topological spaces. He considered the lattice of completely homogeneous fuzzy topologies also [18].

In this chapter we continue the study of completely homogeneous L-fuzzy topological spaces. Some properties of completely homogeneous L-fuzzy topological space are discussed. We characterize Alexandroff discrete completely homogeneous L- fuzzy topological spaces when L is a complete chain. Also we study the completely homogeneous L-fuzzy topological spaces when $L = \{0, a, 1\}$.

5.1 Completely Homogeneous L- fuzzy Topological Spaces

In his paper on Some problems on Lattice of L-Fuzzy Topologies [18], Johnson T. P. introduced the concept of completely homogeneous L-fuzzy topological spaces and is defined as below.

Definition 5.1.1. [18] Completely homogeneous L-fuzzy topological spaces An L-fuzzy topological space (X, δ) is called a completely homogeneous if every bijection of X onto itself is an L-fuzzy homeomorphism.

So an L-fuzzy topological space is completely homogeneous on X if and only if the group of L- fuzzy homeomorphisms of L-fuzzy topological space onto itself is the symmetric group $S(X)$.

Here we are listing some examples of completely homogeneous L-fuzzy topological spaces.

Example 5.1.2. From the definition of completely homogeneous L-fuzzy topological space we immediately see that the following are completely homogeneous.

1. Indiscrete space $\{\underline{0}, \underline{1}\}$.

2. Discrete space 2^X.

3. Discrete L-fuzzy space L^X.

Notations

Throughout this chapter we use the following notations.

If (X, δ) is an L-fuzzy topological space, then define

1. $\overline{\delta} = \delta \setminus \{\underline{1}, \underline{0}\}$.

2. $\Lambda_f = \{f(x) : x \in X\}$.

3. $\Lambda_\delta = \{f(x) : x \in X, f \in \overline{\delta}\}$.

4. $\Lambda_f^X = \{g : X \to \Lambda_f\}$.

If L is a F-lattice, then define

1. $L^c = \{l \in L :\ l$ is comparable to each element in $L\}$.

2. $L' = \{l \in L^c :\ $ if $l_1 > l$ and $l_2 > l$, then $l_1 \wedge l_2 > l\}$.

   When L is a complete chain, we have that $L' = L$.

5.2 Properties of Completely Homogeneous L-fuzzy Topological Spaces

In this section we study some properties of completely homogeneous L-fuzzy topological spaces.

First we recall the definition of level topologies of an L-fuzzy topology.

Definition 5.2.1. [28] Level Topology

Let X be a set and $f \in L^X$. We define for $c \in L \setminus \{1\}$, the set $f_{[c]} = \{x \in X : f(x) > c\}$. Then $f_{[c]}$ is called a c- level of f.

Let δ be an L- fuzzy topology on X. Then for each $c \in L' \setminus \{1\}$, the family

$$T_{[c]} = \{f_{[c]} : f \in \delta\}$$

is a topology on X and is called $c-$ level topology of X.

The relation between the group of all L- fuzzy homeomorphisms $LFH(X, \delta)$ of an L-fuzzy topological space (X, δ) and the group of homeomorphisms of its level topologies is given in the following Theorem.

Theorem 5.2.2. *Let (X, δ) be an L-fuzzy topological space. Then the group of all L- fuzzy homeomorphisms $LFH(X, \delta)$ of an L-fuzzy topological space (X, δ) is a subgroup of the group of homeomorphisms of each c-level topology where $c \in L' \setminus \{1\}$.*

Proof. We have $h \in LFH(X, \delta)$ if and only if h is a bijection and both $h(f)$ and $h^{-1}(f)$ are in δ for all $f \in \delta$. Let $(X, T_{[c]})$ be a level topology of (X, δ) and $U \in T_{[c]}$ where $c \in L' \setminus \{1\}$. Then $U = f_{[c]}$ for some f in δ and

$$\begin{aligned} h(U) &= h(f_{[c]}) \\ &= \{h(x) : f(x) > c\} \end{aligned}$$

$$
\begin{aligned}
&= \{x \in X : f(h^{-1}(x)) > c\} \\
&= \{x \in X : f \circ h^{-1}(x) > c\} \\
&= \{x \in X : h(f)(x) > c\} \\
&= h(f)_{[c]}
\end{aligned}
$$

and hence $h(U) \in T_{[c]}$. Similarly

$$
\begin{aligned}
h^{-1}(U) &= h^{-1}(f_{[c]}) \\
&= \{h^{-1}(x) : f(x) > c\} \\
&= \{x \in X : f(h(x)) > c\} \\
&= \{x \in X : f \circ h(x) > c\} \\
&= \{x \in X : h^{-1}(f)(x) > c\} \\
&= (h^{-1}(f))_{[c]}.
\end{aligned}
$$

So h is a homeomorphism on $(X, T_{[c]})$. Thus the group of L- fuzzy homeomorphisms of an L-fuzzy topological space is a subgroup of the group of homeomorphisms of the level topologies. □

Remark 5.2.3. The inclusion in the Proposition 5.2.2 may proper.

Example 5.2.4. Let $X = \{a, b, c\}$, $L = \{1, .5, 0\}$ with usual order and δ be the L- fuzzy topology having base

$$\mathcal{B} = \{f_1, f_2, f_3, g_1, g_2, g_3\}$$

where

$$f_1(a) = .5, f_1(b) = 0, f_1(c) = 0,$$

$$f_2(a) = 0, f_2(b) = .5, f_2(c) = 0,$$

$$f_3(a) = 0, f_3(b) = 0, f_3(c) = .5,$$

$$g_1(a) = 1, g_1(b) = .5, g_1(c) = 0,$$

$$g_2(a) = 0, g_2(b) = 1, g_2(c) = .5,$$

$$\text{and } g_3(a) = .5, g_3(b) = 0, g_3(c) = 1.$$

It is easy to verify that $LFH(X, \delta) = \{(a, b, c), (a, c, b), I\}$ where I is the identity permutation on X and the level topologies $T_{[\frac{1}{2}]}$ and $T_{[0]}$ are the discrete topology on X. So the group of homeomorphisms of the level topologies are the symmetric group $S(X)$.

Theorem 5.2.5. *[42] Let (X, δ) be an L-fuzzy topological space which is completely homogeneous, then all the level topologies of δ are completely homogeneous.*

Corollary 5.2.6. *Let (X, δ) be an L-fuzzy topological space and $\Lambda_f \subseteq \{0, c\}$ for every $f \in \delta \setminus \{\underline{1}\}$, where c is a non-zero element in L. Then the group L-*

fuzzy homeomorphisms of an L-fuzzy topological space is equal to the group of homeomorphisms of the level topology $T_{[0]}$.

Proof. We have the group of L- fuzzy homeomorphisms $LFH(X, \delta)$ of an L-fuzzy topological space (X, δ) is a subgroup of the group of homeomorphisms $H(X, T_{[0]})$ of the level topology. Let $h \in H(X, T_{[0]})$ and $f \in \delta$. Then $f_{[0]} \in T_{[0]}$ and consequently $h(f_{[0]})$ and $h^{-1}(f_{[0]})$ are in $T_{[0]}$. But $h(f_{[0]}) = (h \circ f)_{[0]}$. Since f takes only one non zero value, $h(f) \in \delta$. Similarly we can prove that $h^{-1}(f) \in \delta$. Thus h is an L- fuzzy homeomorphism and so $H(X, T_{[0]}) \subseteq LFH(X, \delta)$. Hence the theorem. $\square$

An immediate consequence of the Theorem 5.2.5 is the following.

Remark 5.2.7. Observe that if (X, δ) is an L-fuzzy topological space and $\Lambda_f \subseteq \{0, c\}$ for every $f \in \delta \setminus \{\underline{1}\}$, where c is a non-zero element in L, then the L-fuzzy topological space (X, δ) is completely homogeneous if and only if the level topology $T_{[0]}$ is completely homogeneous.

The next definition appears in [14].

Definition 5.2.8. $t-$complete Subset
A subset A of L is said to be $t-$complete if A is closed under finite (nonempty) meet and arbitrary join operations.

Remark 5.2.9. Using a $t-$complete subset of L, we can define more than one completely homogeneous L- fuzzy topology on X. Let A be a t-complete subset of L such that $1 \in A$. Define

1. $\delta_1 = \{f \in L^X : f(x) \in A \text{ for all } x \in X\}$.

2. $\delta = \{\underline{l} : l \in A\}$.

Here both δ_1 and δ_2 are clearly completely homogeneous L-fuzzy topology on X.

For any $t-$complete subset A of L, there exist at least two completely homogeneous L- fuzzy topologies on X.

Remark 5.2.10. If L is a finite chain, then any subset of L containing 0 is a t-complete subset of L. Hence we can define two more completely homogeneous L-fuzzy topologies on X for each t-complete subset A of L such that $1 \in A$.

$$\delta = \{f \in L^X : f(x) \in A \setminus \{0\} \text{ for all } x \in X\} \cup \{\underline{0}\}.$$

$$\delta' = \{f \in L^X : f(x) \in A \setminus \{1\} \text{ for all } x \in X\} \cup \{\underline{1}\}.$$

This is not true in the case of a general lattice L. The following example illustrates this.

Example 5.2.11. Let L be the diamond type lattice $\{0, a, b, 1\}$ where $a \wedge b = 0$ and $a \vee b = 1$

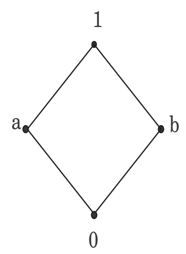


Figure 5.1: Diamond-type lattice

Then $\delta = \{f \in L^X : f(x) \in A \setminus \{0\} \text{ for all } x \in X\} \cup \{\underline{0}\}$ and $\delta' = \{f \in L^X : f(x) \in A \setminus \{1\} \text{ for all } x \in X\} \cup \{\underline{1}\}$ are not even L- fuzzy topology on X where A is L itself.

Let δ be an L-fuzzy topology on X. Then clearly $\{f(x) : x \in X, f \in \delta\}$ is a $t-$complete subset of L. Then by Remark 5.2.9, corresponding to each L-fuzzy topology δ on X, we can define at least two completely homogeneous L-fuzzy topology on X.

Definition 5.2.12. [30] Stratification

Let(X, δ) be an L-fuzzy topology on X. Then the L- fuzzy topology δ' on X generated by $\delta \cup \{\underline{\alpha} : \alpha \in L\}$ is called the stratification of δ and (X, δ') is called the stratification of (X, δ).

Proposition 5.2.13. *Let (X, δ) be an L-fuzzy topological space. Then δ is a completely homogeneous L-fuzzy topology on X if and only if the stratification of δ is a completely homogeneous L-fuzzy topology on X.*

Proof. Let (X, δ') be the stratification of an L-fuzzy topological space (X, δ). Then every $f \in \delta'$ is of the form $g \vee \underline{\alpha}$ or $g \wedge \underline{\beta}$ where $g \in \delta$ and $\alpha, \beta \in L$. So for any $h \in S(X)$, $h(f) = h(g) \vee \underline{\alpha}$ or $h(g) \wedge \underline{\beta}$.

First assume that (X, δ) is a completely homogeneous L- fuzzy topological space. Then $h(g)$ and $h^{-1}(g)$ are in δ for all $g \in \delta$. So $h(f) \in \delta'$. Similarly we can prove that $h^{-1}(f) \in \delta'$. Thus δ' is a completely homogeneous L-fuzzy topological space on X. Now assume that δ' is completely homogeneous L-fuzzy topological space, then clearly δ is a completely homogeneous L-fuzzy topology on X. $\square$

In a completely homogeneous topological space, supersets of nonempty open sets are open [34]. But this is not true in the case of an L- fuzzy topology. See the following example.

Example 5.2.14. Let $X = \{a, b\}$ and $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ with the usual ordering and the involution $'$ given by $l' = 1 - l$ for all $l \in L$.

Define

$$\delta = \{\underline{0}, \underline{1}, a_{\frac{1}{4}}, b_{\frac{1}{4}}, \underline{\frac{1}{4}}, \underline{\frac{1}{2}}, \underline{\frac{3}{4}}, f, g\}.$$

where f and g are L- fuzzy sets of X defined by

$$f(a) = \frac{1}{2},\ f(b) = \frac{3}{4}$$
$$\text{and } g(a) = \frac{3}{4},\ g(b) = \frac{1}{2}$$

Here δ is a completely homogeneous L- fuzzy topology. Let h be an L- fuzzy set defined by

$$h(a) = \frac{1}{2},\, h(b) = \frac{1}{4}.$$

Here h is a super set of $a_{\frac{1}{4}}$ but $h \notin \delta$.

Thus supersets of nonempty L-fuzzy open sets need not be L-fuzzy open in a completely homogeneous L-fuzzy topological space.

Now we prove that super sets of L-fuzzy open sets having the same range are open in a completely homogeneous L- fuzzy topological space as we see in the next Theorem.

Theorem 5.2.15. *Let (X, δ) be a completely homogeneous L- fuzzy topological space and $f \in \delta$ such that $f \neq \underline{0}$. Let g be an L- fuzzy set such that $g \geq f$ and $\Lambda_g \subseteq \Lambda_f$. Then $g \in \delta$.*

Proof. If $g = f$, there is nothing to prove. So assume that $f < g$. Let $Y = \{x \in X : f(x) < g(x)\}$. Now choose two points y and z such that $y \in Y$ and $z \in X$ where $f(z) = g(y)$. We can choose such a point z since $f \neq \underline{0}$ and range of g is a subset of range of f. Now define $f_y = f \circ h_y$ where h_y is a function from X to X such that

$$h_y(x) = \begin{cases} z & \text{if } x = y \\ y & \text{if } x = z \\ x & \text{for all } x \in X \setminus \{y, z\}. \end{cases}$$

Then $f_y \in \delta$ and hence $f \vee f_y \in \delta$. Observe that

$$f_y(x) = \begin{cases} f(x) & \text{for all } x \in X \setminus \{y, z\} \\ f(z) & \text{if } x = y \\ f(y) & \text{if } x = z. \end{cases}$$

Consequently

$$(f \vee f_y)(x) = \begin{cases} f(x) & \text{for all } x \in X \setminus \{y, z\} \\ f(y) \vee f(z) & \text{if } x = y \\ f(z) \vee f(y) & \text{if } x = z. \end{cases}$$

$$= \begin{cases} f(x) & \text{for all } x \in X \setminus \{y\} \\ g(y) & \text{if } x = y. \end{cases}$$

Similarly for each y in Y, we define f_y such that $(f \vee f_y) \in \delta$. Since δ is an L-fuzzy topology, we have $\bigvee_{y \in Y} (f \vee f_y) \in \delta$. Now

$$\bigvee_{y \in Y} (f \vee f_y)(x) = \begin{cases} f(x) & \text{for all } x \in X \setminus Y \\ g(x) & \text{if } x \in Y. \end{cases}$$

$$= g(x) \text{ for all } x \in X$$

Thus $\bigvee_{y \in Y} (f \vee f_y) = g$ and hence $g \in \delta$. This completes the proof. $\square$

That is if (X, δ) is a completely homogeneous L-fuzzy topological space, then

$\{g \in L^X : g \geq f \text{ and } \Lambda_g \subseteq \Lambda_f\} \subseteq \delta$ for all $f \in \overline{\delta}$.

Remark 5.2.16. Converse of the Theorem 5.2.15 is not true.

Example 5.2.17. Let $X = \{x, y\}$ and $L = \{0, \frac{1}{2}, 1\}$ with the usual order. Consider the following L-fuzzy topology

$$\delta = \{\underline{0}, \underline{1}, \underline{\frac{1}{2}}, x_{\frac{1}{2}}, x_1, y^{\frac{1}{2}}, x^{\frac{1}{2}}\}$$

on X. Here for every element f in $\delta \setminus \{\underline{0}\}$, $\{g \in L^X : g \geq f \text{ and } \Lambda_g \subseteq \Lambda_f\} \subset \delta$. But δ is not a completely homogeneous L-fuzzy topology on X since $LFH(X, \delta) = \{I\}$.

5.3 Completely Homogeneous Alexandroff Discrete L-fuzzy Topological Space

Recall that a topological space (X, τ) is an Alexandroff discrete space if arbitrary intersection of open sets are open. Note that the only completely homogeneous Alexandroff discrete topologies on a set X are the discrete topology and the indiscrete topology.

In this section we study completely homogeneous Alexandroff discrete L-fuzzy topological spaces when L is a complete chain. An L- fuzzy topological space (X, δ) is said to be Alexandroff discrete L- fuzzy topological space if $\wedge A \in \delta$ for all $A \subseteq \delta$.

Next we give a characterization for a completely homogeneous Alexandroff discrete L- fuzzy topological space when L is a complete chain.

Theorem 5.3.1. *Let (X, δ) be an Alexandroff discrete L-fuzzy topological space where L is a complete chain. Then (X, δ) is a completely homogeneous L-fuzzy topological space on X if and only if $\Lambda_f^X \subseteq \delta$ for all $f \in \delta$.*

Proof. Let (X, δ) be a completely homogeneous Alexandroff discrete L-fuzzy topological space where L be a complete chain and $f \in \delta$. Define $l_1 = \underset{l \in \Lambda_f}{\wedge} l$ and

$$x'_l(y) = \begin{cases} l & y = x \\ l_1 & \text{otherwise.} \end{cases}$$

where $l \in \Lambda_f$ and $l \neq l_1$.

We claim that $x'_l \in \delta$ for all $l \in \Lambda_f$. Since δ is a completely homogeneous Alexandroff L-fuzzy topology on X, we have that $l_1 \in \Lambda_\delta$. Here we consider two cases.

Case(1): $l_1 \in \Lambda_f$.

Since $l_1 \in \Lambda_f$, there exists an element x_0 in X such that $f(x_0) = l_1$. For each $x \in X \setminus \{x_0\}$, define $f_x = f \circ h_x$ where h_x is a function from X onto itself which maps x to x_0, x_0 to x and keeping all other elements fixed. Then

$$f_x(y) = \begin{cases} l_1 & y = x \\ f(x_0) & y = x_0 \\ f(x) & \text{otherwise.} \end{cases}$$

Here $f_x \in \delta$ for all $x \in X \setminus \{x_0\}$. Let $l \in \Lambda_f$. Then there exists an element z in X such that $f(z) = l$. Now $\bigwedge\limits_{x \in X \setminus \{z\}} f_x \in \delta$ and

$$\bigwedge_{x \in X \setminus \{z\}} f_x(y) = \begin{cases} f(x_0) & y \neq z \\ f(z) & y = z. \end{cases}$$

$$= \begin{cases} l_1 & y \neq z \\ l & y = z. \end{cases}$$

$$= x_l'.$$

Thus we get $x_l' \in \delta$.

Case(2): $l_1 \notin \Lambda_f$.

In this case first we construct an L- fuzzy set f' using f such that $l_1 \in \Lambda_{f'}$. Fix some x' in X. Let $f(x') = l$. Now define $f_x = f \circ h_x$ for all $x \in X$ where h_x is a function from X onto itself which maps x to x', x' to x and keeping all other elements fixed. Let $f' = \bigwedge\limits_{x \in X} f_x$. Since (X, δ) is Alexandroff discrete, we have that $f' \in \delta$ and

$$f'(y) = \begin{cases} l_1 & y = x' \\ l \wedge f(x) & \text{otherwise.} \end{cases}$$

Thus we get an L- fuzzy open subset of X which takes the value l_1. If $l \in \Lambda_{f'}$, proceeding as in the Case (1), we can easily prove that $x_l' \in \delta$.

Now suppose that $l \notin \Lambda_{f'}$. This implies that $f^{-1}(l)$ is the singleton set $\{x'\}$. Then choose a point z in X such that $f(z) < f(x')$. We can choose such a point since $l_1 \notin \Lambda_f$. Define $g : X \to L$ by

$$g(y) = \begin{cases} l & y = z \\ f(y) & \text{otherwise.} \end{cases}$$

Then $g > f$ and $\Lambda_g \subseteq \Lambda_f$. So by Theorem 5.2.15, it follows that $g \in \delta$. Now using the L-fuzzy set g, we can construct an L-fuzzy open set g' such that $l_1 \in \Lambda_{g'}$ and $l \in \Lambda_{g'}$. Now we arrive at Case 1 and hence $x'_l \in \delta$.

So in both Cases $x'_l \in \delta$ for all $l \in \Lambda_f$. Now any $f \in \Lambda_f^X$ can be expressed as a join of x'_l. So $\Lambda_f^X \subseteq \delta$ for all $f \in \delta$.

Conversely assume that $\Lambda_f^X \subseteq \delta$ for all $f \in \delta$. So $f \circ h \in \delta$ for all $f \in \delta$. Thus δ is a completely homogeneous L- fuzzy topology on X. Hence the result. □

An L-fuzzy topological space (X, δ) is said to be finite if the underlying set X is finite.

With the characterization theorem of completely homogeneous Alexandroff discrete L- fuzzy topological space, we list finite completely homogeneous L-fuzzy topological spaces when the membership lattice $L = \{0, \frac{1}{2}, 1\}$ with the usual order.

Corollary 5.3.2. *Let X be a finite set and $L = \{0, \frac{1}{2}, 1\}$. Then the only completely homogeneous L-fuzzy topology on X are the following.*

1. *The indiscrete topology,* $\{\underline{1}, \underline{0}\}$.

2. $\{\underline{1}, \underline{0}, \underline{\frac{1}{2}}\}$.

3. *The discrete topology,* 2^X.

4. $\{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ *for all* $x \in X\}$.

5. $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.

6. *The discrete L- fuzzy topology* L^X.

7. $\{\underline{\frac{1}{2}}\} \cup \{f \in \{0, 1\}^X\}$.

8. $\{f \in L^X : f(x) \leq \frac{1}{2}$ *for all* $x \in X\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.

Proof. Let δ be a completely homogeneous L-fuzzy topology on X. If $\Lambda_\delta = \emptyset$, then δ is the indiscrete topology. Clearly Λ_δ cannot be equal to $\{0\}$ or $\{1\}$. If $\Lambda_\delta = \{\frac{1}{2}\}$, then δ is of the form (2). Let $\Lambda_\delta = \{0, \frac{1}{2}\}$. Then there exist at least one f in δ such that $\Lambda_f = \{0, \frac{1}{2}\}$ and hence $\{0, \frac{1}{2}\}^X \subseteq \delta$ by Theorem 5.3.1. In this case $\delta = \{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ for all $x \in X\}$. Similarly if $\Lambda_\delta = \{0, 1\}$, then δ is the discrete topology 2^X and if $\Lambda_\delta = \{\frac{1}{2}, 1\}$, then $\delta = \{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ for all $x \in X\}$.

Now assume that $\Lambda_\delta = \{0, \frac{1}{2}, 1\}$. Suppose there exists an L- fuzzy set f in δ such that $\Lambda_f = \{0, \frac{1}{2}, 1\}$. Then again by Theorem 5.3.1, $\{0, \frac{1}{2}, 1\}^X \subseteq \delta$ and hence $\delta = L^X$. Otherwise there exist no f in δ such that $\Lambda_f = \{0, \frac{1}{2}, 1\}$. But we have $\Lambda_\delta = \{0, \frac{1}{2}, 1\}$. So there exist at least two functions f_1 and f_2

such that either $\Lambda_{f_1} = \{0, \frac{1}{2}\}$ and $\Lambda_{f_1} = \{1, \frac{1}{2}\}$ or $\Lambda_{f_1} = \{0, 1\}$ and $\Lambda_{f_2} = \{\frac{1}{2}\}$. If $\Lambda_{f_1} = \{0, \frac{1}{2}\}$ and $\Lambda_{f_1} = \{1, \frac{1}{2}\}$, we get $\delta = \{f \in L^X : f(x) \leq \frac{1}{2}$ for all $x \in X\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ for all $x \in X\}$. Now assume that $\Lambda_{f_1} = \{0, 1\}$ and $\Lambda_{f_2} = \{\frac{1}{2}\}$. In this case $\delta = \{\underline{\frac{1}{2}}\} \cup \{f \in \{0, 1\}^X\}$. □

Remark 5.3.3. It is not possible to drop the chain condition on the lattice from the hypothesis of the Theorem 5.3.1. The following example illustrates this.

Example 5.3.4. Let $X = \{x, y\}$ and L be the diamond type lattice $L = \{0, a, b, 1\}$. Define

$$\delta = \{\underline{1}, \underline{0}, f_1, f_2\}$$

where

$$f_1(x) = a, f_1(y) = b$$

$$\text{and } f_2(x) = b, f_2(y) = a.$$

Here δ is a completely homogeneous L- fuzzy topology on X, but $\Lambda_{f_1}^X \nsubseteq \delta$.

Chapter 6

Principal Completely Homogeneous L-fuzzy Topological Spaces

6.1 Introduction

In this Chapter, we define the principal completely homogeneous topology generated by a set. The completely homogeneous topology generated by a single set is called principal completely homogeneous topology. We also introduce principal completely homogeneous L- fuzzy topology generated by an L-fuzzy set and study some of its properties. Also we characterize the principal completely homogeneous L-fuzzy topological space generated by an L- fuzzy set when the membership lattice $L = \{0, \frac{1}{2}, 1\}$ with the usual order. .

6.2 Principal Completely Homogeneous Topological Space

Among the crisp topologies R. E Larson [29] characterized the completely homogeneous topological spaces.

Theorem 6.2.1. *[29] The only completely homogeneous topologies on X are the following.*

1. *The indiscrete topology.*
2. *The discrete topology.*
3. *Topologies of the form $T_m = \{G \subseteq X : |X \setminus G| < m\} \cup \{\emptyset\}$ where $\aleph_0 \leq m \leq |X|$.*

Now we define the term principal completely homogeneous topology generated by a set.

Definition 6.2.2. *Principal completely homogeneous topology*

Let $A \subseteq X$. Then the smallest completely homogeneous topology containing A is called the principal completely homogeneous topology generated by A. Here the set A is called the generator of the principal completely homogeneous topology.

Here we investigate the principal completely homogeneous topologies on an arbitrary set X.

Obviously the indiscrete topology is generated by the whole set X and the discrete topology is generated by any singleton. So the indiscrete topology and discrete topology are principal completely homogeneous topologies on X.

Now we consider the completely homogeneous topologies of the form $T_m = \{G \subseteq X : |X \setminus G| < m\} \cup \{\emptyset\}$ where $\aleph_0 \leq m \leq |X|$.

We need the following definition for proving our next theorem.

Definition 6.2.3. [11] Limit Cardinal

The successor of a cardinal m is the least cardinal greater than m. A cardinal is said to be a limit cardinal if it is not the successor of a cardinal.

The cardinals 0, $\aleph_0, \aleph_\omega$ etc. are limit cardinals. An uncountable cardinal $\aleph_\gamma$ will be a limit cardinal if and only if γ is a limit ordinal [11].

In the following theorem we characterize principal completely homogeneous topology generated by a set.

Theorem 6.2.4. *Let X be an infinite set and and $T_m = \{G \subseteq X : |X \setminus G| < m\} \cup \{\emptyset\}$ where $\aleph_0 \leq m \leq |X|$. Then T_m is a principal completely homogeneous topology on X if and only if m is not a limit cardinal.*

Proof. Assume that m is not a limit cardinal, then there exists an infinite cardinal m' such that m is the immediate successor of m'. Choose a subset A of X such that $|X \setminus A| = m'$. Since T_m is completely homogeneous, it follows that $\{B \subseteq X : |B| = |A| \text{ and } |X \setminus B| = |X \setminus A|\} \subseteq T_m$. Also we have super sets of nonempty open sets are open in X. So T_m is a principal completely homogeneous topology generated by A.

Conversely assume that m is a limit cardinal. Suppose that T_m is generated by subset A of X. Then $|X \setminus A| < m$. Then there exists an infinite cardinal number m' such that $|X \setminus A| < m' < m$. Then $A \in T_{m'}$ and hence the principal completely homogeneous topology generated by A is contained in $T_{m'}$, which is a contradiction. This completes the proof. $\square$

Now we list the principal completely homogeneous topologies on an infinite set X.

Corollary 6.2.5. *The only principal completely homogeneous topologies on X are the following.*

1. *The indiscrete topology.*
2. *The discrete topology.*
3. *Topologies of the form $T_m = \{G \subseteq X : |X \setminus G| < m\} \cup \{\emptyset\}$ where $\aleph_0 \leq m \leq |X|$ and m is not a limit cardinal.*

Remark 6.2.6. If X is a finite set, we have the only completely homogeneous topologies on X are the discrete topology and the indiscrete topology. Hence all completely homogeneous topologies on a finite set are principal.

6.3 Principal Completely Homogeneous L-fuzzy Topological Space

We have that the intersection of all completely homogeneous L- fuzzy topologies containing an L- fuzzy set f is a completely homogeneous L-fuzzy topology δ on X. Then $\mathcal{S} = \{foh : h \in S(X)\}$ forms a sub-base for δ (see [16, 18]).

Here we define a principal completely homogeneous L-fuzzy topology.

Definition 6.3.1. *Principal completely homogeneous L- fuzzy topology*

Let $f \in L^X$. Then the smallest completely homogeneous L- fuzzy topology containing f is called the principal completely homogeneous L- fuzzy topology generated by f and is denoted by $CHLFT(f)$.
Here the L- fuzzy set f is called the generator of the principal completely homogeneous L- fuzzy topology.

A completely homogeneous L-fuzzy topological space (X, δ) is called principal completely homogeneous L-fuzzy topological space if $\delta = CHLFT(f)$ for some L- fuzzy set $f \in L^X$.

Here we list some examples of principal completely homogeneous L-fuzzy topological spaces.

1. Indiscrete space $\{\underline{0}, \underline{1}\}$.

2. Discrete Crisp topology 2^X.

3. The L-fuzzy topological space generated by an L-fuzzy point.

Now we investigate the L-fuzzy discrete space L^X is principal completely homogeneous L-fuzzy topology or not.

Proposition 6.3.2. *Let X be any finite set and L be an F-lattice such that $|X| \geq L$. Then L^X is a principal completely homogeneous L-fuzzy topology on X.*

Proof. Let f be an L-fuzzy set such that $\Lambda_f = L$ and $\delta = CHLFT(f)$. Since $|X| \geq L$, such an L-fuzzy set exists. Now we claim that for each $l \in L \setminus \{0\}$, $x_l \in \delta$. Let $l \in L$. Since $\Lambda_f = L$, there exist some x_0 and z in X such that $f(x_0) = 0$ and $f(z) = l$. For each x in X, let h_x be a function from X onto itself which maps x to x_0, x_0 to x and keeping all other elements fixed. Define $f_x = f \circ h_x$. Now

$$f_x(y) = \begin{cases} 0 & y = x \\ f(x) & y = x_0 \\ f(y) & y \in X \setminus \{x, x_0\}. \end{cases}$$

Since X is finite, $\bigwedge\limits_{x \in X \setminus \{z\}} f_x \in \delta$ and

$$\bigwedge_{x \in X \setminus \{z\}} f_x(y) = \begin{cases} l & y = z \\ 0 & \text{otherwise.} \end{cases}$$

$$= z_l.$$

This is true for all $l \in L \setminus \{0\}$. So all L-fuzzy points are in δ. This completes the proof. $\square$

Remark 6.3.3. If $|X| < L$, L^X need not be a principal completely homogeneous L-fuzzy topology.

The following examples illustrate this.

Examples 6.3.4.

1. Let $X = \{a, b\}$ and $L = \{0, \frac{1}{2}, 1\}$ with the usual order. Let $f \in L^X \setminus \{\underline{0}, \underline{1}, \underline{\frac{1}{2}}\}$. If $\Lambda_f = \{0, \frac{1}{2}\}$, then $CHLFT(f) = \{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}\}$. If $\Lambda_f = \{1, \frac{1}{2}\}$, then $CHLFT(f) = \{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}\}$. If $\Lambda_f = \{0, 1\}$, then $CHLFT(f)$ is the discrete topology 2^X. This follows that there exist no L-fuzzy set f such that $CHLFT(f) = L^X$. So L^X is not a principal completely homogeneous L-fuzzy topology on X.

2. Let $X = \{a, b, c\}$ and L be the diamond type lattice. Let $f : X \to L$ be the L-fuzzy set defined by $f(a) = 0, f(b) = l_1$ and $f(c) = l_2$. Here we can show that $CHLFT(f) = L^X$. Let h_1 and h_2 be two permutations on X defined by $h_1(a) = b$, $h_1(b) = a$ and $h_1(c) = c$ and $h_2(a) = c$, $h_2(b) = b$ and $h_2(c) = a$. Then $f \circ h_1$ and $f \circ h_2$ are in $CHLFT(f)$. Now $f \wedge (f \circ h_1) = c_{l_2}$ and $f \wedge (f \circ h_2) = b_{l_1}$. So x_{l_1} and x_{l_2} are in $CHLFT(f)$ for all $x \in X$. Now we have that $a_{l_1} \vee a_{l_2} = a_1$. This follows that all the L-fuzzy points are in $CHLFT(f)$ and hence $CHLFT(f) = L^X$.

6.4 Properties of Principal Completely Homogeneous L-fuzzy Topological Space

We now prove two important properties of the principal completely homogeneous L-fuzzy topological space.

For proving the subsequent theorems, we need the following set theoretic results.

Let P and Q be two subsets of set X and $|P| = |Q|$, it does not necessarily follow that there exists a bijection of X which maps P onto Q. In order to exist such a function, we must also have that $|X \setminus P| = |X \setminus Q|$. If X is an infinite set, it is possible to choose P and Q such that $P \cup Q = X$, $P \cap Q = \emptyset$ and $|P| = |X| = |Q|$ since for any infinite cardinal number α, we have $\alpha + \alpha = \alpha$ [44].

Theorem 6.4.1. *Let X be an infinite set and f be an L-fuzzy subset of X such that $|f^{-1}(0)| = |X|$. Then $CHLFT(f)$ is $\Lambda_f^X \cup \{\underline{1}\}$.*

Proof. Let $\delta = CHLFT(f)$, $X_1 = f^{-1}(0)$ and $X_2 = X \setminus X_1$. Given that $|X_1| = |X|$. Now we consider the following cases.

Case (1) : $|X_2| = |X|$.

Here $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$ and $|X_1| = |X| = |X_2|$. Hence there exists a bijection h of X which maps X_1 onto X_2 and X_2 onto X_1. Since (X, δ) is a completely homogeneous L-fuzzy topological space, $h^{-1}(f) = f \circ h \in \delta$. Let $g = f \circ h$ and $l \in \Lambda_f$, $l \neq 0$. Then there exists x_0 in X_2 such that

$f(x_0) = l$. We have

$$\begin{aligned} g^{-1}(0) &= (f \circ h)^{-1}(0) \\ &= (h^{-1} \circ f^{-1})(0) \\ &= h^{-1}(X_1) \\ &= X_2. \end{aligned}$$

So we can choose y_0 in X_1 such that $g(y_0) = f(x_0)$. Now consider the L-fuzzy set $g \circ h_{x_0}$ where h_{x_0} is a function from X onto itself which maps x_0 to y_0, y_0 to x_0 and keeping all the other elements fixed. Set $f_1 = f$ and $f_2 = g \circ h_{x_0}$. Then

$$(f_1 \wedge f_2)(x) = \begin{cases} l & x = x_0 \\ 0 & \text{otherwise.} \end{cases}$$

$$= x_{0_l}$$

Case(2) : $|X_2| < |X|$.

If $X_2 = \emptyset$, there is nothing to prove. Suppose that $X_2 \neq \emptyset$. In this case we can find a subset Y_1 of X_1 such that $|X_2| = |Y_1|$. Then $|X \setminus X_2| = |X| = |X \setminus Y_1|$. So there exists a bijection h from X onto X such that $h(Y_1) = X_2$ and $h(X_2) = Y_1$. Then as in the case(1), consider $g = f \circ h$. Now $(f \circ h)(x) = f(h(x)) \neq 0$ for $x \in Y_1$. Now let $l \in \Lambda_f \setminus \{0\}$. Then there exists x_0 in X_2 such that $f(x_0) = l$. We have $X_2 \subseteq g^{-1}(0)$. So we can choose y_0 in Y_1 such that $g(y_0) = f(x_0)$. Now consider the L- fuzzy set $g \circ h_{x_0}$ where h_{x_0} is the function as defined in the case (1) above. Set $f_1 = f$ and $f_2 = g \circ h_{x_0}$.

Then

$$(f_1 \wedge f_2)(x) = \begin{cases} l & x = x_0 \\ 0 & \text{otherwise.} \end{cases}$$

$$= x_{0_l}$$

So in both cases, $x_{0_l} \in \delta$. Since x_0 is arbitrary, $\{x_l : x \in X\} \subseteq \delta$. This is true for all $l \in \Lambda_f$ and hence $\delta = {\Lambda_f}^X \cup \{\underline{1}\}$.

□

Theorem 6.4.2. *Let (X, δ) be a completely homogeneous L-fuzzy topological space where X is an infinite set and $\delta = CHLFT(f)$ for some $f \in L^X$ such that $|f^{-1}(0)| < \alpha$, $\aleph_0 \leq \alpha < |X|$. Then $|g^{-1}(0)| < \alpha$ for all $g \in \delta \setminus \{\underline{0}\}$.*

Proof. Let $g \in \delta \setminus \{\underline{0}\}$. Since $\delta = CHLFT(f)$, $\mathcal{S} = \{f \circ h : h \in S(X)\}$ forms a sub-base for δ. So g can be written as $\bigvee_{i \in I} g_i$ where I is an index set and for each $i \in I$, g_i can be written as the meet of finitely many members of $\mathcal{S}$, say $s_1^i \wedge s_2^i \ldots \wedge s_{r_i}^i$ where $r_i \in N$ and $s_j^i \in \mathcal{S}$, $1 \leq j \leq r_i$. Now $(s_1^i \wedge s_2^i \ldots \wedge s_{r_i}^i)^{-1}(0) = (s_1^i)^{-1}(0) \cup (s_2^i)^{-1}(0) \cup \ldots \cup (s_{r_i}^i)^{-1}(0)$. We have $|(s_j^i)(0)| < \alpha$ for all j, $1 \leq j \leq r_i$. Since α is an infinite cardinal, we get $|(g_i^{-1})(0)| < \alpha$. Now $|g^{-1}(0)| = |(\bigvee_{i \in I} g_i)^{-1}(0)| = |\bigcap_{i \in I} (g_i^{-1}(0))| < \alpha$. Hence the theorem. □

Theorem 6.4.3. *Let X be an infinite set. Then δ is a completely homogeneous L-fuzzy topology on X where $\Lambda_\delta = \{0, l\}$, $l \in L \setminus \{0, 1\}$ if and only if δ is one of the following.*

1. $\{\underline{1}\} \cup \{0, l\}^X$.

2. $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$ *where* $\aleph_0 \leq \alpha \leq |X|$.

Proof. Clearly the above two L-fuzzy topologies are completely homogeneous. Now assume that δ is a completely homogeneous L-fuzzy topological space and $\Lambda_\delta = \{0, l\}$. Then there exist at least one $f \in \delta$ such that $\{f(x) : x \in X\} = \{0, l\}$. Let $X_1 = f^{-1}(0)$. If $|X_1| = |X|$, then by the Theorem 6.4.1, $\delta = \{\underline{1}\} \cup \{0, l\}^X$. Otherwise $|X_1| < |X|$. In this case consider the level topology $T_{[0]}$ of δ. Then $T_{[0]}$ is a completely homogeneous topology on X by Theorem 5.2.6 and $X \setminus X_1$ is open in $T_{[0]}$. So $T_{[0]} = \{A \subseteq X : |X \setminus A| < \alpha\} \cup \{\emptyset\}$ where $\aleph_0 \leq \alpha \leq |X|$. Hence $\delta = \{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$. $\square$

Corollary 6.4.4. *Let X be an infinite set and $f \in L^X$ where $\Lambda_f = \{0, l\}$, $l \neq 0$. Then $\delta = CHLFT(f)$ if and only if δ is one of the following.*

1. $\{\underline{1}\} \cup \{0, l\}^X$.

2. $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$ *where* $\aleph_0 \leq \alpha \leq |X|$ *and α is not a limit cardinal.*

Proof. Let $f \in L^X$ where $\Lambda_f = \{0, l\}$, $l \neq 0$. By the Theorem 6.4.3 we have that the only completely homogeneous topologies on X are the following.

1. $\{\underline{1}\} \cup \{f \in \{0, l\}^X\}$.

2. $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$.

We have the L-fuzzy topology $\{\underline{1}\} \cup \{f \in \{0, l\}^X\}$ is generated by any L-fuzzy point x_l in L^X. Let $\delta = \{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$. Now we claim that $\delta_\alpha = \{\underline{0}, \underline{1}\} \cup \{f \in \{0, l\}^X : |f^{-1}(0)| < \alpha\}$ is a principal completely homogeneous L-fuzzy topology if and only if α is not a limit cardinal.

Assume that α is not a limit cardinal. Then there exists a cardinal α_1 such that α is the immediate successor of α_1. Now choose an L-fuzzy set $f \in \{0, l\}^X$ such that $|f^{-1}(0)| = \alpha_1$. Since (X, δ) is completely homogeneous, it implies that $\{g \in \{0, l\}^X : |g^{-1}(0) = |f^{-1}(0)| \text{ and } |g^{-1}(l) = |f^{-1}(l)|\} \subseteq \delta_\alpha$. Also we have $\{g \in \{0, l\}^X : g \geq f\} \subseteq \delta_\alpha$. So δ_α is a principal completely homogeneous L-fuzzy topology on X generated by f.

Conversely assume that α is a limit cardinal. Suppose that δ_α is generated by an L-fuzzy set f of X. Then $|f^{-1}(0)| < \alpha$. Then there exists an infinite cardinal number α' such that $|f^{-1}(0)| < \alpha' < \alpha$. Then $f \in \delta_{\alpha'}$ and hence the principal completely homogeneous L-fuzzy topology generated by f is contained in $\delta_{\alpha'}$, which is a contradiction. This completes the proof.

□

Now we consider principal completely homogeneous L-fuzzy topology on a finite set.

Example 6.4.5. Let $X = \{a, b, c\}$, $L = \{0, \frac{1}{2}, 1\}$ and $f \in L^X$ defined by

$$f(a) = 0, f(b) = \frac{1}{2} \text{ and } f(c) = 1.$$

Then $\{f \circ h : h \in S(X)\}$ form a sub-base for $CHLFT(f)$. So $x_{\frac{1}{2}}$ and x_1 belongs to $CHLFT(f)$ for all $x \in X$ and hence $CHLFT(f) = L^X$.

Now we observe that there exists non principal completely homogeneous L-topology on a finite set. See the following example.

Example 6.4.6. Let $L = \{0, \frac{1}{2}, 1\}$ with usual order and $X = \{a, b\}$. Now define

$$\delta = \{\underline{0}, \underline{1}, a_{\frac{1}{2}}, b_{\frac{1}{2}}, \underline{\frac{1}{2}}, a^{\frac{1}{2}}, b^{\frac{1}{2}}\}.$$

Then δ is a completely homogeneous L-fuzzy topology on X. We have

$$CHLFT(a_{\frac{1}{2}}) = \{\underline{0}, \underline{1}, a_{\frac{1}{2}}, b_{\frac{1}{2}}, \underline{\frac{1}{2}}\},$$
$$CHLFT(a^{\frac{1}{2}}) = \{\underline{0}, \underline{1}, a^{\frac{1}{2}}, b^{\frac{1}{2}}, \underline{\frac{1}{2}}\}.$$

Here

$$\delta \;=\; CHLFT(a_{\frac{1}{2}}) \cup CHLFT(a^{\frac{1}{2}}).$$

Note that (X, δ) is a finite completely homogeneous L-fuzzy topological space. But as you see here (X, δ) is not a principal completely homogeneous L-fuzzy topological space.

Let X be any set and $L = \{0, a, 1\}$ with the order $0 < a < 1$. Define $\delta = \{0, a\}^X \cup \{a, 1\}^X$. It is easy to prove that δ is a completely homogeneous L-fuzzy topology on X. Any F-lattice $L \neq \{0, 1\}$ contains a sublattice isomorphic to $L = \{0, a, 1\}$. If $L \neq \{0, 1\}$, there exists non principal completely homogeneous L-fuzzy topology on X.

Remark 6.4.7. Let X be a finite set. Then every completely homogeneous L- fuzzy topology on X is principal if and only if $L = \{0, 1\}$.

6.5 Principal Completely Homogeneous L-fuzzy Topological Space when $L = \{0, \frac{1}{2}, 1\}$

Here we consider the principal completely homogeneous L-fuzzy topological space generated by an L- fuzzy set when the membership lattice $L = \{0, \frac{1}{2}, 1\}$ with the usual order.

Remark 6.5.1. Let X be an infinite set and $f \in L^X$ such that $\Lambda_f = \{0, \frac{1}{2}\}$. Then by the Corollary 6.4.4, we have that δ is a principal completely homogeneous L- fuzzy topology generated by f if and only if δ is one of the following.

1. $\{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ for all $x \in X\}$.
2. $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, \frac{1}{2}\}^X : |f^{-1}(0)| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Theorem 6.5.2. *Let X be an infinite set and $f \in L^X$ such that $\Lambda_f = \{1, \frac{1}{2}\}$. Then δ is a principal completely homogeneous L- fuzzy topology generated by f on X if and only if δ is one of the following.*

1. $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.

2. $\{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$ *where* $\aleph_0 \leq \alpha \leq |X|$ *and α is not a limit cardinal.*

Proof. Let $X_1 = f^{-1}(\frac{1}{2})$, $X_2 = f^{-1}(1)$ and $X_1 \cup X_2 = X$. Now we take the following three cases.

Case(1): $|X_1| = |X| = |X_2|$.

In this case we show that δ is of type (1). Since $|X_1| = |X| = |X_2|$, there exists a bijection h from X to X such that h maps X_1 onto X_2 and X_2 onto X_1. Then

$$(f \circ h)(x) = \begin{cases} 1 & \text{if } x \in X_1 \\ \frac{1}{2} & \text{if } x \in X_2. \end{cases}$$

Now choose two points $x_0 \in X_2$ and $y_0 \in X_1$ and define $h' : X \to X$ as follows.

$$h'(x) = \begin{cases} x_0 & \text{if } x = y_0 \\ y_0 & \text{if } x = x_0 \\ x & \text{otherwise.} \end{cases}$$

Consider $g = f \circ h \circ h'$. Since δ is a completely homogeneous L-fuzzy topology on X, we get $g \in \delta$ and

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } x \in (X_2 \setminus \{x_0\}) \cup \{y_0\} \\ 1 & \text{if } x \in (X_1 \setminus \{y_0\}) \cup \{x_0\}. \end{cases}$$

Now let $f_{x_0} = g \wedge f$. Then $f_{x_0} \in \delta$ and

$$f_{x_0} = \begin{cases} 1 & \text{if } x = x_0 \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Thus for each $x \in X$ there is an L- fuzzy set f_x in δ such that f takes the value 1 at x and $\frac{1}{2}$ at all other points in X.

Let $\mu = \{f \in L^X : f(x) \geq \frac{1}{2}\}$. Now we claim that $\mu \subseteq \delta$. Consider $f \in \mu$. Let $Y = \{x \in X : f(x) = 1\}$. Then $f = \bigvee\limits_{x \in Y} f_x$ and hence $f \in \delta$. Thus we get $\delta = \mu$.

Case(2): $|X_1| = |X|$ and $|X_2| < |X|$.

Since (X, δ) is completely homogeneous, by applying Theorem 5.2.15, $\{g \in L^X : g \geq f \text{ and } \Lambda_g \subseteq \Lambda_f\} \subseteq \delta$. So any f in L^X such that $|X_1| = |X| = |X_2|$ are in δ. So by case (1), $\delta = \mu$.

Case(3): $|X_1| < |X|$ and $|X_2| = |X|$.

Since $\delta = CHLFT(f)$, $\{f \circ h : h \in S(X)\}$ is a subbase for δ. So any element g in δ is of the form $g = \bigvee\limits_{i \in I} (\bigwedge\limits_{j=1}^{n_i} f_{ij})$ where $f_{ij} = f \circ h_{ij}$, $h_{ij} \in S(X)$. Now for each $i \in I$, $(\bigwedge\limits_{j=1}^{n_i} f_{ij})^{-1}(\frac{1}{2}) = (f_{i1})^{-1}(\frac{1}{2}) \cup (f_{i2})^{-1}(\frac{1}{2}) \cup \ldots f_{in_i}^{-1}(\frac{1}{2})$. So we get $|(\bigwedge_{j=1}^{n_i} f_{ij})^{-1}(\frac{1}{2})| \leq |X_1|$.

Now

$$\begin{aligned} g^{-1}(\frac{1}{2}) &= (\bigvee_{i\in I}(\bigwedge_{j=1}^{n_i} f_{ij}))^{-1}(\frac{1}{2}) \\ &= \bigcap_{i\in I}(\bigwedge_{j=1}^{n_i} f_{ij})^{-1}(\frac{1}{2}) \\ &= \bigcap_{i\in I}\left(\bigcup_{j=1}^{n_i} f_{ij}^{-1}(\frac{1}{2})\right). \end{aligned}$$

and hence

$$\begin{aligned} |g^{-1}(\frac{1}{2})| &= |\bigcap_{i\in I}\left(\bigcup_{j=1}^{n_i} f_{ij}^{-1}(\frac{1}{2})\right)| \\ &\leq |\bigcup_{j=1}^{n_i} f_{ij}^{-1}(\frac{1}{2})| \\ &\leq |X_1|. \end{aligned}$$

Thus $\delta = \{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Conversely assume δ is one of the form

1. $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ for all $x \in X\}$,
2. $\{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Then it is easy to verify that these are the completely homogeneous L-fuzzy topologies on X. Hence the theorem. $\square$

Theorem 6.5.3. *Let X be an infinite set and $f \in L^X$ where $\Lambda_f = \{1, \frac{1}{2}, 0\}$. Then δ is a principal completely homogeneous L-fuzzy topological space generated by f on X if and only if δ is one of the following.*

1. *The discrete L-fuzzy topology L^X.*
2. $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < \alpha\}$.
3. $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| = |f^{-1}(\frac{1}{2})| < \alpha\}$.
4. $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < |f^{-1}(\frac{1}{2})| < \alpha\}$,

   where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Proof. Obviously the above L-fuzzy topologies are principal completely homogeneous. Conversely let $f \in L^X$, $X_1 = f^{-1}(0)$, $X_2 = f^{-1}(\frac{1}{2})$ and $X_3 = f^{-1}(1)$. If $|X_1| = |X|$, then by Theorem 6.4.1, it follows that $\delta = L^X$. Otherwise $|X_1| = \alpha < |X|$. In this case by Theorem 6.4.2, we have

$$|f^{-1}(0)| < \alpha \text{ for every } f \in \delta \setminus \{\underline{0}\} \tag{6.1}$$

where $\aleph_0 \leq \alpha \leq |X|$. Since $|X_1| < |X|$, $|X_2 \cup X_3| = |X|$.

Case(1): $|X_2| = |X|$ and $|X_3| \leq |X|$.

In this case, by Theorem 5.2.15, there exists an L- fuzzy set g in δ such that $|g^{-1}(\frac{1}{2})| = |g^{-1}(1)| = |X|$ and $X = g^{-1}(\frac{1}{2}) \cup g^{-1}(1)$. Then by Theorem 6.5.2, it follows that

$$\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2} \text{ for all } x \in X\} \subseteq \delta. \tag{6.2}$$

From Equations (6.1) and (6.2), we get $\delta = \{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$ and α is not a limit cardinal.

Case(2): $|X_2| < |X|$ and $|X_3| = |X|$.

Let $|X_1| < |X_2|$. In this case $\delta = \{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < |f^{-1}(\frac{1}{2})| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$. Suppose that $|X_1| = |X_2|$. Then $\delta = \{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| = |f^{-1}(\frac{1}{2})| < \alpha\}$ where $\aleph_0 < \alpha \leq |X|$ and α is not a limit cardinal.

If $|X_2| < |X_1|$, then by Theorem 5.2.15 there exists an L-fuzzy set g in δ such that $|g^{-1}(0)| < |g^{-1}(\frac{1}{2})|$. So in this case also we get $\delta = \{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < |f^{-1}(\frac{1}{2})| < \alpha\}$ where $\aleph_0 \leq \alpha \leq |X|$.

□

Thus we determined the principal completely homogeneous L- fuzzy topologies on an infinite set X when $L = \{0, \frac{1}{2}, 1\}$. We conclude this section by listing all the principal completely homogeneous L-fuzzy topologies on X when $L = \{0, \frac{1}{2}, 1\}$.

Corollary 6.5.4. *Let X be an infinite set. Then the only principal completely homogeneous L-topologies on X when $L = \{0, \frac{1}{2}, 1\}$ are the following.*

1. *The trivial L-fuzzy topology,* $\{\underline{0}, \underline{1}\}$.

2. $\{\underline{0}, \underline{\frac{1}{2}}, \underline{1}\}$.

3. *The discrete crisp topology,* 2^X.

4. *L-topologies of the form* $\{\chi_G : |X \setminus G| < \alpha\} \cup \{\underline{0}\}$.

5. $\{\underline{1}\} \cup \{f \in L^X : f(x) \leq \frac{1}{2}$ *for all* $x \in X\}$.

6. $\{\underline{0}, \underline{1}\} \cup \{f \in \{0, \frac{1}{2}\}^X : |f^{-1}(0)| < \alpha\}$.

7. $\{\underline{0}\} \cup \{f \in L^X : f(x) \geq \frac{1}{2}$ *for all* $x \in X\}$.

8. $\{\underline{0}\} \cup \{f \in \{1, \frac{1}{2}\}^X : |f^{-1}(\frac{1}{2})| < \alpha\}$.

9. *The discrete L-fuzzy topology* L^X.

10. $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| = |f^{-1}(\frac{1}{2})| < \alpha\}$.

11. $\{\underline{0}\} \cup \{f \in L^X : |f^{-1}(0)| < |f^{-1}(\frac{1}{2})| < \alpha\}$

    where $\aleph_0 \leq \alpha \leq |X|$ *and* α *is not a limit cardinal.*

Proof. Proof follows from Remark 6.5.1 and Theorems 6.5.2 and 6.5.3. $\square$

Chapter 7

Conclusion

In this thesis we studied the group of homeomorphisms of a topological space and the group of L- fuzzy homeomorphisms of an L-fuzzy topological space. One of the questions we tried to answer in this thesis is "Which permutation groups can be represented as the group of homeomorphisms of a topological space or the group of L-fuzzy homeomorphisms of an L-fuzzy topological space ". We determined the $t-$representability of some interesting permutation groups and an analogous study carried out in the case of L-fuzzy topological spaces. The properties of t-representable and L_f-representable permutation groups are investigated.

Further we continued the works of T. P. Johnson on completely homogeneous L-fuzzy topological spaces. We studied some properties of completely homogeneous L-fuzzy topological spaces. A characterization of completely homogeneous Alexandroff discrete L-fuzzy topological space is obtained. In addition to the study of completely homogeneous L-fuzzy topological spaces, the concept of principal completely homogeneous $L-$ fuzzy topological space is introduced and investigation on the properties of these spaces are conducted.

7.1 Further Scope of Research

Here we describe some open problems that are to be solved.

Some of the problems studied in this thesis has obtained only a partial solution. We have proved several results on finite permutation groups, but in the case of infinite permutation groups analogous problems remain unsolved. For example, we proved that the direct sum of finite $t-$representable permutation group is $t-$representable. But the condition under which the direct sum of infinite $t-$representable permutation groups become $t-$representable is not yet obtained. We determined the $t-$representability of finite transitive permutation groups, but we are not able to determine the $t-$representability of all infinite transitive permutation groups. We determined t-representability of some cyclic permutation groups. Characterization of t-representable cyclic permutation groups is an open problem.

We studied the t-representability of transitive and maximal subgroups of symmetric group. The L_f-representability of these permutation group remain unsolved

We proved a characterization theorem of completely homogeneous Alexandroff discrete L-fuzzy topological spaces when L is a complete chain, but we are not able to give a complete characterization for a completely homogeneous L-

topological space. We characterized principal completely homogeneous L-fuzzy topologies when $L = \{0, \frac{1}{2}, 1\}$ and the characterization of the same in the case of general F-lattice is still open.

Bibliography

[1] Abian A. : *The Theory of Sets and Transfinite Arithmetic*, W. B. Saunders Company, London, (1965).

[2] Andima S. J. and Thron W. J. : *Order Induced Topological Properties*, Pacific Journal of Mathematics, Vol-75, No. 2, (1978).

[3] Al Gour S. : *Local Homogeneity in Fuzzy Topological Spaces*, International Journal of Mathematics and Mathematical Sciences, $1-14$, (2006).

[4] Babusundar. S. : *Some Lattice Problems in Fuzzy Set Theory and Fuzzy Topology*, Thesis for Ph.D. Degree, Cochin University of Science and Technology, (1989).

[5] Ball R. W. : *Maximal Subgroups of Symmetric Groups*, Transactions of American Mathematical Society, 121, 393-407, (1966).

[6] Baer R. : *Die Kompositionsreihe der Gruppe aller eineindeutigen Abbildungen einer unendlichen Menge auf sich*, Studia Mathematica, Vol-5, No.1,

15-17, (1934).

[7] Brazil M., Covington J., Penttila T., Praeger C. E. and Woods A. R. : *Maximal Subgroups of Infinite Symmetric Groups*, Proceedings of the London Mathematical Soceity , Vol-68, No. 3, 77-111, (1994).

[8] Chang C. L. : *Fuzzy Topological Space*, The Journal of Mathematical Analysis and Applications, Vol-24, 182-190, (1968).

[9] De Groot J. , Wille R. J. : *Rigid Continua and Topological Group Pictures*, Archiv der Mathematik, Vol-9, 441-446, (1958).

[10] De Groot J. : *Groups Represented by Homeomorphism Groups I*, Mathematische Annalen, Vol-138, 80-102, (1959).

[11] Devlin K. : *The Joy of Sets: Fundamentals of Contemporary Set Theory*, Springer-Verlag, New York, (1993).

[12] Dixon J. D. , Mortimer B. : *Permutation Groups*, Graduate Text in Mathematics, Springer-Verlag, New York, (1996).

[13] Fraleigh J. B. : *A First Course in Abstract Algebra*, Fifth Edition, Addison Wesley Longman Inc., (1994).

[14] Ginsburg J. :*A Structure Theorem in Finite Topology*, Canadian Mathematical Bulletin, Vol-26, No.1, 121-122, (1983).

[15] Gouguen J. A. : *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, Vol-18, 145-174, (1967).

[16] Johnson T. P. : *Some Problems on Lattices of Fuzzy Topologies and Related Topics* , Thesis for Ph. D. Degree, Cochin University of Science and Technology, (1990).

[17] Johnson T. P. : *The Group of Fuzzy Homeomorphisms*, Fuzzy sets and Systems, Vol-45, 355-358, (1992).

[18] Johnson T. P. : *Some Problems on Lattice of Fuzzy Topologies*, Proceedings of the National Seminar on Fuzzy Mathematics and Applications, Department of Mathematics, Union Christian College, Alwaye, India, 85-87, (1999).

[19] Johnson T. P. : *The Group of L-fuzzy Homeomorphisms and the Group of Permutations on the Ground Set*, Journal of Mathematical Analysis and Applications, Vol-245, 423-429, (2000).

[20] Jose A. and Mathew S. C. : *Strongly Homogeneous Fuzzy Topological Spaces*, International Journal of Fuzzy Mathematics and Systems, Vol-3, 263–271, (2011).

[21] Jose V. K. : *A Study on Lattice of L-topologies,* Thesis for Ph. D. Degree, Mahathma Gandhi University, (2009).

[22] Jose V. K. and Johnson T. P. : *Complete Homogeneity and Reversibility in L-topological Spaces*, Far East Journal of Mathematical Sciences, Vol-16, 357-362, (2005).

[23] Joshi K. D. : *Introduction to General Topology*, New Age International (P) Limited, (2003).

[24] Kannan V. and Rajagopalan M. : *Constructions and Applications of Rigid Spaces I*, Advances in Mathematics, Vol-29, No.1, 89-130, (July 1978).

[25] Kannan V. and Rajagopalan M. : *Constructions and Applications of Rigid Spaces II*, American Journal of Mathematics, Vol-100, No.6. , 1139-1172, (Dec. 1978).

[26] Kannan V. and Rajagopalan M. : *Constructions and Applications of Rigid Spaces III*, Canadian Journal of Mathematics, Vol-30, 926-932, (1968).

[27] Kannan V. and Ramachandran P. T. : *Heriditariy Homogeneous Spaces*, Proceedings of International Seminar on Recent Trends in Topology and its Applications, St. Joseph's College Irinjalakuda, 48-52, (2009).

[28] Klein A. J. : α *- Closure in Fuzzy Topology,* Rocky Mountain Journal of Mathematics, Vol- 11, 553-560, (1981).

[29] Larson R. E. : *Minimum and Maximum Topological Spaces*, Bulletin De L' Academie Polonaise des Sciences, Vol. XVIII, No.12, 707-710, (1970).

[30] Liu Y. M. and Luo M. K. : *Fuzzy Topology*, World Scientific, Singapore, (1997).

[31] Macpherson H. D. and Neumann P. M. : *Subgroups of Infinite Symmetric Groups*, Journal of the London Mathematical Society, Vol-42, No. 2, 64-84, (1990).

[32] Meenaxi B., Dugald M., Rognvaldur G. M. and Peter M.N. : *Notes on Infinite Permutation Groups*, Hindusthan Book Agency, (1997).

[33] Nakaoda F. and Oda N. : *Some Applications of Minimal Open Sets*, International Journal of Mathematics and Mathematical Sciences, Vol-27, No. 8, 471-476, (2001).

[34] Ramachandran P. T. : *Some Problems in Set Topology Relating Group of Homeomorphisms and Order*, Thesis for Ph. D. Degree, Cochin University of Science and Technology, (1985).

[35] Ramachandran P. T. : *Groups of Homeomorphisms and Normal Subgroups of the Group of Permutations*, International Journal of Mathematics and Mathematical Sciences, Vol-14, No.3, 475-480, (1991).

[36] Ramachandran P. T. : *A Note on the Group of L- fuzzy Homeomorphisms*, Proceedings of the International Conference on Analysis and Applications, Irinjalakkuda, 166-175(2000).

[37] Ramachandran P.T. : *The Group of Homeomorphisms and the L-fuzzy homeomorphisms*, Proceedings of the National Conference on Mathematical Modelling, Kottayam, India,117-121 (2002).

[38] Richman F. : *Maximal Subgroups of Infinite Symmetric Groups*, Canadian Mathematical Bulletin, Vol-10, 375-381, (1967).

[39] Sini P. and Ramachandran P. T. : *On the Group of Homeomorphisms*, Bulletin of KMA, Vol-9, No.1, 55-63, (2012).

[40] Sini P. : *On L_f-representability of Permutation Groups*, Proceedings of International Conference IC-AMMN 2016, FISAT, Angamaly, 93-96, (2016).

[41] Sini P. and Ramachandran P. T. : *On $t-$representability of Cyclic Subgroups of Symmetric Group- I*, International Journal of Pure and Applied Mathematics, Vol-106, No.3, 851-857, (2016).

[42] Sini P., Dhanya P. M. and Ramachandran P. T. : *On Completely Homogeneous L-topological Spaces*, Annals of Fuzzy Mathematics and Informatics, Vol. 12, No. 5, 703-718, (2016).

[43] Sreeja M. :*Some Problems Related to Homogeneity in Topological Spaces*, Thesis for Ph. D Degree, University of Calicut, (2013).

[44] Stoll R. R. : *Set Theory and Logic*, Dover Publications Inc., New York, (1961).

[45] Willard S. : *General Topology*, Dover Publications, INC. New York, (2004).

[46] Zadeh L. A. : *Fuzzy sets*, Information and Control, California, Vol.8, 338-353, (1965).

Appendix

List of Publications

1. Sini P. and Ramachandran P. T. : *On the Group of Homeomorphisms*, Bulletin of KMA, Volume 9, No. 1, 55-63 (2012) ISSN: 0973-2721.

2. Dhanya P. M and Sini P. : *On Homogeneous Generalized Topological Spaces*, Proceeding of the International Conference ICAMMN-16, FISAT, Angamally (2016).

3. Sini P. : *On L_f-representability of Permutation Groups*, Proceedings of International Conference IC-AMMN 2016, FISAT, Angamaly, (2016).

4. Sini P. and Ramachandran P. T. : *On $t-$representability of Cyclic Subgroups of Symmetric Group- I*, International Journal of Pure and Applied Mathematics, Volume 106, No.3, 851-857 (2016) ISSN: 1311-8080(Print version), ISSN: 1314-3395(Online version) doi: 10.12732/ijpam.v.106.13.11.

5. Kavitha T., Sini P. and Ramachandran P. T. : *On the Group of Closure Isomorphisms*, Journal of Advanced Studies in Topology, 7(3)132-136 (2016) ISSN: 2090-8288, doi: 10.20454/jast.2016.1061.

6. Sini P., Dhanya P. M. and Ramachandran P. T. : *On Completely Homogeneous L-topological Spaces*, Annals of Fuzzy Mathematics and Informatics, 12(5) 703-718 (2016) ISSN: 2093-9310(Print version), ISSN: 2287-6235(Online version).

****