A STUDY OF DISTANCE-RELATED SETS IN PRODUCTS OF DIGRAPHS

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DECLARATION

I, Mary Shalet T J, hereby declare that the thesis entitled "A STUDY OF DISTANCE-RELATED SETS IN PRODUCTS OF DIGRAPHS" is a bonafide record of research done by me under the supervision of Dr. G N Prasanth, Assistant Professor, Department of Mathematics, Government College Chittur and this thesis or any part thereof has not been submitted by me for the award of any other degree, diploma, title or recognition before.

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List of Symbols

$2^{V(D)}$	power set of $V(D)$
$[n]$	$\{1, 2, \ldots, n\}$
$\emptyset$	empty set
$\exists$	there exists
$\forall$	for every
$\implies$	implies
$\iff$	if and only if
$\in$	is an element of
$\notin$	is not an element of
$\overrightarrow{C_n}$	directed cycle on n vertices
$\overrightarrow{d}(u, v)$	directed distance between vertices u and v
$\overrightarrow{P_n}$	directed path on n vertices
$\subseteq$	is a subset of
$\widehat{=}$	supposed to be equal to, but not
$\xi_D(x)$	dicycle distance of vertex x in digraph D
C_n	cycle on n vertices
$d(u, v)$	distance between vertices u and v
$D_1 \boxtimes D_2$	strong product of digraphs D_1 and D_2
$D_1 \circ D_2$	lexicographic product of digraphs D_1 and D_2
$D_1 \square D_2$	Cartesian product of digraphs D_1 and D_2
$D_1 \times D_2$	direct product of digraphs D_1 and D_2

$E(D)$	edge set of digraph D
$I_D[u,v]$	geodetic interval between vertices u and v in digraph D
$I[S]$	geodetic closure of S
K_n	complete graph with n vertices
$\mathrm{m}\partial(D)$	m-boundary of digraph D
$\mathrm{mCen}(D)$	m-center of digraph D
$\mathrm{mCt}(D)$	m-contour of digraph D
$\mathrm{md}(u,v)$	m-distance between vertices u and v
$\mathrm{mdiam}(D)$	m-diameter of digraph D
$\mathrm{mecc}(v)$	m-eccentricity of vertex v
$\mathrm{mrad}(D)$	m-radius of digraph D
$\mathrm{mEcc}(D)$	m-eccentricity of digraph D
$\mathrm{mEcc}(W)$	m-eccentricity of a subset W of vertex set of D
$\mathrm{mPer}(D)$	m-periphery of digraph D
$N_D(v)$	open neighborhood of vertex v in digraph D
$N_D[v]$	closed neighborhood of vertex v in digraph D
P_n	path on n vertices
$V(D)$	vertex set of digraph D
$V_1 \times V_2$	Cartesian product of vertex sets V_1 and V_2

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Chapter 1

Introduction

1.1 Background of the Thesis

The theory of digraphs has undergone rapid development in recent times. The study of various graph theoretical parameters in products of digraphs becomes significant in this context, as many of the large digraphs turn out to be the product of two or more smaller digraphs. It is not very surprising that digraphs have immense applications in several areas of science and sociology.

A network is a simplified representation of a *system* which captures only the basics of connection patterns [56]. It represents the interactions between the parts of a system. Most common examples of networks in the modern era are the internet and the World Wide Web. Internet is a network in which a collection of computers is linked by wired or wireless data connections. The World Wide Web is a network of information stored on web pages that are connected by hyperlinks which are purely software constructs.

An undirected network can be represented using vertices and edges connecting the vertices; that is, a graph. A directed network is a network in which each edge has a direction, pointing from one vertex to another. They can be represented as digraphs. The World Wide Web is an example for a directed network. The connections in the Web are directed; that is, from a link given on page A, one can arrive at page B, but it is not necessary that page B contains a link back to A.

Sociology can be considered as the field which initiated the empirical study of networks. The concepts of pattern and communication were used in sociology even before 1950, see [5]. Many of the mathematical tools which are now used to analyse networks owes its origin to the study of social networks. Social networks are a collection of people or organisations and connections are the social interactions between them. Facebook and LinkedIn are some examples of online social networking services.

The study of one-way problem was initiated by Robbins in the article [60]. It deals with the problem of arranging one-way traffic in a city. For road-traffic networks, the advantages of a one-way street network over a two-way street pattern was presented by Stemley in the article [66].

Another scientific field in which networks find their application is biology. In the nervous system, the neurons are connected in intricate communication networks so as to convey sensory information from sensory neurons to the central nervous system and to convey commands from the central nervous system to the various organs [50]. Ecological food webs also form directed biological networks as certain species predate others in a unidirectional way [67].

Other best known networks are telephone networks, peer-to-peer networks, collaboration networks, and disease transmission networks.

Different types of graph products are defined for which the vertex set is the Cartesian product of the vertex sets of its factors. These products differ only in the definition of the adjacency between the vertices. Digraph products can be defined analogous to graph products. In either case, only twenty of the above-mentioned products satisfy the associative property [36]. A study of products of digraphs was initiated by Hammack, Imrich, and Klavžar in 'Handbook of Product Graphs', and an extensive study was conducted by Hammack in 'Digraphs Products' [37]. The terminology and notation that we use in this thesis follow mainly the articles [36] and [37] and the basic graph theory terminology and notation we use here are based on West, D.B., 2001. Introduction to graph theory (Vol. 2) [71]. Note that the Cartesian product, the strong product, the lexicographic product, and the direct product are termed as the four standard graph (digraph) products.

In 1960, Gert Sabidussi introduced the Cartesian product and the strong product of graphs [63]. The connectedness of Cartesian product was discussed by Harary and Trauth in [41]. The direct product (or tensor product) of graphs was introduced by Weichsel, and he characterized connectedness in the direct product of two graphs [70]. Weichsel called it the Kronecker product, because the adjacency matrix of the product is the Kronecker product of the adjacency matrices of the factors. The lexicographic product of graphs was introduced as the composition of graphs by Harary [38].

The direct product was extended to digraphs by Mc Andrew, who called it 'the product' [51]. He gave the number of components in the product of 'k' strongly connected digraphs. The lexicographic product of digraphs was introduced by J W Moon [52].

The distance notion plays a key role in the theory of graphs and digraphs. The problem of sending a message from one node to another in a communication network using the least number of intermediate nodes is equivalent to the problem of finding a shortest path in a graph or a digraph.

The distance between two vertices of a graph is defined as the number of edges in the shortest path between the vertices. If there is no such path, then the distance is defined to be infinity. If two vertices of a graph are joined by an edge, then they are said to be neighbors of each other. A graph is said to be connected if any two vertices of the graph are joined by a path. It can be seen that 'the distance between two vertices' satisfies all the properties of a metric in the case of connected graphs.

The number of vertices in a graph (digraph) is called the order of the graph (digraph). The number of edges in a graph (digraph) is called the size of the graph (digraph).

The eccentricity of a vertex of a graph is the maximum distance between the vertex and any other vertex of the graph. The minimum of the eccentricities of vertices is called the radius, and the maximum is called the diameter of the graph. The eccentricities of the vertices in a connected graph are termed as the associated number of points of the graph by Harary in [40].

In the place of edges of a graph, a digraph has directed edges and so the paths involved are directed paths. A digraph is strongly connected if any two vertices of the digraph are joined by directed paths in either direction. The directed distance from a vertex u to a vertex v in a digraph, denoted by $\overrightarrow{d}(u,v)$ is defined as the length of the shortest directed path from u to v. As this is not generally the same as $\overrightarrow{d}(v,u)$, the directed distance in a strongly connected digraph is not a metric. In Fig. 1.1, $\overrightarrow{d}(u,u_1) = 1$ whereas $\overrightarrow{d}(u_1,u) = 4$.

Even though the eccentricity of a vertex, the radius and the diameter of a digraph D can be defined with respect to directed distance, the inequality $\operatorname{diam}(D) \leq 2\operatorname{rad}(D)$ is not satisfied. This was proved by Chartrand and Tian in [24] and so they introduced two metrics in digraphs, namely, the maximum distance and the sum distance in strongly connected digraphs. As the name indicates, the maximum distance $\operatorname{md}(u,v)$ between two vertices u and v is the maximum of the directed distances in either direction. The sum distance $\operatorname{sd}(u,v)$ between the vertices u and v is the sum of the directed distances in either direction.

Later, Chartrand et al. defined another metric called the strong distance [21] for strongly connected digraphs. The strong distance between two vertices u and v in a digraph D is defined as the size of the smallest strongly connected subdigraph of D

containing both u and v. Consider the digraph in Fig. 1.1 which is taken from [27]. Here, the smallest strongly connected subdigraph containing both u and v is the union of the directed paths P and Q, where $P = u, u_1, u_2, u_3, u_4, v$, $Q = v, u_1, u_2, u_3, u_4, u$, and hence is of size 7. But, P is not the shortest $u - v$ path in the digraph and Q is not the shortest $v - u$ path. Since the first priority in a communication system is efficient communication between the nodes in either direction, strong distance is not a suitable metric in such networks.

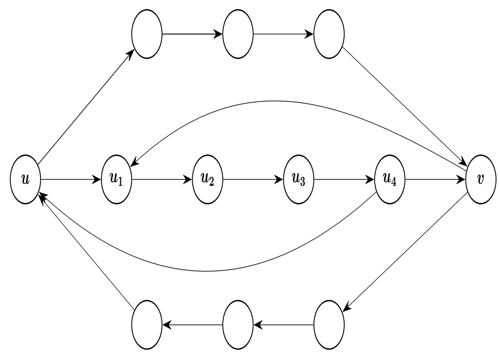

Figure 1.1: Example for a directed network

The metrics md and sd find applications in different types of communication networks depending on the nature of data flow. In digraphs involving communication in either direction independent of each other, the metric md is the right choice. This is because minimizing md results in minimizing the distance between the vertices of the digraph in either direction. Biological neural networks is an example of such a communication network. The sensory neurons convey electrical signals from the input such as the eyes or nerve endings in the hand, the brain processes it and produces output and the motor neurons carry the reverse signal such as reacting to light or heat.

Studies in metric graph theory are motivated by the concepts such as boundary sets in geometry. Convexity, which is a fundamental concept in geometry, is extended to metric spaces. Connected graphs form metric spaces using the metric 'the length of the shortest path' in the set of vertices of the graph. A vertex subset S of a graph G is said to be convex, if it contains the vertices in all the shortest paths connecting any pair of vertices in S. An important tool for the study of metric properties of connected graphs is the interval function. A characterization of the interval function is given by Mulder in [54].

A set S of vertices of a graph G is a geodetic set if every vertex of G lies in some shortest $u-v$ path for $u, v \in S$. The cardinality of a minimum geodetic set in G is the geodetic number of G. The geodetic number of a graph was introduced by Harary et al. [39]. The geodetic number of the Cartesian product and the lexicographic product of graphs was studied in [12] and [13], respectively.

Several distance-related sets are defined in graphs. These definitions can be extended to digraphs. The center is the set of vertices which are close to all of the remaining vertices of the graph; that is, the set of vertices with minimum eccentricity. In social networks, a central vertex can be identified as a recognized leader.

The periphery, contour, eccentricity, and boundary sets are four types of distance-related boundary-type sets of a graph G. This term was first coined by Caćeres et al. in [18]. They are denoted by $\mathrm{Per}(G)$, $\mathrm{Ct}(G)$, $\mathrm{Ecc}(G)$ and $\partial(G)$, respectively. The peripheral vertices of a graph are the vertices which have the maximum eccentricity in the graph. A vertex u is said to be a contour vertex of the graph if the eccentricity of every neighbor does not exceed the eccentricity of u. The eccentricity and the boundary sets of a graph consist of the eccentric vertices and the boundary vertices of the graph, respectively. A vertex v is an eccentric vertex of a graph if there is a vertex u in the graph such that eccentricity of u is the distance from u to v. A vertex v is a boundary vertex of a graph G if there is a vertex u in the graph such that no neighbor of v is further away from u than v.

In [24], Chartrand defined the m-eccentricity of a vertex and the m-radius, m-diameter, m-center, and m-periphery of a digraph with respect to the metric 'md'. All the other boundary-type sets in digraphs that we have listed above can be defined similarly.

It is very difficult to identify the various distance-related sets in large digraphs. If it is possible to factorize the digraph into smaller digraphs, it will be easy to identify these sets, provided they are related to the corresponding sets in the factors. A graph is said to be prime with respect to a given graph product if it is nontrivial and cannot be represented as the product of two nontrivial graphs. The problem of prime factorization of digraphs under various products play a significant role in the study of digraphs.

Of the four standard graph (digraph) products, the direct product presents several issues related to connectivity. The direct product of two connected graphs is connected only if at least one of them contains an odd cycle [70]. As the study is based on distance between vertices, only the products which are strongly connected are of interest. Hence the direct product is not considered in this study of distance-related sets.

1.2 Historical Review

The evolution of the theory of digraphs can be traced back to the article on one-way problem in the year 1939 by Robbins [60]. Some of the earliest works in the field of digraphs are [4, 7, 8, 61, 62, 65]. These works dealt with tournaments, planar digraphs, and acyclic digraphs. During 1980's, the abstract algebra approach applied to graphs was extended to digraphs. The article 'Distance transitive digraphs' by C W Lam [47] is one of the earliest works of this sort. The study of Hamiltonian paths and cycles in Cayley digraphs was the most significant study using the group theoretic approach, see [26, 72]. Later, many research articles which apply the linear algebra approach on digraphs were published [1, 14, 29, 31, 43, 45, 68]. During this period many research was also done in the fields like domination, decomposition, and factorization in digraphs [3, 9, 28, 34, 44, 57, 69, 76]. Many articles dealing with products of digraphs have come out in the recent years. Some of these articles are related to determining the chromatic number and domination number in various products of digraphs, see [48, 49, 64]. There are many articles in which the Hamiltonian and related properties of various products of digraphs were the topic of study, see for instance [11, 58, 73].

Alex Bavelas introduced the concept of centrality index in social networks [5]. Several studies have been done related to central vertices of a graph [16, 46, 59]. The peripheral vertices and eccentric vertices of a graph are part of the literature before the publication of the book 'Distance in Graphs' [15]. The peripheral vertices were studied by Bielak and Syslo in [10], and eccentric vertices by Chartrand et al. in [23]. In [20], Chartrand et al. defined a new boundary-type set called the boundary of a graph and studied the relationship between the periphery, eccentricity, and boundary. The contour set was defined by Ca̒ceres et al. in [19]. All these sets find applications in problems involving facility location [22], and rebuilding in graphs [17].

Sabidussi showed that every connected graph G has a prime factorization under Cartesian multiplication that is unique up to the order and isomorphisms of the factors [63]. After this, some faster factorization algorithms for graphs were developed.

Afterwards, Feigenbaum proved that directed graphs have unique prime factorizations under Cartesian multiplication and that the prime factorizations of weakly connected digraphs can be found in polynomial time [32]. This was improved to a linear time approach by Crespelle et al. in [25].

Directed graphs have unique prime factorizations under strong product also. Marc Hellmuth and Tilen Marc developed a polynomial-time algorithm for determining the prime factor decomposition of strong product of digraphs [42].

Digraphs can be factored as prime factors with respect to the direct product and lexicographic product also, but the prime factorizations under the direct product and lexicographic product are not unique [36].

1.3 Towards the Thesis

In the article [17], Caćeres et al. proved a proposition which presented the results related to the four boundary-type sets of two connected graphs. The question which arose was whether these results hold in the case of strongly connected digraphs with respect to the metric md. It was found that in general, the results did not hold. But, if the digraph satisfies an additional property which is defined as two-sided eccentricity property, then two of the results in the proposition hold.

In [13], Brešar et al. proved the relationship between boundary-type sets of Cartesian product of two graphs and that of their factor graphs. On examining the boundary-type sets of the Cartesian product of two digraphs with respect to the metric md, it turned out that the results related to the periphery and the contour hold if one of the digraphs satisfies the two-sided eccentricity property. Also, the center of the Cartesian product of two digraphs with respect to the metric md was studied. Further, the boundary-type sets and the center of the strong product and the lexicographic product of digraphs were investigated.

Now we present a real life situation involving directed distance between vertices. The design and analysis of road transportation networks has an inevitable role in city planning. In the representation of road networks using graphs or digraphs, the vertices usually represent road intersections, and the edges represent roads [56]. When one-way traffic is introduced in a road network, the edges between the vertices are all directed edges and can be represented only using a digraph D. The necessary condition in planning one-way routes is that each vertex must be reachable from every other vertex. Thus it is clear that there will be a directed path from every vertex to every other vertex of the digraph. The problem of finding the shortest path between two vertices u and v in the digraph D is different from that of graphs. This difference is due to the lack of symmetry in the case of directed distance, that is, $\overrightarrow{d}(u, v) \neq \overrightarrow{d}(v, u)$ for two vertices u and v in $V(D)$. The distance $\mathrm{md}(u, v) = \max\{\overrightarrow{d}(u, v), \overrightarrow{d}(v, u)\}$ gives the maximum of the directed distances between the vertices in either direction. Applying the metric md in such situations seems to be the right choice in the practical sense, because when we think of minimizing the distance between two places in a one-way setup, the directed distances between the vertices in both directions should be minimized.

Finding the maximum distance md between two vertices in large digraphs is a cumbersome task. We have seen in Section 1.2 that there are several algorithms to factorize large digraphs into smaller factor graphs. The results presented in this thesis can be applied to minimize the complexity in finding the distance md between two vertices in the three types of digraph products. The results regarding the boundary-type sets and the center of digraph products contribute to the analysis of road networks and thereby help to apply the remedial measures that ensure the most efficient road transportation system.

1.4 Gist of the Thesis

Distance is one of the most important concepts in Graph Theory. This thesis aims to study the five distance-related sets in the Cartesian, strong and lexicographic product of digraphs.

In the present world, we come across various networks; that is linkages connecting specific objects. The connections may be virtual or physical. The examples vary from acquaintances in social networks to signal transmission in communication networks. If the connections in a network are unidirectional, then the network can be represented using a digraph.

The directed distance $\overrightarrow{d}$ defined in a digraph is not symmetric and hence it is not a metric. The distance studied in this thesis is the metric 'maximum distance' between two vertices of a digraph, abbreviated as md. The results regarding the boundary-type sets and the center of the Cartesian, strong, and lexicographic product of digraphs with respect to the metric md are derived. The four boundary-type sets considered are the m-periphery, the m-contour, the m-eccentricity and the m-boundary.

The thesis is organized as follows. The basic definitions, terminology, and notation used in this thesis are presented in Chapter 2.

In Chapter 3, the geodetic interval and the four boundary-type sets of digraphs with respect to the metric md are dealt with and it is investigated whether the results corresponding to that of usual graphs obtained in the article [17] hold. The significance of the two-sided eccentricity property of digraphs is also presented.

The Cartesian product of digraphs is studied in Chapter 4. The four boundary-type sets and the center of the Cartesian product of two strongly connected digraphs are investigated. It is derived that if one of the digraphs satisfies the two-sided eccentricity property, then the results for the periphery and the contour sets of the Cartesian product of two strongly connected digraphs are the same as that in the case of graphs. The results for the periphery, the contour, and the center are extended to a finite number of digraphs when all except one of the digraphs have the two-sided eccentricity property. Also, the results for the four boundary-type sets of a finite number of digraphs is derived when all except one of the digraphs are either directed cycles or symmetric digraphs.

In Chapter 5, the strong product of digraphs is studied and the results related to the four boundary-type sets and the center for the strong product of two arbitrary strongly connected digraphs are derived. The results for the center and periphery of the strong product are extended to the case of a finite number of arbitrary digraphs.

The lexicographic product of digraphs is the subject of study in Chapter 6. The concept of the class of digraphs which satisfy the 'dicycle distance less than eccentricity' property or in short the DDLE property is introduced. The dicycle distance of a vertex is the length of the shortest dicycle containing the vertex. A digraph is said to satisfy the DDLE property, if the dicycle distance of every vertex is less than its eccentricity. In Section 6.3, the boundary-type sets except the boundary of the lexicographic product $D_1 \circ D_2$, when D_1 is a DDLE digraph are derived, and the four boundary-type sets of the lexicographic product $D_1 \circ D_2$ are derived for the cases when D_1 is a dicycle on n vertices and a symmetric digraph. In Section 6.4, further concepts are introduced to give the expressions for the center and the periphery of any two strongly connected digraphs that cover all the possible cases. The DDLE property is defined for each vertex and similarly, the 'dicycle distance equal to eccentricity' property for a vertex is defined. Based on these definitions, the expressions for the center of lexicographic product of any two strong digraphs are obtained, depending on the radii of the two digraphs. Also, the expressions for the periphery of the lexicographic product of two strong digraphs are obtained, depending on the diameters of the two digraphs.

Finally, in Chapter 7, the concluding remarks and some problems for further study are given.

Chapter 2

Preliminaries

2.1 Introduction

The purpose of this chapter is to list the definitions, terminology, and notation that are used in this thesis. Most of the terms used are part of the standard graph theoretic terminology, and a few terms will be introduced later as and when the situation arises.

2.2 Graph Theory Terminology

A *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its *end-vertices*. Each vertex is indicated by a point, and each edge by a line joining the points representing ends. A graph is finite if its vertex set and edge set are finite. Every graph mentioned in this thesis is finite. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. It is written as $H \subseteq G$ and is read as G contains H.

A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. When u and v are the endpoints of an edge, they are adjacent and are neighbors. The *neighborhood* of v, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v.

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The path and cycle with n vertices are denoted by P_n and C_n, respectively. A *complete* *graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n.

A graph G is *connected* if for each pair of vertices u and v in G, there is at least one path joining u and v; otherwise, G is disconnected. A *maximal connected subgraph* of G is a subgraph that is connected and is not contained in any other connected subgraph of G. The *components* of a graph G are its maximal connected subgraphs. A component (or graph) is *trivial* if it has no edges; otherwise it is nontrivial. If vertex v is an endpoint of edge e, then v and e are incident.

The *degree* of a vertex v in a graph G, written $deg_G(v)$ or $deg(v)$, is the number of edges incident to v, except that each loop at v counts twice. An *isolated vertex* is a vertex of degree 0.

2.3 Digraph Terminology

The main definitions concerning digraphs used in the thesis are in line with the terminology used in [2].

Definition 2.3.1. A *directed graph* or a *digraph* $D = (V, E)$ or D consists of a non-empty finite set $V(D)$ called the vertex set of D and a finite set $E(D)$ of ordered pairs of distinct elements of $V(D)$ called the edge set of D. The elements of $V(D)$ and $E(D)$ are called vertices and edges, respectively. For an edge (u, v), the first vertex u of the ordered pair is the *tail* of the edge and the second vertex v is the *head*; together they are the end-vertices.

This definition of a digraph does not allow loops (edges whose head and tail coincide) or multiple edges (edges with same tail and same head). If (u, v) is an edge with tail u and head v, then u is said to be adjacent to v and v is said to be adjacent from u.

Definition 2.3.2. A *subdigraph* of a digraph D is a graph D' such that $V(D') \subseteq V(D)$ and $E(D') \subseteq E(D)$ and the assignment of endpoints to edges in D' is the same as in D. It is written as $D' \subseteq D$ and is read as D contains D'.

For $n \in N$, the notation $[n]$ indicates the set $\{1, \dots, n\}$.

Definition 2.3.3. A *directed walk* is an alternating sequence

$$W = x_1 a_1 x_2 a_2 x_3 \dots x_{k-1} a_{k-1} x_k$$

of vertices x_i and arcs a_i from D such that x_i and x_{i+1} are the tail and head of a_i, respectively, for every $i \in [k-1]$.

If the vertices of the directed walk W are distinct, then W is a *directed path*.

If the vertices $x_1, x_2, \ldots, x_{k-1}$ are distinct, $k \geq 3$ and $x_1 = x_k$, then W is a *directed cycle* (*dicycle*).

Definition 2.3.4. A digraph is *strongly connected* or *strong* if, for each ordered pair (u, v) of vertices, there is a directed path from u to v.

Definition 2.3.5. The underlying graph U of a digraph D is the simple graph with vertex set $V(D)$ and the edge $xy \in E(U)$ if and only if either $(x, y) \in E(D)$ or $(y, x) \in E(D)$.

Definition 2.3.6. A digraph is said to be *weakly connected* if its underlying graph is connected.

Definition 2.3.7. A digraph $D = (V, E)$ is said to be a symmetric digraph if for every edge $(x, y) \in E(D)$ there is also an edge $(y, x) \in E(D)$.

Definition 2.3.8. The *length* of a path is defined to be the number of edges in the path.

Definition 2.3.9. Let u and v be vertices of a strongly connected digraph D. A shortest directed $u - v$ path is called a *directed u-v geodesic*.

Definition 2.3.10. The number of edges in a directed $u - v$ geodesic is called the *directed distance* $\overrightarrow{d}(u, v)$. If there is no such geodesic, then $\overrightarrow{d}(u, v)$ is taken to be infinity.

The directed distance is positive and satisfies the triangle inequality, but it is not a metric because $\overrightarrow{d}(u, v) \neq \overrightarrow{d}(v, u)$ in general, and so it does not satisfy the symmetry property. The metric 'maximum distance', abbreviated as md was introduced by Chartrand and Tian in [24]. It is also known as m-distance.

Definition 2.3.11. In a strongly connected digraph, the metric *maximum distance* is defined by $\mathrm{md}(u, v) = \max\{\overrightarrow{d}(u, v), \overrightarrow{d}(v, u)\}$.

It can be seen that md is a metric. The proof is from [24]. By definition, md is positive and symmetric. Let $u, v, w \in V(D)$. Without loss of generality, suppose that

$\max\{\overrightarrow{d}(u,v), \overrightarrow{d}(v,u)\} = \overrightarrow{d}(u,v)$. Then

$$\begin{aligned}
\mathrm{md}(u,v) &= \max\{\overrightarrow{d}(u,v), \overrightarrow{d}(v,u)\} \\
&= \overrightarrow{d}(u,v) \\
&\leq \overrightarrow{d}(u,w) + \overrightarrow{d}(w,v) \\
&\leq \max\{\overrightarrow{d}(u,w), \overrightarrow{d}(w,u)\} + \max\{\overrightarrow{d}(w,v), \overrightarrow{d}(v,w)\} \\
&= \mathrm{md}(u,w) + \mathrm{md}(w,v).
\end{aligned}$$

Thus the distance md satisfies the triangle inequality and so is a metric in a strongly connected digraph. For a graph G, md is the usual distance between the vertices of G. Every edge uv in a graph corresponds to two directed edges (u,v) and (v,u) in a digraph. Hence, a graph is a symmetric digraph.

The interval function of a connected graph was defined and extensively studied by Mulder [53]. The interval function of a connected graph G is the mapping $I_G : V(G) \times V(G) \to 2^{V(G)}$ defined by $I_G[u,v] = \{w \in V(G) : d(u,w) + d(w,v) = d(u,v)\}$ for all $u,v \in V(G)$. Each set $I_G[u,v]$ is called a geodetic interval or interval in G. The directed geodetic relation of a strongly connected digraph was defined by Nebeskỳ in [55]. In this article, Nebeskỳ rephrased the definition of interval function of a graph by introducing a ternary relation Γ_G termed as geodetic relation of G on $V(G)$ which is defined as follows:

$\Gamma_G(u,w,v)$ if and only if $d(u,w) + d(w,v) = d(u,v)$ for all $u,v,w \in V(D)$. Thus, $I_G[u,v] = \{w \in V(G) : \Gamma_G(u,w,v)\}$. Hence, the directed geodetic relation of a strongly connected digraph is defined as follows:

Let D be a strong digraph, and let $V(D)$, $E(D)$ and d_D denote the vertex set of D, the edge set of D and the (directed) distance function of D, respectively. The directed geodetic relation of D is the ternary relation Γ_D^{dir} on $V(D)$ defined by

$\Gamma_D^{dir}(u,w,v)$ if and only if $d_D(u,w) + d_D(w,v) = d_D(u,v)$ for all $u,v,w \in V(D)$.

Let d denote any metric in a strong digraph. Analogous to the interval function of a graph, the interval function of a strong digraph D with respect to the metric d can be defined as the mapping $I_D : V(D) \times V(D) \to 2^{V(D)}$ defined by $I_D[u,v] = \{w \in V(D) : \Gamma(u,w,v)\}$ for all $u,v \in V(D)$. So we define the geodetic interval between two vertices u and v in a digraph D with respect to the metric md.

Definition 2.3.12. The *geodetic interval* between two vertices u and v in a digraph D denoted by $I_D[u,v]$ is defined as $I_D[u,v] = \{w \in V(D) : \mathrm{md}(u,w) + \mathrm{md}(w,v) = \mathrm{md}(u,v)\}$.

If the digraph D is clear from the context, $I_D[u,v]$ may be denoted by $I[u,v]$. Thus all the related concepts can be extended to digraphs.

Definition 2.3.13. For $S \subseteq V(D)$, the *geodetic closure* $I[S]$ of S is the union of all geodetic intervals $I[u,v]$ over all pairs $u, v \in S$. So $I[S] = \bigcup_{u,v \in S} I[u,v]$. Geodetic closure of S is also known as interval of S.

Definition 2.3.14. A subset S of vertices of a digraph D is called a *geodetic set* if $I[S] = V(D)$.

Definition 2.3.15. A subset S of vertices of a digraph D is called *convex* if $I[u,v] \subseteq S$ for any pair of vertices $u, v \in S$.

Definition 2.3.16. For a vertex v in a digraph $D = (V, E)$, the neighborhoods are defined as follows:
$N_D^+(v) = \{w \in V : (v,w) \in E\}$, $N_D^-(v) = \{u \in V : (u,v) \in E\}$.
The sets $N_D^+(v)$, $N_D^-(v)$, and $N_D(v) = N_D^+(v) \bigcup N_D^-(v)$ are called the out-neighborhood, in-neighborhood, and neighborhood of v. These neighborhoods are called open neighborhoods of v. The *closed neighborhood* of v, $N_D[v]$ is defined by $N_D[v] = N_D(v) \cup \{v\}$. If the digraph D is clear from the context, the open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[v]$, respectively.

The following definitions are from [24].

The *m-eccentricity* of a vertex $u \in V(D)$ denoted as $\mathrm{mecc}_D(u)$ is defined by $\mathrm{mecc}_D(u) = \max\ \{\mathrm{md}(u,v) | v \in V(D)\}$. If the digraph D is clear from the context, then $\mathrm{mecc}_D(u)$ is denoted as $\mathrm{mecc}(u)$.

Definition 2.3.17. The *m-radius*, $\mathrm{mrad}(D)$ of a digraph D is defined by
$\mathrm{mrad}(D) = \min\{\mathrm{mecc}(v) : v \in V(D)\}$.

Definition 2.3.18. The *m-diameter*, $\mathrm{mdiam}(D)$ of a digraph D is defined by
$\mathrm{mdiam}(D) = \max\{\mathrm{mecc}(v) : v \in V(D)\}$.

A vertex $v \in V(D)$ is called an *m-central vertex* of D if its m-eccentricity is the minimum over all vertices in D; that is, if the m-eccentricity of v is equal to the m-radius of D.

Definition 2.3.19. The *m-center*, $\mathrm{mCen}(D)$ of a strongly connected digraph D is the set of all of its m-central vertices;

$$\begin{aligned}\mathrm{mCen}(D) &= \{v \in V(D) : \mathrm{mecc}(v) \leq \mathrm{mecc}(u), \forall u \in V(D)\} \\ &= \{v \in V(D) : \mathrm{mecc}(v) = \mathrm{mrad}(D)\}.\end{aligned}$$

In a strongly connected digraph, the m-distance between every pair of vertices and the m-eccentricity of every vertex is finite.

Now the definitions for the boundary-type sets of a digraph D with respect to the metric 'maximum distance' are introduced. Most of the definitions are analogous to the definitions in [20] and the definition of m-contour is analogous to the definition of contour in [19].

Let D be a strong digraph and $u, v \in V(D)$. The vertex v is said to be an m-boundary vertex of u if no neighbor of v is further away from u than v.

A vertex $v \in V(D)$ is called an *m-boundary vertex* of D if it is the m-boundary vertex of some vertex $u \in V(D)$.

Definition 2.3.20. The *m-boundary*, $\mathrm{m}\partial(D)$ of a strong digraph D is the set of all of its m-boundary vertices;
$\mathrm{m}\partial(D) = \{v \in V(D) | \exists u \in V(D), \forall w \in N(v) : \mathrm{md}(u, w) \leq \mathrm{md}(u, v)\}$.

Given $u, v \in V(D)$, the vertex v is called an *m-eccentric vertex* of u if no vertex in $V(D)$ is further away from u than v; that is, if $\mathrm{md}(u, v) = \mathrm{mecc}(u)$.

A vertex $v \in V(D)$ is called an *m-eccentric vertex* of D if it is the m-eccentric vertex of some vertex $u \in V(D)$.

Definition 2.3.21. The *m-eccentricity*, $\mathrm{mEcc}(D)$ of a digraph D is the set of all of its m-eccentric vertices;
$\mathrm{mEcc}(D) = \{v \in V(D) | \exists u \in V(D), \mathrm{mecc}(u) = \mathrm{md}(u, v)\}$.

In a similar way, the *m-eccentricity* of any proper subset W of the vertex set $V(D)$ can be defined; $\mathrm{mEcc}(W) = \{v \in V(D) | \exists u \in W, \mathrm{mecc}(u) = \mathrm{md}(u, v)\}$.

A vertex $v \in V(D)$ is called an *m-peripheral vertex* of digraph D if no vertex in $V(D)$ has an m-eccentricity greater than $\mathrm{mecc}(v)$, that is, if the m-eccentricity of v is exactly equal to the m-diameter of D.

Definition 2.3.22. The *m-periphery*, $\mathrm{mPer}(D)$ of a strongly connected digraph D is the set of all of its m-peripheral vertices;

$$\begin{aligned} \mathrm{mPer}(D) &= \{v \in V(D) : \ \mathrm{mecc}(u) \leq \mathrm{mecc}(v), \forall u \in V(D)\} \\ &= \{v \in V(D) : \ \mathrm{mecc}(v) = \mathrm{mdiam}(D)\}. \end{aligned}$$

A vertex $v \in V(D)$ is called an *m-contour vertex* of digraph D if no neighbor vertex of v has an m-eccentricity greater than $\mathrm{mecc}(v)$.

Definition 2.3.23. The *m-contour*, $\mathrm{mCt}(D)$ of a digraph D is the set of all of its m-contour vertices;
$\mathrm{mCt}(D) = \{v \in V(D) | \mathrm{mecc}(u) \leq \mathrm{mecc}(v), \forall u \in N(v)\}$.

As given in [17] in the case of a graph, the following results hold for a digraph also.

1. $\mathrm{mPer}(D) \subseteq \mathrm{mCt}(D) \cap \mathrm{mEcc}(D)$
2. $\mathrm{mEcc}(D) \cup \mathrm{mCt}(D) \subseteq \mathrm{m}\partial(D)$.

This is because an m-peripheral vertex has the largest m-eccentricity among the vertices of D and hence it is an m-contour vertex of D as well as an m-eccentric vertex of an m-peripheral vertex in D. If v is an m-eccentric vertex of a vertex u, then v is also an m-boundary vertex of u. If v is an m-contour vertex, then $\mathrm{mecc}(u) \leq \mathrm{mecc}(v)$ for all $u \in N(v)$. Hence there exists some vertex $w \in V(D)$ such that $\mathrm{md}(w, u) \leq \mathrm{md}(w, v)$ for all $u \in N(v)$ and hence v is an m-boundary vertex of w.

The m-eccentricity of a vertex of a digraph is one-sided, in the sense that the m-distance to the farthest vertex may occur only in one direction unlike in the case of usual graphs. So we introduce the following definition for strongly connected digraphs.

Definition 2.3.24. A strongly connected digraph D is said to satisfy the *two-sided eccentricity property*, if for all $u_i \in V(D)$, there exist vertices $u_j, u_k \in V(D)$ (not necessarily distinct) such that $\mathrm{mecc}(u_i) = \overrightarrow{d}(u_i, u_j) = \overrightarrow{d}(u_k, u_i)$.

Directed cycles $\overrightarrow{C_n}$ $(n \geq 2)$ constitute an infinite family of digraphs that satisfy the two-sided eccentricity property. Let C_n be a directed cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. Then $\overrightarrow{d}_D(v_i, v_{i-1}) = \overrightarrow{d}_D(v_{i+1}, v_i) = n - 1$, where $v_0 = v_n$ and $v_{n+1} = v_1$. Thus $\mathrm{mecc}_{C_n}(v_i) = n - 1$, for $i = 1, 2, \ldots, n$

The following definitions are from [37].

Definition 2.3.25. The *Cartesian product* of two digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ with vertex sets $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ is a digraph $D = D_1 \Box D_2$ with vertex set $V(D) = V_1 \times V_2$ in which there is an edge from vertex (u_i, v_r) to the vertex (u_j, v_s) if either $u_i = u_j$ and $(v_r, v_s) \in E_2$ or $v_r = v_s$ and $(u_i, u_j) \in E_1$.

In a similar manner, the Cartesian product of n digraphs can be defined.

Definition 2.3.26. The Cartesian product of n digraphs $D_1, \ldots, D_n$ is the digraph $D = D_1 \Box D_2 \Box \cdots \Box D_n$ with vertex set $V(D) = \{(x_1, x_2, \ldots, x_n) | x_i \in V(D_i)\}$ and there is an edge from the vertex $x = (x_1, x_2 \ldots, x_n)$ to the vertex $y = (y_1, y_2, \ldots, y_n)$, when $(x_i, y_i) \in E(D_i)$ for exactly one index i $(i = 1, \ldots, n)$ and $x_j = y_j$ for each index $j \neq i$.

Cartesian product of digraphs is commutative and associative [36].

Definition 2.3.27. The *direct product* $D_1 \times D_2$ of two digraphs D_1 and D_2 with vertex sets $V(D_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_n\}$ is the digraph having the vertex set $V(D_1) \times V(D_2)$ and with arc set $E(D_1 \times D_2)$ defined as follows. $(u_i, v_r), (u_j, v_s) \in E(D_1 \times D_2)$ if $(u_i, u_j) \in E(D_1)$ and $(v_r, v_s) \in E(D_2)$.

Definition 2.3.28. The direct product of n digraphs $D_1, \ldots, D_n$ is the digraph $D = D_1 \times D_2 \times \cdots \times D_n$ with vertex set $V(D) = \{(x_1, x_2, \ldots, x_n) | x_i \in V(D_i)\}$ and there is an edge from the vertex $x = (x_1, x_2 \ldots, x_n)$ to the vertex $y = (y_1, y_2, \ldots, y_n)$, when $(x_i, y_i) \in E(D_i)$ for every index i $(i = 1, \ldots, n)$.

Definition 2.3.29. The *strong product* $D_1 \boxtimes D_2$ of two digraphs D_1 and D_2 with vertex sets $V(D_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_n\}$ is the digraph having the vertex set $V(D_1) \times V(D_2)$ and arc set $E(D_1 \boxtimes D_2)$ defined as follows. There is an edge from the vertex (u_i, v_r) to (u_j, v_s) in $D_1 \boxtimes D_2$ if either

1. $(u_i, u_j) \in E(D_1)$, $v_r = v_s$, or
2. $u_i = u_j$, $(v_r, v_s) \in E(D_2)$, or
3. $(u_i, u_j) \in E(D_1)$, $(v_r, v_s) \in E(D_2)$.

The strong product of digraphs is commutative and associative [36].

Definition 2.3.30. The strong product of n digraphs $D_1, \ldots, D_n$ is the digraph $D = D_1 \boxtimes D_2 \boxtimes \cdots \boxtimes D_n$ with vertex set $V(D) = \{(x_1, x_2, \ldots, x_n) | x_i \in V(D_i)\}$ and there is an edge from a vertex $(x_1, x_2 \ldots, x_n)$ to the vertex $(y_1, y_2, \ldots, y_n)$ provided $x_i = y_i$ or $(x_i, y_i) \in E(D_i)$ for all $i \in [n]$ and $(x_i, y_i) \in E(D_i)$ for at least one $i \in [n]$.

Definition 2.3.31. The *lexicographic product* of two digraphs D_1 and D_2 is the digraph $D_1 \circ D_2$, having the vertex set $V(D_1) \times V(D_2)$ and arc set defined as follows. There is an edge from the vertex (u_i, v_r) to the vertex (u_j, v_s) in $D_1 \circ D_2$ if either

1. $(u_i, u_j) \in E(D_1)$, or
2. $u_i = u_j$, $(v_r, v_s) \in E(D_2)$.

The lexicographic product of directed graphs is not commutative [36].

Definition 2.3.32. The lexicographic product of n digraphs $D_1, \ldots, D_n$ is the digraph $D = D_1 \circ D_2 \circ \cdots \circ D_n$ with vertex set $V(D) = \{(x_1, x_2, \ldots, x_n) | x_i \in V(D_i)\}$ and for which there is an edge from a vertex $(x_1, x_2, \ldots, x_n)$ to the vertex $(y_1, y_2, \ldots, y_n)$, when $(x_i, y_i) \in E(D_i)$ for exactly one index $i \in [n]$ and $x_j = y_j$ for each index $1 \leq j < i$.

The trivial digraph K_1 is a unit for the Cartesian product, the strong product and the lexicographic product; that is, for any digraph D, $K_1 \Box D = D$, $K_1 \boxtimes D = D$ and $K_1 \circ D = D$. This can be verified by taking the single vertex in K_1 as 1 and identifying the vertex $(1, x) = x = (x, 1)$ for all vertices x in D. But for the direct product, the unit is K_1^*, where K_1^* is a loop on one vertex; see [37].

Definition 2.3.33. A digraph D is *prime* over the product $*$ if D is non-trivial , and for any factoring $D = D_1 * D_2$, one factor D_1 is isomorphic to D and the other is K_1.

We require the following definitions to analyze the structural properties that distinguish the four products defined above from other products of graphs and digraphs.

Definition 2.3.34. A digraph *homomorphism* $\phi : D \to D'$ is a map $\phi : V(D) \to V(D')$ for which $(x, y) \in E(D)$ implies $(\phi(x), \phi(y)) \in E(D')$.

A map $\phi : V(D) \to V(D')$ contracts an edge $(x, y) \in E(D)$ if $\phi(x) = \phi(y)$.

ϕ is called a *weak homomorphism* if $(x, y) \in E(D)$ implies $(\phi(x), \phi(y)) \in E(D')$ or $\phi(x) = \phi(y)$.

Thus a weak homomorphism either preserves edges or contracts edges. Clearly, every digraph is weakly homomorphic to K_1, the complete graph with a single vertex. This idea parallels the concept of trivial homomorphism in group theory.

Definition 2.3.35. For each $k \in [n]$, the *projection* $\pi_k : V(D_1) \times \cdots \times V(D_n) \to V(D_k)$ is defined as $\pi_k(x_1, \ldots, x_n) = x_k$.

Each projection $\pi_k : D_1 \Box \cdots \Box D_n \to D_K$ and $\pi_k : D_1 \boxtimes \cdots \boxtimes D_n \to D_K$ is a weak homomorphism. Each projection $\pi_k : D_1 \times \cdots \times D_n \to D_K$ is indeed a homomorphism. In general, only the first projection $\pi_1 : D_1 \circ \cdots \circ D_n \to D_1$ of a lexicographic product is a weak homomorphism [37].

There are twenty associative products defined for graphs and digraphs having the vertex set as the Cartesian product of the vertex sets of the factors [36]. Of these, the only products for which at least one projection is a weak homomorphism (or homomorphism) and which employ the adjacency structure of both the factors are the Cartesian product, the strong product, the direct product, and the lexicographic product; see [36]. Hence these four products are called the standard products for graphs and digraphs.

Our study is restricted to strongly connected digraphs. Because of the connectivity issues of the direct product given in Section 1.1, we restrict our study in this thesis to the remaining three standard digraph products.

Chapter 3

Geodetic Interval and Boundary-type Sets of a Digraph

3.1 Introduction

The geodetic interval between two vertices in a graph have been studied extensively since the interval function was introduced by Mulder [53]. Also, the various boundary-type sets were introduced and studied by Chartrand et al. and Cáceres et al. [19, 20]. In this chapter, the variations in the distance-related properties of digraphs from that of usual graphs are studied. In the case of graphs, the eccentricities of two adjacent vertices differ by at most one. But this is not the case for digraphs with respect to the metric md. The m-eccentricities of two adjacent vertices may differ by more than one. See the digraph D' in the Figure 3.3. Here, y is adjacent to the vertices v and w, $\text{mecc}(y) = 2$ whereas $\text{mecc}(v) = 4$ and $\text{mecc}(w) = 4$. Consequently, many of the distance-related properties of graphs do not hold for digraphs with respect to the metric 'md'. Thus an attempt was made to identify the digraphs other than symmetric digraphs for which the m-eccentricities of two adjacent vertices differ by at most one. It was observed that if a digraph satisfy the two-sided eccentricity property, then the m-eccentricities of two adjacent vertices differ by at most one. Consequently, some of the results related to the geodetic interval and distance-related sets hold for such digraphs.

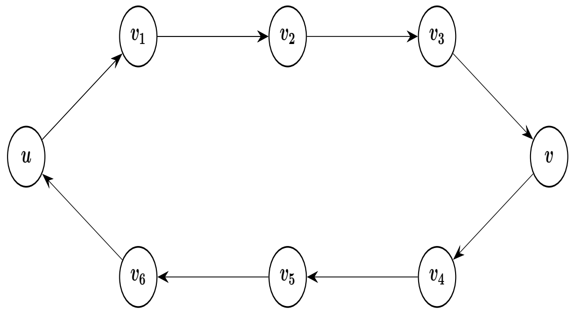

Figure 3.1: Geodetic interval in a directed cycle

3.2 Geodetic Interval in a Digraph

The geodetic interval between two vertices u and v in a digraph D, denoted by $I_D[u,v]$ is defined as $I_D[u,v] = \{w \in V(D) : \text{md}(u,w) + \text{md}(w,v) = \text{md}(u,v)\}$. The geodetic interval between two vertices in a digraph have many variations from that of graphs. If G is a cycle, then for any two non-adjacent vertices u and v, $I[u,v]$ consists of the vertices which lie in a shortest path from u to v. But for the directed cycle $\overrightarrow{C_n}$, $I_{\overrightarrow{C_n}}[u,v] = \{u,v\}$ for any two vertices $u,v \in V(D)$. To see this, consider the digraph $\overrightarrow{C_8}$ in Figure 3.1. In the underlying graph G, $I_G[u,v] = V(G)$, because $d_G(u,v) = 4$ and $d_G(u,v_i) + d_G(v_i,v) = 4$ for $i = 1,2,\ldots,6$. But none of these vertices lie in $I_{\overrightarrow{C_n}}[u,v]$. For example, consider the vertex v_1. $\text{md}_{\overrightarrow{C_n}}[u,v] = \max\{\overrightarrow{d}(u,v), \overrightarrow{d}(v,u)\} = 4$, but $\text{md}_{\overrightarrow{C_n}}(u,v_1) + \text{md}_{\overrightarrow{C_n}}(v_1,v) = 7+5 = 12$. Similar is the case of the other vertices.

3.3 Boundary-type Sets of a Digraph

Cáceres et al. proved the following proposition for a connected graph in [17].

Proposition 3.3.1 (Proposition 8 of [17]). Let $G = (V,E)$ be a connected graph.

1. If $\text{Ct}(G) = \text{Per}(G)$, then $I[\text{Ct}(G)] = V(G)$.
2. If $|\text{Ct}(G)| = |\text{Per}(G)| = 2$, then either $|\partial(G)| = 2$ or $|\partial(G)| \geq 4$.
3. If $|\text{Ecc}(G)| = |\text{Per}(G)| + 1$, then $|\partial(G)| > |\text{Ecc}(G)|$.
4. If $|\text{Ecc}(G)| > |\text{Per}(G)|$, then $|\partial(G)| \geq |\text{Per}(G)| + 2$.

The digraph analogue of Proposition 3.3.1 does not hold with respect to the metric md. To see that (1) and (2) need not hold, consider the digraph D in Figure 3.2. There is only a directed edge from the vertex v to the vertex w even though all other edges are two-way edges. Here $\text{mCt}(D) = \text{mPer}(D) = \text{mEcc}(D) = \{w,z\}$ but

$v \notin I[w,z]$. This is because $\overrightarrow{d}(w,z) = 3$, $\overrightarrow{d}(z,w) = 3$ giving $\mathrm{md}(w,z) = 3$ whereas the $w - z$ directed path passing through v is of length 4 and hence is not a directed $w - z$ geodesic. Also $\mathrm{m}\partial(D) = \{v,w,z\}$ as v is an m-boundary vertex of w while x and y are not m-boundary vertices of any vertex in D.

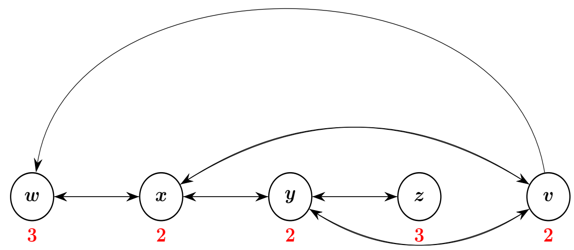

Figure 3.2: Example for a digraph D that violates the results (1) and (2) of Proposition 3.3.1.

To see that (3) and (4) does not hold in general, see the digraph D' in Figure 3.3.

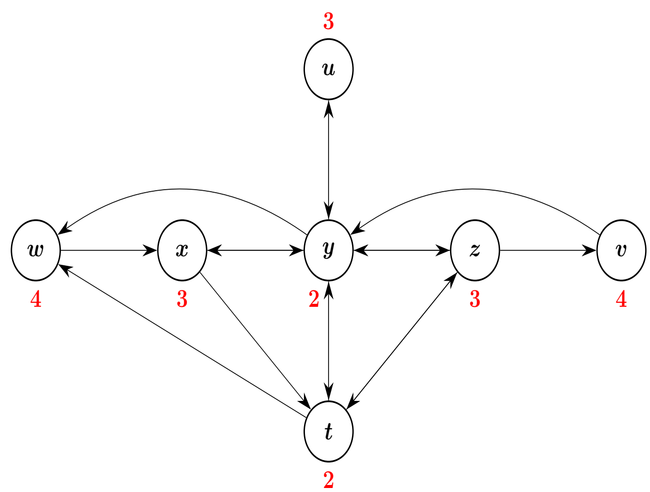

Figure 3.3: Example for a digraph that violates the results (3) and (4) of Proposition 3.3.1.

Here $\mathrm{mPer}(D') = \{w,v\}$, $\mathrm{mEcc}(D') = \{w,v,u\}$, since u is an m-eccentric vertex of the vertex t. $|\mathrm{mEcc}(D')| = |\mathrm{mPer}(D')| + 1$, but $\mathrm{m}\partial(D') = \{w,v,u\}$.

The above variations from that of graphs were the motivation to investigate the cases in which the results related to the boundary-type sets hold. As illustrated in the introduction, the m- eccentricities of adjacent vertices in a digraph differ by more than one. The difference of the m-eccentricities of two adjacent vertices in a digraph D with n vertices can be any number no greater than $\lfloor \frac{n-1}{2} \rfloor$. The upper bound is the largest possible difference between the m-diameter and the m-radius of D. If $|V(D)| = n$, then $\text{mdiam}(D) \leq n-1$. It is proved in [24] that since md is a metric, $\text{mdiam}(D) \leq 2\,\text{mrad}(D)$. Thus $\text{mrad}(D) \geq \dfrac{\text{mdiam}(D)}{2}$. If $\text{mdiam}(D) = n-1$, then $\text{mrad}(D) \geq \lceil \frac{n-1}{2} \rceil$ and hence it follows that $\text{mdiam}(D) - \text{mrad}(D) \leq \lfloor \frac{n-1}{2} \rfloor$. In the following lemma, we prove that if a digraph satisfy the two-sided eccentricity property, then the m-eccentricities of adjacent vertices differ by at most one.

Lemma 3.3.2. Let D be a strong digraph that satisfy the two-sided eccentricity property. Then the m-eccentricities of two adjacent vertices differ by at most one.

Proof. If possible, suppose that the m-eccentricities of two adjacent vertices u and v in the strong digraph D differ by more than one. Let $\text{mecc}(u) = k$ and $\text{mecc}(v) = r$, where $r > k+1$. As u and v are adjacent vertices in the digraph D, three cases arise.
(1) There is a directed edge from u to v.
(2) There is a directed edge from v to u.
(3) There is a directed edge from u to v and a directed edge from v to u.
Since D satisfies the two-sided eccentricity property, there exist vertices $v', v'' \in V(D)$ such that $\overrightarrow{d}(v, v') = r$ and $\overrightarrow{d}(v'', v) = r$.

Case 1: Suppose that $(u, v) \in E(D)$. Let P be a directed geodesic from the vertex v'' to the vertex v of length r. Then there are two possibilities. Either $u \in P$ or $u \notin P$. If $u \in P$, then since P is the shortest directed path from v'' to v, the part of the directed path P from v'' to u must be a shortest directed path from v'' to u. Thus $\overrightarrow{d}(v'', v) = r$ gives $\overrightarrow{d}(v'', u) = r-1$ which contradicts the fact that $\text{mecc}(u) = k$ and $k < r-1$. If $u \notin P$, then since D is strongly connected, there is a directed path from v'' to u. Let P' be a directed geodesic from v'' to u. Since $\text{mecc}(u) = k$, $\overrightarrow{d}(v'', u) \leq k$. If $v \in V(P')$, then $\overrightarrow{d}(v'', v) \leq k-1$. Otherwise, the directed path obtained by concatenating the directed edge (u, v) to the path P', is a directed path from v'' to v of length less than or equal to $k+1$. In either case, length of the directed geodesic from v'' to v is at most $k+1$ which contradicts the fact that $\overrightarrow{d}(v'', v) = r$, since $r > k+1$.

Case 2: Suppose that $(v, u) \in E(D)$. Let Q be a directed geodesic from the vertex v to the vertex v' of length r. Then there are two possibilities. Either $u \in Q$ or $u \notin Q$. If $u \in Q$, then since Q is the shortest directed path from v to v', the part of the directed path Q from v to u must be a shortest directed path from u to v'. Thus $\overrightarrow{d}(v, v') = r$ gives $\overrightarrow{d}(u, v') = r-1$ which contradicts the fact that $\text{mecc}(u) = k$ and $r-1 > k$. If $u \notin Q$, then since D is strongly connected, there is a directed path from u to v'. Let Q' be a directed geodesic from u to v'. Since $\text{mecc}(u) = k$,

$\overrightarrow{d}(u, v') \leq k$. If $v \in V(Q')$, then $\overrightarrow{d}(v, v') \leq k - 1$. Otherwise, the directed path obtained by concatenating the directed edge (v, u) to the path Q', is a directed path from v to v' of length less than or equal to $k+1$. In either case, length of the directed geodesic from v to v' is at most $k+1$ which contradicts the fact that $\overrightarrow{d}(v, v') = r$, since $r > k+1$.

As Case 3 is the combination of the Cases 1 and 2, the result follows. □

The converse of Lemma 3.3.2 does not hold. That is, if the m-eccentricities of two adjacent vertices differ by at most one, the digraph need not satisfy the two-sided eccentricity property. To see this, consider the digraph in Figure 3.4. Here, $\text{mecc}(u_1) = \text{mecc}(u_3) = 2$ and $\text{mecc}(u_2) = 1$, since $\text{md}(u_1, u_2) = \text{md}(u_2, u_3) = 1$ and $\text{md}(u_1, u_3) = 2$. But $\overrightarrow{d}(u_1, u_3) = 1$, $\overrightarrow{d}(u_3, u_1) = 2$. Hence this digraph does not satisfy the two-sided eccentricity property.

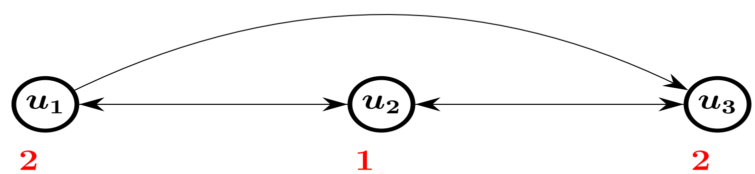

Figure 3.4: Example for a digraph that does not satisfy the two-sided eccentricity property

It can be seen from the digraph D given in Figure 3.2 that even if the digraph satisfy the two-sided eccentricity property, the results (1) and (2) of Proposition 3.3.1 need not hold. But, the results (3) and (4) of Proposition 3.3.1 hold for strong digraphs satisfying the two-sided eccentricity property. This is due to the fact that by Lemma 3.3.2 the m-eccentricities of two adjacent vertices differ by at most one. The proof is similar to that of parts (3) and (4) of Proposition 8 of [17]. However for the sake of completeness, we include it below.

Proposition 3.3.3. Let $D = (V, E)$ be a strongly connected digraph that satisfy the two-sided eccentricity property.

1. If $|\text{mEcc}(D)| = |\text{mPer}(D)| + 1$, then $|\text{m}\partial(D)| > |\text{mEcc}(D)|$.

2. If $|\text{mEcc}(D)| > |\text{mPer}(D)|$, then $|\text{m}\partial(D)| \geq |\text{mPer}(D)| + 2$.

Proof. 1. Let x be the unique vertex in $\text{mEcc}(D)$ which does not lie in $\text{mPer}(D)$. Take W as the set of vertices in D for which x is an m-eccentric vertex. That is, $W = \{y \in V(D) : \text{md}(y, x) = \text{mecc}(y)\}$. Then $W \cap \text{mPer}(D) = \emptyset$, since

$x \notin \text{mPer}(D)$. Also, $W \cap \text{mEcc}(D) = \emptyset$, since $\text{mEcc}(D) = \text{mPer}(D) \cup \{x\}$. Let $z \in W$ be such that $\text{mecc}(z) = \max\limits_{y \in W} \text{mecc}(y)$.

Claim: z is an m-boundary vertex of x. To this end, suppose that there exists a vertex $w \in N(z)$ such that $\text{md}(w, x) = \text{md}(z, x) + 1$. Since D satisfies the two-sided eccentricity property, the m-eccentricities of two adjacent vertices in D differ by at most one. Since $z \in W$, $\text{md}(z, x) = \text{mecc}(z)$. Thus the only possibility is that $\text{mecc}(w) = \text{mecc}(z) + 1$. This yields a contradiction, for then $w \in W$ has greater m-eccentricity than z. Hence, $z \in \text{m}\partial(D)$.

2. $\text{mPer}(D) \subseteq \text{mEcc}(D) \subseteq \text{m}\partial(D)$. Hence the result follows from part 1 of the proposition.

□

Chapter 4

Boundary-type Sets and Center in Cartesian Product

4.1 Introduction

Cartesian product is the product having the most real life applications among the various graph products and is widely used in metric graph theory. Hypercubes are Cartesian powers of K_2, the complete graph with 2 vertices. Hamming graphs are Cartesian products of complete graphs and grid graphs are Cartesian product of paths.

The boundary-type sets of Cartesian product of graphs was studied in [12] and the center of Cartesian product of graphs was studied in [75]. In this chapter, a similar attempt is made for strongly connected digraphs and the distance considered is md. The relationship between the m-periphery, m-contour, m-eccentricity, m-boundary, and m-center of two digraphs and their Cartesian product is investigated here.

The results in this chapter if combined with methods for prime factorization of digraphs can be applied to find the m-periphery, m-contour, m-eccentricity, m-boundary, and m-center sets of very large strongly connected digraphs.

4.2 Basic Results

The Fundamental Theorem of prime factorization of digraphs under Cartesian Product proved by Feigenbaum in [32] is as follows.

Theorem 4.2.1. [32] For any weakly connected digraph D, there exists a unique $n \geq 1$ and a unique tuple $(D_1, \ldots, D_n)$ of digraphs up to reordering and isomorphism of the factors D_i, such that each D_i has at least two vertices, each D_i is prime for the Cartesian product and $D = D_1 \Box D_2 \Box \cdots \Box D_n$. $(D_1, \ldots, D_n)$ is called the prime decomposition of D.

The following proposition was proved by Harary in [41].

Proposition 4.2.2. $D_1 \Box D_2$ is strongly connected if and only if both D_1 and D_2 are strongly connected.

We can extend this to get the following immediate corollary.

Corollary 4.2.3. $D_1 \Box D_2 \Box \cdots \Box D_n$ is strongly connected if and only if $D_1, D_2, \ldots, D_n$ are strongly connected.

As in the case of graphs, it is easy to see that $N_{D_1 \Box D_2}(u_i, v_r) = [\{u_i\} \times N_{D_2}(v_r)] \bigcup [N_{D_1}(u_i) \times \{v_r\}]$ for any two digraphs D_1 and D_2.

4.3 Distance Between Two Vertices

All the digraphs considered are strongly connected and hence the directed distance between any two vertices is finite.

The distance between two vertices in the Cartesian product of two graphs is given in [36]. Adapting this proof, the directed distance between two vertices in the Cartesian product of n digraphs, $D_1 \Box \cdots \Box D_n$ is proved in the following proposition in [37].

Proposition 4.3.1. [37] In a Cartesian product $D = D_1 \Box \cdots \Box D_n$, the distance between vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ is $\overrightarrow{d}_D((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \Sigma_{1 \leq i \leq n} \overrightarrow{d}_{D_i}(x_i, y_i)$.

Using this result, we derive the formula for the maximum distance between two vertices in the Cartesian product of two digraphs.

Lemma 4.3.2. Let D_1 and D_2 be two strong digraphs with vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively. Then $\mathrm{md}_{D_1 \Box D_2}((u_i, v_r), (u_j, v_s)) = \max\{\overrightarrow{d}_{D_1}(u_i, u_j) + \overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_1}(u_j, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r)\}$ for all $((u_i, v_r), (u_j, v_s) \in V(D_1 \Box D_2)$.

Proof. Let $D = D_1 \Box D_2$. By the definition of maximum distance between two vertices, $\mathrm{md}_D((u_i, v_r), (u_j, v_s)) = \max\{\overrightarrow{d}_D((u_i, v_r), (u_j, v_s)), \overrightarrow{d}_D((u_j, v_s), (u_i, v_r))\}$.

From Proposition 4.3.1, $\overrightarrow{d}_D((u_i,v_r),(u_j,v_s)) = \overrightarrow{d}_{D_1}(u_i,u_j) + \overrightarrow{d}_{D_2}(v_r,v_s)$. Similarly, $\overrightarrow{d}_D((u_j,v_s),(u_i,v_r)) = \overrightarrow{d}_{D_1}(u_j,u_i) + \overrightarrow{d}_{D_2}(v_s,v_r)$. Therefore, $\text{md}_D((u_i,v_r),(u_j,v_s)) = \max\{\overrightarrow{d}_{D_1}(u_i,u_j) + \overrightarrow{d}_{D_2}(v_r,v_s), \overrightarrow{d}_{D_1}(u_j,u_i) + \overrightarrow{d}_{D_2}(v_s,v_r)\}$. □

Figure 4.1 illustrates the determination of distances in $D_1 \Box D_2$. The m-eccentricity of each vertex is displayed near the vertex so that the boundary type sets can be easily identified. In general, the metric 'maximum distance' does not satisfy $\text{md}_{D_1 \Box D_2}((u_i,v_r),(u_j,v_s)) = \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s)$, which is true in the case of Cartesian product of two graphs. Consider the vertices (u_1,v_1) and (u_3,v_3) in the Cartesian product $D = D_1 \Box D_2$.

$$\begin{aligned}\text{md}_D((u_1,v_1),(u_3,v_3)) &= \max\{\overrightarrow{d}_{D_1}(u_1,u_3) + \overrightarrow{d}_{D_2}(v_1,v_3), \overrightarrow{d}_{D_1}(u_3,u_1) + \overrightarrow{d}_{D_2}(v_3,v_1)\}\\ &= \max\{2+1, 1+2\} = 3 \neq \text{md}_{D_1}(u_1,u_3) + \text{md}_{D_2}(v_3,v_1).\end{aligned}$$

Consequently, $\text{mecc}_{D_1 \Box D_2}(u_i,v_r) \neq \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$ unlike in the case of graphs. But we prove that $\text{md}_{D_1 \Box D_2}((u_i,v_r),(u_j,v_s)) \leq \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s)$ and $\text{mecc}_{D_1 \Box D_2}(u_i,v_r) \leq \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$.

Theorem 4.3.3. Let D_1 and D_2 be two strongly connected digraphs. Then $\text{md}_{D_1 \Box D_2}((u_i,v_r),(u_j,v_s)) \leq \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s)$ for all $(u_i,v_r),(u_j,v_s) \in V(D_1 \Box D_2)$.

Proof. $\text{md}_{D_1 \Box D_2}((u_i,v_r),(u_j,v_s)) = \max\{\overrightarrow{d}_{D_1}(u_i,u_j) + \overrightarrow{d}_{D_2}(v_r,v_s), \overrightarrow{d}_{D_1}(u_j,u_i) + \overrightarrow{d}_{D_2}(v_s,v_r)\}$. $\text{md}_{D_1}(u_i,u_j) = \max\{\overrightarrow{d}_{D_1}(u_i,u_j), \overrightarrow{d}_{D_1}(u_j,u_i)\}$ and $\text{md}_{D_2}(v_r,v_s) = \max\{\overrightarrow{d}_{D_2}(v_r,v_s), \overrightarrow{d}_{D_2}(v_s,v_r)\}$. Hence $\overrightarrow{d}_{D_1}(u_i,u_j) \leq \text{md}_{D_1}(u_i,u_j)$, $\overrightarrow{d}_{D_2}(v_r,v_s) \leq \text{md}_{D_2}(v_r,v_s)$, $\overrightarrow{d}_{D_1}(u_j,u_i) \leq \text{md}_{D_1}(u_i,u_j)$ and $\overrightarrow{d}_{D_2}(v_s,v_r) \leq \text{md}_{D_2}(v_r,v_s)$. Thus, $\overrightarrow{d}_{D_1}(u_i,u_j) + \overrightarrow{d}_{D_2}(v_r,v_s) \leq \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s)$ and $\overrightarrow{d}_{D_1}(u_j,u_i) + \overrightarrow{d}_{D_2}(v_s,v_r) \leq \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s)$. Hence $\max\{\overrightarrow{d}_{D_1}(u_i,u_j) + \overrightarrow{d}_{D_2}(v_r,v_s), \overrightarrow{d}_{D_1}(u_j,u_i) + \overrightarrow{d}_{D_2}(v_s,v_r)\} \leq \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s)$ which gives $\text{md}_{D_1 \Box D_2}((u_i,v_r),(u_j,v_s)) \leq \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s)$. □

Corollary 4.3.4. Let D_1 and D_2 be two strongly connected digraphs. Then $\text{mecc}_{D_1 \Box D_2}(u_i,v_r) \leq \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$ for all $u_i \in V(D_1), v_r \in V(D_2)$.

Proof. Let (u_j,v_s) be an eccentric vertex of (u_i,v_r) in $D_1 \Box D_2$. Then $\text{mecc}_{D_1 \Box D_2}(u_i,v_r) = \text{md}_{D_1 \Box D_2}((u_i,v_r),(u_j,v_s)) \leq \text{md}_{D_1}(u_i,u_j) + \text{md}_{D_2}(v_r,v_s) \leq \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$. □

Now we give the necessary and sufficient condition for every vertex $(u_i,v_r) \in V(D_1 \Box D_2)$ to satisfy $\text{mecc}_{D_1 \Box D_2}(u_i,v_r) = \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$.

Proposition 4.3.5. Let D_1 and D_2 be two strongly connected digraphs. A necessary and sufficient condition for every vertex $(u_i, v_r) \in V(D_1 \Box D_2)$ to satisfy $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$ is that either D_1 or D_2 satisfy the two-sided eccentricity property.

Proof. Let $D = D_1 \Box D_2$. To show that the condition is sufficient; without loss of generality suppose that D_1 satisfy the two-sided eccentricity property and u_j, u_k are the eccentric vertices of u_i in D_1 in the two directions respectively. Hence $\mathrm{mecc}_{D_1}(u_i) = \overrightarrow{d}_{D_1}(u_i, u_j) = \overrightarrow{d}_{D_1}(u_k, u_i)$.

Case 1: Suppose $u_j \neq u_k$.

Let $\mathrm{mecc}_{D_1}(u_i) = \ell$. Then $\overrightarrow{d}_{D_1}(u_i, u_j) = \overrightarrow{d}_{D_1}(u_k, u_i) = \ell$. Let $v_r \in V(D_2)$ and v_s be an eccentric vertex of v_r. Consider $(u_i, v_r) \in V(D_1 \Box D_2)$. Let $\mathrm{mecc}_{D_2}(v_r) = \ell'$. Then there are three subcases.

Subcase 1.1: $\overrightarrow{d}_{D_2}(v_r, v_s) = \ell'$ and $\overrightarrow{d}_{D_2}(v_s, v_r) < \ell'$. In this case,

$$\begin{aligned} \mathrm{md}_D((u_i, v_r), (u_j, v_s)) &= \max\{\overrightarrow{d}_{D_1}(u_i, u_j) + \overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_1}(u_j, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r)\} \\ &= \ell + \ell' \\ &= \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r). \end{aligned}$$

Subcase 1.2: $\overrightarrow{d}_{D_2}(v_r, v_s) < \ell'$ and $\overrightarrow{d}_{D_2}(v_s, v_r) = \ell'$. Then

$$\begin{aligned} \mathrm{md}_D((u_i, v_r), (u_k, v_s)) &= \max\{\overrightarrow{d}_{D_1}(u_i, u_k) + \overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_1}(u_k, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r)\} \\ &= \ell + \ell' \\ &= \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r). \end{aligned}$$

Subcase 1.3: $\overrightarrow{d}_{D_2}(v_r, v_s) = \overrightarrow{d}_{D_2}(v_s, v_r) = \ell'$. Then as in the above two subcases

$$\begin{aligned} \mathrm{md}_D((u_i, v_r), (u_k, v_s)) &= \mathrm{md}_D((u_i, v_r), (u_j, v_s)) \\ &= \ell + \ell' \\ &= \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r). \end{aligned}$$

Case 2: Suppose that $u_j = u_k$.
$\overrightarrow{d}_{D_1}(u_i, u_j) = \overrightarrow{d}_{D_1}(u_j, u_i) = \ell$. Then as in the Subcases of Case 1, it can be proved that $\mathrm{md}_{D_1 \Box D_2}((u_i, v_r), (u_j, v_s)) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$, irrespective of whether $\mathrm{md}(v_r, v_s)$ is $\overrightarrow{d}_{D_2}(v_r, v_s)$ or $\overrightarrow{d}_{D_2}(v_s, v_r)$. Thus in all cases, there is a vertex in $D_1 \Box D_2$ whose distance from (u_i, v_r) equals $\mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$.
Since $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) \leq \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$, the result follows.

Now to prove the necessary part, suppose that both D_1 and D_2 do not satisfy the two-sided eccentricity property. So there exists at least one vertex $u_i \in V(D_1)$ and at least one vertex $v_r \in V(D_2)$ such that $\text{mecc}_{D_1}(u_i) = \overrightarrow{d}_{D_1}(u_i, u_j) > \overrightarrow{d}_{D_1}(u_k, u_i)$ for all $u_k \in V(D_1)$ and $\text{mecc}_{D_2}(v_r) = \overrightarrow{d}_{D_1}(v_q, v_r) > \overrightarrow{d}_{D_2}(v_r, v_s)$ for all $v_s \in V(D_2)$. Then for any vertex $(u_k, v_s) \in V(D_1 \Box D_2)$, $\text{md}_{D_1 \Box D_2}((u_i, v_r), (u_k, v_s)) = \max\{\overrightarrow{d}_{D_1}(u_i, u_k) + \overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_1}(u_k, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r)\}$. Now since $\overrightarrow{d}_{D_1}(u_i, u_k) + \overrightarrow{d}_{D_2}(v_r, v_s) < \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$ and $\overrightarrow{d}_{D_1}(u_k, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r) < \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$, it follows that $\text{md}_{D_1 \Box D_2}((u_i, v_r), (u_k, v_s)) < \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$ for all $(u_k, v_s) \in V(D_1 \Box D_2)$ and so $\text{mecc}_{D_1 \Box D_2}(u_i, v_r) < \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$. □

4.4 Boundary-type Sets of Cartesian Product of Digraphs

Boundary-type sets of Cartesian product of two connected graphs have been studied by Brešar et al. in [12]. The following theorem was proved in [12].

Theorem 4.4.1. For any graphs G and H,

1. $\partial(G \Box H) = \partial(G) \times \partial(H)$,
2. $\text{Ct}(G \Box H) = \text{Ct}(G) \times \text{Ct}(H)$,
3. $\text{Ecc}(G \Box H) = \text{Ecc}(G) \times \text{Ecc}(H)$,
4. $\text{Per}(G \Box H) = \text{Per}(G) \times \text{Per}(H)$.

In the general case, the digraph analogue of the theorem is not expected to hold with respect to the metric **maximum distance** owing to the fact that for vertices (u_i, v_r) and (u_j, v_s) in $D_1 \Box D_2$, $\text{md}_{D_1 \Box D_2}((u_i, v_r), (u_j, v_s)) \neq \text{md}(u_i, u_j) + \text{md}(v_r, v_s)$ and $\text{mecc}_{D_1 \Box D_2}(u_i, v_r) \neq \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$. There are a number of variations in the properties of boundary-type vertices of Cartesian product of digraphs from that of usual graphs. The most significant among these variations are given below.

1. If u_i is an eccentric vertex of u_j in D_1 and v_r is an eccentric vertex of v_s in D_2, then (u_i, v_r) need not be an eccentric vertex of (u_j, v_s) in $D_1 \Box D_2$. To see this, consider the digraphs in Figure 4.1. u_1 is an eccentric vertex of u_3 in D_1 and v_3 is an eccentric vertex of v_1 in D_2. But (u_1, v_1) is not an eccentric vertex of (u_3, v_3) since $\text{mecc}_{D_1 \Box D_2}(u_3, v_3) = \text{md}_{D_1 \Box D_2}((u_3, v_3), (u_1, v_2)) = \max\{\overrightarrow{d}_{D_1}(u_3, u_1) + \overrightarrow{d}_{D_2}(v_3, v_2), \overrightarrow{d}_{D_1}(u_1, u_3) + \overrightarrow{d}_{D_2}(v_2, v_3)\} = \max\{1+1, 2+2\} = 4$, whereas $\text{md}_{D_1 \Box D_2}((u_3, v_3), (u_1, v_1)) = \max\{\overrightarrow{d}_{D_1}(u_3, u_1) + \overrightarrow{d}_{D_2}(v_3, v_1), \overrightarrow{d}_{D_1}(u_1, u_3) + \overrightarrow{d}_{D_2}(v_1, v_3)\} = \max\{1+2, 2+1\} = 3$.

2. A vertex (u_i, v_r) may be an m-eccentric vertex in $D_1 \Box D_2$, without satisfying $u_i \in \text{mEcc}(D_1)$ and $v_r \in \text{mEcc}(D_2)$.

   This behaviour of vertices is illustrated in Figure 4.2. It can be seen that u_2 is not an eccentric vertex in D_1 but (u_2, v_1) is an eccentric vertex of (u_4, v_3) in $D_1 \Box D_2$.

The m-diameter of the Cartesian product of digraphs can be seen to be additive in the following sense.

Proposition 4.4.2. Let D_1 and D_2 be two strong digraphs. Then $\text{mdiam}(D_1 \Box D_2) = \text{mdiam}(D_1) + \text{mdiam}(D_2)$.

Proof. Let $\text{mdiam}(D_1 \Box D_2) = k$, $\text{mdiam}(D_1) = n_1$, and $\text{mdiam}(D_2) = n_2$.

Let $(u_i, v_r) \in V(D_1 \Box D_2)$ be such that $\text{mecc}_{D_1 \Box D_2}(u_i, v_r) = k$. By Corollary 4.3.4, $\text{mecc}_{D_1 \Box D_2}(u_i, v_r) \leq \text{mecc}_{D_1}(u_i) + \text{mecc}_{D_2}(v_r)$. Thus $\text{mecc}_{D_1 \Box D_2}(u_i, v_r) \leq \max_{u_i \in V(D_1)}\{\text{mecc}_{D_1}(u_i)\} + \max_{v_r \in V(D_2)}\{\text{mecc}_{D_2}(v_r)\}$. That is, $\text{mdiam}(D_1 \Box D_2) \leq \text{mdiam}(D_1) + \text{mdiam}(D_2)$.

Conversely, let $u_j \in V(D_1)$ be such that $\overrightarrow{d}_{D_1}(u_j, u_k) = n_1$ and let $v_s \in V(D_2)$ be such that $\overrightarrow{d}_{D_2}(v_s, v_t) = n_2$. Thus $\overrightarrow{d}_{D_1}(u_j, u_k) + \overrightarrow{d}_{D_2}(v_s, v_t) = n_1 + n_2$. Hence, $\text{md}_{D_1 \Box D_2}(u_j, v_s), (u_k, v_t) \geq n_1 + n_2$ which implies $\text{mecc}_{D_1 \Box D_2}(u_j, v_s) \geq n_1 + n_2$. Thus $\text{mdiam}(D_1 \Box D_2) \geq \text{mdiam}(D_1) + \text{mdiam}(D_2)$. Hence, $\text{mdiam}(D_1 \Box D_2) = \text{mdiam}(D_1) + \text{mdiam}(D_2)$. □

As a consequence of this result, the following theorem holds for any two strong digraphs D_1 and D_2.

Theorem 4.4.3. Let D_1 and D_2 be two strongly connected digraphs. Then $\text{mPer}(D_1 \Box D_2) \subseteq \text{mPer}(D_1) \times \text{mPer}(D_2)$.

Proof. Let $(u_j, v_s) \in \text{mPer}(D_1 \Box D_2)$. Then $\text{mecc}_{D_1 \Box D_2}(u_j, v_s) = \text{mdiam}(D_1 \Box D_2) = \text{mdiam}(D_1) + \text{mdiam}(D_2)$. Applying the results $\text{mecc}_{D_1 \Box D_2}(u_j, v_s) \leq \text{mecc}_{D_1}(u_j) + \text{mecc}_{D_2}(v_s)$ and $\text{mdiam}(D_1) = \max\limits_{u_i \in V(D_1)}\{\text{mecc}_{D_1}(u_i)\}$, $\text{mdiam}(D_2) = \max\limits_{v_r \in V(D_2)}\{\text{mecc}_{D_2}(v_r)\}$, it follows that $\text{mecc}_{D_1}(u_j) + \text{mecc}_{D_2}(v_s) \geq \text{mecc}_{D_1 \Box D_2}(u_j, v_s) = \max\limits_{u_i \in V(D_1)}\{\text{mecc}_{D_1}(u_i)\} + \max\limits_{v_r \in V(D_2)}\{\text{mecc}_{D_2}(v_r)\}$ and hence necessarily $\text{mecc}_{D_1}(u_j) = \max\limits_{u_i \in V(D_1)}\{\text{mecc}_{D_1}(u_i)\} = \text{mdiam}(D_1)$ and $\text{mecc}_{D_2}(v_s) = \max\limits_{v_r \in V(D_2)}\{\text{mecc}_{D_2}(v_r)\} = \text{mdiam}(D_2)$.

Therefore $u_j \in \text{mPer}(D_1)$ and $v_s \in \text{mPer}(D_2)$. Hence the result. □

Generally, for any two strong digraphs D_1 and D_2,

1. $\text{mPer}(D_1) \times \text{mPer}(D_2) \nsubseteq \text{mPer}(D_1 \Box D_2)$
2. $\text{mCt}(D_1) \times \text{mCt}(D_2) \nsubseteq \text{mCt}(D_1 \Box D_2)$
3. $\text{mEcc}(D_1) \times \text{mEcc}(D_2) \nsubseteq \text{mEcc}(D_1 \Box D_2)$
4. $\text{m}\partial(D_1) \times \text{m}\partial(D_2) \nsubseteq \text{m}\partial(D_1 \Box D_2)$.

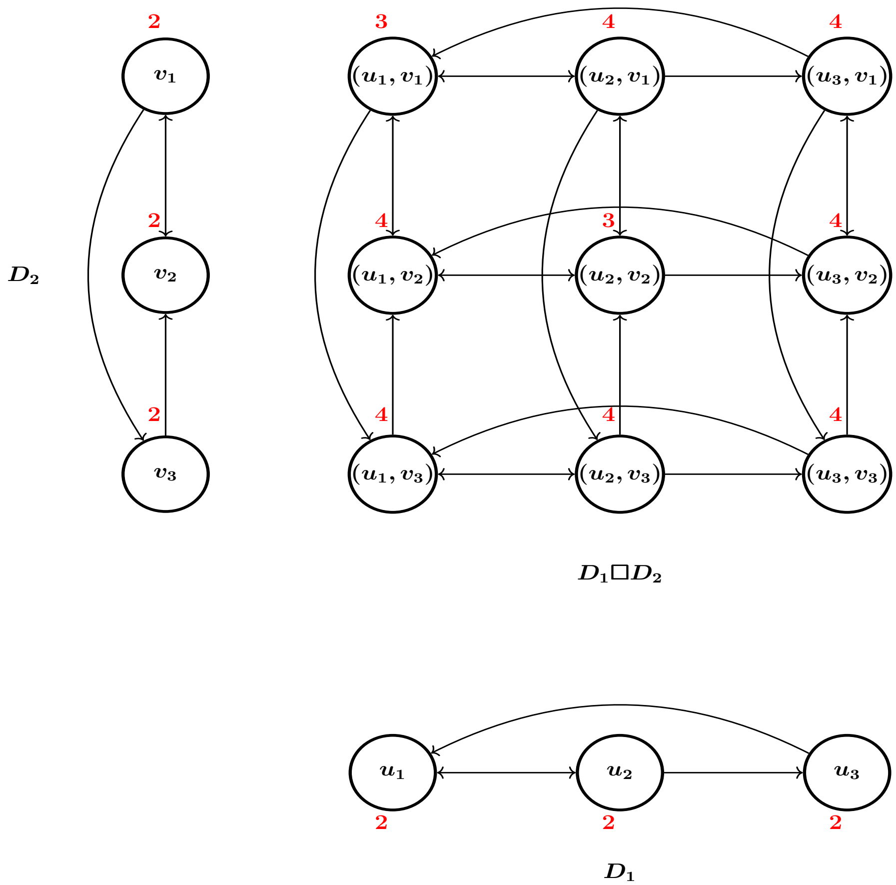

Figure 4.1: An example in which $\text{mPer}(D_1) \times \text{mPer}(D_2) \nsubseteq \text{mPer}(D_1 \Box D_2)$, $\text{mCt}(D_1) \times \text{mCt}(D_2) \nsubseteq \text{mCt}(D_1 \Box D_2)$, $\text{mEcc}(D_1) \times \text{mEcc}(D_2) \nsubseteq \text{mEcc}(D_1 \Box D_2)$, $\text{m}\partial(D_1) \times \text{m}\partial(D_2) \nsubseteq \text{m}\partial(D_1 \Box D_2)$

In Figure 4.1, $\text{mPer}(D_1) = \text{mCt}(D_1) = \text{mEcc}(D_1) = \text{m}\partial(D_1) = \{u_1, u_2, u_3\}$ and $\text{mPer}(D_2) = \text{mCt}(D_2) = \text{mEcc}(D_2) = \text{m}\partial(D_2) = \{v_1, v_2, v_3\}$ but (u_1, v_1) and (u_2, v_2) do not belong to any of the sets $\text{mPer}(D_1 \Box D_2)$, $\text{mCt}(D_1 \Box D_2)$, $\text{mEcc}(D_1 \Box D_2)$, $\text{m}\partial(D_1 \Box D_2)$. This is because the m-eccentricities of these two vertices are 3. The m-eccentric vertices of (u_1, v_1) are (u_3, v_2) and (u_2, v_3) and the m-eccentric vertices of (u_2, v_2) are (u_1, v_3) and (u_3, v_1). Also, they do not form the m-boundary vertices of any vertex in $D_1 \Box D_2$.

For two strongly connected digraphs, generally

1. $\mathrm{mCt}(D_1 \square D_2) \nsubseteq \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)$
2. $\mathrm{mEcc}(D_1 \square D_2) \nsubseteq \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2)$
3. $\mathrm{m}\partial(D_1 \square D_2) \nsubseteq \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2)$.

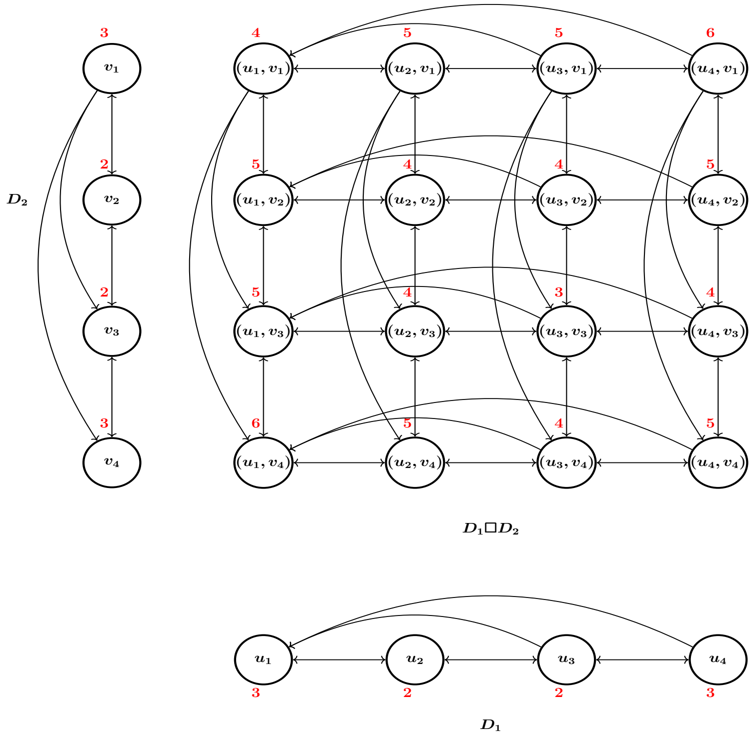

Figure 4.2: An example in which $\mathrm{mCt}(D_1 \square D_2) \nsubseteq \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)$, $\mathrm{mEcc}(D_1 \square D_2) \nsubseteq \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2)$, $\mathrm{m}\partial(D_1 \square D_2) \nsubseteq \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2)$

In Figure 4.2, $\mathrm{mCt}(D_1) = \mathrm{mEcc}(D_1) = \mathrm{m}\partial(D_1) = \{u_1, u_4\}$ and $\mathrm{mCt}(D_2) = \mathrm{mEcc}(D_2) = \mathrm{m}\partial(D_2) = \{v_1, v_4\}$. But, it can be seen that (u_2, v_1) and (u_1, v_2) are in $\mathrm{mCt}(D_1 \square D_2)$. Also, (u_2, v_1) and (u_1, v_2) are both m-eccentric vertices and hence m-boundary vertices of the vertices (u_3, v_3), (u_3, v_4), (u_4, v_3) and (u_4, v_4).

The following proposition deals with the m-periphery and m-contour of the Cartesian product of two strongly connected digraphs, if at least one of them satisfies the two-sided eccentricity property.

Proposition 4.4.4. Let D_1 and D_2 be two strongly connected digraphs such that at least one of D_1 and D_2 satisfies the two-sided eccentricity property.Then

1. $\mathrm{mPer}(D_1 \square D_2) = \mathrm{mPer}(D_1) \times \mathrm{mPer}(D_2)$

2. $\mathrm{mCt}(D_1 \square D_2) = \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)$.

Proof. Given, D_1 and D_2 are two strongly connected digraphs such that at least one of D_1 and D_2 satisfy the two-sided eccentricity property.

1. By theorem 4.4.3, $\mathrm{mPer}(D_1\Box D_2) \subseteq \mathrm{mPer}(D_1) \times \mathrm{mPer}(D_2)$ for every strongly connected digraphs D_1 and D_2. Thus it remains to prove that $\mathrm{mPer}(D_1) \times \mathrm{mPer}(D_2) \subseteq \mathrm{mPer}(D_1\Box D_2)$.

   Let $u_i \in \mathrm{mPer}(D_1)$ and $v_r \in \mathrm{mPer}(D_2)$. Thus $\mathrm{mecc}_{D_1}(u_i) \geq \mathrm{mecc}_{D_1}(u_j)$ for all $u_j \in V(D_1)$ and $\mathrm{mecc}_{D_2}(v_r) \geq \mathrm{mecc}_{D_2}(v_s)$ for all $v_s \in V(D_2)$. Hence $\mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r) \geq \mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_s)$ for all $u_j \in V(D_1)$ and for all $v_s \in V(D_2)$. At least one of D_1 and D_2 satisfies the two-sided eccentricity property. Therefore, by Proposition 4.3.5, $\mathrm{mecc}_{D_1\Box D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$ for all $(u_i, v_r) \in V(D_1\Box D_2)$. This implies, $\mathrm{mecc}_{D_1\Box D_2}(u_i, v_r) \geq \mathrm{mecc}_{D_1\Box D_2}(u_j, v_s)$ for all $(u_j, v_s) \in V(D_1\Box D_2)$ and hence by the definition of m-periphery, it follows that $(u_i, v_r) \in \mathrm{mPer}(D_1\Box D_2)$.

2. Let $(u_i, v_r) \in \mathrm{mCt}(D_1\Box D_2)$. If possible, let $u_i \notin \mathrm{mCt}(D_1)$. Then there is a vertex $u_j \in N_{D_1}(u_i)$ such that $\mathrm{mecc}_{D_1}(u_j) > \mathrm{mecc}_{D_1}(u_i)$. Therefore $\mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_r) > \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$ and hence $\mathrm{mecc}_{D_1\Box D_2}(u_j, v_r) > \mathrm{mecc}_{D_1\Box D_2}(u_i, v_r)$, which is a contradiction, since $(u_j, v_r) \in N_{D_1\Box D_2}(u_i, v_r)$. Similarly it can be shown that $v_r \in \mathrm{mCt}(D_2)$. This implies, $(u_i, v_r) \in \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)$. Hence $\mathrm{mCt}(D_1\Box D_2) \subseteq \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)$.

   Conversely, suppose that $u_i \in \mathrm{mCt}(D_1)$ and $v_r \in \mathrm{mCt}(D_2)$. If possible, let $(u_i, v_r) \notin \mathrm{mCt}(D_1\Box D_2)$. Then there is a vertex $(u_j, v_s) \in N_{D_1\Box D_2}(u_i, v_r)$ such that $\mathrm{mecc}_{D_1\Box D_2}(u_j, v_s) > \mathrm{mecc}_{D_1\Box D_2}(u_i, v_r)$. Since $(u_j, v_s) \in N_{D_1\Box D_2}(u_i, v_r)$, without loss of generality, assume that $u_i = u_j$ and $v_s \in N_{D_2}(v_r)$. Thus $\mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_s) > \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$ which implies, $\mathrm{mecc}_{D_2}(v_s) > \mathrm{mecc}_{D_2}(v_r)$, which is a contradiction. Hence $\mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2) \subseteq \mathrm{mCt}(D_1\Box D_2)$.

□

We have seen the results for $\mathrm{mPer}(D_1\Box D_2)$ and $\mathrm{mCt}(D_1\Box D_2)$, when D_1 and D_2 are two strong digraphs such that at least one of D_1 and D_2 satisfies the two-sided eccentricity property. But only the inclusion $\mathrm{mEcc}(D_1\Box D_2) \subseteq \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2)$ holds when at least one of D_1 and D_2 satisfies the two-sided eccentricity property. Also, nothing could be said about the m-boundary of $D_1\Box D_2$ in connection with the two-sided eccentricity property.

Proposition 4.4.5. Let D_1 and D_2 be two strong digraphs such that at least one of D_1 and D_2 satisfies the two-sided eccentricity property. Then $\mathrm{mEcc}(D_1 \square D_2) \subseteq \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2)$.

Proof. Let $(u_i, v_r) \in \mathrm{mEcc}(D_1 \square D_2)$. Hence, there exists a vertex (u_j, v_q) such that $\mathrm{mecc}_{D_1 \square D_2}(u_j, v_q) = \mathrm{md}_{D_1 \square D_2}((u_j, v_q), (u_i, v_r))$. But $\mathrm{md}_{D_1 \square D_2}((u_j, v_q), (u_i, v_r)) \leq \mathrm{md}_{D_1}(u_j, u_i) + \mathrm{md}_{D_2}(v_q, v_r) \leq \mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_q)$. Since at least one of the digraphs satisfies the two-sided eccentricity property, $\mathrm{mecc}_{D_1 \square D_2}(u_j, v_q) = \mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_q)$. Thus $\mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_q) = \mathrm{md}_{D_1}(u_j, u_i) + \mathrm{md}_{D_2}(v_q, v_r)$ which implies, $\mathrm{mecc}_{D_1}(u_j) = \mathrm{md}_{D_1}(u_j, u_i)$ and $\mathrm{mecc}_{D_2}(v_q) = \mathrm{md}_{D_2}(v_q, v_r)$. This implies, $u_i \in \mathrm{mEcc}(D_1)$ and $v_r \in \mathrm{mEcc}(D_2)$. □

Remark 4.4.6. In general, $\mathrm{mEcc}(D_1 \square D_2) \neq \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2)$ and $\mathrm{m}\partial(D_1 \square D_2) \neq \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2)$ even if both D_1 and D_2 satisfy the two-sided eccentricity property. To see this, consider the digraphs in Figure 4.3. Here, $\mathrm{mEcc}(D_1) = \mathrm{m}\partial(D_1) = \{u_1, u_3, u_4, u_5\}$ and $\mathrm{mEcc}(D_2) = \mathrm{m}\partial(D_2) = \{v_1, v_3, v_4, v_5\}$, but $(u_3, v_3) \notin \mathrm{mEcc}(D_1 \square D_2)$ as (u_3, v_3) is not an m-eccentric vertex or m-boundary vertex of any vertex in $D_1 \square D_2$.

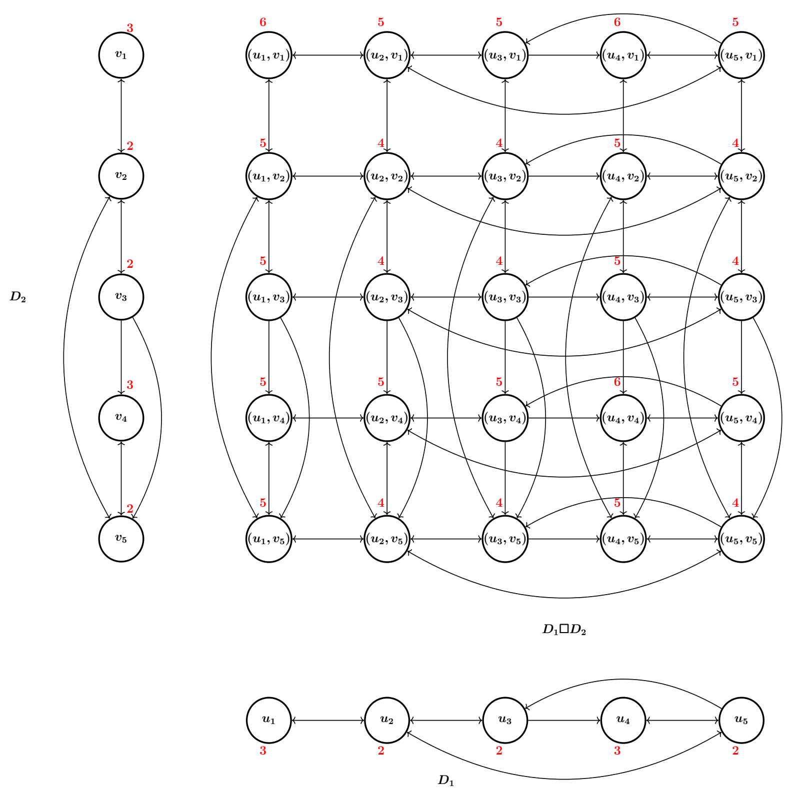

Figure 4.3: An example in which D_1 and D_2 satisfy two-sided eccentricity property, but $\mathrm{mEcc}(D_1 \square D_2) \neq \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2)$, $\mathrm{m}\partial(D_1 \square D_2) \neq \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2)$

But in the following proposition, it is proved that if one of the digraphs is a directed cycle or a Cartesian product of directed cycles, then $\text{mEcc}(D_1 \square D_2) = \text{mEcc}(D_1) \times \text{mEcc}(D_2)$ and $\text{m}\partial(D_1 \square D_2) = \text{m}\partial(D_1) \times \text{m}\partial(D_2)$.

Before proving the proposition, some characteristics of vertices of directed cycles and products of directed cycles are considered. Let D be a directed cycle $u_1 \to u_2 \to \cdots \to u_m \to u_1$. Then $\overrightarrow{d}_D(u_i, u_{i-1}) = \overrightarrow{d}_D(u_{i+1}, u_i) = m-1$, where $u_{m+1} = u_1$. Thus $\text{mecc}_D(u_i) = m-1$, for $i = 1, 2, \ldots, m$ and so $u_i \in \text{mPer}(D)$ and $u_i \in \text{mCt}(D)$ for $i = 1, 2, \ldots, m$. Since each u_i is an m-eccentric vertex and so an m-boundary vertex of u_{i-1} and u_{i+1}, it follows that $u_i \in \text{mEcc}(D)$ and $u_i \in \text{m}\partial(D)$ for $i = 1, 2, \ldots, m$. Thus, $\text{mPer}(D) = \text{mCt}(D) = \text{mEcc}(D) = \text{m}\partial(D) = V(D)$.

Now let D be the Cartesian product of directed cycles D_1 and D_2. Let D_1 be the directed cycle $u_1 \to u_2 \to \cdots \to u_m \to u_1$ and D_2 be the directed cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. Then in $D_1 \square D_2$, $\overrightarrow{d}_{D_1 \square D_2}((u_i, v_r), (u_{i-1}, v_{r-1}) = \overrightarrow{d}_{D_1 \square D_2}((u_{i+1}, v_{r+1}), (u_i, v_r) = (m-1) + (n-1)$, where $u_{m+1} = u_1$, $v_{n+1} = v_1$. Thus $\text{mecc}_{D_1 \square D_2}(u_i, v_r) = m + n - 2$, for $i = 1, 2, \ldots, m$, $r = 1, 2, \ldots, n$ and so $\text{mPer}(D) = \text{mCt}(D) = \text{mEcc}(D) = \text{m}\partial(D) = V(D) = V(D_1) \times V(D_2)$.

Again consider $D_1 \square D_2 \square D_3$ where D_1, D_2, D_3 are directed cycles $u_1 \to u_2 \to \cdots \to u_m \to u_1$, $v_1 \to v_2 \to \cdots \to v_n \to v_1$ and $w_1 \to w_2 \to \cdots \to w_\ell \to w_1$, respectively. As already seen in $D_1 \square D_2$, $\overrightarrow{d}_{D_1 \square D_2}((u_i, v_r), (u_{i-1}, v_{r-1}) = \overrightarrow{d}_{D_1 \square D_2}((u_{i+1}, v_{r+1}), (u_i, v_r)) = m + n - 2 = \text{mecc}_{D_1 \square D_2}(u_i, v_r)$, for all $(u_i, v_r) \in V(D_1 \square D_2)$. Now in $D_1 \square D_2 \square D_3$, $\overrightarrow{d}_{D_1 \square D_2 \square D_3}((u_i, v_r, w_p), (u_{i-1}, v_{r-1}, w_{p-1})) = \overrightarrow{d}_{D_1 \square D_2 \square D_3}((u_{i+1}, v_{r+1}, w_{p+1}), (u_i, v_r, w_p)) = m + n - 2 + \ell - 1 = m + n + \ell - 3 = \text{mecc}_{D_1 \square D_2 \square D_3}(u_i, v_r, w_p)$ for all $(u_i, v_r, w_p) \in V(D_1 \square D_2 \square D_3)$.

Continuing like this, it can be seen that $\text{mPer}(D) = \text{mCt}(D) = \text{mEcc}(D) = \text{m}\partial(D) = V(D)$ for any digraph D which is the Cartesian product of a finite number of directed cycles.

Proposition 4.4.7. If one of the digraphs D_1 and D_2 is a directed cycle or a Cartesian product of directed cycles, then $\text{m}\partial(D_1 \square D_2) = \text{m}\partial(D_1) \times \text{m}\partial(D_2)$ and $\text{mEcc}(D_1 \square D_2) = \text{mEcc}(D_1) \times \text{mEcc}(D_2)$.

Proof. Without loss of generality, suppose that D_1 is the Cartesian product of directed cycles and D_2 is an arbitrary strong digraph. Then $\text{mEcc}(D_1) = \text{m}\partial(D_1) = V(D_1)$. Let $\text{mecc}_{D_1}(u_i) = \ell$, for all $u_i \in V(D_1)$. Also, if v_r is an m-boundary vertex of v_s in D_2, then for every $u_i \in V(D_1)$, (u_i, v_r) is an m-boundary vertex of either (u_j, v_s) or (u_k, v_s) in $D_1 \square D_2$, where $u_j, u_k \in V(D_1)$ are such that $\overrightarrow{d}(u_i, u_j) = \overrightarrow{d}(u_k, u_i) = \ell$, and vice versa.

Similarly, if v_r is an m-eccentric vertex of v_s in D_2, then for every $u_i \in V(D_1)$, (u_i, v_r) is an m-eccentric vertex of either (u_j, v_s) or (u_k, v_s) in $D_1 \Box D_2$ and vice versa.
Thus $\mathrm{m}\partial(D_1 \Box D_2) = V(D_1) \times \mathrm{m}\partial(D_2)$ and $\mathrm{mEcc}(D_1 \Box D_2) = V(D_1) \times \mathrm{mEcc}(D_2)$ and hence the result. □

Now we prove the necessary and sufficient condition for the Cartesian product $D_1 \Box D_2$ to satisfy the two-sided eccentricity property.

Theorem 4.4.8. $D_1 \Box D_2$ satisfies the two-sided eccentricity property if and only if both D_1 and D_2 satisfy the two-sided eccentricity property.

Proof. Suppose that both D_1 and D_2 satisfy the two-sided eccentricity property. For every $u_i \in V(D_1)$, $v_r \in V(D_2)$, there exist vertices $u_j, u_k \in V(D_1)$ such that $\mathrm{mecc}_{D_1}(u_i) = \overrightarrow{d}_{D_1}(u_i, u_j) = \overrightarrow{d}_{D_1}(u_k, u_i)$, and v_q, v_s in $V(D_2)$ such that $\mathrm{mecc}_{D_2}(v_r) = \overrightarrow{d}_{D_2}(v_r, v_q) = \overrightarrow{d}_{D_2}(v_s, v_r)$. By Proposition 4.3.5, $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$, if at least one of D_1 and D_2 satisfy the two-sided eccentricity property. Hence $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r) = \overrightarrow{d}_{D_1}(u_i, u_j) + \overrightarrow{d}_{D_2}(v_r, v_q) = \overrightarrow{d}_{D_1 \Box D_2}((u_i, v_r), (u_j, v_q))$ and also $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r) = \overrightarrow{d}_{D_1}(u_k, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r) = \overrightarrow{d}_{D_1 \Box D_2}((u_k, v_s), (u_i, v_r))$. Thus $D_1 \Box D_2$ satisfies the two-sided eccentricity property.

Now to prove the converse part, suppose that at least one of D_1 and D_2 does not satisfy the two-sided eccentricity property. Assume that D_1 does not satisfy the two-sided eccentricity property. Then there exists a vertex $u_i \in V(D_1)$ such that $\mathrm{mecc}_{D_1}(u_i) = \overrightarrow{d}_{D_1}(u_i, u_j) > \overrightarrow{d}_{D_1}(u_k, u_i)$ for every $u_j, u_k \in V(D_1)$. D_2 may or may not satisfy the two-sided eccentricity property. Let $v_r \in V(D_2)$ such that $\mathrm{mecc}_{D_2}(v_r) = \overrightarrow{d}_{D_2}(v_r, v_q) \geq \overrightarrow{d}_{D_2}(v_s, v_r)$, for every $v_s \in V(D_2)$. In either case, $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \overrightarrow{d}_{D_1}(u_i, u_j) + \overrightarrow{d}_{D_2}(v_r, v_q) > \overrightarrow{d}_{D_1}(u_k, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r)$. That is, $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \overrightarrow{d}_{D_1 \Box D_2}((u_i, v_r), (u_j, v_q)) > \overrightarrow{d}_{D_1 \Box D_2}((u_k, v_s), (u_i, v_r))$, for every $(u_k, v_s) \in V(D_1 \Box D_2)$ and so $D_1 \Box D_2$ does not satisfy the two-sided eccentricity property. Thus $D_1 \Box D_2$ satisfies the two-sided eccentricity property only if both D_1 and D_2 satisfy the two-sided eccentricity property. □

Clearly, Theorem 4.4.8 can be extended to the case of a finite number of digraphs and hence, we get the following corollary.

Corollary 4.4.9. Let $D_1, D_2, \ldots, D_n$ be n strongly connected digraphs such that all except one satisfy the two-sided eccentricity property. Then

1. $\mathrm{mPer}(D_1 \Box D_2 \Box \cdots \Box D_n) = \mathrm{mPer}(D_1) \times \mathrm{mPer}(D_2) \times \cdots \times \mathrm{mPer}(D_n)$,

2. $\text{mCt}(D_1 \square D_2 \square \cdots \square D_n) = \text{mCt}(D_1) \times \text{mCt}(D_2) \times \cdots \times \text{mCt}(D_n)$.

Proof. Since the Cartesian product of digraphs is commutative and associative, $D_1, D_2, \ldots, D_n$ may be arranged such that the first $n-1$ digraphs $D_1, D_2, \ldots, D_{n-1}$ satisfy the two-sided eccentricity property. As Theorem 4.4.8 can be extended for a finite number of digraphs, $D' = D_1 \square D_2 \square \cdots \square D_{n-1}$ is strongly connected and satisfy the two-sided eccentricity property. Hence it follows from Proposition 4.4.4 that $\text{mPer}(D' \square D_n) = \text{mPer}(D') \times \text{mPer}(D_n)$ and $\text{mCt}(D' \square D_n) = \text{mCt}(D') \times \text{mCt}(D_n)$. Since $D_1, D_2, \ldots, D_{n-1}$ satisfy the two-sided eccentricity property, $\text{mPer}(D') = \text{mPer}(D_1) \times \text{mPer}(D_2) \times \cdots \times \text{mPer}(D_{n-1})$ and $\text{mCt}(D') = \text{mCt}(D_1) \times \text{mCt}(D_2) \times \cdots \times \text{mCt}(D_{n-1})$. Hence the result. □

The following proposition deals with the m-distance between two vertices in the Cartesian product of two strong digraphs if one of them is a symmetric digraph.

Proposition 4.4.10. Let D_1 and D_2 be two strong digraphs such that one of them is a symmetric digraph. Then in $D_1 \square D_2$,

$$\text{md}_{D_1 \square D_2}((u_i, v_r), (u_j, v_s)) = \text{md}_{D_1}(u_i, u_j) + \text{md}_{D_2}(v_r, v_s),$$

for every $u_i, u_j \in V(D_1)$ and for every $v_r, v_s \in V(D_2)$.

Proof. Since Cartesian product is commutative, without loss of generality, suppose that D_1 is a symmetric digraph. Then $\overrightarrow{d}_{D_1}(u_i, u_j) = \overrightarrow{d}_{D_1}(u_j, u_i) = \text{md}_{D_1}(u_i, u_j)$ for every $u_i, u_j \in V(D_1)$.

$$\begin{aligned}
\text{md}_{D_1 \square D_2}((u_i, v_r), (u_j, v_s)) &= \max\{\overrightarrow{d}_{D_1}(u_i, u_j) + \overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_1}(u_j, u_i) + \overrightarrow{d}_{D_2}(v_s, v_r)\} \\
&= \max\{\overrightarrow{d}_{D_1}(u_i, u_j) + \overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_1}(u_i, u_j) + \overrightarrow{d}_{D_1}(v_s, v_r)\} \\
&= \overrightarrow{d}_{D_1}(u_i, u_j) + \max\{\overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_2}(v_s, v_r)\} \\
&= \text{md}_{D_1}(u_i, u_j) + \text{md}_{D_2}(v_r, v_s).
\end{aligned}$$

□

Thus we can see that if one of the digraphs is a symmetric digraph, then we get the same results for the boundary-type sets of the Cartesian product of two strong digraphs as in the case of the Cartesian product of two connected graphs.

Proposition 4.4.11. Let D_1 and D_2 be two strong digraphs such that one of them is a symmetric digraph. Then

1. $\mathrm{m}\partial(D_1\Box D_2) = \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2)$,
2. $\mathrm{mEcc}(D_1\Box D_2) = \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2)$,
3. $\mathrm{mPer}(D_1\Box D_2) = \mathrm{mPer}(D_1) \times \mathrm{mPer}(D_2)$,
4. $\mathrm{mCt}(D_1\Box D_2) = \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)$.

Proof. Since Cartesian multiplication of digraphs is commutative, without loss of generality, suppose that D_1 is a symmetric digraph.

1. Let $(u_i, v_r) \in \mathrm{m}\partial(D_1\Box D_2)$. If possible let $u_i \notin \mathrm{m}\partial(D_1)$. Then for every $u_j \in V(D_1)$, there exists $u_k \in N_{D_1}(u_i)$ such that $\mathrm{md}D_1(u_j, u_k) > \mathrm{md}_{D_1}(u_j, u_i)$. Let v_q be an arbitrary vertex in $V(D_2)$. Consider the vertex $(u_k, v_r) \in N_{D_1\Box D_2}(u_i, v_r)$. Then by Proposition 4.4.10, $\mathrm{md}_{D_1\Box D_2}((u_j, v_q),(u_k, v_r)) = \mathrm{md}_{D_1}(u_j, u_k) + \mathrm{md}_{D_2}(v_q, v_r) > \mathrm{md}_{D_1}(u_j, u_i) + \mathrm{md}_{D_2}(v_q, v_r) = \mathrm{md}_{D_1\Box D_2}((u_j, v_q),(u_i, v_r))$ which is a contradiction to $(u_i, v_r) \in \mathrm{m}\partial(D_1\Box D_2)$. Hence $u_i \in \mathrm{m}\partial(D_1)$. Similarly it can be proved that $v_r \in \mathrm{m}\partial(D_2)$.

   Conversely, let $u_i \in \mathrm{m}\partial(D_1)$ and $v_r \in \mathrm{m}\partial(D_2)$. This implies that there exists vertices $u_j \in V(D_1)$ and $v_q \in V(D_2)$ such that for every $u_k \in N_{D_1}(u_i)$, $\mathrm{md}_{D_1}(u_j, u_i) \geq \mathrm{md}_{D_1}(u_j, u_k)$ and for every $v_s \in N_{D_2}(v_r)$, $\mathrm{md}_{D_2}(v_q, v_r) \geq \mathrm{md}_{D_2}(v_q, v_s)$. Consider an arbitrary vertex $(u_k, v_s) \in N_{D_1\Box D_2}(u_i, v_r)$. Without loss of generality assume that u_k is adjacent to u_i in D_1 and $v_r = v_s$ in D_2. Then by Proposition 4.4.10, $\mathrm{md}_{D_1\Box D_2}((u_j, v_q),(u_k, v_s)) = \mathrm{md}_{D_1}(u_j, u_k) + \mathrm{md}_{D_2}(v_q, v_s) \leq \mathrm{md}_{D_1}(u_j, u_i) + \mathrm{md}_{D_2}(v_q, v_r) = \mathrm{md}_{D_1\Box D_2}((u_j, v_q),(u_i, v_r))$ which gives (u_i, v_r) is an m-boundary vertex of (u_j, v_q) in $D_1\Box D_2$.

2. Since D_1 is a symmetric digraph, $\mathrm{mecc}_{D_1\Box D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$, for every $(u_i, v_r) \in V(D_1\Box D_2)$. Let $(u_i, v_r) \in \mathrm{mEcc}(D_1\Box D_2)$. So there exists a vertex (u_j, v_q) such that $\mathrm{mecc}_{D_1\Box D_2}(u_j, v_q) = \mathrm{md}_{D_1\Box D_2}((u_j, v_q),(u_i, v_r))$. Hence $\mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_q) = \mathrm{md}_{D_1}(u_j, u_i) + \mathrm{md}_{D_2}(v_q, v_r)$. This implies, $\mathrm{mecc}_{D_1}(u_j) = \mathrm{md}_{D_1}(u_j, u_i)$ and $\mathrm{mecc}_{D_2}(v_q) = \mathrm{md}_{D_2}(v_q, v_r)$, which implies $u_i \in \mathrm{mEcc}(D_1)$ and $v_r \in \mathrm{mEcc}(D_2)$.

   Conversely if $u_i \in \mathrm{mEcc}(D_1)$ and $v_r \in \mathrm{mEcc}(D_2)$, then there are vertices $u_j \in V(D_1)$ and $v_q \in V(D_2)$ such that $\mathrm{mecc}_{D_1}(u_j) = \mathrm{md}_{D_1}(u_j, u_i)$ and $\mathrm{mecc}_{D_2}(v_q) = \mathrm{md}_{D_2}(v_q, v_r)$. Hence by Proposition 4.4.10, $\mathrm{mecc}_{D_1\Box D_2}(u_j, v_q) = \mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_q) = \mathrm{md}_{D_1}(u_j, u_i) + \mathrm{md}_{D_2}(v_q, v_r) = \mathrm{md}_{D_1\Box D_2}((u_j, v_q),(u_i, v_r))$ which gives (u_i, v_r) is an m-eccentric vertex of (u_j, v_q) in $D_1\Box D_2$.

Since D_1 satisfies the two-sided eccentricity property, $\mathrm{mPer}(D_1\Box D_2) = \mathrm{mPer}(D_1) \times \mathrm{mPer}(D_2)$ and $\mathrm{mCt}(D_1\Box D_2) = \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)$. Hence items 3 and 4 of the proposition holds. □

Combining propositions 4.4.4, 4.4.7 and 4.4.11 yields the following corollary.

Corollary 4.4.12. Let $D_1(V_1, E_1), D_2(V_2, E_2), \ldots, D_n(V_n, E_n)$ be n strong digraphs. If all except one of $D_1, D_2, \ldots, D_n$ are either directed cycles or symmetric digraphs, then

1. $\mathrm{mPer}(D_1 \Box D_2 \Box \cdots \Box D_n) = \mathrm{mPer}(D_1) \times \mathrm{mPer}(D_2) \times \cdots \times \mathrm{mPer}(D_n)$,
2. $\mathrm{mCt}(D_1 \Box D_2 \Box \cdots \Box D_n) = \mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2) \times \cdots \times \mathrm{mCt}(D_n)$,
3. $\mathrm{mEcc}(D_1 \Box D_2 \Box \cdots \Box D_n) = \mathrm{mEcc}(D_1) \times \mathrm{mEcc}(D_2) \times \cdots \times \mathrm{mEcc}(D_n)$,
4. $\mathrm{m}\partial(D_1 \Box D_2 \Box \cdots \Box D_n) = \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2) \times \cdots \times \mathrm{m}\partial(D_n)$.

4.5 Center of Cartesian Product of Digraphs

For any two graphs G and H, it is evident that $\mathrm{Cen}(G \Box H) = \mathrm{Cen}(G) \times \mathrm{Cen}(H)$. But it can be seen that the result $\mathrm{mCen}(D_1 \Box D_2) = \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$ is not generally true in the case of two strong digraphs. For the digraphs D_1 and D_2 in Figure 4.4, $\mathrm{mCen}(D_1) = \{u_2, u_3\}$, $\mathrm{mCen}(D_2) = \{v_2, v_3\}$, and $\mathrm{mCen}(D_1 \Box D_2) = \{(u_2, v_2), (u_3, v_2), (u_2, v_3), (u_3, v_3), (u_3, v_4)\}$. Hence $\mathrm{mCen}(D_1 \Box D_2) \nsubseteq \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$. Also, it can be seen from Figure 4.1 that in general, $\mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2) \nsubseteq \mathrm{mCen}(D_1 \Box D_2)$, in general as $\mathrm{mCen}(D_1) = \{u_1, u_2, u_3\}$, $\mathrm{mCen}(D_2) = \{v_1, v_2, v_3\}$, but $\mathrm{mCen}(D_1 \Box D_2) = \{(u_1, v_1), (u_2, v_2)\}$.

We prove that if at least one of D_1 and D_2 satisfies the two-sided eccentricity property, then $\mathrm{mCen}(D_1 \Box D_2) = \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$.

Proposition 4.5.1. Let D_1 and D_2 be two strongly connected digraphs such that at least one of D_1 and D_2 satisfies the two-sided eccentricity property. Then $\mathrm{mCen}(D_1 \Box D_2) = \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$.

Proof. First, to show that $\mathrm{mCen}(D_1 \Box D_2) \subseteq \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$, suppose that $(u_i, v_r) \in \mathrm{mCen}(D_1 \Box D_2)$. Then $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \mathrm{mrad}(D_1 \Box D_2) = \min\limits_{(u_j, v_s) \in V(D_1 \Box D_2)} \mathrm{mecc}_{D_1 \Box D_2}(u_j, v_s) = \min\limits_{u_j \in V(D_1), v_s \in V(D_2)} [\mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_s)]$. Since at least one of D_1 and D_2 satisfies the two-sided eccentricity property, by Proposition 4.3.5, $\mathrm{mecc}_{D_1 \Box D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$. Thus it follows that $\mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r) = \min\limits_{u_j \in V(D_1)} \mathrm{mecc}_{D_1}(u_j) + \min\limits_{v_s \in V(D_2)} \mathrm{mecc}_{D_2}(v_s)$.

Then necessarily $\mathrm{mecc}_{D_1}(u_i) = \min\limits_{u_j \in V(D_1)} \mathrm{mecc}_{D_1}(u_j)$ and $\mathrm{mecc}_{D_2}(v_r) = \min\limits_{v_s \in V(D_2)} \mathrm{mecc}_{D_2}(v_s)$.

Therefore $u_i \in \mathrm{mCen}(D_1)$ and $v_r \in \mathrm{mCen}(D_2)$.

Now, to prove that $\mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2) \subseteq \mathrm{mCen}(D_1 \square D_2)$, let $u_i \in \mathrm{mCen}(D_1)$ and $v_r \in \mathrm{mCen}(D_2)$. Thus, $\mathrm{mecc}_{D_1}(u_i) \leq \mathrm{mecc}_{D_1}(u_j)$ for all $u_j \in V(D_1)$ and $\mathrm{mecc}_{D_2}(v_r) \leq \mathrm{mecc}_{D_2}(v_s)$ for all $v_s \in V(D_2)$. Hence $\mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r) \leq \mathrm{mecc}_{D_1}(u_j) + \mathrm{mecc}_{D_2}(v_s)$ for all $u_j \in V(D_1)$ and for all $v_s \in V(D_2)$. But by Proposition 4.3.5, $\mathrm{mecc}_{D_1 \square D_2}(u_i, v_r) = \mathrm{mecc}_{D_1}(u_i) + \mathrm{mecc}_{D_2}(v_r)$ for all $(u_i, v_r) \in V(D_1 \square D_2)$. This implies, $\mathrm{mecc}_{D_1 \square D_2}(u_i, v_r) \leq \mathrm{mecc}_{D_1 \square D_2}(u_j, v_s)$ for all $(u_j, v_s) \in V(D_1 \square D_2)$, and hence $(u_i, v_r) \in \mathrm{mCen}(D_1 \square D_2)$. □

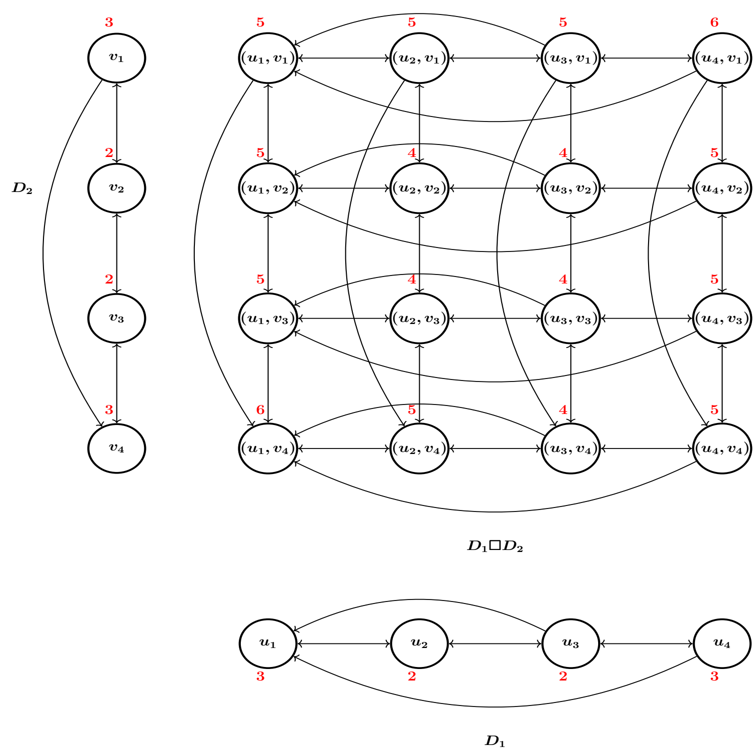

Figure 4.4: An example in which $\mathrm{mCen}(D_1 \square D_2) \nsubseteq \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$

Extending Proposition 4.5.1 to a finite number of digraphs, we get the following corollary.

Corollary 4.5.2. Let $D_1, \ldots, D_n$ be n strongly connected digraphs such that at most one of these digraphs does not satisfy the two-sided eccentricity property. Then

$$\mathrm{mCen}(D_1 \square \cdots \square D_n) = \mathrm{mCen}(D_1) \times \cdots \times \mathrm{mCen}(D_n).$$

Proof. Since Cartesian product of digraphs is commutative and associative, without loss of generality, suppose that $D_1, \ldots, D_{n-1}$ satisfy the two-sided eccentricity property, and D_n does not satisfy the two-sided eccentricity property. Then

$D'_2 = D_1 \square D_2$ is strongly connected and satisfies the two-sided eccentricity property, by Theorem 4.4.8. Also, by Proposition 4.5.1, $\mathrm{mCen}(D_1 \square D_2) = \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$ and $\mathrm{mCen}(D_1 \square D_2 \square D_3) = \mathrm{mCen}(D'_2 \square D_3) = \mathrm{mCen}(D'_2) \times \mathrm{mCen}(D_3) = \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2) \times \mathrm{mCen}(D_3)$. Now, let $D'_3 = D_1 \square D_2 \square D_3 = D'_2 \square D_3$. Then D'_3 is strongly connected and satisfies the two-sided eccentricity property, and so $\mathrm{mCen}(D_1 \square D_2 \square D_3 \square D_4) = \mathrm{mCen}(D'_3 \square D_4) = \mathrm{mCen}(D'_3) \times \mathrm{mCen}(D_4) = \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2) \times \mathrm{mCen}(D_3) \times \mathrm{mCen}(D_4)$, and so on. Continuing like this, taking $D'_{n-1} = D_1 \square \cdots \square D_{n-1}$, it follows that $\mathrm{mCen}(D_1 \square \cdots \square D_n) = \mathrm{mCen}(D'_{n-1}) \times \mathrm{mCen}(D_n) = \mathrm{mCen}(D_1) \times \cdots \times \mathrm{mCen}(D_n)$. □

Chapter 5

Boundary-type Sets and Center in Strong Product

5.1 Introduction

The boundary-type sets and the center of the strong product of digraphs is studied in this chapter. In Section 5.2, the expressions for the m-distance between two vertices, the m-eccentricity of a vertex, the m-radius, and the m-diameter of the strong product of two strongly connected digraphs are derived. In Section 5.3, the expressions for the four boundary-type sets of the strong product of two strongly connected digraphs are obtained. Finally, in Section 5.4, the m-center and the m-periphery of the strong product of a finite number of strongly connected digraphs are obtained.

The open neighborhood $N(v)$ can be replaced by the closed neighborhood $N[v]$ in the definitions of the boundary and the contour sets. This does not affect the definitions, but it is necessary for proving the relationship between the m-boundary and the m-contour sets of the strong product of two digraphs and its factors.

5.2 Distance Between Two Vertices

The distance between two vertices (g, h) and (g', h') in the strong product $G \boxtimes H$ of two graphs G and H is given in [36] as follows:
$d_{G \boxtimes H}((g, h), (g', h')) = \max\{d_G(g, g'), d_H(h, h')\}$.
So in the case of two digraphs D_1 and D_2, it follows that the directed distance $\overrightarrow{d}_{D_1 \boxtimes D_2}((u_i, v_r), (u_j, v_s)) = \max\{\overrightarrow{d}_{D_1}(u_i, u_j), \overrightarrow{d}_{D_2}(v_r, v_s)\}$. Hence we derive the following results for the metric md.

Lemma 5.2.1. Let D_1 and D_2 be two strongly connected digraphs. Then

$$\text{md}_{D_1 \boxtimes D_2}((u_i, v_r), (u_j, v_s)) = \max\{\text{md}_{D_1}(u_i, u_j), \text{md}_{D_2}(v_r, v_s)\},$$
$$\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}.$$

Proof. $\text{md}_{D_1 \boxtimes D_2}((u_i, v_r), (u_j, v_s))$
$= \max\{\overrightarrow{d}_{D_1 \boxtimes D_2}((u_i, v_r), (u_j, v_s)), \overrightarrow{d}_{D_1 \boxtimes D_2}((u_j, v_s), (u_i, v_r))\}$
$= \max\{\max\{\overrightarrow{d}_{D_1}(u_i, u_j), \overrightarrow{d}_{D_2}(v_r, v_s)\}, \max\{\overrightarrow{d}_{D_1}(u_j, u_i), \overrightarrow{d}_{D_2}(v_s, v_r)\}\}$
$= \max\{\max\{\overrightarrow{d}_{D_1}(u_i, u_j), \overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_1}(u_j, u_i), \overrightarrow{d}_{D_2}(v_s, v_r)\}\}$
$= \max\{\max\{\overrightarrow{d}_{D_1}(u_i, u_j), \overrightarrow{d}_{D_1}(u_j, u_i)\}, \max\{\overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_2}(v_s, v_r)\}\}$
$= \max\{\text{md}_{D_1}(u_i, u_j), \text{md}_{D_2}(v_r, v_s)\}$.
Hence it follows that
$\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r)$
$= \max\{\text{md}_{D_1 \boxtimes D_2}((u_i, v_r), (u_j, v_s)) : (u_j, v_s) \in V(D_1 \boxtimes D_2)\}$
$= \max\{\max\{\text{md}_{D_1}(u_i, u_j), \text{md}_{D_2}(v_r, v_s)\} : u_j \in V(D_1), v_s \in V(D_2)\}$
$= \max\{\max\{\text{md}_{D_1}(u_i, u_j) : u_j \in V(D_1)\}, \max\{\text{md}_{D_2}(v_r, v_s) : v_s \in V(D_2)\}\}$
$= \max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}$. □

Corollary 5.2.2. Let D_1 and D_2 be two strongly connected digraphs. Then

$$\text{mrad}(D_1 \boxtimes D_2) = \max\{\text{mrad}(D_1), \text{mrad}(D_2)\},$$
$$\text{mdiam}(D_1 \boxtimes D_2) = \max\{\text{mdiam}(D_1), \text{mdiam}(D_2)\}.$$

Proof.

$$\begin{aligned}
\text{mrad}(D_1 \boxtimes D_2) &= \min_{(u_i, v_r) \in V(D_1 \boxtimes D_2)} \{\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r)\} \\
&= \min_{\substack{u_i \in V(D_1), \\ v_r \in V(D_2)}} \{\max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}\} \\
&= \max\{\min_{u_i \in V(D_1)} \{\text{mecc}_{D_1}(u_i)\}, \min_{v_r \in V(D_2)} \{\text{mecc}_{D_2}(v_r)\}\} \\
&= \max\{\text{mrad}(D_1), \text{mrad}(D_2)\}. \\
\text{mdiam}(D_1 \boxtimes D_2) &= \max_{(u_i, v_r) \in V(D_1 \boxtimes D_2)} \{\text{mecc}(u_i, v_r)\} \\
&= \max_{\substack{u_i \in V(D_1), \\ v_r \in V(D_2)}} \{\max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}\} \\
&= \max\{\max_{u_i \in V(D_1)} \{\text{mecc}(u_i)\}, \max_{v_r \in V(D_2)} \{\text{mecc}(v_r)\}\} \\
&= \max\{\text{mdiam}(D_1), \text{mdiam}(D_2)\}.
\end{aligned}$$

□

The following results are given in the book 'Handbook of Product Graphs' [36].

1. The strong product of two directed graphs is strongly connected if and only if both the digraphs are strongly connected.

2. If G and H are two graphs, then $N_{G\boxtimes H}[(g,h)] = N_G[g] \times N_H[h]$.

For a fixed vertex $(u_i, v_r) \in V(D_1 \boxtimes D_2)$, the only possible vertices in $N_{D_1\boxtimes D_2}[(u_i, v_r)]$ are those vertices in $N_{G\boxtimes H}[(u_i, v_r)] = N_G[u_i] \times N_H[v_r] = N_{D_1}[u_i] \times N_{D_2}[v_r]$, where G and H are the underlying graphs of D_1 and D_2, respectively. If $u_k \in N_{D_1}(u_i)$ and $v_q \in N_{D_2}(v_r)$, then the vertex (u_k, v_q) need not belong to $N_{D_1\boxtimes D_2}(u_i, v_r)$. But for all vertices $(u_j, v_s) \in V(D_1 \boxtimes D_2)$, at least one of the following inequalities must be satisfied. That is, either $\mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_k, v_q)) \leq \mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_k, v_r))$ or $\mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_k, v_q)) \leq \mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_i, v_q))$. This is because $\mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_k, v_q)) = \max\{\mathrm{md}_{D_1}(u_j, u_k), \mathrm{md}_{D_2}(v_s, v_q)\}$. It is either $\mathrm{md}_{D_1}(u_j, u_k)$ or $\mathrm{md}_{D_2}(v_s, v_q)$ and in any case, it cannot be greater than both $\mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_k, v_r))$ and $\mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_i, v_q))$. Hence, if $\mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_i, v_r)) \geq \mathrm{md}_{D_1\boxtimes D_2}((u_j, v_s), (u_k, v_q))$ for all vertices $(u_k, v_q) \in N_{D_1\boxtimes D_2}[(u_i, v_r)]$, then $\max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} \geq \{\mathrm{md}_{D_1}(u_j, u_k), \mathrm{md}_{D_2}(v_s, v_q)\}$ for all vertices $u_k \in N_{D_1}[u_i]$ and $v_q \in N_{D_2}[v_r]$. Hence it follows that, if $\mathrm{mecc}_{D_1\boxtimes D_2}(u_i, v_r) \geq \mathrm{mecc}_{D_1\boxtimes D_2}(u_k, v_q)$ for all vertices in $N_{D_1\boxtimes D_2}[(u_i, v_r)]$, then $\max\{\mathrm{mecc}_{D_1}(u_i), \mathrm{mecc}_{D_2}(v_r)\} \geq \max\{\mathrm{mecc}_{D_1}(u_k), \mathrm{mecc}_{D_2}(v_q)\}$ for all vertices $u_k \in N_{D_1}[u_i]$, $v_q \in N_{D_2}[v_r]$. These results are applied in the upcoming proofs.

5.3 Boundary-type Sets of Strong Product of Digraphs

In [18], Cáceres et al. presented a description of the boundary-type sets of two graphs and their description of the boundary is as follows.

For two graphs G and H, $\partial(G \boxtimes H) = \big[\partial(G) \times V(H)\big] \bigcup \big[V(G) \times \partial(H)\big]$.
But this result does not hold in the case of two digraphs with respect to the metric md.

Consider the strong product, $D_1 \boxtimes D_2$ of the digraphs D_1 and D_2 in Figure 5.1. The eccentricity of each vertex is displayed near the vertex.

$\text{mPer}(D_1) = \text{mEcc}(D_1) = \text{mCt}(D_1) = \{u_1, u_4\}$, $\text{mPer}(D_2) = \text{mEcc}(D_2) = \text{mCt}(D_2) = \{v_1, v_2\}$, and $\text{mPer}(D_1 \boxtimes D_2) = \text{mEcc}(D_1 \boxtimes D_2) = \text{mCt}(D_1 \boxtimes D_2) = \{(u_1, v_1), (u_4, v_1), (u_1, v_2), (u_4, v_2)\}$. Also, $\text{m}\partial(D_1) = \{u_1, u_4\}$, $\text{m}\partial(D_2) = \{v_1, v_2\}$, and $\text{m}\partial(D_1 \boxtimes D_2) = \{(u_1, v_1), (u_4, v_1), (u_1, v_2), (u_4, v_2)\}$. The reason for $(u_2, v_1), (u_2, v_2), (u_3, v_1), (u_3, v_2) \notin \text{m}\partial(D_1 \boxtimes D_2)$ is explained after the proof of Theorem 5.3.1.

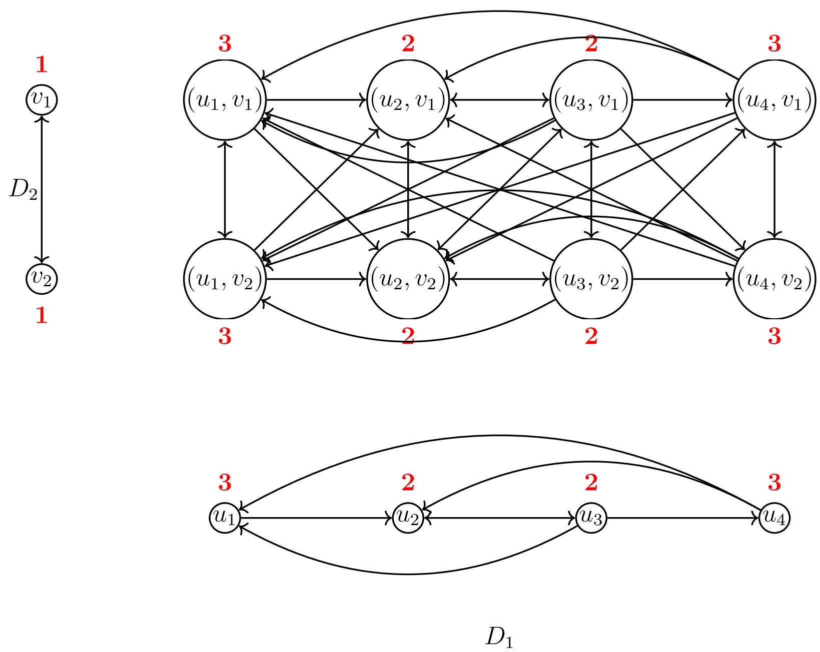

Figure 5.1: Example for strong product of digraphs

The results concerning the boundary-type sets of the strong product of two strongly connected digraphs are presented below. In all these results, D_1 and D_2 can be interchanged due to the commutativity of strong product of digraphs.

Claim: $\text{m}\partial(D_1 \boxtimes D_2) \subseteq [\text{m}\partial(D_1) \times V(D_2)] \bigcup [V(D_1) \times \text{m}\partial(D_2)]$.

To this end, let $(u_i, v_r) \in \text{m}\partial(D_1 \boxtimes D_2)$. Then there exists a vertex $(u_j, v_s) \in V(D_1 \boxtimes D_2)$ such that $\text{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_i, v_r)) \geq \text{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_k, v_q))$ for every $(u_k, v_q) \in N_{D_1 \boxtimes D_2}[(u_i, v_r)]$. This implies, $\max\{\text{md}_{D_1}(u_j, u_i), \text{md}_{D_2}(v_s, v_r)\} \geq \max\{\text{md}_{D_1}(u_j, u_k), \text{md}_{D_2}(v_s, v_q)\}$ for every $u_k \in N_{D_1}[u_i]$ and for every $v_q \in N_{D_2}[v_r]$. Hence $\text{md}_{D_1}(u_j, u_i) \geq \text{md}_{D_1}(u_j, u_k)$ for every $u_k \in N_{D_1}[u_i]$, or $\text{md}_{D_2}(v_s, v_r) \geq \text{md}_{D_2}(v_s, v_q)$ for every $v_q \in N_{D_2}[v_r]$. Thus, $u_i \in \text{m}\partial(D_1)$ or $v_r \in \text{m}\partial(D_2)$ or both. That is, if $(u_i, v_r) \in \text{m}\partial(D_1 \boxtimes D_2)$, then at least one of the vertices u_i and v_r must be a boundary vertex in the corresponding factor graph.

The following theorem gives the expression for the m-boundary of the strong product of two strongly connected digraphs.

Theorem 5.3.1. Let D_1 and D_2 be two strongly connected digraphs.
Then $\mathrm{m}\partial(D_1 \boxtimes D_2) = A_1 \bigcup A_2 \bigcup A_3$, where
$A_1 = \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2)$,
$A_2 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \in \mathrm{m}\partial(D_1),\ v_r \notin \mathrm{m}\partial(D_2)$, and $\exists v_t \in V(D_2)$ such that $\mathrm{md}_{D_2}(v_t, v_q) \leq \mathrm{mecc}_{D_1}(u_i),\ \forall v_q \in N_{D_2}[v_r]\}$, and
$A_3 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \notin \mathrm{m}\partial(D_1),\ v_r \in \mathrm{m}\partial(D_2)$, and $\exists u_\ell \in V(D_1)$ such that $\mathrm{md}_{D_1}(u_\ell, u_k) \leq \mathrm{mecc}_{D_2}(v_r),\ \forall u_k \in N_{D_1}[u_i]\}$.

Proof. Suppose that $(u_i, v_r) \in \mathrm{m}\partial(D_1 \boxtimes D_2)$.

Then there exists a vertex $(u_j, v_s) \in V(D_1 \boxtimes D_2)$ such that $\mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_i, v_r)) \geq \mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_k, v_q))$ for all vertices $(u_k, v_q) \in N_{D_1 \boxtimes D_2}[(u_i, v_r)]$. Since $\mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_i, v_r)) = \max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\}$ and $\mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_k, v_q)) = \max\{\mathrm{md}_{D_1}(u_j, u_k), \mathrm{md}_{D_2}(v_s, v_q)\}$, it follows that $\max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} \geq \max\{\mathrm{md}_{D_1}(u_j, u_k), \mathrm{md}_{D_2}(v_s, v_q)\}$ for all $u_k \in N_{D_1}[u_i]$, $v_q \in N_{D_2}[v_r]$. Four cases are distinguished.

Case 1: $\max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} = \mathrm{md}_{D_1}(u_j, u_i)$ and $\mathrm{md}_{D_2}(v_s, v_r) \geq \mathrm{md}_{D_2}(v_s, v_q)$ for all $v_q \in N_{D_2}[v_r]$

Case 2: $\max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} = \mathrm{md}_{D_1}(u_j, u_i)$ and $\mathrm{md}_{D_2}(v_s, v_r) \geq \mathrm{md}_{D_2}(v_s, v_q)$ does not hold for all $v_q \in N_{D_2}[v_r]$

Case 3: $\max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} = \mathrm{md}_{D_2}(v_s, v_r)$ and $\mathrm{md}_{D_1}(u_j, u_i) \geq \mathrm{md}_{D_1}(u_j, u_k)$ for all $u_k \in N_{D_1}[u_i]$

Case 4: $\max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} = \mathrm{md}_{D_2}(v_s, v_r)$ and $\mathrm{md}_{D_1}(u_j, u_i) \geq \mathrm{md}_{D_1}(u_j, u_k)$ does not hold for all $u_k \in N_{D_1}[u_i]$

In Cases 1 and 3, $\mathrm{md}_{D_1}(u_j, u_i) \geq \mathrm{md}_{D_1}(u_j, u_k)$ for all $u_k \in N_{D_1}[u_i]$ and $\mathrm{md}_{D_2}(v_s, v_r) \geq \mathrm{md}_{D_2}(v_s, v_q)$ for all $v_q \in N_{D_2}[v_r]$. So $u_i \in \mathrm{m}\partial(D_1)$, $v_r \in \mathrm{m}\partial(D_2)$, and hence $(u_i, v_r) \in A_1$.

In Case 2, $u_i \in \mathrm{m}\partial(D_1)$ and v_r is not a boundary vertex of v_s in D_2. If there exists any vertex v_t such that v_r is a boundary vertex of v_t, then $(u_i, v_r) \in A_1$. Otherwise, since $v_r \notin \mathrm{m}\partial(D_2)$, for every vertex $v_t \in V(D_2)$, there exists some vertex $v_q \in N_{D_2}[v_r]$ such that $\mathrm{md}_{D_2}(v_t, v_r) < \mathrm{md}_{D_2}(v_t, v_q)$. Hence if (u_i, v_r) is a boundary vertex of a vertex (u_ℓ, v_t) in $D_1 \boxtimes D_2$, then $\mathrm{md}_{D_1 \boxtimes D_2}((u_\ell, v_t), (u_i, v_r)) = \max\{\mathrm{md}_{D_1}(u_\ell, u_i), \mathrm{md}_{D_2}(v_t, v_r)\} = \mathrm{md}_{D_1}(u_\ell, u_i) > \mathrm{md}_{D_2}(v_t, v_r)$, for otherwise

$\mathrm{md}_{D_1}(u_\ell, u_i) \leq \mathrm{md}_{D_2}(v_t, v_r)$ and hence $\mathrm{md}_{D_1 \boxtimes D_2}((u_\ell, v_t), (u_i, v_r)) = \mathrm{md}_{D_2}(v_t, v_r) < \mathrm{md}_{D_2}(v_t, v_q) = \mathrm{md}_{D_1 \boxtimes D_2}((u_\ell, v_t), (u_i, v_q))$, where $(u_i, v_q) \in N_{D_1 \boxtimes D_2}[(u_i, v_r)]$.

Let $(u_k, v_q) \in N_{D_1 \boxtimes D_2}[(u_i, v_r)]$. Then $\mathrm{md}_{D_1 \boxtimes D_2}((u_\ell, v_t), (u_k, v_q)) = \max\{\mathrm{md}_{D_1}(u_\ell, u_k), \mathrm{md}_{D_2}(v_t, v_q)\}$. If (u_i, v_r) is a boundary vertex of (u_ℓ, v_t), then $\max\{\mathrm{md}_{D_1}(u_\ell, u_i), \mathrm{md}_{D_2}(v_t, v_r)\} \geq \max\{\mathrm{md}_{D_1}(u_\ell, u_k), \mathrm{md}_{D_2}(v_t, v_q)\}$. So the necessary condition for the vertex (u_i, v_r) such that $u_i \in \mathrm{m}\partial(D_1)$ and $v_r \notin \mathrm{m}\partial(D_2)$ to be a boundary vertex of the vertex (u_ℓ, v_t) in $D_1 \boxtimes D_2$ is $\mathrm{md}_{D_1}(u_\ell, u_i) \geq \mathrm{md}_{D_2}(v_t, v_q)$ for all $v_q \in N_{D_2}[v_r]$. Since $\mathrm{mecc}_{D_1}(u_i) \geq \mathrm{md}_{D_1}(u_\ell, u_i)$ for all $u_\ell \in V(D_1)$, the necessary condition becomes $\mathrm{mecc}_{D_1}(u_i) \geq \mathrm{md}_{D_2}(v_t, v_q)$ for all $v_q \in N_{D_2}[v_r]$. Thus, $(u_i, v_r) \in A_2$.

Thus in Case 2, $(u_i, v_r) \in A_1 \cup A_2$.

In Case 4, $v_r \in \mathrm{m}\partial(D_2)$ and u_i is not a boundary vertex of u_j in D_1. As in Case 2, it follows that $(u_i, v_r) \in A_1 \cup A_3$.

Thus in all cases, $\mathrm{m}\partial(D_1 \boxtimes D_2) \subseteq A_1 \bigcup A_2 \bigcup A_3$.

Conversely, suppose that $(u_i, v_r) \in A_1 \bigcup A_2 \bigcup A_3$. First let $(u_i, v_r) \in A_1$. Then $u_i \in \mathrm{m}\partial(D_1)$ and $v_r \in \mathrm{m}\partial(D_2)$. So there exists vertices $u_j \in V(D_1)$, $v_s \in V(D_2)$ such that $\mathrm{md}_{D_1}(u_j, u_i) \geq \mathrm{md}_{D_1}(u_j, u_k)$ for every $u_k \in N_{D_1}[u_i]$, and $\mathrm{md}_{D_2}(v_s, v_r) \geq \mathrm{md}_{D_2}(v_s, v_q)$ for every $v_q \in N_{D_2}[v_r]$. Hence in $D_1 \boxtimes D_2$, $\mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_i, v_r)) = \max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} \geq \max\{\mathrm{md}_{D_1}(u_j, u_k), \mathrm{md}_{D_2}(v_s, v_q)\} = \mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_k, v_q))$ for all vertices $(u_k, v_q) \in N_{D_1 \boxtimes D_2}[(u_i, v_r)]$. Thus, $A_1 \subseteq \mathrm{m}\partial(D_1 \boxtimes D_2)$.

Now let $(u_i, v_r) \in A_2$. Then $u_i \in \mathrm{m}\partial(D_1), v_r \notin \mathrm{m}\partial(D_2)$ and there exists some vertex $v_t \in V(D_2)$ such that $\mathrm{md}_{D_2}(v_t, v_q) \leq \mathrm{mecc}_{D_1}(u_i)$, for all $v_q \in N_{D_2}[v_r]$. Since $u_i \in \mathrm{m}\partial(D_1)$, there exists at least one vertex $u_j \in V(D_1)$ such that $\mathrm{md}_{D_1}(u_j, u_i) \geq \mathrm{md}_{D_1}(u_j, u_k)$ for every $u_k \in N_{D_1}[u_i]$. Of these vertices, let u_b be a vertex such that $\mathrm{md}_{D_1}(u_b, u_i) = \mathrm{mecc}_{D_1}(u_i)$. Hence in $D_1 \boxtimes D_2$, $\mathrm{md}_{D_1 \boxtimes D_2}((u_b, v_t), (u_i, v_r)) = \max\{\mathrm{md}_{D_1}(u_b, u_i), \mathrm{md}_{D_2}(v_t, v_r)\} \geq \max\{\mathrm{md}_{D_1}(u_b, u_k), \mathrm{md}_{D_2}(v_t, v_q)\} = \mathrm{md}_{D_1 \boxtimes D_2}((u_b, v_t), (u_k, v_q))$ for all $(u_k, v_q) \in N_{D_1 \boxtimes D_2}[(u_i, v_r)]$, since $\mathrm{md}_{D_2}(v_t, v_q) \leq \mathrm{mecc}_{D_1}(u_i) = \mathrm{md}_{D_1}(u_b, u_i)$ for all $v_q \in N_{D_2}[v_r]$. Thus, (u_i, v_r) is a boundary vertex of (u_b, v_t) in $D_1 \boxtimes D_2$ and hence $A_2 \subseteq \mathrm{m}\partial(D_1 \boxtimes D_2)$.

By analogous arguments and since the strong product of digraphs is commutative, it follows that $A_3 \subseteq \mathrm{m}\partial(D_1 \boxtimes D_2)$.

Hence $A_1 \bigcup A_2 \bigcup A_3 \subseteq \mathrm{m}\partial(D_1 \boxtimes D_2)$. □

Now consider Figure 5.1. Here, $\mathrm{mecc}_{D_2}(v_1) = \mathrm{mecc}_{D_2}(v_2) = 1$. $N_{D_1}[u_2] = N_{D_1}[u_3] = \{u_1, u_2, u_3, u_4\}$, $\mathrm{md}_{D_1}(u_1, u_4) = 3$, $\mathrm{md}_{D_1}(u_1, u_2) = \mathrm{md}_{D_1}(u_1, u_3) = \mathrm{md}_{D_1}(u_2, u_4) = \mathrm{md}_{D_1}(u_3, u_4) = 2$, and $\mathrm{md}_{D_1}(u_2, u_3) = 1$. $u_2 \notin \mathrm{m}\partial(D_1)$ and hence (u_2, v_1), $(u_2, v_2) \notin \mathrm{m}\partial(D_1 \boxtimes D_2)$, since there is no vertex $u_\ell \in V(D_1)$

such that $\mathrm{md}_{D_1}(u_\ell, u_k)) \leq 1$ for all $u_k \in N_{D_1}[u_2]$. For similar reasons, (u_3, v_1), $(u_3, v_2) \notin \mathrm{m}\partial(D_1 \boxtimes D_2)$.

Consider the strong product of two connected graphs. Since the distance between any two distinct vertices are dealt with, it doesn't matter whether the graphs are simple or not; that is, whether they contain loops or parallel edges. So the result for any two connected nontrivial (not equal to K_1) graphs is deduced in Corollary 5.3.3.

Remark 5.3.2. The description for the boundary set of the strong product of two graphs G and H presented in [18] holds only for the product of two **nontrivial graphs** G and H. To this end, let $H = K_1 = (\{v\}, \emptyset)$. $\mathrm{m}\partial(K_1) = \{v\}$ (since all vertices of a complete graph are boundary vertices of the graph), and hence $\mathrm{m}\partial(G) \widehat{=} \mathrm{m}\partial(G \boxtimes K_1) = \big[\mathrm{m}\partial(G) \times \{v\}\big] \bigcup \big[V(G) \times \{v\}\big] \widehat{=} V(G)$, which is not true in general. (The notation $\widehat{=}$ denotes 'supposed to be equal to, but is not so'.)

Corollary 5.3.3. Let D_1 and D_2 be two nontrivial connected graphs. Then

$$\mathrm{m}\partial(D_1 \boxtimes D_2) = \big[\mathrm{m}\partial(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times \mathrm{m}\partial(D_2)\big].$$

Proof. By Theorem 5.3.1, if D_1 and D_2 are two strongly connected digraphs, then $\mathrm{m}\partial(D_1 \boxtimes D_2) = A_1 \bigcup A_2 \bigcup A_3$. Since D_1 and D_2 are given to be two nontrivial graphs, $\mathrm{mecc}_{D_1}(u_i) \geq 1$ for all $u_i \in V(D_1)$, $\mathrm{mecc}_{D_2}(v_r) \geq 1$ for all $v_r \in V(D_2)$, $\mathrm{md}_{D_1}(u_i, u_k) = 1$ for all $u_k \in N_{D_1}(u_i)$, and $\mathrm{md}_{D_2}(v_r, v_q) = 1$ for all $v_q \in N_{D_2}(v_r)$. Thus, $A_1 = \mathrm{m}\partial(D_1) \times \mathrm{m}\partial(D_2)$, $A_2 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : \; u_i \in \mathrm{m}\partial(D_1), \; v_r \notin \mathrm{m}\partial(D_2)$, and $\exists v_t \in V(D_2)$ such that $\mathrm{md}_{D_2}(v_t, v_q) \leq \mathrm{mecc}_{D_1}(u_i), \; \forall v_q \in N_{D_2}(v_r)\} = \mathrm{m}\partial(D_1) \times V(D_2)$, and $A_3 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : \; u_i \notin \mathrm{m}\partial(D_1), \; v_r \in \mathrm{m}\partial(D_2)$, and $\exists u_\ell \in V(D_1)$ such that $\mathrm{md}_{D_1}(u_\ell, u_k) \leq \mathrm{mecc}_{D_2}(v_r), \; \forall u_k \in N_{D_1}(u_i)\} = V(D_1) \times \mathrm{m}\partial(D_2)$.

Therefore, $\mathrm{m}\partial(D_1 \boxtimes D_2) = A_1 \cup A_2 \cup A_3 = \big[\mathrm{m}\partial(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times \mathrm{m}\partial(D_2)\big]$. □

The results for the m-periphery and the m-eccentricity of the strong product of two strongly connected digraphs are derived in the following two theorems. We can see that they are the same as the descriptions for the m-periphery and the m-eccentricity of the strong product of two graphs which are given in [18].

Theorem 5.3.4. Let D_1 and D_2 be two strongly connected digraphs.

1. If $\mathrm{mdiam}(D_1) < \mathrm{mdiam}(D_2)$, then $\mathrm{mPer}(D_1 \boxtimes D_2) = V(D_1) \times \mathrm{mPer}(D_2)$.

2. If $\text{mdiam}(D_1) = \text{mdiam}(D_2)$, then $\text{mPer}(D_1 \boxtimes D_2) = \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times \text{mPer}(D_2)\big]$.

Proof. 1. Let $\text{mdiam}(D_2) = n$. Let $v_r \in \text{mPer}(D_2)$.

Then for all $u_i \in V(D_1)$, $\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\} = n$. Hence $(u_i, v_r) \in \text{mPer}(D_1 \boxtimes D_2)$. Also if $v_r \notin \text{mPer}(D_2)$, then since $\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) < n$, $(u_i, v_r) \notin \text{mPer}(D_1 \boxtimes D_2)$. Hence it follows that $\text{mPer}(D_1 \boxtimes D_2) = V(D_1) \times \text{mPer}(D_2)$.

2. Let $\text{mdiam}(D_1) = \text{mdiam}(D_2) = n$. If $u_i \in \text{mPer}(D_1)$, then for all $v_r \in V(D_2)$, $(u_i, v_r) \in \text{mPer}(D_1 \boxtimes D_2)$, since $\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\} = n$. Hence $(u_i, v_r) \in \text{mPer}(D_1 \boxtimes D_2)$. Similarly, if $v_r \in \text{mPer}(D_2)$, then for all $u_i \in V(D_1)$, $(u_i, v_r) \in \text{mPer}(D_1 \boxtimes D_2)$. Hence it follows that $\big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times \text{mPer}(D_2)\big] \subseteq \text{mPer}(D_1 \boxtimes D_2)$.

Conversely, if $(u_i, v_r) \in \text{mPer}(D_1 \boxtimes D_2)$, then $\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) = \max\{\text{mdiam}(D_1), \text{mdiam}(D_2)\} = n$. Thus, at least one of $\text{mecc}_{D_1}(u_i)$ and $\text{mecc}_{D_2}(v_r)$ must be necessarily equal to n. Hence $u_i \in \text{mPer}(D_1)$ or $v_r \in \text{mPer}(D_2)$, and therefore, $\text{mPer}(D_1 \boxtimes D_2) \subseteq \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times \text{mPer}(D_2)\big]$.

□

Theorem 5.3.5. Let D_1 and D_2 be two strongly connected digraphs.

1. If $\text{mrad}(D_1) = \text{mrad}(D_2)$, then

$$\text{mEcc}(D_1 \boxtimes D_2) = \big[\text{mEcc}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times \text{mEcc}(D_2)\big].$$

2. If $\text{mrad}(D_1) < \text{mrad}(D_2)$, then

$$\text{mEcc}(D_1 \boxtimes D_2) = \big[\text{mEcc}(W) \times V(D_2)\big] \bigcup \big[V(D_1) \times \text{mEcc}(D_2)\big],$$

where $W = \{u_i \in V(D_1) : \text{mecc}(u_i) \geq \text{mrad}(D_2)\}$.

Proof. 1. First step is to prove that $\text{mEcc}(D_1 \boxtimes D_2) \subseteq \big[\text{mEcc}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times \text{mEcc}(D_2)\big]$. Let $(u_i, v_r) \in \text{mEcc}(D_1 \boxtimes D_2)$. Then there exists a vertex $(u_j, v_s) \in V(D_1 \boxtimes D_2)$ such that $\text{mecc}_{D_1 \boxtimes D_2}(u_j, v_s) = \text{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_i, v_r)) = \max\{\text{md}_{D_1}(u_j, u_i), \text{md}_{D_2}(v_s, v_r)\}$. Since $\text{mecc}_{D_1 \boxtimes D_2}(u_j, v_s) = \max\{\text{mecc}_{D_1}(u_j), \text{mecc}_{D_2}(v_s)\}$, $\text{mecc}_{D_1}(u_j) \geq \text{md}_{D_1}(u_j, u_i)$, and $\text{mecc}_{D_2}(v_s) \geq \text{md}_{D_2}(v_s, v_r)$, it follows that at least one of the two cases $\mathrm{mecc}_{D_1}(u_j) = \mathrm{md}_{D_1}(u_j, u_i)$ and $\mathrm{mecc}_{D_2}(v_s) = \mathrm{md}_{D_2}(v_s, v_r)$ must hold. So it is necessary that either u_i is an m-eccentric vertex of u_j, or v_r is an m-eccentric vertex of v_s.

Hence $(u_i, v_r) \in \left[\mathrm{mEcc}(D_1) \times V(D_2)\right] \bigcup \left[V(D_1) \times \mathrm{mEcc}(D_2)\right]$.

Let $\mathrm{mrad}(D_1) = \mathrm{mrad}(D_2) = n$. Let $u_i \in \mathrm{mEcc}(D_1)$. Then there exists a vertex $u_j \in V(D_1)$ such that $\mathrm{mecc}_{D_1}(u_j) = \mathrm{md}_{D_1}(u_j, u_i)$. Consider the vertex $(u_i, v_r) \in V(D_1 \boxtimes D_2)$, where v_r is an arbitrary vertex in D_2. Since $\mathrm{mrad}(D_2) = n$, there exists a vertex $v_s \in V(D_2)$ such that $\mathrm{mecc}_{D_2}(v_s) = n$. Hence $\mathrm{md}_{D_2}(v_s, v_r) \leq n$ and so $\mathrm{mecc}_{D_1 \boxtimes D_2}(u_j, v_s) = \max\{\mathrm{mecc}_{D_1}(u_j), \mathrm{mecc}_{D_2}(v_s)\} = \max\{\mathrm{mecc}_{D_1}(u_j), n\} = \mathrm{mecc}_{D_1}(u_j)$. Thus, $\mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_i, v_r)) = \max\{\mathrm{md}_{D_1}(u_j, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} = \mathrm{mecc}_{D_1}(u_j) = \mathrm{mecc}_{D_1 \boxtimes D_2}(u_j, v_s)$. Hence (u_i, v_r) is an m-eccentric vertex of (u_j, v_s). Thus if $u_i \in \mathrm{mEcc}(D_1)$, then $(u_i, v_r) \in \mathrm{mEcc}(D_1 \boxtimes D_2)$ for all $v_r \in V(D_2)$. Similarly, it can proved that if $v_q \in \mathrm{mEcc}(D_2)$, then $(u_k, v_q) \in \mathrm{mEcc}(D_1 \boxtimes D_2)$ for all $u_k \in V(D_1)$.

Hence $\left[\mathrm{mEcc}(D_1) \times V(D_2)\right] \bigcup \left[V(D_1) \times \mathrm{mEcc}(D_2)\right] \subseteq \mathrm{mEcc}(D_1 \boxtimes D_2)$, and so the result holds.

2. Let $\mathrm{mrad}(D_1) < \mathrm{mrad}(D_2) = n$. Let $u_i \in V(D_1)$, $v_r \in V(D_2)$. Here two cases arise:

Case 1. $v_r \in \mathrm{mEcc}(D_2)$.

Then there exists a vertex $v_s \in V(D_2)$ such that $\mathrm{mecc}_{D_2}(v_s) = \mathrm{md}_{D_2}(v_s, v_r)$. Let $u_p \in V(D_1)$ be such that $\mathrm{mecc}_{D_1}(u_p) = \mathrm{mrad}(D_1)$. Then since $\mathrm{mrad}(D_2) > \mathrm{mecc}_{D_1}(u_p)$, $\mathrm{mecc}_{D_1 \boxtimes D_2}(u_p, v_s) = \max\{\mathrm{mecc}_{D_1}(u_p), \mathrm{mecc}_{D_2}(v_s)\} = \mathrm{mecc}_{D_2}(v_s)$. Also, $\mathrm{md}_{D_1 \boxtimes D_2}((u_p, v_s), (u_i, v_r)) = \max\{\mathrm{md}_{D_1}(u_p, u_i), \mathrm{md}_{D_2}(v_s, v_r)\} = \mathrm{mecc}_{D_2}(v_s)$. Thus, (u_i, v_r) is an m-eccentric vertex of (u_p, v_s). So in this case, $V(D_1) \times \mathrm{mEcc}(D_2) \subseteq \mathrm{mEcc}(D_1 \boxtimes D_2)$.

Case 2. $v_r \notin \mathrm{mEcc}(D_2)$.

Let $v_q \in V(D_2)$ be such that $\mathrm{mecc}_{D_2}(v_q) = \mathrm{mrad}(D_2)$. Let $u_k \in \mathrm{mEcc}(W)$. Then there exists a vertex $u_p \in V(D_1)$ such that $\mathrm{mecc}_{D_1}(u_p) \geq \mathrm{mrad}(D_2)$ and $\mathrm{mecc}_{D_1}(u_p) = \mathrm{md}_{D_1}(u_p, u_k)$. Then $\mathrm{md}_{D_1 \boxtimes D_2}((u_p, v_q), (u_k, v_r)) = \max\{\mathrm{md}_{D_1}(u_p, u_k), \mathrm{md}_{D_2}(v_q, v_r)\} = \mathrm{md}_{D_1}(u_p, u_k) = \mathrm{mecc}_{D_1}(u_p) = \mathrm{mecc}_{D_1 \boxtimes D_2}(u_p, v_q)$ and hence (u_k, v_r) is an m-eccentric vertex of (u_p, v_q). Hence in this case, $\mathrm{mEcc}(W) \times V(D_2) \subseteq \mathrm{mEcc}(D_1 \boxtimes D_2)$.

Thus, $\left[\mathrm{mEcc}(W) \times V(D_2)\right] \bigcup \left[V(D_1) \times \mathrm{mEcc}(D_2)\right] \subseteq \mathrm{mEcc}(D_1 \boxtimes D_2)$.

Conversely, let $(u_k, v_r) \in \mathrm{mEcc}(D_1 \boxtimes D_2)$. Then there exists a vertex $(u_j, v_s) \in V(D_1 \boxtimes D_2)$ such that $\mathrm{mecc}_{D_1 \boxtimes D_2}(u_j, v_s) = \mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_k, v_r)) = \max\{\mathrm{md}_{D_1}(u_j, u_k), \mathrm{md}_{D_2}(v_s, v_r)\} = \max\{\mathrm{mecc}_{D_1}(u_j), \mathrm{mecc}_{D_2}(v_s)\}$. If $v_r \in \mathrm{mEcc}(D_2)$, then $(u_k, v_r) \in V(D_1) \times \mathrm{mEcc}(D_2)$.
Hence suppose that $(u_k, v_r) \in \mathrm{mEcc}(D_1 \boxtimes D_2)$ and $v_r \notin \mathrm{mEcc}(D_2)$. Then for all $v_s \in V(D_2)$, $\mathrm{mecc}_{D_2}(v_s) > \mathrm{md}_{D_2}(v_s, v_r)$. Thus, $\mathrm{mecc}_{D_1 \boxtimes D_2}(u_j, v_s) = \mathrm{mecc}_{D_1}(u_j) = \mathrm{md}_{D_1}(u_j, u_k)$. If possible, suppose that $u_k \notin \mathrm{mEcc}(W)$. Thus, there is no vertex u_j in D_1 such that $\mathrm{mecc}_{D_1}(u_j) = \mathrm{md}_{D_1}(u_j, u_k)$ and $\mathrm{mecc}_{D_1}(u_j) \geq \mathrm{mrad}(D_2)$. Hence if u_k is an m-eccentric vertex of u_j in D_1, then $\mathrm{md}_{D_1}(u_j, u_k) < \mathrm{mrad}(D_2)$. $\mathrm{mrad}(D_1 \boxtimes D_2) = \max\{\mathrm{mrad}(D_1), \mathrm{mrad}(D_2)\} = \mathrm{mrad}(D_2)$. Thus, (u_k, v_r) cannot be the m-eccentric vertex of any vertex $(u_j, v_s) \in D_1 \boxtimes D_2$, since $\mathrm{md}_{D_1 \boxtimes D_2}((u_j, v_s), (u_k, v_r)) = \max\{\mathrm{md}_{D_1}(u_j, u_k), \mathrm{md}_{D_2}(v_s, v_r)\} \neq \mathrm{mecc}_{D_1 \boxtimes D_2}(u_j, v_s)$ in this case. This is a contradiction, and hence $u_k \in \mathrm{mEcc}(W)$. Hence $(u_k, v_r) \in \mathrm{mEcc}(W) \times V(D_2)$. Hence $\mathrm{mEcc}(D_1 \boxtimes D_2) \subseteq \big[\mathrm{mEcc}(W) \times V(D_2)\big] \bigcup \big[V(D_1) \times \mathrm{mEcc}(D_2)\big]$.

□

Now we derive the expression for the m-contour of $D_1 \boxtimes D_2$.

Theorem 5.3.6. Let D_1 and D_2 be two strongly connected digraphs. Then $\mathrm{mCt}(D_1 \boxtimes D_2) = A_1 \cup A_2 \cup A_3$, where $A_1 = \big[\mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)\big]$,
$A_2 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : \; u_i \in \mathrm{mCt}(D_1), \; v_r \notin \mathrm{mCt}(D_2)$, and $\mathrm{mecc}_{D_2}(v_q) \leq \mathrm{mecc}_{D_1}(u_i)$ for all $v_q \in N_{D_2}[v_r]\}$,
$A_3 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : \; u_i \notin \mathrm{mCt}(D_1), \; v_r \in \mathrm{mCt}(D_2)$, and $\mathrm{mecc}_{D_1}(u_k) \leq \mathrm{mecc}_{D_2}(v_r)$ for all $u_k \in N_{D_1}[u_i]\}$.

Proof. $(u_i, v_r) \in \mathrm{mCt}(D_1 \boxtimes D_2)$ if and only if $\mathrm{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) \geq \mathrm{mecc}_{D_1 \boxtimes D_2}(u_k, v_q)$ for all $(u_k, v_q) \in N_{D_1 \boxtimes D_2}[(u_i, v_r)]$;
if and only if $\max\{\mathrm{mecc}_{D_1}(u_i), \mathrm{mecc}_{D_2}(v_r)\} \geq \max\{\mathrm{mecc}_{D_1}(u_k), \mathrm{mecc}_{D_2}(v_q)\}$ for all $u_k \in N_{D_1}[u_i]$ and $v_q \in N_{D_2}[v_r]$;
if and only if one of the following three cases holds.

Case 1: $\max\{\mathrm{mecc}_{D_1}(u_i), \mathrm{mecc}_{D_2}(v_r)\} = \mathrm{mecc}_{D_1}(u_i) = \mathrm{mecc}_{D_2}(v_r)$. Then $\mathrm{mecc}_{D_1}(u_i) \geq \mathrm{mecc}_{D_1}(u_k)$ and $\mathrm{mecc}_{D_2}(v_r) \geq \mathrm{mecc}_{D_2}(v_q)$ for all $u_k \in N_{D_1}[u_i]$ and $v_q \in N_{D_2}[v_r]$.

Case 2: $\max\{\mathrm{mecc}_{D_1}(u_i), \mathrm{mecc}_{D_2}(v_r)\} = \mathrm{mecc}_{D_1}(u_i) > \mathrm{mecc}_{D_2}(v_r)$. Then $\mathrm{mecc}_{D_1}(u_i) \geq \mathrm{mecc}_{D_1}(u_k)$ for all $u_k \in N_{D_1}[u_i]$ and $\mathrm{mecc}_{D_2}(v_r) < \mathrm{mecc}_{D_1}(u_i)$,

$\text{mecc}_{D_2}(v_q) \leq \text{mecc}_{D_1}(u_i)$ for all $v_q \in N_{D_2}(v_r)$.

Case 3: $\max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\} = \text{mecc}_{D_2}(v_r) > \text{mecc}_{D_1}(u_i)$. Then $\text{mecc}_{D_2}(v_r) \geq \text{mecc}_{D_2}(v_q)$ for all $v_q \in N_{D_2}[v_r]$ and $\text{mecc}_{D_1}(u_i) < \text{mecc}_{D_2}(v_r)$, $\text{mecc}_{D_1}(u_k) \leq \text{mecc}_{D_2}(v_r)$ for all $u_k \in N_{D_1}(u_i)$.

In Case 1, $(u_i, v_r) \in \text{mCt}(D_1) \times \text{mCt}(D_2)$.

In Case 2, $(u_i, v_r) \in A_2 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \in \text{mCt}(D_1),\ v_r \notin \text{mCt}(D_2),$ and $\text{mecc}_{D_2}(v_q) \leq \text{mecc}_{D_1}(u_i)$ for all $v_q \in N_{D_2}[v_r]\}$.

In Case 3, $(u_i, v_r) \in A_3 = \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \notin \text{mCt}(D_1),\ v_r \in \text{mCt}(D_2),$ and $\text{mecc}_{D_1}(u_k) \leq \text{mecc}_{D_2}(v_r)$ for all $u_k \in N_{D_1}[u_i]\}$.

Thus, $\text{mCt}(D_1 \boxtimes D_2) = A_1 \bigcup A_2 \bigcup A_3$. □

Consider the contour of the strong product of two connected graphs. As in the case of the boundary set, the result for the contour holds even when the two graphs are not simple.

Corollary 5.3.7. Let D_1 and D_2 be two connected graphs. Then $\text{mCt}(D_1 \boxtimes D_2)$
$= \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \in \text{mCt}(D_1),\ v_r \notin \text{mCt}(D_2),$ and $\text{mecc}_{D_2}(v_r) < \text{mecc}_{D_1}(u_i)\}$
$\bigcup \{(u_i, v_r) \in V(D_1 \boxtimes D_2) : u_i \notin \text{mCt}(D_1),\ v_r \in \text{mCt}(D_2),$ and $\text{mecc}_{D_1}(u_i) < \text{mecc}_{D_2}(v_r)\}$
$\bigcup \big[\text{mCt}(D_1) \times \text{mCt}(D_2)\big]$.

Proof. By Theorem 5.3.6, when D_1 and D_2 are two strongly connected digraphs, $\text{mCt}(D) = A_1 \bigcup A_2 \bigcup A_3$. Since D_1 and D_2 are two connected graphs, the m-eccentricity of two adjacent vertices differ by at most one. Let $D = D_1 \boxtimes D_2$.
Hence $A_1 = \text{mCt}(D_1) \times \text{mCt}(D_2)$,

$$\begin{aligned} A_2 =& \{(u_i, v_r) \in V(D) : u_i \in \text{mCt}(D_1), v_r \notin \text{mCt}(D_2), \text{ and} \\ & \text{mecc}_{D_2}(v_q) \leq \text{mecc}_{D_1}(u_i) \text{ for all } v_q \in N_{D_2}[v_r]\} \\ =& \{(u_i, v_r) \in V(D) : u_i \in \text{mCt}(D_1), v_r \notin \text{mCt}(D_2), \text{ and} \\ & \text{mecc}_{D_2}(v_r) + 1 \leq \text{mecc}_{D_1}(u_i)\} \\ =& \{(u_i, v_r) \in V(D) : u_i \in \text{mCt}(D_1), v_r \notin \text{mCt}(D_2), \text{ and } \text{mecc}_{D_2}(v_r) < \text{mecc}_{D_1}(u_i)\}, \end{aligned}$$

and $A_3 = \{(u_i, v_r) \in V(D) : u_i \notin \text{mCt}(D_1),\ v_r \in \text{mCt}(D_2),$ and $\text{mecc}_{D_1}(u_i) < \text{mecc}_{D_2}(v_r)\}$, since $\max\limits_{u_k \in N_{D_1}[u_i]} \{\text{mecc}_{D_1}(u_k)\} = \text{mecc}_{D_1}(u_i) + 1$. Thus the expression for $\text{mCt}(D_1 \boxtimes D_2)$ when D_1 and D_2 are two connected graphs is given by

$$\{(u_i, v_r) \in V(D) : u_i \in \mathrm{mCt}(D_1),\ v_r \notin \mathrm{mCt}(D_2),\ \text{and } \mathrm{mecc}_{D_2}(v_r) < \mathrm{mecc}_{D_1}(u_i)\}$$
$$\bigcup\{(u_i, v_r) \in V(D) : u_i \notin \mathrm{mCt}(D_1),\ v_r \in \mathrm{mCt}(D_2),\ \text{and } \mathrm{mecc}_{D_1}(u_i) < \mathrm{mecc}_{D_2}(v_r)\}$$
$$\bigcup[\mathrm{mCt}(D_1) \times \mathrm{mCt}(D_2)].$$

□

5.4 Center and Periphery of the Strong Product of a Finite Number of Digraphs

The directed distance between two vertices in the strong product of n strongly connected digraphs is obtained from [37] as follows.

Proposition 5.4.1. [37] For the strong product $D = D_1 \boxtimes \cdots \boxtimes D_n$, the directed distance between two vertices $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in V(D)$ is $\overrightarrow{d}_D((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\limits_{1 \le i \le n} \overrightarrow{d}_{D_i}(x_i, y_i)$.

Based on this result, we derive the formula for m-distance between two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and the m-eccentricity of a vertex in the strong product $D = D_1 \boxtimes \cdots \boxtimes D_n$.

Proposition 5.4.2. Let $D_1, \ldots, D_n$ be n strongly connected digraphs. Then

$$\mathrm{md}_{D_1 \boxtimes \cdots \boxtimes D_n}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \le i \le n} \{\mathrm{md}_{D_i}(x_i, y_i)\},$$
$$\mathrm{mecc}_{D_1 \boxtimes \cdots \boxtimes D_n}(x_1, \ldots, x_n) = \max_{1 \le i \le n} \{\mathrm{mecc}_{D_i}(x_i)\}.$$

Proof. Let $D = D_1 \boxtimes \cdots \boxtimes D_n$. By the definition of m-distance, $\mathrm{md}_D((x_1, \ldots, x_n), (y_1, \ldots, y_n))$

$$\begin{aligned}
&= \max\{\overrightarrow{d}_D((x_1, \ldots, x_n), (y_1, \ldots, y_n)), \overrightarrow{d}_D((y_1, \ldots, y_n), (x_1, \ldots, x_n))\} \\
&= \max\left\{\max_{1 \le i \le n}\{\overrightarrow{d}_{D_i}(x_i, y_i)\}, \max_{1 \le i \le n}\{\overrightarrow{d}_{D_i}(y_i, x_i)\}\right\} \\
&= \max\left\{\max_{1 \le i \le n}\{\overrightarrow{d}_{D_i}(x_i, y_i), \overrightarrow{d}_{D_i}(y_i, x_i)\}\right\} \\
&= \max_{1 \le i \le n}\{\max\{\overrightarrow{d}_{D_i}(x_i, y_i), \overrightarrow{d}_{D_i}(y_i, x_i)\}\} \\
&= \max_{1 \le i \le n}\{\mathrm{md}_{D_i}(x_i, y_i)\}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{mecc}_D(x_1,\ldots,x_n) &= \max\left\{\mathrm{md}_D((x_1,\ldots,x_n),(y_1,\ldots,y_n)) :\ (y_1,\ldots,y_n)\in V(D)\right\}\\
&= \max\left\{\max_{1\le i\le n}\left\{\mathrm{md}_{D_i}(x_i,y_i) :\ y_i\in V(D_i)\right\}\right\}\\
&= \max_{1\le i\le n}\left\{\max\left\{\mathrm{md}_{D_i}(x_i,y_i) :\ y_i\in V(D_i)\right\}\right\}\\
&= \max_{1\le i\le n}\left\{\mathrm{mecc}_{D_i}(x_i)\right\}.
\end{aligned}$$

□

Thus, the m-radius and the m-diameter of $D_1 \boxtimes \cdots \boxtimes D_n$ is obtained as follows.

Proposition 5.4.3. Let $D_1,\ldots,D_n$ be n strongly connected digraphs. Then

$$\begin{aligned}
\mathrm{mrad}(D_1 \boxtimes \cdots \boxtimes D_n) &= \max_{1\le i\le n}\{\mathrm{mrad}(D_i)\},\\
\mathrm{mdiam}(D_1 \boxtimes \cdots \boxtimes D_n) &= \max_{1\le i\le n}\{\mathrm{mdiam}(D_i)\}.
\end{aligned}$$

Proof. Let $D = D_1 \boxtimes \cdots \boxtimes D_n$.

$$\begin{aligned}
\mathrm{mrad}(D) &= \min_{(x_1,\ldots,x_n)\in V(D)}\{\mathrm{mecc}_D(x_1,\ldots,x_n)\}\\
&= \min_{(x_1,\ldots,x_n)\in V(D)}\left\{\max_{1\le i\le n}\ \mathrm{mecc}_{D_i}(x_i)\right\}\\
&= \max_{1\le i\le n}\left\{\min_{x_i\in V(D_i)}\ \mathrm{mecc}_{D_i}(x_i)\right\}\\
&= \max_{1\le i\le n}\{\mathrm{mrad}(D_i)\}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{mdiam}(D) &= \max_{(x_1,\ldots,x_n)\in V(D)}\{\mathrm{mecc}_D(x_1,\ldots,x_n)\}\\
&= \max_{(x_1,\ldots,x_n)\in V(D)}\left\{\max_{1\le i\le n}\ \mathrm{mecc}_{D_i}(x_i)\right\}\\
&= \max_{1\le i\le n}\left\{\max_{x_i\in V(D_i)}\ \mathrm{mecc}_{D_i}(x_i)\right\}\\
&= \max_{1\le i\le n}\{\mathrm{mdiam}(D_i)\}.
\end{aligned}$$

□

The strong product of n digraphs is strongly connected if and only if all the n digraphs are strongly connected [36]. Also, since strong product of digraphs is commutative, we can interchange the order of the digraphs in the product.

5.4.1 Center of the Strong Product

The m-center of the strong product of two strongly connected digraphs D_1 and D_2 are obtained here. Further, the m-center of the strong product of a finite number of strongly connected digraphs is derived. These results hold for graphs also, with m-distance replaced by the usual distance in graphs.

Theorem 5.4.4. Let D_1 and D_2 be two strongly connected digraphs with vertex sets $V(D_1) = \{u_1, \ldots, u_m\}$ and $V(D_2) = \{v_1, \ldots, v_n\}$.

1. If $\text{mrad}(D_1) < \text{mrad}(D_2)$, then $\text{mCen}(D_1 \boxtimes D_2) = A \times \text{mCen}(D_2)$, where $A = \{u_i \in V(D_1) : \text{mecc}(u_i) \leq \text{mrad}(D_2)\}$.

2. If $\text{mrad}(D_1) = \text{mrad}(D_2)$, then $\text{mCen}(D_1 \boxtimes D_2) = \text{mCen}(D_1) \times \text{mCen}(D_2)$.

Proof. 1. Let $\text{mrad}(D_2) = n$. Then $\text{mrad}(D_1 \boxtimes D_2) = n$.
$(u_i, v_r) \in \text{mCen}(D_1 \boxtimes D_2)$ if and only if $\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\} = n$;
if and only if $\text{mecc}_{D_1}(u_i) \leq n$ and $\text{mecc}_{D_2}(v_r) = n$;
if and only if $u_i \in A$ and $v_r \in \text{mCen}(D_2)$;
if and only if $(u_i, v_r) \in A \times \text{mCen}(D_2)$. Hence it follows that $\text{mCen}(D_1 \boxtimes D_2) = A \times \text{mCen}(D_2)$.

2. Let $\text{mrad}(D_1) = \text{mrad}(D_2) = n$. Then $\text{mrad}(D_1 \boxtimes D_2) = n$.
$(u_i, v_r) \in \text{mCen}(D_1 \boxtimes D_2)$ if and only if $\text{mecc}_{D_1 \boxtimes D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\} = n$;
if and only if $\text{mecc}_{D_1}(u_i) = n$ and $\text{mecc}_{D_2}(v_r) = n$;
if and only if $u_i \in \text{mCen}(D_1)$ and $v_r \in \text{mCen}(D_2)$;
if and only if $(u_i, v_r) \in \text{mCen}(D_1) \times \text{mCen}(D_2)$. Hence it follows that $\text{mCen}(D_1 \boxtimes D_2) = \text{mCen}(D_1) \times \text{mCen}(D_2)$.

□

Corollary 5.4.5. Let D_1 and D_2 be two strongly connected digraphs such that $\text{mdiam}(D_1) \leq \text{mrad}(D_2)$, then $\text{mCen}(D_1 \boxtimes D_2) = V(D_1) \times \text{mCen}(D_2)$.

Theorem 5.4.6. Let $D_1, \ldots, D_n$ be n strongly connected digraphs which are arranged such that $\text{mrad}(D_1) \leq \cdots \leq \text{mrad}(D_n)$. Let $\text{mrad}(D_k) = \text{mrad}(D_{k+1}) = \cdots = \text{mrad}(D_n) = a$ for some $k \geq 1$. Then $\text{mCen}(D_1 \boxtimes \cdots \boxtimes D_n) = A_1 \times \cdots \times A_{k-1} \times \text{mCen}(D_k) \times \cdots \times \text{mCen}(D_n)$, where $A_i = \{x_i \in V(D_i) : \text{mecc}(x_i) \leq a\}$.

Proof. $\text{mrad}(D_1 \boxtimes \cdots \boxtimes D_n) = \max\{\text{mrad}(D_1), \ldots, \text{mrad}(D_n)\} = a$. Hence $(x_1, \ldots, x_n) \in \text{mCen}(D_1 \boxtimes \cdots \boxtimes D_n)$
if and only if $\text{mecc}(x_1, \ldots, x_n) = \max\{\text{mecc}(x_1), \ldots, \text{mecc}(x_n)\} = a$;
if and only if $\text{mecc}(x_i) \leq a$ for $i = 1, 2, \ldots, k-1$, and $\text{mecc}(x_i) = a$ for $i = k, k+1, \ldots, n$;
if and only if $x_i \in A_i$ for $i = 1, 2, \ldots, k-1$, and $x_i \in \text{mCen}(D_i)$ for $i = k, k+1, \ldots, n$;
if and only if $(x_1, \ldots, x_n) \in A_1 \times \cdots \times A_{k-1} \times \text{mCen}(D_k) \times \cdots \times \text{mCen}(D_n)$. □

5.4.2 Periphery of the Strong Product

We have derived the expression for the m-periphery of the strong product of two strongly connected digraphs. In this section, we consider the m-periphery of the strong product of a finite number of strongly connected digraphs.

Theorem 5.4.7. Let $D_1, \ldots, D_n$ be n strongly connected digraphs which are arranged such that $\text{mdiam}(D_1) \leq \cdots \leq \text{mdiam}(D_n)$. Let $\text{mdiam}(D_k) = \text{mdiam}(D_{k+1}) = \cdots = \text{mdiam}(D_n) = a$ for some $k \geq 1$. Then $\text{mPer}(D_1 \boxtimes \cdots \boxtimes D_n) = \big[V(D_1) \times \cdots \times V(D_{k-1}) \times \text{mPer}(D_k) \times V(D_{k+1}) \times \cdots \times V(D_n)\big] \bigcup \big[V(D_1) \times \cdots \times V(D_{k-1}) \times V(D_k) \times \text{mPer}(D_{k+1}) \times \cdots \times V(D_n)\big] \bigcup \cdots \bigcup \big[V(D_1) \times \cdots \times V(D_{k-1}) \times V(D_k) \times V(D_{k+1}) \times \cdots \times \text{mPer}(D_n)\big]$.

Proof. $\text{mdiam}(D_1 \boxtimes \cdots \boxtimes D_n) = \max\{\text{mdiam}(D_1), \ldots, \text{mdiam}(D_n)\} = a$. Hence $(x_1, \ldots, x_n) \in \text{mPer}(D_1 \boxtimes \cdots \boxtimes D_n)$
if and only if $\text{mecc}_{D_1 \boxtimes \cdots \boxtimes D_n}(x_1, \ldots, x_n) = \max\{\text{mecc}_{D_1}(x_1), \ldots, \text{mecc}_{D_n}(x_n)\} = a$;
if and only if $\text{mecc}_{D_i}(x_i) = a$ for at least one of $i = k+1, k+2, \ldots, n$;
if and only if at least one of x_i, $i = k, k+1, \ldots, n$ lies in $\text{mPer}(D_i)$;
if and only if

$$
\begin{aligned}
(x_1, \ldots, x_n) \quad & \in \big[V(D_1) \times \cdots \times V(D_{k-1}) \times \text{mPer}(D_k) \times V(D_{k+1}) \times \cdots \times V(D_n)\big] \\
& \bigcup \big[V(D_1) \times \cdots \times V(D_{k-1}) \times V(D_k) \times \text{mPer}(D_{k+1}) \times \cdots \times V(D_n)\big] \\
& \bigcup \cdots \bigcup \big[V(D_1) \times \cdots \times V(D_{k-1}) \times V(D_k) \times V(D_{k+1}) \times \cdots \times \text{mPer}(D_n)\big].
\end{aligned}
$$

□

Chapter 6

Boundary-type Sets and Center in Lexicographic Product

6.1 Introduction

The significance of the lexicographic product is that if the edge relations of the two graphs are order relations, then the edge relation of their lexicographic product is the corresponding lexicographic order [36]. The study of lexicographic product of two graphs was initiated by Frank Harary in 1959. In [38], he defined a binary operation on graphs, which was called composition, such that the group of the composition of two graphs is permutationally equivalent to the composition of their groups. This problem was suggested by the work of Frucht [33] who gave the result for the case that the first graph in the composition is totally disconnected.

The chapter is organized as follows. The m-distance between two vertices and the m-eccentricity of a vertex in the lexicographic product of two digraphs are derived in Section 6.2. The four boundary-type sets of lexicographic product of two digraphs is investigated for certain classes of digraphs in Section 6.3. The concept of a DDLE digraph is introduced and the results related to the four boundary-type sets of the lexicographic product $D_1 \circ D_2$ of two digraphs D_1 and D_2 are considered in the following cases. The case when D_1 is a digraph satisfying the DDLE property is considered in Subsection 6.3.1. The cases when D_1 is a directed cycle and a symmetric digraph are dealt with in Subsections 6.3.2 and 6.3.3, respectively. The concepts of vertices satisfying the DDLE property and the DDEE property are introduced in Section 6.4 to obtain the expressions for the center and periphery of two strong digraphs. The center and periphery of the lexicographic product of two strongly connected digraphs is investigated in Subsections 6.4.1 and 6.4.2, respectively. Finally, in Section 6.5, the center and periphery of two graphs are considered. The results for the center and periphery of two graphs was already given in [75]. Some minor mistakes are pointed out, and the actual expressions for the center and the periphery are provided with proofs.

6.2 Distance Between Two Vertices

The distance between two vertices (u_i, v_r) and (u_j, v_s) in the lexicographic product $G \circ H$ of a connected graph G and a graph H is obtained from [36] as:

$$d_{G\circ H}((u_i, v_r), (u_j, v_s)) = \begin{cases} d_G(u_i, u_j) & \text{if } u_i \neq u_j, \\ \min\{2, d_H(v_r, v_s)\} & \text{if } u_i = u_j. \end{cases}$$

Prior to the definition of the distance between two vertices in the lexicographic product of two digraphs, several other notions from [37] need to be introduced.

Definition 6.2.1. Let D be a digraph. Given a vertex x of a digraph D, the dicycle distance of x in D, denoted by $\xi_D(x)$, is the length of a shortest dicycle in D containing x, or infinity if no such dicycle exists.

Consider two digraphs D_1 and D_2 with vertex sets $V(D_1) = \{u_1, u_2, \ldots, u_m\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_n\}$, respectively. Let $(u_i, v_r), (u_j, v_s) \in V(D_1 \circ D_2)$. The formula for the directed distance from (u_i, v_r) to (u_j, v_s), $\overrightarrow{d}_{D_1\circ D_2}((u_i, v_r), (u_j, v_s))$ is obtained from [37] as follows:

$$\overrightarrow{d}_{D_1\circ D_2}((u_i, v_r), (u_j, v_s)) = \begin{cases} \overrightarrow{d}_{D_1}(u_i, u_j) & \text{if } u_i \neq u_j, \\ \min\{\xi_{D_1}(u_i), \overrightarrow{d}_{D_2}(v_r, v_s)\} & \text{if } u_i = u_j. \end{cases}$$

Now we derive the expressions for the m-distance between two vertices and the m-eccentricity of a vertex in $V(D_1 \circ D_2)$.

$$\begin{aligned} &\mathrm{md}_{D_1\circ D_2}((u_i, v_r), (u_j, v_s)) \\ &\quad = \max\{\overrightarrow{d}_{D_1\circ D_2}((u_i, v_r), (u_j, v_s)), \overrightarrow{d}_{D_1\circ D_2}((u_j, v_s), (u_i, v_r))\} \\ &\quad = \begin{cases} \max\{\overrightarrow{d}_{D_1}(u_i, u_j), \overrightarrow{d}_{D_1}(u_j, u_i)\} & \text{if } u_i \neq u_j, \\ \min\{\xi_{D_1}(u_i), \max\{\overrightarrow{d}_{D_2}(v_r, v_s), \overrightarrow{d}_{D_2}(v_s, v_r)\}\} & \text{if } u_i = u_j. \end{cases} \\ &\quad = \begin{cases} \mathrm{md}_{D_1}(u_i, u_j) & \text{if } u_i \neq u_j, \\ \min\{\xi_{D_1}(u_i), \mathrm{md}_{D_2}(v_r, v_s)\} & \text{if } u_i = u_j. \end{cases} \end{aligned}$$

The approach in [37] is followed to draw all the digraph products. This alignment is helpful to find the m-eccentricity of vertices in the lexicographic product of two digraphs. Consider the lexicographic product $D = D_1 \circ D_2$ of the digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ with vertex sets $V_1 = \{u_1, \ldots, u_m\}$ and $V_2 = \{v_1, \ldots, v_n\}$. The first digraph D_1 is aligned parallel to the x-axis and the second digraph D_2 is aligned parallel to the y-axis, D_1 below and D_2 to the left of the vertex set $V(D) = V_1 \times V_2$ so that the vertex (u_i, v_r) in the product projects vertically to $u_i \in V_1$ and horizontally to $v_r \in V_2$, see Figure 6.1. Thus, for a vertex $(u_i, v_r) \in V(D_1 \circ D_2)$, the maximum possible distance to any other vertex in the horizontal direction is $\mathrm{mecc}_{D_1}(u_i)$ and that in the vertical direction is $\min\{\xi_{D_1}(u_i), \mathrm{mecc}_{D_2}(v_r)\}$. If $\mathrm{mecc}_{D_1}(u_i) = 1$, then $\xi_{D_1}(u_i) = 2$ and hence the expression reduces to $\min\{\mathrm{mecc}_{D_2}(v_r), 2\}$. Hence it follows that

$$\mathrm{mecc}_{D_1 \circ D_2}(u_i, v_r) = \begin{cases} \min\{\mathrm{mecc}_{D_2}(v_r), 2\} & \text{if } \mathrm{mecc}_{D_1}(u_i) = 1, \\ \max\{\mathrm{mecc}_{D_1}(u_i), \min\{\xi_{D_1}(u_i), \mathrm{mecc}_{D_2}(v_r)\}\} & \text{if } \mathrm{mecc}_{D_1}(u_i) \geq 2. \end{cases}$$

As the digraphs under consideration are clear from the vertex labelling, $\mathrm{md}_{D_1}(u_i, u_j)$ may be denoted by $\mathrm{md}(u_i, u_j)$, $\mathrm{md}_{D_1 \circ D_2}((u_i, v_r), (u_j, v_s))$ by $\mathrm{md}((u_i, v_r), (u_j, v_s))$, and $\mathrm{mecc}_{D_1 \circ D_2}(u_i, v_r)$ by $\mathrm{mecc}(u_i, v_r)$.

For every vertex x of a strongly connected digraph D with at least two vertices, there exists a dicycle in D containing x. So $\xi_D(x)$ is finite for every vertex x in D. Also, $\mathrm{mecc}_D(x)$ is finite for every vertex x of a strongly connected digraph D.

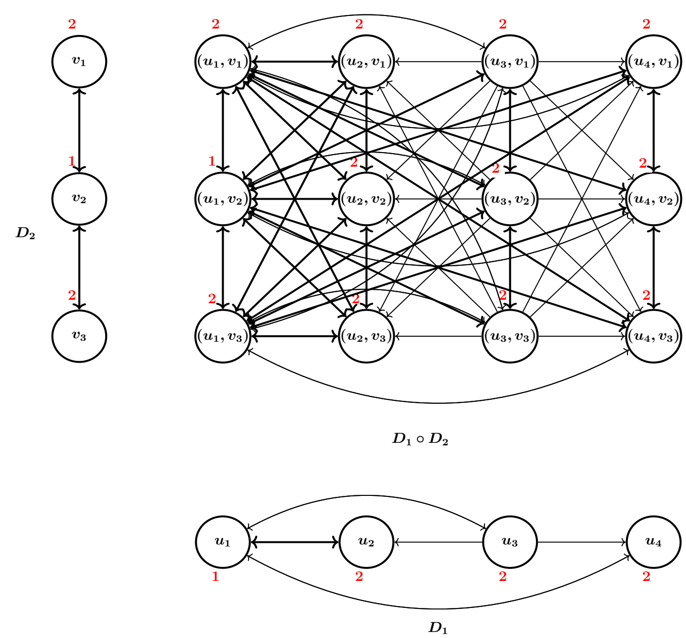

Figure 6.1: Example for lexicographic product of digraphs

Consider the lexicographic product of the digraphs D_1 and D_2 in Figure 6.1. The number in red color beside each vertex indicates the m-eccentricity of the vertex. To reduce the complexity of the drawing without affecting the understanding of the lexicographic product, a bi-directed edge is drawn instead of two edges with opposite directions and D_2 is taken as a symmetric digraph. The lexicographic product can be obtained from D_1 by substituting a copy of D_{2_u} of D_2 for every vertex u of D_1 and then assigning edges joining all vertices of D_{2_u} to all vertices of $D_{2_{u'}}$ if $uu' \in E(D_1)$ [36].

6.3 Boundary-type Sets of Lexicographic Product of Digraphs

We make the following definition to begin with the study of the boundary vertices of the lexicographic product of two digraphs.

Definition 6.3.1. A strong digraph D is said to satisfy the **dicycle distance less than eccentricity property** or in short the **DDLE property**, if for every vertex $x \in V(D)$, $\text{mecc}_D(x) > \xi_D(x)$. A digraph D which satisfy the DDLE property is called a **DDLE digraph**.

In the general case, the m-eccentricity of a vertex $(u_i, v_r) \in V(D_1 \circ D_2)$ depends upon the values of $\text{mecc}_{D_1}(u_i)$, $\text{mecc}_{D_2}(v_r)$, and $\xi_{D_1}(u_i)$. But if D_1 is a DDLE digraph, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mecc}_{D_1}(u_i)$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$ as illustrated in Section 6.3.1. Hence all the boundary-type sets of the lexicographic product $D_1 \circ D_2$, except $\partial(D_1 \circ D_2)$ depend only on the digraph D_1. This is the significance of a DDLE digraph.

If D_1 is a DDLE digraph, then $\text{mecc}_{D_1}(u_i) > 2$ for all $u_i \in V(D_1)$, and hence $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mecc}_{D_1}(u_i) > \xi_{D_1}(u_i)$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$. Let G and H be the underlying graphs of the digraphs D_1 and D_2, respectively. From [36], $N_{G \circ H}(u_i, v_r) = \big[N_G(u_i) \times V(H)\big] \bigcup \big[\{u_i\} \times N_H(v_r)\big]$. The neighbors of (u_i, v_r) are exactly the same in $D_1 \circ D_2$ and its underlying graph $G \circ H$. Thus it follows that $N_{D_1 \circ D_2}(u_i, v_r) = \big[N_{D_1}(u_i) \times V(D_2)\big] \bigcup \big[\{u_i\} \times N_{D_2}(v_r)\big]$. The lexicographic product of two digraphs D_1 and D_2 is strongly connected if and only if D_1 is strongly connected [36]. All digraphs D for which the m-radius is greater than 2 and such that there is at least one two-way edge incident to every vertex of the digraph are DDLE digraphs. Examples of digraphs that do not satisfy the DDLE property are the directed cycles $\overrightarrow{C_n}$ and the complete graphs K_n, where $n \geq 2$.

6.3.1 $D_1 \circ D_2$, D_1 is a DDLE Digraph

As D_1 is a DDLE digraph, $\text{mecc}_{D_1}(u_i) > \xi_{D_1}(u_i)$ for all $u_i \in V(D_1)$. Also since $\text{mecc}_{D_1}(u_i) > 2$, $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \min\{\xi_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}\} = \text{mecc}_{D_1}(u_i)$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$. Thus if D_1 is a DDLE digraph, then the following results regarding the m-periphery, m-contour and m-eccentricity sets of $D_1 \circ D_2$ are obtained. Since the DDLE property is related to the m-eccentricity of a vertex, nothing could be inferred about the m-boundary, $\text{m}\partial(D_1 \circ D_2)$.

Proposition 6.3.2. Let D_1 be a strongly connected DDLE digraph and D_2 be an arbitrary digraph. Then

1. $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2)$,
2. $\text{mCt}(D_1 \circ D_2) = \text{mCt}(D_1) \times V(D_2)$,
3. $\text{mEcc}(D_1 \circ D_2) = \text{mEcc}(D_1) \times V(D_2)$.

Proof. Given D_1 is a DDLE digraph. Let $u_i \in V(D_1)$ and $v_r \in V(D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mecc}_{D_1}(u_i)$.

$$
\begin{aligned}
1.(u_i, v_r) \in \text{mPer}(D_1 \circ D_2) \iff & \text{mecc}_{D_1 \circ D_2}(u_i, v_r) \geq \text{mecc}_{D_1 \circ D_2}(u_k, v_q) \\
& \text{ for all } (u_k, v_q) \in V(D_1 \circ D_2) \\
\iff & \text{mecc}_{D_1}(u_i) \geq \text{mecc}_{D_1}(u_k) \text{ for all } u_k \in V(D_1) \\
\iff & u_i \in \text{mPer}(D_1).
\end{aligned}
$$

Thus, $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2)$.

2. $N_{D_1 \circ D_2}(u_i, v_r) = \left[N_{D_1}(u_i) \times V(D_2)\right] \bigcup \left[\{u_i\} \times N_{D_2}(v_r)\right]$.

$$
\begin{aligned}
(u_i, v_r) \in \text{mCt}(\text{D}_1 \circ \text{D}_2) \iff & \text{mecc}_{D_1 \circ D_2}(u_i, v_r) \geq \text{mecc}_{D_1 \circ D_2}(u_k, v_q) \\
& \text{ for all } (u_k, v_q) \in N_{D_1 \circ D_2}(u_i, v_r) \\
\iff & \text{mecc}_{D_1}(u_i) \geq \text{mecc}_{D_1}(u_k) \text{ for all } u_k \in N_{D_1}(u_i) \\
\iff & u_i \in \text{mCt}(D_1).
\end{aligned}
$$

Hence it follows that $\text{mCt}(D_1 \circ D_2) = \text{mCt}(D_1) \times V(D_2)$.

3. $(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$ if and only if there exists $(u_j, v_s) \in V(D_1 \circ D_2)$ such that $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mecc}_{D_1}(u_j) = \text{md}_{D_1 \circ D_2}((u_j, v_s), (u_i, v_r))$. Since D_1 is a DDLE digraph, $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mecc}_{D_1}(u_j)$, and $u_j \neq u_i$. Thus, $\text{md}_{D_1 \circ D_2}((u_j, v_s), (u_i, v_r)) = \text{md}_{D_1}(u_j, u_i)$. Hence it follows that $(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$ if and only if there exists a vertex u_j in D_1 such that $\text{mecc}_{D_1}(u_j) = \text{md}_{D_1}(u_j, u_i)$; if and only if $u_i \in \text{mEcc}(D_1)$. Thus, $\text{mEcc}(D_1 \circ D_2) = \text{mEcc}(D_1) \times V(D_2)$. □

A digraph D is said to be symmetric if $(u, v) \in E(D)$ if and only if $(v, u) \in E(D)$, and so the maximum distance md is the usual distance d and likewise m-eccentricity is the usual eccentricity and so on. Thus the prefix m can be avoided for boundary-type sets also. If D is a connected symmetric digraph, then $\xi_D(x) = 2$ for all $x \in V(D)$. Hence the DDLE property for D is $\mathrm{ecc}(x) > 2$ for all $x \in V(D)$; that is, $\mathrm{rad}(D) > 2$.

Thus, an immediate corollary follows from Proposition 6.3.2.

Corollary 6.3.3. Let D_1 be a connected symmetric digraph with $\mathrm{rad}(D_1) > 2$ and D_2 be an arbitrary digraph. Then

1. $\mathrm{mPer}(D_1 \circ D_2) = \mathrm{Per}(D_1) \times V(D_2)$,
2. $\mathrm{mCt}(D_1 \circ D_2) = \mathrm{Ct}(D_1) \times V(D_2)$,
3. $\mathrm{mEcc}(D_1 \circ D_2) = \mathrm{Ecc}(D_1) \times V(D_2)$.

6.3.2 $\overrightarrow{C_n} \circ D_2$

In this section, we derive the expressions for all the four boundary-type sets of the lexicographic product $\overrightarrow{C_n} \circ D_2$.

Proposition 6.3.4. Let $\overrightarrow{C_n}$ be the dicycle on n vertices and D_2 be an arbitrary digraph.

1. If $\mathrm{mrad}(D_2) \geq n$ or $\mathrm{mdiam}(D_2) < n$, then $\mathrm{m}\partial(\overrightarrow{C_n} \circ D_2) = \mathrm{mCt}(\overrightarrow{C_n} \circ D_2) = \mathrm{mEcc}(\overrightarrow{C_n} \circ D_2) = \mathrm{mPer}(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times V(D_2)$.
2. If $\mathrm{mrad}(D_2) < n$ and $\mathrm{mdiam}(D_2) \geq n$, then $\mathrm{mPer}(\overrightarrow{C_n} \circ D_2) = \mathrm{mCt}(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times I$, and $\mathrm{mEcc}(\overrightarrow{C_n} \circ D_2) = \mathrm{m}\partial(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times V(D_2)$, where $I = \{v_r \in V(D_2) : \mathrm{mecc}_{D_2}(v_r) \geq n\}$.

Proof. $\mathrm{mecc}_{D_1}(u_i) = n - 1$ and $\xi_{\overrightarrow{C_n}}(u_i) = n$ for all $u_i \in \overrightarrow{C_n}$. Hence

$$\mathrm{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_r) = \begin{cases} n-1 & \text{if } \mathrm{mecc}_{D_2}(v_r) \leq n-1, \\ n & \text{if } \mathrm{mecc}_{D_2}(v_r) \geq n. \end{cases}$$

1. If $\mathrm{mrad}(D_2) \geq n$, then $\mathrm{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_r) = n$ for all $(u_i, v_r) \in V(\overrightarrow{C_n} \circ D_2)$. If $\mathrm{mdiam}(D_2) < n$, then $\mathrm{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_r) = n - 1$ for all $(u_i, v_r) \in V(\overrightarrow{C_n} \circ D_2)$. So in both the cases, $\mathrm{m}\partial(\overrightarrow{C_n} \circ D_2) = \mathrm{mCt}(\overrightarrow{C_n} \circ D_2) = \mathrm{mEcc}(\overrightarrow{C_n} \circ D_2) = \mathrm{mPer}(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times V(D_2)$.

2. If $\text{mrad}(D_2) < n$ and $\text{mdiam}(D_2) \geq n$, then $\text{mPer}(\overrightarrow{C_n} \circ D_2)$ consists of all those vertices (u_i, v_r) such that $\text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_r) = n$. Hence $\text{mPer}(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times I$. Since $\text{mPer}(\overrightarrow{C_n} \circ D_2) \subseteq \text{mCt}(\overrightarrow{C_n} \circ D_2)$, $V(\overrightarrow{C_n}) \times I \subseteq \text{mCt}(\overrightarrow{C_n} \circ D_2)$. If $v_r \in V(D_2)$ is such that $\text{mecc}_{D_2}(v_r) < n$, then $\text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_r) = n - 1$ for all $u_i \in V(\overrightarrow{C_n})$. $N_{\overrightarrow{C_n}}(u_i) \times V(D_2) \subseteq N_{\overrightarrow{C_n} \circ D_2}(u_i, v_r)$. Since $\text{mdiam}(D_2) \geq n$, there exists a vertex $(u_k, v_q) \in N_{\overrightarrow{C_n} \circ D_2}(u_i, v_r)$ such that $\text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_k, v_q) = n$. Hence if $\text{mecc}_{D_2}(v_r) < n$, then $(u_i, v_r) \notin \text{mCt}(\overrightarrow{C_n} \circ D_2)$. Hence $\text{mCt}(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times I$.

   Let $u_i \in V(\overrightarrow{C_n})$ and $v_r \in V(D_2)$. If $\text{mecc}_{D_2}(v_r) < n$, then $\text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_r) = n - 1$ and there exists $u_j \neq u_i$ such that $\text{md}_{\overrightarrow{C_n} \circ D_2}((u_j, v_r), (u_i, v_r)) = n - 1 = \text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_j, v_r)$ and hence (u_i, v_r) is an m-eccentric vertex of (u_j, v_r). If $\text{mecc}_{D_2}(v_r) \geq n$, then $\text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_r) = n$ and there exists a vertex $v_s \in V(D_2)$ such that $\text{md}_{D_2}(v_s, v_r) \geq n$ and so $\text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_s) = n$. Thus $\text{mecc}_{\overrightarrow{C_n} \circ D_2}(u_i, v_s) = \text{md}_{\overrightarrow{C_n} \circ D_2}((u_i, v_s), (u_i, v_r)) = n$ and hence (u_i, v_r) is an m-eccentric vertex of (u_i, v_s). Hence $\text{mEcc}(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times V(D_2)$. Since $\text{mEcc}(\overrightarrow{C_n} \circ D_2) \subseteq \text{m}\partial(\overrightarrow{C_n} \circ D_2)$, $\text{m}\partial(\overrightarrow{C_n} \circ D_2) = V(\overrightarrow{C_n}) \times V(D_2)$.

□

If D_1 is not a DDLE digraph, then the boundary-type sets are no longer characterized by the m-radius or m-diameter of the two digraphs. This is because $\text{mecc}_{D_1 \circ D_2}(u_i, v_r)$ depends on $\xi_{D_1}(u_i)$, in addition to $\text{mecc}_{D_1}(u_i)$ and $\text{mecc}_{D_2}(v_r)$.

To see this, consider the digraphs D_1 and D_1' in Figure 6.2. The m-eccentricity of each vertex is displayed near the vertex. Here, $\text{mdiam}(D_1) = \text{mdiam}(D_1') = 2$. Let D_2 be the symmetric dipath P_4 with labels v_1, v_2, v_3, v_4 in order. Then $\text{mecc}_{D_2}(v_1) = \text{mecc}_{D_2}(v_4) = 3$ and $\text{mecc}_{D_2}(v_2) = \text{mecc}_{D_2}(v_3) = 2$. In D_1, $\xi_{D_1}(u_1) = \xi_{D_1}(u_3) = 3$ and $\xi_{D_1}(u_2) = \xi_{D_1}(u_4) = 2$.

$$\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \begin{cases} \min\{\text{mecc}_{D_2}(v_r), 2\} & \text{if } \text{mecc}_{D_1}(u_i) = 1, \\ \max\{\text{mecc}_{D_1}(u_i), \min\{\xi_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}\} & \text{if } \text{mecc}_{D_1}(u_i) \geq 2. \end{cases}$$

Hence in $D_1 \circ D_2$, $\text{mecc}(u_1, v_1) = \text{mecc}(u_3, v_1) = \text{mecc}(u_1, v_4) = \text{mecc}(u_3, v_4) = 3$ and the m-eccentricity of all the other vertices is 2. Thus, $\text{mPer}(D_1 \circ D_2) = \{(u_1, v_1), (u_3, v_1), (u_1, v_4), (u_3, v_4)\}$. In D_1', $\text{mecc}(u_1') = 1$ and hence $\text{mecc}(u_1', v_1) = 2$. The m-eccentricities of every other vertex is 2, and dicycle distance of every vertex is 2. Since the m-eccentricity of every vertex in D_2 is greater than or equal to 2, the m-eccentricity of every vertex in $D_1' \circ D_2$ is 2 and hence $\text{mPer}(D_1' \circ D_2) = V(D_1) \times V(D_2)$. Hence, the remaining discussion on boundary-type sets as a whole is restricted to the case when D_1 is a symmetric digraph. In Subsection 6.4.2, the expression for

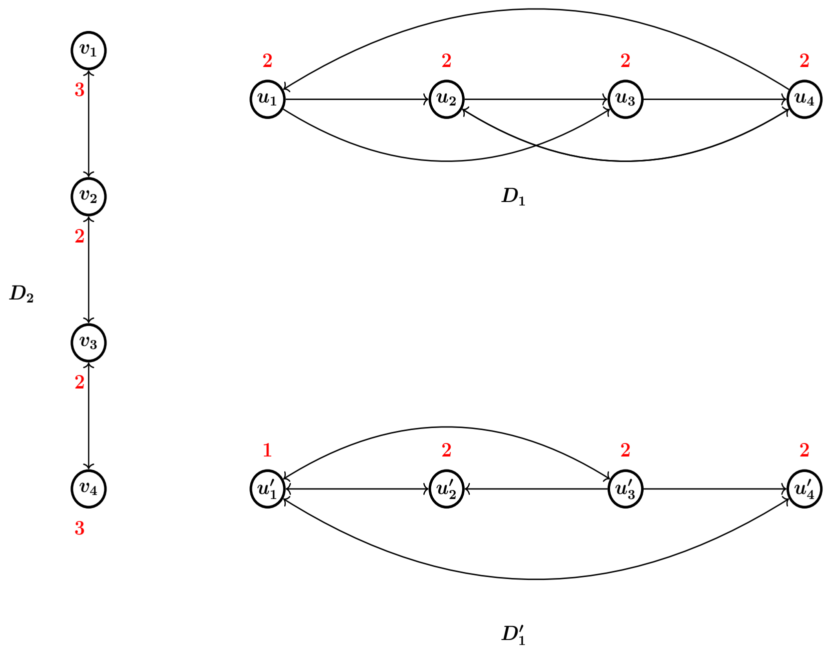

Figure 6.2: Two digraphs D_1 and D_1' with diameter 2, but with different m-periphery for lexicographic product with $D_2 = P_4$

the m-periphery of the lexicographic product of two strongly connected digraphs is derived.

6.3.3 $D_1 \circ D_2$, D_1 is a Symmetric Digraph

Suppose that D_1 is a symmetric digraph. Then $\xi_{D_1}(u_i) = 2$ for all $u_i \in V(D_1)$.

Thus, $\text{md}_{D_1 \circ D_2}((u_i, v_r), (u_j, v_s)) = \begin{cases} \text{md}_{D_1}(u_i, u_j) & \text{if } u_i \neq u_j, \\ \min\{2, \text{md}_{D_2}(v_r, v_s)\} & \text{if } u_i = u_j. \end{cases}$

and $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \begin{cases} \text{ecc}_{D_1}(u_i) & \text{if } \text{ecc}_{D_1}(u_i) \geq 2 \\ \min\{\text{mecc}_{D_2}(v_r), 2\} & \text{if } \text{ecc}_{D_1}(u_i) = 1. \end{cases}$

The m-distance between two vertices in $D_1 \circ D_2$, where D_1 is a symmetric digraph is analogous to the distance between two vertices in the case of the lexicographic product of two symmetric digraphs. So all the results for the boundary type sets happens to be the same. The description of the boundary-type sets of the lexicographic product of two symmetric digraphs is given in [18]. In this section, the results for the boundary-type sets are proved, when D_1 is a symmetric digraph and D_2 is an arbitrary digraph. It is proved in [24] that $\text{mdiam}(D) \leq 2\ \text{mrad}(D)$, for all digraphs D.

Now we consider the m-periphery, m-contour, and m-eccentricity of the lexicographic product $K_n \circ D_2$, where K_n is the complete graph on n vertices. If $D_1 = K_n$, $n \geq 2$, then $\text{ecc}(u_i) = 1$ for all $u_i \in V(D_1)$.

Hence, $\text{md}_{K_n \circ D_2}((u_i, v_r), (u_j, v_s)) = \begin{cases} 1 & \text{if } u_i \neq u_j, \\ \min\{2, \text{md}_{D_2}(v_r, v_s)\} & \text{if } u_i = u_j. \end{cases}$

Also, $\text{mecc}_{K_n \circ D_2}(u_i, v_r) = \min\{\text{mecc}_{D_2}(v_r), 2\}$.

Proposition 6.3.5. Let K_n be the complete symmetric digraph on n vertices and D_2 be an arbitrary digraph. Then

1. $$\text{mPer}(K_n \circ D_2) = \text{mCt}(K_n \circ D_2) = \begin{cases} V(K_n) \times V(D_2) & \text{if } \text{mrad}(D_2) \geq 2 \text{ or } D_2 = K_m, \\ V(K_n) \times B' & \text{if } \text{mrad}(D_2) = 1 \text{ and } D_2 \neq K_m, \end{cases}$$

   where $B' = V(D_2) \backslash \text{mCen}(D_2)$.

2. $\text{mEcc}(K_n \circ D_2) = V(K_n) \times V(D_2)$.

Proof.

1. If $\text{mrad}(D_2) \geq 2$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 2$ for all $(u_i, v_r) \in V(K_n \circ D_2)$. If $D_2 = K_m$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 1$ for all $(u_i, v_r) \in V(K_n \circ D_2)$. Thus in both the cases, $\text{mPer}(K_n \circ D_2) = \text{mCt}(K_n \circ D_2) = V(K_n) \times V(D_2)$.

   If $\text{mrad}(D_2) = 1$ and $D_2 \neq K_m$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 1$ for all $v_r \in \text{mCen}(D_2)$ and $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 2$ for all $v_r \in V(D_2) \backslash \text{mCen}(D_2)$. Thus, $\text{mPer}(K_n \circ D_2) = V(K_n) \times (V(D_2) \backslash \text{mCen}(D_2))$. If $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 1$, then $(u_i, v_r) \notin \text{mCt}(K_n \circ D_2)$, because there exists $v_q \in N(v_r)$ such that $\text{mecc}_{D_2}(v_q) = 2$ and hence $\text{mecc}_{D_1 \circ D_2}(u_k, v_q) = 2$, where $(u_k, v_q) \in N_{D_1 \circ D_2}(u_i, v_r)$. Since every vertex with m-eccentricity 2 is in $\text{mPer}(K_n \circ D_2)$, it follows that $\text{mPer}(K_n \circ D_2) = \text{mCt}(K_n \circ D_2) = V(K_n) \times (V(D_2) \backslash \text{mCen}(D_2))$.

2. First, suppose that $\text{mrad}(D_2) = 1$. If $v_s \in \text{mCen}(D_2)$, then $\text{mecc}_{D_2}(v_s) = 1$, and hence $\text{mecc}_{D_1 \circ D_2}(u_i, v_s) = 1 = \text{md}_{K_n \circ D_2}((u_i, v_s), (u_i, v_r))$ for all $(u_i, v_r) \in V(K_n \circ D_2)$.

   Now, if $\text{mrad}(D_2) \geq 2$, then for all $v_s \in V(D_2)$, $\text{mecc}_{D_2}(v_s) \geq 2$, and hence $\text{mecc}_{D_1 \circ D_2}(u_i, v_s) = 2 = \text{md}_{K_n \circ D_2}((u_i, v_s), (u_i, v_r))$ for all $(u_i, v_r) \in V(K_n \circ D_2)$.

   Hence in both the cases, $\text{mEcc}(K_n \circ D_2) = V(K_n) \times V(D_2)$.

□

From Corollary 6.3.3, if $\text{mrad}(D_1) > 2$, then $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2)$. In the next two propositions, $\text{mPer}(D_1 \circ D_2)$ is obtained for $\text{mdiam}(D_1) \geq 3$ and $\text{mdiam}(D_1) = 2$. $\text{mdiam}(D_1) = 1$ is the case when $D_1 = K_n$, which was already discussed.

Proposition 6.3.6. Let D_1 be a connected symmetric digraph with $\text{diam}(D_1) \geq 3$, and D_2 be an arbitrary digraph. Then $\text{mPer}(D_1 \circ D_2) = \text{Per}(D_1) \times V(D_2)$.

Proof. If $\text{diam}(D_1) = n \geq 3$, then $\text{mdiam}(D_1 \circ D_2) = n$, since $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = n$ for all vertices (u_i, v_r) in $D_1 \circ D_2$ such that $u_i \in \text{Per}(D_1), v_r \in V(D_2)$ and $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) < n$ for the remaining vertices. Hence $\text{mPer}(D_1 \circ D_2) = \text{Per}(D_1) \times V(D_2)$. □

Proposition 6.3.7. Let D_1 be a connected symmetric digraph with $\text{diam}(D_1) = 2$, and D_2 be an arbitrary digraph. Then

$$\text{mPer}(D_1 \circ D_2) = \begin{cases} V(D_1) \times V(D_2) & \text{if } \text{mrad}(D_2) \geq 2, \\ \big[\text{Per}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times B'\big] & \text{if } \text{mrad}(D_2) = 1, \end{cases}$$

where $B' = V(D_2) \backslash \text{mCen}(D_2)$.

Proof. If $\text{diam}(D_1) = 2$ and $\text{mrad}(D_2) \geq 2$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 2$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$. Hence in this case, $\text{mPer}(D_1 \circ D_2) = V(D_1) \times V(D_2)$.

If $\text{diam}(D_1) = 2$ and $\text{mrad}(D_2) = 1$, then

$$\begin{aligned} (u_i, v_r) \in \text{mPer}(D_1 \circ D_2) &\iff \text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 2 \\ &\iff \text{either } \text{ecc}(u_i) = 2 \text{ or } \text{mecc}_{D_2}(v_r) = 2 \\ &\iff \text{either } u_i \in \text{Per}(D_1) \text{ or } v_s \in V(D_2) \backslash \text{mCen}(D_2). \end{aligned}$$

Hence $\text{mPer}(D_1 \circ D_2) = \big[\text{Per}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \times B'\big]$. □

The m-contour and the m-eccentricity sets of $D_1 \circ D_2$ depends on the m-radii of both D_1 and D_2, unless $\text{mrad}(D_1) \geq 3$.

Proposition 6.3.8. Let D_1 be a connected symmetric digraph different from K_n and D_2 be an arbitrary digraph. Then

$$\text{mCt}(D_1 \circ D_2) = \begin{cases} \text{Ct}(D_1) \times V(D_2) & \text{if } \text{rad}(D_1) \geq 2, \\ V(D_1) \times V(D_2) & \text{if } \text{rad}(D_1) = 1 \text{ and } \text{mrad}(D_2) \geq 2, \\ \big[(V(D_1) \backslash \text{Cen}(D_1)) \times V(D_2)\big] \bigcup \big[\text{Cen}(D_1) \times B'\big] & \text{if } \text{rad}(D_1) = 1 \text{ and } \text{mrad}(D_2) = 1, \end{cases}$$

where $B' = V(D_2) \backslash \text{mCen}(D_2)$.

Proof. Let $u_i \in V(D_1)$ and $v_r \in V(D_2)$.

Suppose that $\text{rad}(D_1) \geq 2$. If $\text{rad}(D_1) > 2$, then by Corollary 6.3.3, $\text{mCt}(D_1 \circ D_2) = \text{Ct}(D_1) \times V(D_2)$. Let $\text{rad}(D_1) = 2$. Then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{ecc}_{D_1}(u_i)$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$. Also, $N_{D_1 \circ D_2}(u_i, v_r) = \big[N_{D_1}(u_i) \times V(D_2)\big] \bigcup \big[\{u_i\} \times N_{D_2}(v_r)\big]$. If $u_i \in \text{Ct}(D_1)$, then $\text{ecc}_{D_1}(u_i) \geq \text{ecc}_{D_1}(u_k)$ for all $u_k \in N_{D_1}(u_i)$. Thus, $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) \geq \text{mecc}_{D_1 \circ D_2}(u_k, v_s)$ for all $(u_k, v_s) \in N_{D_1 \circ D_2}(u_i, v_r)$. Hence $\text{Ct}(D_1) \times V(D_2) \subseteq \text{mCt}(D_1 \circ D_2)$. If $u_i \notin \text{Ct}(D_1)$, then there exists $u_q \in N_{D_1}(u_i)$ such that $\text{ecc}_{D_1}(u_q) > \text{ecc}_{D_1}(u_i)$. Hence there exists $(u_q, v_s) \in N_{D_1 \circ D_2}(u_i, v_r)$ such that $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) < \text{mecc}_{D_1 \circ D_2}(u_q, v_s)$ and thus $(u_i, v_r) \notin \text{mCt}(D_1 \circ D_2)$. Hence $\text{mCt}(D_1 \circ D_2) = \text{Ct}(D_1) \times V(D_2)$.

Suppose that $\text{rad}(D_1) = 1$ and $\text{mrad}(D_2) \geq 2$. Thus, $\text{diam}(D_1) \leq 2$, and $\text{ecc}_{D_1}(u_i) = 1$ or 2. Since $\text{mecc}_{D_2}(v_r) \geq 2$ for all $v_r \in V(D_2)$, $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 2$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$. Hence in this case, $\text{mCt}(D_1 \circ D_2) = V(D_1) \times V(D_2)$.

Consider the case, $\text{rad}(D_1) = \text{mrad}(D_2) = 1$. In this case, $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 1$ or 2 for all $(u_i, v_r) \in V(D_1 \circ D_2)$. Let $u_i \in V(D_1)$ and $v_r \in V(D_2)$. If $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 2$, then $(u_i, v_r) \in \text{mCt}(D_1 \circ D_2)$. $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 2$ only in the following two cases.

Case 1: $\text{ecc}_{D_1}(u_i) = 2$ and $\text{mecc}_{D_2}(v_r) \geq 1$

Case 2: $\text{ecc}_{D_1}(u_i) = 1$ and $\text{mecc}_{D_2}(v_r) = 2$

Since $D_1 \neq K_n$, $\text{diam}(D_1) = 2$ and so the first possibility is $u_i \in V(D_1) \backslash \text{Cen}(D_1)$ and $v_r \in V(D_2)$. The second possibility is $u_i \in \text{Cen}(D_1)$ and $v_r \in V(D_2) \backslash \text{mCen}(D_2)$. $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = 1$ if and only if $\text{ecc}_{D_1}(u_i) = \text{mecc}_{D_2}(v_r) = 1$. As $D_1 \neq K_n$ and $\text{ecc}_{D_1}(u_i) = 1$, there is at least one $u_k \in N_{D_1}(u_i)$ such that $\text{ecc}(u_k) = 2$. Hence $\text{mecc}(u_k, v_r) = 2$ and since $(u_k, v_r) \in N_{D_1 \circ D_2}(u_i, v_r)$, it follows that $(u_i, v_r) \notin \text{mCt}(D_1 \circ D_2)$. Hence in this case, $\text{mCt}(D_1 \circ D_2) = \big[(V(D_1) \backslash \text{Cen}(D_1)) \times V(D_2)\big] \bigcup \big[\text{Cen}(D_1) \times (V(D_2) \backslash \text{mCen}(D_2))\big]$.

□

To derive the expressions for $\text{mEcc}(D_1 \circ D_2)$ for the cases when D_1 is either a symmetric digraph with $\text{mrad}(D_1) = 2$ or a symmetric digraph with $\text{mrad}(D_1) = 1$ and $D_1 \neq K_n$, we require the following lemma.

Lemma 6.3.9. Let D_1 be a connected symmetric digraph and D_2 be an arbitrary digraph. Then $\text{Ecc}(D_1) \times V(D_2) \subseteq \text{mEcc}(D_1 \circ D_2)$.

Proof. Let $u_i \in V(D_1)$ and $v_r \in V(D_2)$. Whenever $u_i \in \text{Ecc}(D_1)$, there exists a vertex $u_j \in V(D_1)$ such that $\text{ecc}_{D_1}(u_j) = d_{D_1}(u_j, u_i)$. If $\text{rad}(D_1) \geq 2$, then in $D_1 \circ D_2$,

$\text{mecc}_{D_1\circ D_2}(u_j, v_r) = \text{ecc}_{D_1}(u_j) = d_{D_1}(u_j, u_i) = \text{md}_{D_1\circ D_2}((u_j, v_r), (u_i, v_r))$ and hence $(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$. If $\text{rad}(D_1) = 1$ and if $u_i \in \text{Ecc}(D_1)$, let $u_j \in V(D_1)$ be such that $\text{ecc}_{D_1}(u_j) = 1 = d_{D_1}(u_j, u_i)$. Then there are two cases.

If $\text{mecc}_{D_2}(v_r) = 1$, then $\text{mecc}_{D_1\circ D_2}(u_j, v_r) = \text{md}_{D_1\circ D_2}((u_j, v_r), (u_i, v_r)) = 1$ and so $(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$. If $\text{mecc}_{D_2}(v_r) \geq 2$, then there exists a vertex $v_s \in V(D_2)$ such that in $D_1 \circ D_2$, $\text{mecc}_{D_1\circ D_2}(u_i, v_s) = \text{md}_{D_1\circ D_2}((u_i, v_s), (u_i, v_r)) = 2$ and so $(u_i, v_r) \in \text{mEcc}(D_1\circ D_2)$. So in both the cases, $\text{Ecc}(D_1)\times V(D_2) \subseteq \text{mEcc}(D_1\circ D_2)$. □

Proposition 6.3.10. Let D_1 be a connected symmetric digraph with $\text{rad}(D_1) = 2$, and D_2 be an arbitrary digraph. Then

$$\text{mEcc}(D_1 \circ D_2) = \begin{cases} \left[\text{Ecc}(D_1) \times V(D_2)\right] \bigcup \left[\text{Cen}(D_1) \times B'\right] & \text{if } \text{mrad}(D_2) = 1, \\ \left[\text{Ecc}(D_1) \bigcup Cen(D_1)\right] \times V(D_2) & \text{if } \text{mrad}(D_2) \geq 2, \end{cases}$$

where $B' = V(D_2)\backslash\text{mCen}(D_2)$.

Proof. By Lemma 6.3.9, $\text{Ecc}(D_1) \times V(D_2) \subseteq \text{mEcc}(D_1 \circ D_2)$. Now, it is enough to find the vertices $(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$ such that $u_i \notin \text{Ecc}(D_1)$.
First, suppose that $\text{mrad}(D_2) = 1$. If $u_i \in \text{Cen}(D_1)$, then $\text{ecc}(u_i) = 2$ and hence $\text{mecc}(u_i, v_r) = 2$ for all $v_r \in V(D_2)$. If $v_r \notin \text{mCen}(D_2)$, then $\text{mecc}(v_r) \geq 2$ and so there exists a vertex $v_s \in V(D_2)$ such that $\text{md}_{D_2}(v_s, v_r) \geq 2$ and hence $\text{md}_{D_1\circ D_2}((u_i, v_s), (u_i, v_r)) = 2$. But, if $v_r \in \text{mCen}(D_2)$, $\text{md}_{D_2}(v_s, v_r) = 1$ for all $v_s \in V(D_2)$. Hence there exists no vertex (u_i, v_s) in $D_1 \circ D_2$ such that $\text{md}_{D_1\circ D_2}((u_i, v_s), (u_i, v_r)) = 2$. Thus if $u_i \notin \text{Ecc}(D_1)$, then $(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$ if and only if $u_i \in \text{Cen}(D_1)$ and $v_r \notin \text{mCen}(D_2)$. Hence $\text{mEcc}(D_1 \circ D_2) = \left[\text{Ecc}(D_1) \times V(D_2)\right] \bigcup \left[\text{Cen}(D_1) \times (V(D_2)\backslash\text{mCen}(D_2))\right]$.

Next, suppose that $\text{mrad}(D_2) \geq 2$. Let $v_r \in V(D_2)$. Since $\text{rad}(D_1) = 2$ and $\text{mrad}(D_2) \geq 2$, $\text{mecc}(u_i, v_s) = \begin{cases} \text{ecc}(u_i) & \text{if } u_i \notin \text{Cen}(D_1), \\ 2 & \text{if } u_i \in \text{Cen}(D_1). \end{cases}$
$(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$ if and only if there exists a vertex $(u_j, v_s) \in V(D_1 \circ D_2)$ such that $\text{mecc}(u_j, v_s) = \text{md}_{D_1\circ D_2}((u_j, v_s), (u_i, v_r))$. If $v_s \in V(D_2)$ is such that $\text{md}_{D_2}(v_s, v_r) \geq 2$, then $\text{mecc}_{D_1\circ D_2}(u_i, v_s) = 2 = \text{md}_{D_1\circ D_2}((u_i, v_s), (u_i, v_r))$ for all $u_i \in \text{Cen}(D_1)$. Thus besides $\text{Ecc}(D_1) \times V(D_2)$, all elements (u_i, v_r), where $u_i \in \text{Cen}(D_1)$ are also m-eccentric vertices in $D_1 \circ D_2$. For vertices u_i such that $u_i \notin \text{Ecc}(D_1)$ and $u_i \notin \text{Cen}(D_1)$, $\text{ecc}(u_i) > 2$ and there exist no vertex $u_j \in V(D_1)$ such that $\text{ecc}(u_j) = d_{D_1}(u_j, u_i)$. Hence in $D_1 \circ D_2$, there exist no vertex (u_j, v_s) such that $\text{mecc}_{D_1\circ D_2}(u_j, v_s) = \text{md}_{D_1\circ D_2}((u_j, v_s), (u_i, v_r))$. That is, (u_i, v_r) is not an m-eccentric vertex of any vertex (u_j, v_s) in $D_1 \circ D_2$. Thus $\text{mEcc}(D_1 \circ D_2) = \left[\text{Ecc}(D_1) \bigcup \text{Cen}(D_1)\right] \times V(D_2)$. □

Proposition 6.3.11. Let D_1 be a connected symmetric digraph with $\text{rad}(D_1) = 1$ different from K_n and D_2 be an arbitrary digraph. Then

$$\text{mEcc}(D_1 \circ D_2) = \begin{cases} \big[\text{Ecc}(D_1) \times V(D_2)\big] \bigcup \big[\text{Cen}(D_1) \times B'\big] & \text{if mrad}(D_2) = 1 \text{ and} \\ & |\text{Cen}(D_1)| = |\text{mCen}(D_2)| = 1 \\ \text{Ecc}(D_1) \times V(D_2) & \text{if mrad}(D_2) = 1 \text{ and} \\ & |\text{Cen}(D_1)| \geq 2 \\ \big[\text{Ecc}(D_1) \bigcup \text{Cen}(D_1)\big] \times V(D_2) & \text{if either mrad}(D_2) \geq 2 \text{ or} \\ & \text{mrad}(D_2) = 1, |\text{Cen}(D_1)| = 1, \\ & \text{and } |\text{mCen}(D_2)| \geq 2 \end{cases}$$

where $B' = V(D_2) \backslash \text{mCen}(D_2)$.

Proof. By Lemma 6.3.9, $\text{mEcc}(D_1) \times V(D_2) \subseteq \text{mEcc}(D_1 \circ D_2)$.
Thus, it is enough to find the remaining vertices in $\text{Ecc}(D_1 \circ D_2)$ in each of the four cases.

Case 1: $\text{mrad}(D_2) = 1$ and $|\text{Cen}(D_1)| = |\text{mCen}(D_2)| = 1$. Let $v_r \in V(D_2)$.

$$\text{mecc}_{D_1 \circ D_2}(u_i, v_s) = \begin{cases} \text{ecc}_{D_1}(u_i) & \text{if } u_i \notin \text{Cen}(D_1), \\ \min\{\text{mecc}_{D_2}(v_s), 2\} & \text{if } u_i \in \text{Cen}(D_1). \end{cases}$$

Thus if $u_i \notin \text{Ecc}(D_1)$, then $(u_i, v_r) \in \text{mEcc}(D_1 \circ D_2)$ only if $u_i \in \text{Cen}(D_1)$ and $v_r \notin \text{mCen}(D_2)$. For, if $v_s \in V(D_2)$ is such that $\text{md}_{D_2}(v_r, v_s) = 2$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_s) = 2$. Thus $\text{mecc}_{D_1 \circ D_2}(u_i, v_s) = \text{md}_{D_1 \circ D_2}((u_i, v_s), (u_i, v_r)) = 2$ for $u_i \in \text{mCen}(D_1)$ and $v_r \notin \text{mCen}(D_2)$, and there exists only one vertex in $D_1 \circ D_2$ having m-eccentricity 1 (the single vertex in $\text{Cen}(D_1) \times \text{mCen}(D_2)$). This vertex cannot be the m-eccentric vertex of any vertex in $D_1 \circ D_2$. Hence $\text{mEcc}(D_1 \circ D_2) = \big[\text{Ecc}(D_1) \times V(D_2)\big] \bigcup \big[\text{Cen}(D_1) \times (V(D_2) \backslash \text{mCen}(D_2))\big]$.

Case 2: $\text{mrad}(D_2) = 1$ and $|\text{Cen}(D_1)| \geq 2$. If $u_i \in \text{Cen}(D_1)$, then there exists another vertex $u_j \in \text{Cen}(D_1)$ such that $\text{ecc}(u_j) = d_{D_1}(u_j, u_i) = 1$. Hence $u_i \in \text{Ecc}(D_1)$. Thus, $\text{mEcc}(D_1 \circ D_2) = \text{Ecc}(D_1) \times V(D_2)$.

Case 3: $\text{mrad}(D_2) = 1$, $|\text{Cen}(D_1)| = 1$ and $|\text{mCen}(D_2)| \geq 2$. For each $v_r \in \text{mCen}(D_2)$, there exists another vertex $v_s \in \text{mCen}(D_2)$ such that $\text{mecc}(v_s) = \text{md}_{D_2}(v_s, v_r) = 1$. Thus, if $u_i \in \text{Cen}(D_1)$, then $\text{mecc}(u_i, v_s) = \text{md}_{D_1 \circ D_2}((u_i, v_s), (u_i, v_r)) = 1$. That is, every vertex (u_i, v_r) such that $u_i \in \text{Cen}(D_1)$, $v_r \in \text{mCen}(D_2)$ is an m-eccentric vertex in $D_1 \circ D_2$. But all vertices in $\text{Cen}(D_1) \times (V(D_2) \backslash \text{mCen}(D_2))$ are in $\text{mEcc}(D_1 \circ D_2)$, as proved in the first case. Since no other vertex could be an m-eccentric vertex in $D_1 \circ D_2$, it follows that $\text{mEcc}(D_1 \circ D_2) = \big[\text{Ecc}(D_1) \bigcup \text{Cen}(D_1)\big] \times V(D_2)$.

Case 4: $\text{mrad}(D_2) \geq 2$.

Then $\text{mecc}_{D_1 \circ D_2}(u_i, v_s) = \begin{cases} \text{ecc}_{D_1}(u_i) & \text{if } u_i \notin \text{Cen}(D_1), \\ 2 & \text{if } u_i \in \text{Cen}(D_1). \end{cases}$

Hence if $u_i \in \text{Cen}(D_1)$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_s) = \text{md}_{D_1 \circ D_2}((u_i, v_s), (u_i, v_r)) = 2$ for all $v_r \in V(D_2)$. So the vertices $(u_i, v_r) \in \text{Cen}(D_1) \times V(D_2)$ are the other vertices in $\text{mEcc}(D_1 \circ D_2)$. Thus, $\text{mEcc}(D_1 \circ D_2) = \big[\text{Ecc}(D_1) \bigcup \text{Cen}(D_1)\big] \times V(D_2)$. □

The m-boundary of $D_1 \circ D_2$ when D_1 is a connected symmetric digraph is considered below.

Proposition 6.3.12. Let D_1 be a connected symmetric digraph and D_2 be an arbitrary digraph. Then

1. $\text{m}\partial(D_1 \circ D_2) = V(D_1) \times V(D_2)$, if $\text{mrad}(D_2) \geq 2$ or $\text{mrad}(D_2) = 1$ and $|\text{mCen}(D_2)| \geq 2$.

2. $\text{m}\partial(D_1 \circ D_2) = \big[V(D_1) \times B'\big] \bigcup \big[\partial(D_1) \times \text{mCen}(D_2)\big]$, if $\text{mrad}(D_2) = 1$ and $|\text{mCen}(D_2)| = 1$, where $B' = V(D_2) \backslash \text{mCen}(D_2)$.

Proof. Let $u_i \in V(D_1)$ and $v_r \in V(D_2)$.

1. Suppose that $\text{mrad}(D_2) \geq 2$.

   Then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \begin{cases} \text{ecc}_{D_1}(u_i) & \text{if } \text{ecc}_{D_1}(u_i) \geq 2, \\ 2 & \text{if } \text{ecc}_{D_1}(u_i) = 1. \end{cases}$

   Since $\text{md}_{D_1 \circ D_2}((u_j, v_s), (u_i, v_r)) = \begin{cases} d_{D_1}(u_j, u_i) & \text{if } u_i \neq u_j, \\ 1 & \text{if } u_i = u_j \text{ and } \text{md}_{D_2}(v_s, v_r) = 1, \\ 2 & \text{if } u_i = u_j \text{ and } \text{md}_{D_2}(v_s, v_r) > 1, \end{cases}$

   it follows that if $v_s \in V(D_2)$ is a vertex such that $\text{md}_{D_2}(v_s, v_r) > 1$, then $\text{md}_{D_1 \circ D_2}((u_i, v_s), (u_i, v_r)) = 2$. If $(u_k, v_q) \in N_{D_1 \circ D_2}(u_i, v_r)$, then

   $\text{md}_{D_1 \circ D_2}((u_i, v_s), (u_k, v_q)) = \begin{cases} 1 & \text{if } u_k \in N_{D_1}(u_i), \\ 1 & \text{if } u_k = u_i \text{ and } \text{md}_{D_2}(v_s, v_q) = 1, \\ 2 & \text{if } u_k = u_i \text{ and } \text{md}_{D_2}(v_s, v_q) > 1, \end{cases}$

   as $N_{D_1 \circ D_2}(u_i, v_r) = \big[N_{D_1}(u_i) \times V(D_2)\big] \bigcup \big[\{u_i\} \times N(v_r)\big]$. Hence (u_i, v_r) is a boundary vertex of (u_i, v_s). Thus, $V(D_1) \times V(D_2) \subseteq \text{m}\partial(D_1 \circ D_2)$.

   Now suppose that $\text{mrad}(D_2) = 1$ and $|\text{mCen}(D_2)| \geq 2$. Let $v_a \in \text{mCen}(D_2)$. So $\text{md}_{D_2}(v_a, v_r) = 1$ for all $v_r \in V(D_2)$. Then $\text{md}_{D_1 \circ D_2}((u_i, v_a), (u_i, v_r)) = 1$, and $\text{md}_{D_1 \circ D_2}((u_i, v_a), (u_k, v_q)) = 1$ for all $(u_k, v_q) \in N_{D_1 \circ D_2}(u_i, v_r)$, since

$d_{D_1}(u_i, u_k) = 1$ for all $u_k \in N_{D_1}(u_i)$. Since $|\text{mCen}(D_2)| \geq 2$, there exist $v_b \in \text{mCen}(D_2)$ such that $\text{md}_{D_2}(v_b, v_a) = 1$. Thus we get $\text{md}_{D_1 \circ D_2}((u_i, v_b), (u_i, v_a)) = 1$, and also $\text{md}_{D_1 \circ D_2}((u_i, v_b), (u_k, v_q)) = 1$ for all $(u_k, v_q) \in N_{D_1 \circ D_2}(u_i, v_a)$. So in this case also, $V(D_1) \times V(D_2) \subseteq \text{m}\partial(D_1 \circ D_2)$.

2. Suppose that $\text{mrad}(D_2) = 1$ and $|\text{mCen}(D_2)| = 1$. So there is only one vertex v_a such that $\text{md}_{D_2}(v_a, v_r) = 1$ for all $v_r \in V(D_2)$. Hence every vertex $(u_i, v_r) \in V(D_1 \circ D_2)$ is a boundary vertex of (u_i, v_a). Also, (u_i, v_a) is not a boundary vertex in $D_1 \circ D_2$ unless u_i is a boundary vertex in D_1. For, if $u_i \in \partial(D_1)$, then u_i is a boundary vertex of $u_j \in V(D_1)$, and hence (u_i, v_a) is a boundary vertex of (u_j, v_a) in $D_1 \circ D_2$. So in this case, $\text{m}\partial(D_1 \circ D_2) = \big[V(D_1) \times (V(D_2) \backslash \text{mCen}(D_2))\big] \bigcup \big[\partial(D_1) \times \text{mCen}(D_2)\big]$.

□

6.4 Center and Periphery of the Lexicographic Product

The study of the center and periphery of the lexicographic product of any two strong digraphs is the highlight of this chapter. The expressions for the m-center of the lexicographic product of any two strong digraphs is obtained as various cases, depending on the m-radii of D_1 and D_2 in Subsection 6.4.1. In the previous sections, the various boundary-type sets of lexicographic product of digraphs were studied in certain particular cases. The expressions for the m-periphery of the lexicographic product of any two strong digraphs are obtained in Subsection 6.4.2.

Two new concepts are to be introduced to obtain the m-center and the m-periphery of two arbitrary strongly connected digraphs D_1 and D_2. The DDLE property already defined for a digraph as a whole is now defined for each vertex in the digraph. Also, the 'dicycle distance equal to eccentricity property' or in short the DDEE property for a vertex in the digraph is defined.

Definition 6.4.1. Let D be a digraph. A vertex $x \in V(D)$ is said to satisfy the **dicycle distance less than eccentricity property** (**DDLE property**), if $\text{mecc}(x) > \xi_D(x)$.

Definition 6.4.2. Let D be a digraph. A vertex $x \in V(D)$ is said to satisfy the **dicycle distance equal to eccentricity property** (**DDEE property**), if $\text{mecc}(x) = \xi_D(x)$.

Lemma 6.4.3. Let D be any digraph. If $x_1 \in V(D)$ is such that x_1 does not satisfy the DDLE property and the DDEE property, then $\xi_D(x_1) = \text{mecc}(x_1) + 1$.

Proof. Given $x_1 \in V(D)$ does not satisfy the DDLE property and the DDEE property. So $\xi_D(x_1) \geq \text{mecc}(x_1)+1$. Let the shortest dicycle containing x_1 be $x_1 a_1 x_2 \dots x_k a_k x_1$. Suppose that $\xi_D(x_1) \geq \text{mecc}(x_1) + 2$. Then $\vec{d}(x_1, x_k) \geq \text{mecc}(x_1) + 1$, which gives $\text{md}(x_1, x_k) \geq \text{mecc}(x_1) + 1$, which is not possible. Hence $\xi_D(x_1) = \text{mecc}(x_1) + 1$. □

6.4.1 Center of the Lexicographic Product of Two Digraphs

The m-center of the lexicographic product of two strongly connected digraphs D_1 and D_2 with vertex sets $V(D_1) = \{u_1, \dots, u_m\}$, $V(D_2) = \{v_1, \dots, v_n\}$ is investigated here. It is seen that the m-center of the lexicographic product depends upon the m-radii of the two digraphs. The following sets are defined to obtain the explicit representations for the m-center in various cases.
$A = \{u_i \in V(D_1) : \text{mecc}_{D_1}(u_i) = \text{mrad}(D_1) + 1\}$,
$B = \{u_i \in V(D_1) \backslash \text{mCen}(D_1) : \xi_{D_1}(u_i) = \text{mrad}(D_1) + 1\}$,
$E = \{v_r \in V(D_2) : \text{mecc}_{D_2}(v_r) \leq \text{mrad}(D_1)\}$,
$F = \{u_i \in \text{mCen}(D_1) : u_i \text{ is either DDLE or DDEE}\}$.

First, we consider the case when $\text{mrad}(D_1) = 1$.

Proposition 6.4.4. Let D_1 and D_2 be strongly connected digraphs. If $\text{mrad}(D_1) = 1$, then

$$\text{mCen}(D_1 \circ D_2) = \begin{cases} \text{mCen}(D_1) \times \text{mCen}(D_2) & \text{if mrad}(D_2) = 1, \\ \left[A' \times \text{mCen}(D_2)\right] \bigcup \left[(\text{mCen}(D_1) \bigcup B) \times V(D_2)\right] & \text{if mrad}(D_2) = 2, \\ \left[(\text{mCen}(D_1) \bigcup B\right] \times V(D_2) & \text{if mrad}(D_2) \geq 3, \end{cases}$$

where $A' = V(D_1) \backslash \text{mCen}(D_1)$.

Proof. As md is a metric, it satisfies the inequality $\text{mdiam}(D_1) \leq 2\ \text{mrad}(D_1)$. Thus $\text{mecc}_{D_1}(u_i) = 2$ for all vertices $u_i \in V(D_1) \backslash \text{mCen}(D_1)$ and hence in this case, equation (6.1) reduces to

$$\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \begin{cases} \min\{\text{mecc}_{D_2}(v_r), 2\} & \text{if mecc}_{D_1}(u_i) = 1, \\ \max\{2, \min\{\xi_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}\} & \text{if mecc}_{D_1}(u_i) = 2. \end{cases}$$

Case 1: $\mathrm{mrad}(D_2) = 1$.
Since $\mathrm{mrad}(D_1) = \mathrm{mrad}(D_2) = 1$, it follows that $\mathrm{mrad}(D_1 \circ D_2) = 1$. Let $(u_j, v_s) \in \mathrm{mCen}(D_1 \circ D_2)$. Then $\mathrm{mecc}_{D_1 \circ D_2}(u_j, v_s) = 1$ and hence $\mathrm{mecc}_{D_1}(u_j) = \mathrm{mecc}_{D_2}(v_s) = 1$. Thus, $(u_j, v_s) \in \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$.
Conversely, if $(u_j, v_s) \in \mathrm{mCen}(D_1) \times \mathrm{mCen}(D_2)$, then $u_j \in \mathrm{mCen}(D_1)$ and $v_s \in \mathrm{mCen}(D_2)$ and hence $\mathrm{mecc}_{D_1 \circ D_2}(u_j, v_s) = 1$. Therefore, $(u_j, v_s) \in \mathrm{mCen}(D_1 \circ D_2)$.
Case 2: $\mathrm{mrad}(D_2) = 2$.
Since $\mathrm{mrad}(D_1) = 1$, and $\mathrm{mrad}(D_2) = 2$, it follows that $\mathrm{mrad}(D_1 \circ D_2) = 2$. If $(u_j, v_s) \in \mathrm{mCen}(D_1 \circ D_2)$, then $\mathrm{mecc}_{D_1 \circ D_2}(u_j, v_s) = 2$ and this happens only in the following cases.

1. $u_j \in \mathrm{mCen}(D_1)$ and v_s is any vertex in $V(D_2)$; that is, $(u_j, v_s) \in \mathrm{mCen}(D_1) \times V(D_2)$

2. $u_j \notin \mathrm{mCen}(D_1)$ and $v_s \in \mathrm{mCen}(D_2)$; that is, $(u_j, v_s) \in (V(D_1) \backslash \mathrm{mCen}(D_1)) \times \mathrm{mCen}(D_2)$

3. $u_j \notin \mathrm{mCen}(D_1)$, $\xi_{D_1}(u_j) = 2$, and v_s is any vertex in $V(D_2)$; that is, $(u_j, v_s) \in B \times V(D_2)$

Thus, $(u_j, v_s) \in \big[\mathrm{mCen}(D_1) \times V(D_2)\big] \bigcup \big[(V(D_1) \backslash \mathrm{mCen}(D_1)) \times \mathrm{mCen}(D_2)\big] \bigcup \big[B \times V(D_2)\big]$.
Conversely, if (u_j, v_s) belongs to any one of the sets $\mathrm{mCen}(D_1) \times V(D_2)$, $(V(D_1) \backslash \mathrm{mCen}(D_1)) \times \mathrm{mCen}(D_2)$ or $B \times V(D_2)$, then $\mathrm{mecc}_{D_1 \circ D_2}(u_j, v_s) = 2$. Therefore, $(u_j, v_s) \in \mathrm{mCen}(D_1 \circ D_2)$. Hence $\big[\mathrm{mCen}(D_1) \times V(D_2)\big] \bigcup \big[(V(D_1) \backslash \mathrm{mCen}(D_1)) \times \mathrm{mCen}(D_2)\big] \bigcup \big[B \times V(D_2)\big] \subseteq \mathrm{mCen}(D_1 \circ D_2)$. Thus, $\mathrm{mCen}(D_1 \circ D_2) = \big[\mathrm{mCen}(D_1) \times V(D_2)\big] \bigcup \big[(V(D_1) \backslash \mathrm{mCen}(D_1)) \times \mathrm{mCen}(D_2)\big] \bigcup \big[B \times V(D_2)\big] = \big[(V(D_1) \backslash \mathrm{mCen}(D_1)) \times \mathrm{mCen}(D_2)\big] \bigcup \big[(\mathrm{mCen}(D_1) \bigcup B) \times V(D_2)\big]$.
Case 3: $\mathrm{mrad}(D_2) \geq 3$.
Since $\mathrm{mrad}(D_1) = 1$ and $\mathrm{mrad}(D_2) \geq 3$, it follows that $\mathrm{mrad}(D_1 \circ D_2) = 2$. Let $(u_j, v_s) \in \mathrm{mCen}(D_1 \circ D_2)$. Then $\mathrm{mecc}_{D_1 \circ D_2}(u_j, v_s) = 2$. $\mathrm{mecc}_{D_1 \circ D_2}(u_j, v_s) = 2$ only in the following cases.

1. $u_j \in \mathrm{mCen}(D_1)$ and v_s is any vertex in $V(D_2)$; that is, $(u_j, v_s) \in \mathrm{mCen}(D_1) \times V(D_2)$

2. $u_j \notin \mathrm{mCen}(D_1)$, $\xi_{D_1}(u_j) = 2$, and v_s is any vertex in $V(D_2)$

Thus, $(u_j, v_s) \in \big[\mathrm{mCen}(D_1) \times V(D_2)\big] \bigcup \big[B \times V(D_2)\big]$.
Conversely, if (u_j, v_s) belongs to any one of the sets $\mathrm{mCen}(D_1) \times V(D_2)$ and $B \times$

$V(D_2)$, then $\text{mecc}_{D_1\circ D_2}(u_j, v_s) = 2$ and hence $\big[\text{mCen}(D_1) \times V(D_2)\big] \bigcup \big[B \times V(D_2)\big] \subseteq \text{mCen}(D_1 \circ D_2)$. Thus, $\text{mCen}(D_1 \circ D_2) = \big[\text{mCen}(D_1) \times V(D_2)\big] \bigcup \big[B \times V(D_2)\big] = \big[\text{mCen}(D_1) \bigcup B\big] \times V(D_2)$. □

Now, we consider the case when $\text{mrad}(D_1) \geq 2$.

Proposition 6.4.5. Let D_1 and D_2 be two strongly connected digraphs and $\text{mrad}(D_1) \geq 2$.

1. If $\text{mrad}(D_2) \leq \text{mrad}(D_1)$, then $\text{mCen}(D_1 \circ D_2) = \big[\text{mCen}(D_1) \times E\big] \bigcup \big[F \times V(D_2)\big]$.

2. If $\text{mrad}(D_2) = \text{mrad}(D_1) + 1$, then

$$\text{mCen}(D_1 \circ D_2) = \begin{cases} F \times V(D_2) & \text{if } F \text{ is non-empty,} \\ \big[A \times \text{mCen}(D_2)\big] \bigcup \big[X \times V(D_2)\big] & \text{otherwise,} \end{cases}$$

   where $X = \text{mCen}(D_1) \bigcup (A \cap B)$.

3. If $\text{mrad}(D_2) \geq \text{mrad}(D_1) + 2$, then

$$\text{mCen}(D_1 \circ D_2) = \begin{cases} F \times V(D_2) & \text{if } F \text{ is non-empty,} \\ \big[\text{mCen}(D_1) \bigcup (A \cap B)\big] \times V(D_2) & \text{otherwise.} \end{cases}$$

Proof. $\text{mecc}_{D_1\circ D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \min\{\xi_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}\}$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$.

1. $\text{mrad}(D_2) \leq \text{mrad}(D_1)$.
   Since $\text{mecc}_{D_1\circ D_2}(u_i, v_r) \geq \text{mecc}_{D_1}(u_i)$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$, it follows that $\text{mrad}(D_1 \circ D_2)$ is $\text{mrad}(D_1)$. Let $(u_j, v_s) \in \text{mCen}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1\circ D_2}(u_j, v_s) = \text{mrad}(D_1)$. $\text{mecc}_{D_1\circ D_2}(u_j, v_s) = \text{mrad}(D_1)$ only in the following cases.

   (a) $u_j \in \text{mCen}(D_1)$ and $v_s \in E$

   (b) $u_j \in F$ and v_s is any vertex in $V(D_2)$

   Thus, $(u_j, v_s) \in \big[\text{mCen}(D_1) \times E\big] \bigcup \big[F \times V(D_2)\big]$.
   Conversely, if (u_j, v_s) belongs to any one of the sets $\text{mCen}(D_1) \times E$ or $F \times V(D_2)$, then $\text{mecc}_{D_1\circ D_2}(u_j, v_s) = \text{mrad}(D_1)$ and hence $\big[\text{mCen}(D_1) \times E\big] \bigcup \big[F \times V(D_2)\big] \subseteq \text{mCen}(D_1 \circ D_2)$. Thus, $\text{mCen}(D_1 \circ D_2) = \big[\text{mCen}(D_1) \times E\big] \bigcup \big[F \times V(D_2)\big]$.

2. $\text{mrad}(D_2) = \text{mrad}(D_1) + 1$.

Case 1: F is non-empty.

Since there is at least one vertex $u \in \text{mCen}(D_1)$ such that $\xi_{D_1}(u) \leq \text{mecc}_{D_1}(u)$, it follows that the minimum possible m-eccentricity of a vertex in $D_1 \circ D_2$ is $\text{mrad}(D_1)$; that is, $\text{mrad}(D_1 \circ D_2) = \text{mrad}(D_1)$. Let $(u_j, v_s) \in \text{mCen}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1)$. $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1)$ only in the case $u_j \in F$ and v_s is any vertex in $V(D_2)$. Thus, $(u_j, v_s) \in F \times V(D_2)$.

Conversely, if (u_j, v_s) belongs to the set $F \times V(D_2)$, then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1)$ and hence $F \times V(D_2) \subseteq \text{mCen}(D_1 \circ D_2)$. Thus, $\text{mCen}(D_1 \circ D_2) = F \times V(D_2)$.

Case 2: F is empty.

For all vertices $u \in \text{mCen}(D_1)$, $\xi_{D_1}(u) > \text{mecc}_{D_1}(u)$ ($\xi_{D_1}(u) = \text{mecc}_{D_1}(u) + 1 = \text{mrad}(D_1) + 1$) and since $\text{mrad}(D_2) = \text{mrad}(D_1) + 1$, $\min\{\text{mecc}_{D_2}(v)\} = \text{mrad}(D_1) + 1$. So it follows that if $u \in \text{mCen}(D_1)$, $\text{mecc}_{D_1 \circ D_2}(u, v) = \max\{\text{mecc}_{D_1}(u), \min\{\xi_{D_1}(u), \text{mecc}_{D_2}(v)\}\} = \max\{\text{mrad}(D_1), \text{mrad}(D_1) + 1\}\}$ for every vertex $v \in V(D_2)$. So the minimum possible m-eccentricity of a vertex in $D_1 \circ D_2$ is $\text{mrad}(D_1) + 1$; that is, $\text{mrad}(D_1 \circ D_2) = \text{mrad}(D_1) + 1$.

Let $(u_j, v_s) \in \text{mCen}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1) + 1$. $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1) + 1$ is attained by vertices in $V(D_1 \circ D_2)$ in exactly three different ways.

(a) If $u_i \in A$ and $v_r \in \text{mCen}(D_2)$, then $\text{mecc}_{D_1}(u_i) = \text{mrad}(D_1) + 1$ and $\text{mecc}_{D_2}(v_r) = \text{mrad}(D_1) + 1$. Thus, $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mrad}(D_1) + 1$ for all $u_i \in A$, $v_r \in \text{mCen}(D_2)$.

(b) If $u_i \in \text{mCen}(D_1)$, then since F is empty, $\xi_{D_1}(u_i) = \text{mecc}_{D_1}(u_i) + 1$ and hence $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mrad}(D_1) + 1$ for all $u_i \in \text{mCen}(D_1)$, $v_r \in V(D_2)$.

(c) If $u_i \in A \cap B$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mrad}(D_1) + 1$ for all $v_r \in V(D_2)$.

Since $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) > \text{mrad}(D_1) + 1$ for all other vertices $(u_j, v_s) \in V(D_1 \circ D_2)$, it follows that $\text{mCen}(D_1 \circ D_2) \subseteq \big[A \times \text{mCen}(D_2)\big] \bigcup \big[(\text{mCen}(D_1) \bigcup (A \cap B)) \times V(D_2)\big]$.

Conversely, if (u_j, v_s) belongs to any one of the sets $A \times \text{mCen}(D_2)$, $\text{mCen}(D_1) \times V(D_2)$, and $(A \cap B) \times V(D_2)$, then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1) + 1$ and hence $\big[A \times \text{mCen}(D_2)\big] \bigcup \big[(\text{mCen}(D_1) \bigcup (A \cap B)) \times V(D_2)\big] \subseteq \text{mCen}(D_1 \circ D_2)$. Thus if F is empty, $\text{mCen}(D_1 \circ D_2) = \big[A \times \text{mCen}(D_2)\big] \bigcup \big[(\text{mCen}(D_1) \bigcup (A \cap B)) \times V(D_2)\big]$.

3. $\text{mrad}(D_2) \geq \text{mrad}(D_1) + 2$.

Case 1: F is non-empty.

As in the case $\text{mrad}(D_2) = \text{mrad}(D_1) + 1$, it follows that $\text{mrad}(D_1 \circ D_2) = \text{mrad}(D_1)$. Let $(u_j, v_s) \in \text{mCen}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1)$. $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1)$ only in the case $u_j \in F$ and v_s is any vertex in $V(D_2)$. Thus, $(u_j, v_s) \in F \times V(D_2)$.

Conversely, if (u_j, v_s) belongs to the set $F \times V(D_2)$, then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1)$ and hence $F \times V(D_2) \subseteq \text{mCen}(D_1 \circ D_2)$. Thus, $\text{mCen}(D_1 \circ D_2) = F \times V(D_2)$.

Case 2: F is empty.

Similar to the case $\text{mrad}(D_2) = \text{mrad}(D_1) + 1$, $\text{mrad}(D_1 \circ D_2)$ is $\text{mrad}(D_1) + 1$. Let $(u_j, v_s) \in \text{mCen}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1) + 1$. Here $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1) + 1$ is attained by vertices in $V(D_1 \circ D_2)$ in only two different ways. This is because the vertices in $\text{mCen}(D_2)$ have m-radius greater than or equal to $\text{mrad}(D_1) + 2$ and so $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1) + 1$ only in the following two cases.

(a) If $u_i \in A \cap B$, then $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mrad}(D_1) + 1$ for all $v_r \in V(D_2)$.

(b) If $u_i \in \text{mCen}(D_1)$, then since F is empty, $\xi_{D_1}(u_i) = \text{mecc}_{D_1}(u_i) + 1$ and hence $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \text{mrad}(D_1) + 1$ for all $u_i \in \text{mCen}(D_1)$, $v_r \in V(D_2)$.

Since $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) > \text{mrad}(D_1) + 1$ for all other vertices $(u_i, v_r) \in V(D_1 \circ D_2)$, it follows that $\text{mCen}(D_1 \circ D_2) \subseteq \left[\text{mCen}(D_1) \bigcup (A \cap B)\right] \times V(D_2)$.

Conversely, if (u_j, v_s) belongs to any one of the sets $\text{mCen}(D_1) \times V(D_2)$ or $(A \cap B) \times V(D_2)$, then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mrad}(D_1) + 1$ and hence $\left[\text{mCen}(D_1) \bigcup (A \cap B)\right] \times V(D_2) \subseteq \text{mCen}(D_1 \circ D_2)$. Thus, $\text{mCen}(D_1 \circ D_2) = \left[\text{mCen}(D_1) \bigcup (A \cap B)\right] \times V(D_2)$.

□

6.4.2 Periphery of the Lexicographic Product of Two Digraphs

The m-periphery of the lexicographic product of two strongly connected digraphs D_1 and D_2 with vertex sets $V(D_1) = \{u_1, \ldots, u_m\}$ and $V(D_2) = \{v_1, \ldots, v_n\}$ is investigated here. The m-periphery of the lexicographic product of digraphs D_1 and D_2 is already obtained in Section 6.3 in certain cases.

Expressions for the m-periphery of the lexicographic product are obtained, depending upon the m-diameters of the two digraphs. The following sets are defined for this purpose.
$G = \{u_i \in \text{mPer}(D_1) : u_i \text{ is neither DDLE nor DDEE}\}$,
$H = \{u_i \in V(D_1) : \big[\text{mecc}_{D_1}(u_i) = \text{mdiam}(D_1) - 1\big] \ \& \ \big[\xi_{D_1}(u_i) = \text{mdiam}(D_1)\big]\}$,
$I = \{v_r \in V(D_2) : \text{mecc}_{D_2}(v_r) \geq \text{mdiam}(D_1) + 1\}$,
$J = \{v_r \in V(D_2) : \text{mecc}_{D_2}(v_r) \geq \text{mdiam}(D_1)\}$.

Proposition 6.4.6. Let D_1 and D_2 be two strongly connected digraphs.

1. If $\text{mdiam}(D_2) \geq \text{mdiam}(D_1)$, then

$$\text{mPer}(D_1 \circ D_2) = \begin{cases} G \times I & \text{if } G \text{ and } I \text{ are non-empty,} \\ \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big] & \text{otherwise.} \end{cases}$$

2. If $\text{mdiam}(D_2) < \text{mdiam}(D_1)$, then $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2)$.

Proof. $\text{mecc}_{D_1 \circ D_2}(u_i, v_r) = \max\{\text{mecc}_{D_1}(u_i), \min\{\xi_{D_1}(u_i), \text{mecc}_{D_2}(v_r)\}\}$ for all $(u_i, v_r) \in V(D_1 \circ D_2)$.

1. $\text{mdiam}(D_2) \geq \text{mdiam}(D_1)$.
   In this case, the maximum m-eccentricity of the vertices in $V(D_1 \circ D_2)$ is either $\text{mdiam}(D_1) + 1$ or $\text{mdiam}(D_1)$ depending on whether G and I are non-empty or not.
   Case 1: G and I are non-empty.
   Here, the maximum possible m-eccentricity of a vertex in $D_1 \circ D_2$, which is $\text{mdiam}(D_1) + 1$ is attained. So $\text{mdiam}(D_1 \circ D_2) = \text{mdiam}(D_1) + 1$. Let $(u_j, v_s) \in G \times I$. Then $\xi_{D_1}(u_j) = \text{mecc}_{D_1}(u_j) + 1 = \text{mdiam}(D_1) + 1$ and $\text{mecc}_{D_2}(v_s) \geq \text{mdiam}(D_1) + 1$. Thus, $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \max\{\text{mdiam}(D_1), \min\{\text{mdiam}(D_1) + 1, \text{mecc}_{D_2}(v_s)\}\} = \text{mdiam}(D_1) + 1$. Hence $(u_j, v_s) \in \text{mPer}(D_1 \circ D_2)$. Thus $G \times I \subseteq \text{mPer}(D_1 \circ D_2)$.
   Conversely, let $(u_j, v_s) \in \text{mPer}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mdiam}(D_1) + 1$. The m-eccentricity $\text{mdiam}(D_1) + 1$ is attained only by those vertices $u_j \in \text{mPer}(D_1)$ for which $\xi_{D_1}(u_j) = \text{mecc}_{D_1}(u_j) + 1$, and $v_s \in V(D_2)$ for which $\text{mecc}_{D_2}(v_s) \geq \text{mdiam}(D_1) + 1$. Since the m-eccentricity of all other vertices is less than $\text{mdiam}(D_1) + 1$, $\text{mPer}(D_1 \circ D_2) \subseteq G \times I$.
   Hence $\text{mPer}(D_1 \circ D_2) = G \times I$.
   Case 2: Either G or I is empty.

If G is empty, then since no vertex in $V(D_1 \circ D_2)$ has m-eccentricity $\text{mdiam}(D_1) + 1$, it follows that $\text{mdiam}(D_1 \circ D_2) = \text{mdiam}(D_1)$. If G is non-empty and I is empty, then also $\text{mdiam}(D_1 \circ D_2) = \text{mdiam}(D_1)$ (since there is no vertex $v_r \in V(D_2)$ for which $\text{mecc}_{D_2}(v_r) \geq \text{mdiam}(D_1) + 1$). So $\text{mdiam}(D_1 \circ D_2) = \text{mdiam}(D_1)$.

Let $(u_j, v_s) \in \text{mPer}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mdiam}(D_1)$. In this case, the m-eccentricity of $\text{mdiam}(D_1)$ is attained by two types of vertices in $V(D_1 \circ D_2)$. These are the vertices in $\text{mPer}(D_1) \times V(D_2)$ and all the vertices in $H \times J$ and hence $(u_j, v_s) \in \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big]$. Thus, $\text{mPer}(D_1 \circ D_2) \subseteq \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big]$.

Conversely, if $(u_j, v_s) \in \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big]$, then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mdiam}(D_1)$ and hence $(u_j, v_s) \in \text{mPer}(D_1 \circ D_2)$. Thus, $\big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big] \subseteq \text{mPer}(D_1 \circ D_2)$. Hence $\text{mPer}(D_1 \circ D_2) = \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big]$.

2. $\text{mdiam}(D_2) < \text{mdiam}(D_1)$.

   Since $\text{mecc}_{D_2}(v_r) < \text{mdiam}(D_1)$ for all $v_r \in V(D_2)$, $\text{mdiam}(D_1 \circ D_2) = \text{mdiam}(D_1)$. Let $(u_j, v_s) \in \text{mPer}(D_1 \circ D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mdiam}(D_1)$, which is attained by all vertices in $\text{mPer}(D_1) \times V(D_2)$. Since $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) < \text{mdiam}(D_1)$ for all other vertices $(u_j, v_s) \in V(D_1 \circ D_2)$, it follows that $\text{mPer}(D_1 \circ D_2) \subseteq \text{mPer}(D_1) \times V(D_2)$.

   Conversely, let $(u_j, v_s) \in \text{mPer}(D_1) \times V(D_2)$. Then $\text{mecc}_{D_1 \circ D_2}(u_j, v_s) = \text{mdiam}(D_1)$ and hence $(u_j, v_s) \in \text{mPer}(D_1 \circ D_2)$. Thus, $\text{mPer}(D_1) \times V(D_2) \subseteq \text{mPer}(D_1 \circ D_2)$. Hence $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2)$.

□

It can be seen that the results obtained for the m-periphery of the lexicographic product of any two digraphs D_1 and D_2 are in line with the results obtained when D_1 is either a DDLE digraph, or a dicycle, or a symmetric digraph. To see this, consider the results of Proposition 6.4.6 in the following five cases.

1. D_1 is a DDLE digraph.

   If $\text{mdiam}(D_2) < \text{mdiam}(D_1)$, then $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2)$. If $\text{mdiam}(D_2) \geq \text{mdiam}(D_1)$, then $\text{mPer}(D_1 \circ D_2) = \big[\text{mPer}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big]$, since G is empty. As H is empty, this expression also reduces to $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2)$.

2. D_1 is the dicycle $\overrightarrow{C_n}$.
   Here $\text{mdiam}(\overrightarrow{C_n}) = n - 1$. Thus, there arises the following three cases.
   (a) If $\text{mrad}(D_2) \geq n$, then $\text{mdiam}(D_2) \geq \text{mdiam}(\overrightarrow{C_n})$ and hence, $\text{mPer}(D_1 \circ D_2) = G \times I = V(\overrightarrow{C_n}) \times V(D_2)$.
   (b) If $\text{mdiam}(D_2) < n$; that is, $\text{mdiam}(D_2) < \text{mdiam}(D_1)$. Then $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2) = V(\overrightarrow{C_n}) \times V(D_2)$.
   (c) If $\text{mrad}(D_2) < n$ and $\text{mdiam}(D_2) \geq n$, then since G and I are non-empty, $\text{mPer}(D_1 \circ D_2) = G \times I = V(\overrightarrow{C_n}) \times I$.

3. D_1 is the complete graph K_n.
   Here $\text{diam}(D_1) = 1$ and hence $\text{mdiam}(D_2) \geq \text{diam}(D_1)$. $G = V(D_1)$ and $I = \{v_r \in V(D_2) : \text{mecc}_{D_2}(v_r) \geq 2\}$. Thus there are three cases. Either $D_2 = K_m$, or $\text{mrad}(D_2) \geq 2$, or $\text{mrad}(D_2) = 1$, $\text{mdiam}(D_2) = 2$ (since $\text{mdiam}(D_2) \leq 2\,\text{mrad}(D_2)$).
   (a) If $D_2 = K_m$, then I is empty and hence $\text{mPer}(D_1 \circ D_2) = \text{mPer}(D_1) \times V(D_2) = V(K_n) \times V(D_2)$, since H is empty.
   (b) If $\text{mrad}(D_2) \geq 2$, then $I = V(D_2)$. Thus $\text{mPer}(D_1 \circ D_2) = V(K_n) \times V(D_2)$.
   (c) If $\text{mrad}(D_2) = 1$ and $\text{mdiam}(D_2) = 2$, then $I = V(D_2) \backslash \text{mCen}(D_2)$. Thus $\text{mPer}(D_1 \circ D_2) = V(K_n) \times [V(D_2) \backslash \text{mCen}(D_2)]$.

4. D_1 is a symmetric digraph with $\text{diam}(D_1) \geq 3$.
   $\xi_{D_1}(u_i) = 2$ for all $u_i \in V(D_1)$. Hence G and H are empty.
   (a) If $\text{mdiam}(D_2) < \text{diam}(D_1)$, then $\text{mPer}(D_1 \circ D_2) = \text{Per}(D_1) \times V(D_2)$.
   (b) If $\text{mdiam}(D_2) \geq \text{diam}(D_1)$, then since G and H are empty, $\text{mPer}(D_1 \circ D_2) = \text{Per}(D_1) \times V(D_2)$.

5. D_1 is a symmetric digraph with $\text{diam}(D_1) = 2$.
   In this case, three subcases arise.
   (a) If $\text{mrad}(D_2) \geq 2$, then $\text{mdiam}(D_2) \geq \text{diam}(D_1)$. G is empty. Thus $\text{mPer}(D_1 \circ D_2) = \big[\text{Per}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big] = \big[\text{Per}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \backslash \text{Per}(D_1) \times V(D_2)\big] = V(D_1) \times V(D_2)$.
   (b) If $\text{mrad}(D_2) = 1$ and $\text{mdiam}(D_2) = 2$, then $\text{mdiam}(D_2) \geq \text{diam}(D_1)$. G is empty. Thus $\text{mPer}(D_1 \circ D_2) = \big[\text{Per}(D_1) \times V(D_2)\big] \bigcup \big[H \times J\big] = \big[\text{Per}(D_1) \times V(D_2)\big] \bigcup \big[V(D_1) \backslash \text{Per}(D_1) \times V(D_2)\big] = V(D_1) \times (V(D_2) \backslash \text{mCen}(D_2))$.

(c) If $\text{mrad}(D_2) = 1$ and $\text{mdiam}(D_2) = 1$; that is $D_2 = K_m$, then $m\text{diam}(D_2) < \text{diam}(D_1)$. Thus $\text{mPer}(D_1 \circ D_2) = \text{Per}(D_1) \times V(D_2)$. Here B' is empty. So the result agrees with the earlier result.

6.5 Center and Periphery of the Lexicographic Product of Two Graphs - An Erratum

Finally, we consider the center and periphery of the lexicographic product of two graphs. The descriptions for the center and periphery of two graphs is given in [75] by Zarah Yarahmadi and Sirous Moradi. Some minor mistakes in these descriptions are rectified. The eccentricity of a vertex in the lexicographic product $G \circ H$ of two graphs G and H (notation used in the article is $G[H]$) is given in article [30] which deals with the eccentric connectivity index for several families of composite graphs. For a vertex $(u, v) \in V(G[H])$,

$$\text{ecc}_{G[H]}(u, v) = \begin{cases} 1 & \text{if } \text{ecc}_G(u) = 1, \text{ecc}_H(v) = 1, \\ 2 & \text{if } \text{ecc}_G(u) = 1, \text{ecc}_H(v) \geq 2, \\ \text{ecc}_G(u) & \text{if } \text{ecc}_G(u) \geq 2. \end{cases} \tag{6.1}$$

Based on the results 6.1, the expressions for the center and the periphery of lexicographic product of two graphs is given in [75]. Item 6 of Theorem 3.2 of the article [75] is as follows.

Theorem 6.5.1. Let G and H be graphs. Then

$$\text{Cen}(G[H]) = \begin{cases} \text{Cen}(G) \times \text{Cen}(H) & \text{if } \text{rad}(G) = 1 \text{ and } \text{rad}(H) = 1, \\ \text{Cen}(G) \times V(H) & \text{otherwise.} \end{cases} \tag{6.2}$$

Similarly, item 6 of Theorem 3.12 of the article [75] is as follows.

Theorem 6.5.2. Let G and H be graphs. Then

$$Per(G[H]) = \begin{cases} V(G) \times V(H) & \text{if } \text{diam}(G) = 1 \text{ and } \text{diam}(H) = 1, \\ V(G) \times A & \text{if } \text{diam}(G) = 1 \text{ and } \text{diam}(H) \geq 1, \\ Per(G) \times V(H) & \text{if } \text{diam}(G) \geq 2, \end{cases} \tag{6.3}$$

where $A = \{b \in V(H) : \text{ecc}_H(b) \geq 2\}$.

Consider the lexicographic product of the graphs G and H in Figure 6.3. Here, $\mathrm{rad}(G) = 1$ and $\mathrm{rad}(H) = 2$ and $\mathrm{Cen}(G \circ H) = V(G) \times V(H)$. According to Theorem 6.5.1 from article [75], $\mathrm{Cen}(G \circ H) = \mathrm{Cen}(G) \times V(H)$, which is not true in this example.

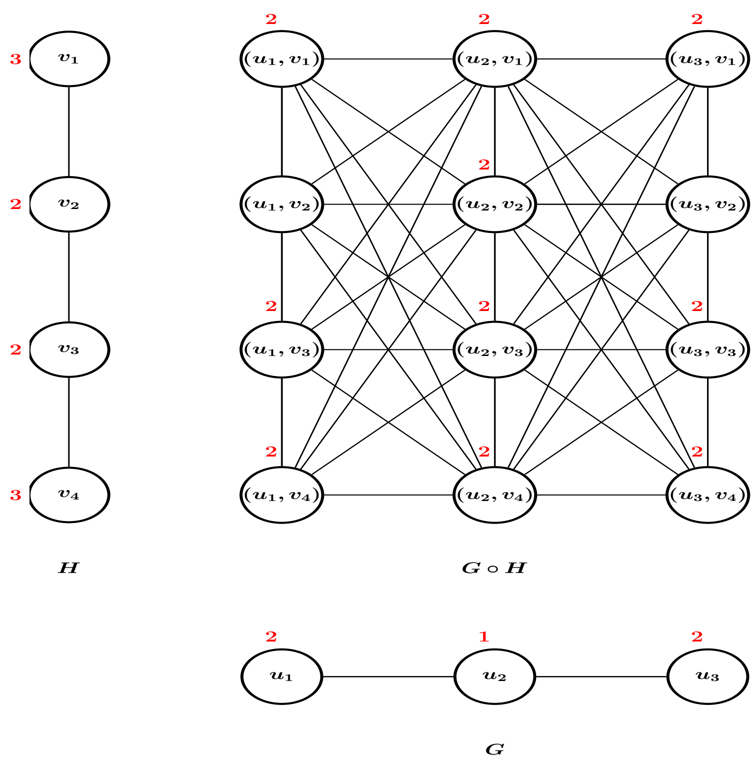

Figure 6.3: An example for $G \circ H$ with $\mathrm{rad}(G) = 1$ and $\mathrm{rad}(H) \geq 2$.

Thus we can see that the case when $\mathrm{rad}(G) = 1$ and $\mathrm{rad}(H) \geq 2$ was not mentioned in Theorem 6.5.1. So we give the corrected version of Theorem 6.5.1 below.

Theorem 6.5.1′. *Let G and H be two graphs. Then*

$$\mathrm{Cen}(G \circ H) = \begin{cases} \mathrm{Cen}(G) \times \mathrm{Cen}(H) & \text{if } \mathrm{rad}(G) = 1 \text{ and } \mathrm{rad}(H) = 1, \\ V(G) \times V(H) & \text{if } \mathrm{rad}(G) = 1 \text{ and } \mathrm{rad}(H) \geq 2, \\ \mathrm{Cen}(G) \times V(H) & \text{otherwise.} \end{cases}$$

Proof. Consider $\mathrm{Cen}(G \circ H)$ in different cases using result 6.1. Note that when $\mathrm{rad}(G) = 1$, $\mathrm{diam}(G) \leq 2$.

1. If $\mathrm{rad}(G) = 1$ and $\mathrm{rad}(H) = 1$, then $\mathrm{ecc}_{G \circ H}(u, v) = 1$ for all $(u, v) \in \mathrm{Cen}(G) \times \mathrm{Cen}(H)$ and $\mathrm{ecc}_{G \circ H}(u, v) = 2$ for all $(u, v) \notin \mathrm{Cen}(G) \times \mathrm{Cen}(H)$. Hence $\mathrm{Cen}(G \circ H) = \mathrm{Cen}(G) \times \mathrm{Cen}(H)$.

2. If $\text{rad}(G) = 1$ and $\text{rad}(H) \geq 2$, then $\text{ecc}_{G\circ H}(u, v) = 2$ for all $(u, v) \in V(G) \times V(H)$. Therefore, $\text{Cen}(G \circ H) = V(G) \times V(H)$.

3. If $\text{rad}(G) \geq 2$, then $\text{ecc}_{G\circ H}(u, v) = \text{ecc}_G(u)$ for all $(u, v) \in V(G) \times V(H)$ and hence it follows that $\text{Cen}(G \circ H) = \text{Cen}(G) \times V(H)$.

□

Now consider the periphery of $G \circ H$ for the graphs G and H in Figure 6.3. Here, $\text{Per}(G \circ H) = V(G) \times V(H)$. As per Theorem 6.5.2 from article [75], $\text{Per}(G \circ H) = \text{Per}(G) \times V(H)$ which is not true. Thus for periphery of $G \circ H$, the case $\text{diam}(G) = 2$ has to be considered separately. The corrected version of Theorem 6.5.2 is as follows.

Theorem 6.5.2′. *Let G and H be two graphs. Then*

$$\text{Per}(G \circ H) = \begin{cases} V(G) \times V(H) & \text{if } \text{diam}(G) = 1,\ \text{diam}(H) = 1, \\ V(G) \times A & \text{if } \text{diam}(G) = 1,\ \text{diam}(H) \geq 2, \\ \big[\text{Per}(G) \times V(H)\big] \bigcup \big[V(G) \times V(H) \backslash \text{Cen}(H)\big] & \text{if } \text{diam}(G) = 2,\ \text{rad}(H) = 1, \\ V(G) \times V(H) & \text{if } \text{diam}(G) = 2,\ \text{rad}(H) \geq 2, \\ \text{Per}(G) \times V(H) & \text{if } \text{diam}(G) > 2, \end{cases}$$

where $A = \{b \in V(H) : \text{ecc}_H(b) \geq 2\}$.

Proof. Using result 6.1, $\text{Per}(G \circ H)$ is determined in all cases.

1. If $\text{diam}(G) = 1$ and $\text{diam}(H) = 1$, then $\text{ecc}_{G\circ H}(u, v) = 1$ for all $(u, v) \in V(G) \times V(H)$. Hence when both G and H are complete graphs, $\text{Per}(G \circ H) = V(G) \times V(H)$.

2. If $\text{diam}(G) = 1$ and $\text{diam}(H) \geq 2$, then $\text{ecc}_{G\circ H}(u, v) = 2$ for all $(u, v) \in V(G) \times A$. Therefore, $\text{Per}(G \circ H) = V(G) \times A$.

3. If $\text{diam}(G) = 2$ and $\text{rad}(H) = 1$, then $\text{ecc}_{G\circ H}(u, v) = 1$ only when $\text{ecc}_G(u) = 1$ and $\text{ecc}_H(v) = 1$; that is if $u \notin \text{Per}(G)$ and $v \in \text{Cen}(H)$. Thus, if $u \in \text{Per}(G)$ or $v \notin Cen(H)$, then $\text{ecc}_{G\circ H}(u, v) = 2$. Hence it follows that $\text{Per}(G \circ H) = \big[\text{Per}(G) \times V(H)\big] \bigcup \big[V(G) \times V(H) \backslash \text{Cen}(H)\big]$.

4. If $\text{diam}(G) = 2$ and $\text{rad}(H) \geq 2$, then $\text{ecc}_{G\circ H}(u, v) = 2$ for all $(u, v) \in V(G) \times V(H)$. Therefore, $\text{Per}(G \circ H) = V(G) \times V(H)$.

5. If $\text{diam}(G) > 2$, then since $\text{rad}(G) > 1$, $\text{ecc}_{G \circ H}(u, v) = \text{ecc}_G(u)$ for all $(u, v) \in V(G) \times V(H)$. So it follows that $\text{Per}(G \circ H) = \text{Per}(G) \times V(H)$.

□

Chapter 7

Concluding Remarks

In this thesis, the four boundary-type sets and the center of the three products, namely, the Cartesian, strong, and lexicographic products of two strongly connected digraphs are subjected to study. The results are extended to a finite number of digraphs in the case of the two commutative products, the Cartesian and the strong product. The results obtained for digraphs are compared to the earlier results obtained for graphs. It is ensured that all the possible cases coming under the scope of this research is covered.

In the study of large networks, the determination of the boundary-type sets and center has important applications. In certain cases involving designing of one-way networks, it helps in an optimal construction of a network. Even in the case of internet and other biological networks, the information about boundary-type sets can be used to determine both the local and global efficiency of the network. The center vertices of a digraph can be considered as the most prominent vertices. Determination of the center vertices in large digraphs also has several applications which varies from selecting a leader in a community to finding the location for emergency facilities. The results for various digraph products, combined with prime factorization of digraphs make the determination procedure simpler. This is accomplished by first applying any of the algorithms for finding the prime factor decomposition and then applying these results to find the boundary-type sets and center, in all possible cases.

Problems for Further Study

- A study of the boundary-type sets and the center of the various digraph products with respect to the metric 'sum distance'.

- A study of the boundary-type sets and the center of the various digraph products with respect to the metric ‘strong distance’.

- A study of the boundary-type sets and the center of the other digraph products like corona product and cardinal product with respect to the metric ‘maximum distance’.

- A study of the boundary-type sets and the center of the direct product with respect to the metric ‘maximum distance’ on the class of digraphs with loops allowed.

- The linear algebra approach may be extended to digraphs to study the spectral properties based on the metric md. A similar study was recently conducted based on the sum distance in digraphs [74].

- Eccentric sequences of graphs and digraphs have been already studied [6, 35]. A study of m-eccentric sequences of digraphs is aimed at.

- A study of self-centered digraphs and the problem of embedding a digraph D in a digraph D' containing D such that $\mathrm{mCen}(D')$ is isomorphic to D. Such studies have already been made in the case of graphs.

List of Publications/Presentations

List of Publications

4. Prasanth G. Narasimha-Shenoi, and Mary Shalet Thottungal Joseph. "Center of Cartesian and strong product of digraphs." Journal of the Ramanujan Mathematical Society 36, no. 4 (2021): 267-273.

3. Bijo S. Anand, Manoj Changat, Prasanth G. Narasimha-Shenoi, and Mary Shalet Thottungal Joseph. "Boundary-type sets of strong product of directed graphs." ARS MATHEMATICA CONTEMPORANEA 20, no. 2 (2021): 275-288.

2. Manoj Changat, Prasanth G. Narasimha-Shenoi, and Mary Shalet Thottungal Joseph. "Lexicographic Product of Digraphs and Related Boundary-Type Sets." In Conference on Algorithms and Discrete Applied Mathematics, pp. 234-246. Springer, Cham, 2021.

1. Manoj Changat, Prasanth G. Narasimha-Shenoi, Mary Shalet Thottungal Joseph, and Ram Kumar. "Boundary Vertices of Cartesian Product of Directed Graphs." International Journal of Applied and Computational Mathematics 5, no. 1 (2019): 1-19.

List of Presentations

4. Presented the paper entitled "Lexicographic Product of Digraphs and Related Boundary-type Sets" in the Seventh International Conference on Algorithms and Discrete Applied Mathematics held during February 11-13, 2021 in the Department of Computer Science and Engineering, IIT Ropar, India.

3. Presented the paper entitled "Center of Cartesian and Strong Product of Digraphs" in the International Conference on Number Theory and Discrete Math-

ematics held during December 11-14, 2020 organized by Ramanujan Mathematical Society at Rajagiri School of Engineering & Technology.

2. Presented the paper entitled “Some distance related properties of digraphs and their Cartesian product” in the International Conference on Graph Connections held during August 6-8, 2020 organized by Department of Mathematics and Statistics and IQAC, BCM College, Kottayam in association with Mahatma Gandhi University, Kottayam.

1. Presented the paper entitled “Directed Graphs and its Boundary Vertices” in the National conference on Discrete Structures and its Applications held during September 29-30, 2016 organized by Department of Mathematics, LBS Institute of Technology for Women, Thiruvanathapuram.

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