A STUDY ON SOME CLASSES OF BOUNDED AND UNBOUNDED OPERATORS

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By

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DECLARATION

I hereby declare that the thesis entitled "**A STUDY ON SOME CLASSES OF BOUNDED AND UNBOUNDED OPERATORS** " is a bonafide record of the original research work done by me under the guidance and supervision of Dr. Shine Lal E, Associate Professor, Department of Mathematics, University College, Thiruvananthapuram and that this thesis or any part thereof has not previously formed the basis for the award of any degree either to this University or to any other University or Institution. The manuscript has been subjected to plagiarism check by **iThenticate** software.

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CERTIFICATE

This is to certify that the thesis entitled "**A STUDY ON SOME CLASSES OF BOUNDED AND UNBOUNDED OPERATORS** " submitted by **Ramya P,** to the University of Calicut, for the award of the degree of **Doctor of Philosophy** in Mathematics is a bonafide record of the original research work carried out by her under my guidance. The contents of this thesis, in full or in parts, have not been submitted to this University or to any other University or Institution for the award of any degree. The manuscript has been subjected to plagiarism check by **iThenticate** software.

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Chapter 1

Introduction and preliminaries

1.1 Introduction

Operator theory is a main branch of Mathematics, which mainly focuses on the study of bounded and unbounded linear operators defined on a normed linear space. A continuous linear operator on a normed linear space is called a bounded linear operator. An operator which is not bounded is called an unbounded operator. It is well known that the class of all closed densely defined operators is a subclass of the set of all unbounded operators. Self adjoint densely defined operators play an important role in the field of quantum mechanics ([12]).

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space. A linear operator T on $\mathcal{H}$ is said to be bounded if there exist a constant $c > 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in \mathcal{H}$ ([37]). Let $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators defined on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$,

$$\|T\| = \sup \{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$$

([37]). Let $T \in \mathcal{B}(\mathcal{H})$. Then there exist a unique operator $S \in \mathcal{B}(\mathcal{H})$ such that $\langle Tx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in \mathcal{H}$. The operator S is called the adjoint of T and is denoted by T^*. An operator T is said to be normal if $T^*T = TT^*$ and is selfadjoint if $T^* = T$ ([37]). It is clear that every self- adjoint operators are normal. T is said to be positive if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$ and it is denoted by $T \geq 0$.

In the last two or three decades, there was much progress in the study of non normal classes of operators. Some of the non normal classes operators are hyponormal, M-hyponormal, paranormal, $*$-paranormal operators etc. P. R. Halmos introduced hyponormal operator, which is an extension of normal operator. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$ ([16]). In ([48, 50]), I. H. Sheth and J. G. Stampfli studied some characterizations of hyponormal operators and its spectrum. Later, J. G. Stampfli introduced a class of M-hyponormal operators, which contains the class of hyponormal operators. For $M > 0$. T is said to be *M-hyponormal* if $\|(T - \lambda I)^*x\| \leq M\|(T - \lambda I)x\|$ for all $\lambda \in \mathbb{C}$ and for all $x \in \mathcal{H}$ ([55]). In ([44, 55]), M. Radjabalipour and B. L. Wadhwa studied some properties of M-hyponormal operators. Note that M-hyponormal operators are not normaloid. T is said to be *dominant* if for each $\lambda \in \mathbb{C}$, there exist a constant $M(\lambda) \geq 0$ such that $\|(T - \lambda I)^*x\| \leq M(\lambda)\|(T - \lambda I)x\|$ for all $x \in \mathcal{H}$. It is clear that every M-hyponormal operators are dominant. In ([51]), some spectral properties of dominant operators were studied. It is clear that

$$selfadjoint \subset normal \subset hyponormal \subset M - hyponormal \subset dominant$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *posinormal* if $TT^* \leq \lambda^2 T^*T$ for some constant $\lambda \geq 0$ ([47]). T is said to be *polynomially (P)-posinormal* if $P(T)P(T^*) \leq \lambda^2 T^*T$, where $P(z)$ is a polynomial with zero constant term and for some constant $\lambda \geq 0$ ([26]). It is clear that

$$hyponormal \subset posinormal \subset polynomially\ (P) - posinormal$$

T is said to be a *normaloid* if $\|T\| = \sup\ \{|\lambda| : \lambda \in \sigma(T)\}$, where $\sigma(T)$ is the spectrum of T. It can be seen that every hyponormal operators are normaloid ([50]), but every dominant operators need not be normaloid.

T. Furuta introduced a new class of operators, paranormal operators, which contains the class of hyponormal operators and are normaloid ([13, 28]). T is said to be *paranormal* if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ ([13]).

T is said to be $*$-*paranormal* if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ ([41]). It can be seen that every $*$-paranormal operators are normaloid, but need not be paranormal ([53]). In ([53]), K. Tanahashi and A. Uchiyama studied some characterizations and spectral properties of $*$-paranormal operators.
In ([5]), P. Dharmarha and S. Ram introduced (m, n)-paranormal and $(m, n)^*$-paranormal operators, which are extensions of paranormal and $*$-paranormal operators respectively. For $m \in \mathbb{R}^+$, $n \in \mathbb{N}$. T is said to be (m, n)- paranormal if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ ([6]), and is said to be $(m, n)^*$- paranormal if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ ([5]).
B. P. Duggal, C. S. Kubrusly and N. Levan introduced class $\mathcal{Q}$ operator, which is an extensions of paranormal operator. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *class* $\mathcal{Q}$ if $T^{*2}T^2 - 2T^*T + I \geq 0$ ([10]).

Many authors studied the properties of closed densely defined operators on a Hilbert space ([1, 30, 31]). In ([31]), S. H. Kulkarni, M. T. Nair and G. Ramesh studied certain spectral characterizations of such operators. J. Janas introduced densely defined hyponormal operator and studied its properties ([23]). Some properties of closed densely defined paranormal operator are studied by N. Bala and G. Ramesh ([1]).

"If a bounded linear map interwines two normal operators, then it interwines their adjoint" is known as the famous Putnam - Fuglede theorem ([42]). It is known that every subnormal operators need not satisfies Putnam - Fuglede theorem. Let $A, B \in \mathcal{B}(\mathcal{H})$ be normal operators. If $AX = XB^*$ for some $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB$. This is known as asymmetric Putnam- Fuglede theorem. In ([14, 15]), T. Furuta proved asymmetric Putnam - Fuglede theorem for bounded subnormal operators. Asymmetric Putnam - Fuglede theorem for closed densely defined hyponormal and closed densely defined subnormal operators are proved by Stochel ([52]).

In this thesis, we introduce some new classes of operators k-quasi (m, n)-paranormal, k-quasi $(m, n)^*$-paranormal, (m, n)-class $\mathcal{Q}$, (m, n)-class $\mathcal{Q}^*$, k-quasi (m, n)-class $\mathcal{Q}$ and k-quasi (m, n)-class $\mathcal{Q}^*$ operators which contains some well known classes of operators. Also we introduced totally $(m, n)^*$-paranormal operator, which is having nice characteristics, like transilation invariance and finiteness. Moreover we studied some properties of polynomially P- posinormal operators namely, finiteness, spectral continuity etc. Finally, we introduced a closed densely defined M- hyponormal operator and proved asymmetric Fuglede- Putnam theorem for this class.

1.2 Outline of the thesis

Thesis is divided in to seven chapters.

In the second chapter, we introduce two classes of operators namely k-quasi $(m, n)^*$-paranormal operators and k-quasi (m, n)-paranormal operators, which includes the classes of $(m, n)^*$-paranormal and (m, n)-paranormal operators respectively. We proved some characteristics of the operators and its 2×2 matrix representation.

In the third chapter, we define (m, n)-class $\mathcal{Q}$ and (m, n)-class $\mathcal{Q}^*$ operators, which contains the classes of (m, n)-paranormal and $(m, n)^*$- paranormal operators respectively. Also we characterize the classes of composition operators of (m, n)-class $\mathcal{Q}$ and (m, n)-class $\mathcal{Q}^*$ operators on L^2 space.

In the fourth chapter, we introduce k-quasi (m, n)-class $\mathcal{Q}$ and k-quasi (m, n)-class $\mathcal{Q}^*$ operators, which are the extensions of k-quasi (m, n)-paranormal and k-quasi $(m, n)^*$-paranormal operators respectively. Here we give some examples and some properties of these classes of operators. Also we characterize the composition operators of these classes on L^2 space.

In the fifth chapter, we define two new classes of operator namely totally $(m, n)^*$-paranormal and k-quasi totally $(m, n)^*$-paranormal operators. We study its spectral continuity, finiteness and 2×2 matrix representation.

In the sixth chapter, we mainly deal with totally P-posinormal operators. In this we study its spectral properties, Riesz projection, spectral continuity and range kernel orthoganality.

In the seventh chapter, we define a closed densely defined M- hyponormal operator which contains closed densely defined hyponormal operator. We prove asymmetric Fuglede-Putnam theorem for closed densely defined M- hyponormal and closed densely defined subnormal operators.

1.3 Preliminaries

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space over the field of complex numbers $\mathbb{C}$. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be bounded if and only if T is continuous on $\mathcal{H}$ ([37]). Let $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators defined on $\mathcal{H}$. It is well known that for $T \in \mathcal{B}(\mathcal{H})$, there is a unique operator $T_1 \in \mathcal{B}(\mathcal{H})$ such that $\langle Tx, y\rangle = \langle x, T_1 y\rangle$ for all $x, y \in \mathcal{H}$. The operator T_1 is called adjoint of T, and is denoted as T^*. Let $N(T)$ and $R(T)$ denote the kernel and range of T respectively. The point spectrum of T, $\sigma_p(T)$, approximate point spectrum of T, $\sigma_a(T)$, residual spectrum of T, $\sigma_r(T)$ and resolvent set of T, $\rho(T)$ are defined as:
$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-one}\}$
$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}$
$\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is one-one and } R(T - \lambda I) \text{ is not dense in } \mathcal{H}\}$
$\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})\}$.
The compliment of $\rho(T)$ is called spectrum of T and is denoted by $\sigma(T)$. The joint point spectrum, $\sigma_{jp}(T)$ and the joint approximate point spectrum, $\sigma_{ja}(T)$ are defined as:
$\sigma_{jp}(T) = \{\lambda \in \mathbb{C} : \lambda \in \sigma_p(T) \text{ and } \bar{\lambda} \in \sigma_p(T^*)\}$
$\sigma_{ja}(T) = \{\lambda \in \mathbb{C} : \lambda \in \sigma_a(T) \text{ and } \bar{\lambda} \in \sigma_a(T^*)\}$.
Now we give some well known results in connection with spectra of operators.

Theorem 1.3.1. *([17]) Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T) = \sigma_a(T) \cup \{\bar{\lambda} : \lambda \in \sigma_a(T^*)\}$.*

Theorem 1.3.2. *([29]) Let $T \in \mathcal{B}(\mathcal{H})$. Then $\partial\sigma(T) \subset \sigma_a(T)$, where $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$.*

Now we give some characterizations and examples of some well known classes of bounded operators on $\mathcal{H}$ namely subnormal, hyponormal, posinormal etc.

Let $T \in \mathcal{B}(\mathcal{H})$. T is said to be *subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a normal operator S on $\mathcal{K}$ such that $Tx = Sx$ for all $x \in \mathcal{H}$. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{2}{3}x_2, \frac{3}{4}x_3, \cdots)$ is subnormal ([28]). Recall that T is said to be a *normaloid* if $\|T\| = \sup\ \{|\lambda| : \lambda \in \sigma(T)\}$.

For example, $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, x_4, \cdots)$ is normaloid.

An operator $T \in \mathcal{B}(\mathcal{H})$ is *hyponormal* if and only if $T^*T \geq TT^*$ ([16]). For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$ is hyponormal.

T is *M-hyponormal* if and only if $M^2(T - \lambda I)^*(T - \lambda I) - (T - \lambda I)(T - \lambda I)^* \geq 0$ for all $\lambda \in \mathbb{C}$, for some $M > 0$ ([55]). In particular, by taking $M = 1$ and $\lambda = 0$ in the above relation, we get $T^*T \geq TT^*$. That is, 1-hyponormal operators are hyponormal. It is known that hyponormal operators preserves transilation invariant property ([50]). Hence every hyponormal operators are 1-hyponormal. Thus the class of all hyponormal operators is a subclass of the classes of all M- hyponormal operators. But the converse need not be true. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, 2x_2, x_3, x_4, \cdots)$ is M-hyponormal for any $M > 1$, but not hyponormal.

Recall that T is said to be *dominant* if for each $\lambda \in \mathbb{C}$, there exist a constant $M(\lambda) \geq 0$ such that $\|(T - \lambda I)^*x\| \leq M(\lambda)\|(T - \lambda I)x\|$ for all $x \in \mathcal{H}$ ([7]). If $M(\lambda) = M$ for all $\lambda \in \mathbb{C}$, then the dominant operator is M-hyponormal. That is, every M-hyponormal operators are dominant. But the converse need not be true. For example, let $\{e_n\}$ be an orthonormal basis of a Hilbert space $l^2(\mathbb{Z})$. Consider a bilateral weighted shift operator $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ defined by $Te_n = 2^{-|n|}e_{n+1}$, for all $n \in \mathbb{Z}$. Then T is dominant but not M-hyponormal for any $M > 0$ ([44]).

It is evident that

$$selfadjoint \subset normal \subset hyponormal \subset M - hyponormal \subset dominant$$

Recall that $T \in \mathcal{B}(\mathcal{H})$ is *posinormal* if $TT^* \leq \lambda^2 T^*T$ for some constant $\lambda \geq 0$ ([47]). For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, 3x_2, x_3, x_4, \cdots)$ is posinormal.

Recall that $T \in \mathcal{B}(\mathcal{H})$ is *polynomially (P)-posinormal* if $P(T)P(T^*) \leq \lambda^2 T^*T$, where $P(z)$ is a polynomial with zero constant term and for some constant $\lambda \geq 0$ ([26]). If $P(z) = z$, then polynomially (P)-posinormal operator become posinormal.

For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, 5x_2, x_3, x_4, \cdots)$ is polynomially (P)-posinormal. In General

$$hyponormal \subset M - hyponormal \subset posinormal \subset polynomially\ (P) - posinormal$$

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be *paranormal* if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ ([13]). It is known that T is paranormal if and only if $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I > 0$, for all $\lambda > 0$ ([28]). Note that every paranormal operators are normaloid ([28]).

Lemma 1.3.1. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_k\}, (k = 1, 2, ...)$, defined by $Te_k = \alpha_k\ e_{k+1}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is paranormal if and only if*

$$|\alpha_k|^2 \leq |\alpha_k|\ |\alpha_{k+1}|, \forall\ k \in \mathbb{N}.$$

Proof. Since $Te_k = \alpha_k\ e_{k+1}$, we have $T^2e_k = \alpha_k\ \alpha_{k+1}\ e_{k+2},\ \forall\ k \in \mathbb{N}$. Now,

$$\begin{aligned} T \text{ is } \text{ paranormal} &\Leftrightarrow \|Tx\|^2 \leq \|T^2x\|\|x\|, \forall x \in \mathcal{H}. \\ &\Leftrightarrow \|Te_k\|^2 \leq \|T^2e_k\|\|e_k\|, \forall k \in \mathbb{N}. \\ &\Leftrightarrow |\alpha_k|^2 \leq |\alpha_k|\ |\alpha_{k+1}|, \forall k \in \mathbb{N}. \end{aligned}$$

□

Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, x_4, \cdots)$. Using Lemma 1.3.1, we can see that T is paranormal.
Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be $*$-*paranormal* if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ ([41]).

Lemma 1.3.2. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_k\}, (k = 1, 2, ...)$, defined by $Te_k = \alpha_k\ e_{k+1}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is $*$-paranormal if and only if*

$$|\alpha_k|^2 \leq |\alpha_{k+1}|\ |\alpha_{k+2}|, \forall\ k \in \mathbb{N}.$$

Proof. Since $Te_k = \alpha_k\ e_{k+1}$, we have $T^2 e_k = \alpha_k\ \alpha_{k+1}\ e_{k+2},\ \forall\ k \in \mathbb{N}$ and $T^* e_k\ =\ \overline{\alpha_{k-1}}\ e_{k-1},\ \forall\ k \geq 2$. Now,

$$\begin{aligned} T \text{ is } *\text{-paranormal} &\Leftrightarrow \|T^*x\|^2 \leq \|T^2x\|\|x\|, \forall x \in \mathcal{H}. \\ &\Leftrightarrow \|T^*e_k\|^2 \leq \|T^2e_k\|\|e_k\|, \forall k \in \mathbb{N}. \\ &\Leftrightarrow |\alpha_k|^2 \leq |\alpha_{k+1}|\ |\alpha_{k+2}|, \forall k \in \mathbb{N}. \end{aligned}$$

□

Consider $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by $T(x_1, x_2, x_3, \cdots) = (0, \sqrt{2}x_1, x_2, 2x_3, 2x_4, \cdots)$. From Lemma 1.3.2, we get T is $*$-paranormal.

Theorem 1.3.3. *(Weighted Arithmetic Mean–Geometric Mean Inequality)([33])*

If $0 \leq c_i \in \mathbb{R}$ $(i = 1, 2, \cdots n)$ *and* $0 \leq \lambda_i \in \mathbb{R}$ $(i = 1, 2, \cdots n)$ *such that* $\sum_{i=1}^{n} \lambda_i = 1$, *then*

$$\prod_{k=1}^{n} c_k^{\lambda_k} \leq \sum_{k=1}^{n} \lambda_k c_k \tag{1.1}$$

Lemma 1.3.3. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then* T *is a* $*$*-paranormal operator if and only if* $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0$, *for all* $\lambda > 0$.

Proof.

$$\begin{aligned} T \text{ is a } *\text{-paranormal operator } &\Leftrightarrow \|T^*x\|^2 \leq \|T^2x\|\|x\|,\ \forall x \in \mathcal{H}. \\ &\Leftrightarrow \langle T^*x, T^*x\rangle \leq \langle T^2x, T^2x\rangle^{\frac{1}{2}} \langle x, x\rangle^{\frac{1}{2}},\ \forall x \in \mathcal{H}. \end{aligned}$$

$$T \text{ is } *\text{-paranormal } \Leftrightarrow \langle TT^*x, x\rangle \leq \langle T^{*2}T^2x, x\rangle^{\frac{1}{2}} \langle x, x\rangle^{\frac{1}{2}},\ \forall x \in \mathcal{H}. \tag{1.2}$$

For $\lambda > 0$, from weighted arithmetic mean-geometric mean inequality (1.1), we get

$$\begin{aligned} \frac{1}{2}\langle \lambda^{-1}T^{*2}T^2x, x\rangle + \frac{1}{2}\langle \lambda x, x\rangle &\geq \langle \lambda^{-1}T^{*2}T^2x, x\rangle^{\frac{1}{2}} \langle \lambda x, x\rangle^{\frac{1}{2}},\ \forall x \in \mathcal{H}. \\ &= \langle T^{*2}T^2x, x\rangle^{\frac{1}{2}} \langle x, x\rangle^{\frac{1}{2}},\ \forall x \in \mathcal{H}. \end{aligned}$$

Now from (1.2), we get

$$T \text{ is a } * \text{-paranormal operator } \Leftrightarrow \frac{\lambda^{-1}}{2}\langle T^{*2}T^2x, x\rangle + \frac{\lambda}{2}\langle x, x\rangle \geq \langle TT^*x, x\rangle, \ \forall x \in \mathcal{H}.$$

$$T \text{ is a } * \text{-paranormal operator } \Leftrightarrow T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \geq 0, \ \forall \lambda > 0.$$

□

The classes of paranormal and $*$-paranormal operators are independent. This can been seen as follows:

Consider $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by $T(x_1, x_2, x_3, \cdots) = (0, \sqrt{2}x_1, x_2, 2x_3, 2x_4, \cdots)$. From Lemma 1.3.2, we can see that T is $*$-paranormal. From Lemma 1.3.1, T is not paranormal.

Now consider the operator T defined by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then for any $\lambda > 0$, we have $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I = \begin{pmatrix} (\lambda-1)^2 & 2-2\lambda \\ 2-2\lambda & (\lambda-2)^2+1 \end{pmatrix} \geq 0$.

Hence T is paranormal. But

$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I = \begin{pmatrix} (\lambda-2)^2-3 & 2-2\lambda \\ 2-2\lambda & (\lambda-1)^2+4 \end{pmatrix} < 0$, for $\lambda = 2$. Hence T is not $*$-paranormal.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *class* $\mathcal{Q}$ if $T^{*2}T^2 - 2T^*T + I \geq 0$. It is known that T is paranormal if and only if λT is class $\mathcal{Q}$ for all $\lambda \geq 0$ ([10]). In particular, if $\lambda = 1$, we can see that all paranormal operators are class $\mathcal{Q}$. Hence

$$\textit{hyponormal} \subset \textit{paranormal} \subset \textit{class } \mathcal{Q}$$

In general, every class Q operator need not be a paranormal operator. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \frac{1}{4}x_3, \cdots)$ is class $\mathcal{Q}$, but not paranormal since T is not normaloid ([10]).

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *class* $\mathcal{Q}^*$ if $T^{*2}T^2 - 2TT^* + I \geq 0$ ([58]).

Lemma 1.3.4. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_k\}, (k = 1, 2, ...)$, defined by $Te_k = \alpha_k \ e_{k+1}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is a class $\mathcal{Q}^*$ operator if and only if*

$$|\alpha_{k+1}|^2|\alpha_{k+2}|^2 - 2\ |\alpha_k|^2 + 1 \geq 0, \forall\ k \in \mathbb{N}.$$

Proof. Since $Te_k = \alpha_k \ e_{k+1}$, we have $T^2e_k = \alpha_k \ \alpha_{k+1} \ e_{k+2}$, $TT^*e_k = |\alpha_{k-1}|^2 \ e_k$, $T^{*2}T^2e_k = |\alpha_k|^2|\alpha_{k+1}|^2e_k$, $\forall\ k \in \mathbb{N}$.

$$\begin{aligned} T \text{ is class } \mathcal{Q}^* &\Leftrightarrow T^{*2}T^2 - 2TT^* + I \geq 0. \\ &\Leftrightarrow \big\langle (T^{*2}T^2 - 2TT^* + I)e_k, e_k \big\rangle \geq 0,\ \ \forall k \in \mathbb{N}. \\ &\Leftrightarrow |\alpha_{k+1}|^2|\alpha_{k+2}|^2 - 2\ |\alpha_k|^2 + 1 \geq 0, \forall k \in \mathbb{N}. \end{aligned}$$

□

Consider $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, \sqrt{2}x_1, x_2, 2x_3, \cdots)$. From Lemma 1.3.4, we get T is class $\mathcal{Q}^*$.

Lemma 1.3.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is $*$-paranormal if and only if $\lambda^{\frac{-1}{2}}T$ is class $\mathcal{Q}^*$ for all $\lambda > 0$.*

Proof. Let $\lambda > 0$.

$$\begin{aligned} \lambda^{\frac{-1}{2}}T \text{ is class } \mathcal{Q}^* &\Leftrightarrow (\lambda^{\frac{-1}{2}}T)^{*2}(\lambda^{\frac{-1}{2}}T)^2 - 2(\lambda^{\frac{-1}{2}}T)(\lambda^{\frac{-1}{2}}T)^* + I > 0. \\ &\Leftrightarrow \lambda^{-2}T^{*2}T^2 - 2\lambda^{-1}TT^* + I > 0. \\ &\Leftrightarrow T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I > 0. \\ &\Leftrightarrow T \text{ is } * \text{ - paranormal.} \end{aligned}$$

□

From the above lemma it can be seen that $*$- paranormal $\subset$ *class* $\mathcal{Q}^*$.

In general every class $\mathcal{Q}^*$ operators are need not be $*$- paranormal. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{4}x_2, \frac{1}{4}x_3, \cdots)$.

Here $\alpha_1 = \dfrac{1}{2}$, $\alpha_k = \dfrac{1}{4}$ for k ≥ 2. Using Lemma 1.3.4, we get T is a class $\mathcal{Q}^*$ operator. From Lemma 1.3.2, T is a $*$- paranormal operator if and only if $|\alpha_k|^2 \leq |\alpha_{k+1}|\ |\alpha_{k+2}|$, $\forall\ k \in \mathbb{N}$. If $k = 1$, then the above relation is not satisfied. Hence T is not $*$-paranormal.

Let $m \in \mathbb{R}^+$and $n \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (m, n)- paranormal if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ ([5]).

Lemma 1.3.6. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_k\}, (k = 1, 2, ...)$, defined by $Te_k = \alpha_k\ e_{k+1}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is a (m, n)- paranormal operator if and only if*

$$|\alpha_k|^{n+1} \leq\ m\ |\alpha_k|\ |\alpha_{k+1}|......|\alpha_{k+n}|,\ \forall\ k \in \mathbb{N}.$$

Proof. Since $Te_k = \alpha_k\ e_{k+1}$, we have $T^{n+1}e_k\ =\ \alpha_k\ \alpha_{k+1}.....\alpha_{k+n}\ e_{k+n+1}$. Hence,

$$\begin{aligned} T \text{ is } (m,n)\text{-paranormal} &\Leftrightarrow \|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n,\ \forall\ x \in \mathcal{H}. \\ &\Leftrightarrow \|Te_k\|^{n+1} \leq m\|T^{n+1}e_k\|\|e_k\|^n,\ \forall\ k \in \mathbb{N}. \\ &\Leftrightarrow |\alpha_k|^{n+1} \leq\ m\ |\alpha_k|\ |\alpha_{k+1}|......|\alpha_{k+n}|,\ \forall\ k \in \mathbb{N}. \end{aligned}$$

□

Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, 2x_2, 3x_3, 3x_4, \cdots)$. From Lemma 1.3.6, we get T is (m, n)- paranormal for $m \geq 1$ and $n \geq 2$.

It is known that T is (m, n)- paranormal if and only if

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n T^* T + m^{\frac{2}{n+1}} n\ a^{n+1}\ I \geq 0,$$

for each $a > 0$ ([5]).

In particular if $m = n = 1$, then (m, n)- paranormal operator are paranormal. That is, the class of all paranormal operators is a subclass of class of all (m, n)- paranormal operators.

In general every (m, n)- paranormal operator need not be paranormal. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, \sqrt{2}x_1, x_2, 2x_3, 2x_4, \cdots)$.

Here $\alpha_1 = \sqrt{2}$, $\alpha_2 = 1$, $\alpha_k = 2$ for $k \geq 3$. Using Lemma 1.3.6, T is $(1,2)$-paranormal. But from Lemma 1.3.1, we can see that T is not paranormal.

For $n \in \mathbb{N}$, $T \in \mathcal{B}(\mathcal{H})$ is said to be n^*- paranormal if $\|T^*x\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$ ([45]). Equivalently T is n^*- paranormal if and only if

$$T^{*n+1}T^{n+1} - (n+1)a^n TT^* + n\ a^{n+1}\ I \geq 0, \forall a > 0 \tag{1.3}$$

([45]). Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, 2x_1, 3x_2, 3x_3, 3x_4, \cdots)$ is 2^*- paranormal.

Let $m \in \mathbb{R}^+$, $n \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be $(m,n)^*$- paranormal if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$ ([5]). It is proved in ([5]) that T is $(m,n)^*$- paranormal if and only if

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)a^n TT^* + m^{\frac{2}{n+1}}n\ a^{n+1}\ I \geq 0, \forall a > 0. \tag{1.4}$$

In particular, if $m = n = 1$, $(m,n)^*$- paranormal operators are $*$-paranormal. That is, every $*$-paranormal operators are $(m,n)^*$-paranormal.

In general, every $(m,n)^*$- paranormal operator need not be $*$- paranormal. For example, let T be defined by the matrix

$$T = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Then for any λ, we have $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I = \begin{pmatrix} 1 - 20\lambda + \lambda^2 & 6 - 6\lambda \\ 6 - 6\lambda & 37 - 2\lambda + \lambda^2 \end{pmatrix}$.

If $\lambda = 1$, it is evident that $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I < 0$. Hence T is not $*$-paranormal.

From (1.4), T is a $(100^{\frac{3}{2}}, 2)^*$ - paranormal operator if and only if $100T^{*3}T^3 - 3a^2TT^* + 200a^3 I \geq 0$, for all $a > 0$. Now

$$100T^{*3}T^3 - 3a^2TT^* + 200a^3 I = \begin{pmatrix} 100 - 30a^2 + 200a^3 & 900 - 9a^2 \\ 900 - 9a^2 & 8200 - 3a^2 + 200a^3 \end{pmatrix} \geq 0,$$

for all $a > 0$. Hence T is $(100^{\frac{3}{2}}, 2)^*$ - paranormal.

It is clear that if $m = 1$, $(m, n)^*$- paranormal operators are n^*- paranormal. That is, the class of all n^*- paranormal operators forms a subclass of class of all $(m, n)^*$- paranormal operators. But every $(m, n)^*$- paranormal operators need not be n^*- paranormal.

For proving this, we consider the above example,

$$T = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

We have T is $(100^{\frac{3}{2}}, 2)^*$ -paranormal operator.

From (1.3), T is 2^*-paranormal if and only if $T^{*3}T^3 - 3a^2TT^* + 2a^3I \geq 0, \forall a > 0$.

$$T^{*3}T^3 - 3a^2TT^* + 2a^3I = \begin{pmatrix} 1 - 30a^2 + 2a^3 & 9 - 9a^2 \\ 9 - 9a^2 & 82 - 3a^2 + 2a^3 \end{pmatrix} < 0, \text{ for } a = 1.$$

Hence T is not 2^*- paranormal operator.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a contraction if $T^*T \leq I$. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$ is a contraction. It is evident that, if T is a contraction then T^* is a contraction.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be unitarily equivalent to an operator $B \in \mathcal{B}(\mathcal{H})$ if there exist an unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $B = U^*TU$. For example, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_1 + x_2, x_2)$, is unitarily equivalent to an operator $B : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $B(x_1, x_2) = (x_1 - x_2, x_2)$.

An operator $U \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if there exist a closed subspace M of $\mathcal{H}$ such that $\|Ux\| = \|x\|, \forall x \in M$ and $Ux = 0, \forall x \in M^{\perp}$. In this case, M is said to be the initial space of U and $R(U)$ is said to be the final space of U ([16]).

Now we give the matrix representation of $T \in \mathcal{B}(\mathcal{H})$ as follows:

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. The block matrix representation of $T \in \mathcal{B}(\mathcal{H})$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ is

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{B}(\mathcal{M})$, $B \in \mathcal{B}(\mathcal{M}^{\perp}, \mathcal{M})$, $C \in \mathcal{B}(\mathcal{M}, \mathcal{M}^{\perp})$ and $D \in \mathcal{B}(\mathcal{M}^{\perp})$ ([28]).

For example, let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1,x_2,x_3)=(x_1,x_2,0)$.
Let $\mathcal{M}=\{(x_1,x_2,0): x_1,x_2\in\mathbb{R}\}$, $\mathcal{M}^\perp=\{(0,0,x_3): x_3\in\mathbb{R}\}$. Then

$$T=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} \text{ on } \mathbb{R}^3=\mathcal{M}\oplus\mathcal{M}^\perp.$$

Here $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $B=C=D=0$.

It is well known that $T=\begin{pmatrix}A&B\\B^*&C\end{pmatrix}$ is positive if and only if $A\geq 0, C\geq 0$ and $B=A^{\frac{1}{2}}WC^{\frac{1}{2}}$, for some contraction W ([6]).

A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be *invariant* under $T\in\mathcal{B}(\mathcal{H})$ if $T(\mathcal{M})\subset\mathcal{M}$ ([28]). For example, let $T:\mathbb{R}^3\to\mathbb{R}^3$ be defined by $T(x_1,x_2,x_3)=(x_1+x_2,x_2,x_3)$. Then $\mathcal{M}=\{(x,0,0): x\in\mathbb{R}\}$ is invariant under T.

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. Then the block matrix representation of $T\in\mathcal{B}(\mathcal{H})$ on $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$ is given by

$$T=\begin{pmatrix}A&B\\0&D\end{pmatrix},$$

where $A=T|_{\mathcal{M}}$, $B\in\mathcal{B}(\mathcal{M}^\perp,\mathcal{M})$ and $D\in\mathcal{B}(\mathcal{M}^\perp)$ ([28]).

A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to *reduces* $T\in\mathcal{B}(\mathcal{H})$ if $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under T. Note that a closed subspace $\mathcal{M}$ of $\mathcal{H}$ reduces $T\in\mathcal{B}(\mathcal{H})$ if and only if $\mathcal{M}$ is invariant under T and T^* ([28]).

Theorem 1.3.4. *([20]) Let* $T=\begin{pmatrix}A&B\\0&D\end{pmatrix}$ *be the matrix representation of* T *on* $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$, *where* $\mathcal{M}$ *is a closed subspace of* $\mathcal{H}$ *which is invariant under* T. *If* $\sigma(A)\cap\sigma(D)$ *has no interior point, then* $\sigma(T)=\sigma(A)\cup\sigma(D)$.

Next we define Riesz projection for bounded operators. Let $T \in \mathcal{B}(\mathcal{H})$ and λ be an isolated point of $\sigma(T)$. Then there exist a $D_\lambda = \{z \in \mathbb{C} : |z - \lambda| \leq r\}$ for some $r \geq 0$ with $D_\lambda \cap \sigma(T) = \{\lambda\}$. The operator defined by

$$E_\lambda = \frac{1}{2\pi i} \int\limits_{\partial D_\lambda} (zI - T)^{-1} dz$$

is called *Riesz projection* of T with respect to λ, where ∂D_λ denotes the boundary of D_λ ([29]). The Riesz projection E_λ satisfies the following properties:

Theorem 1.3.5. *([4, 29]) Suppose $T \in \mathcal{B}(\mathcal{H})$. Then the following holds.*

(i) E_λ *is a projection.*

(ii) $R(E_\lambda)$ *and* $N(E_\lambda)$ *are invariant under* T.

(iii) $\sigma(T|_{R(E_\lambda)}) = \{\lambda\}$ *and* $\sigma(T|_{N(E_\lambda)}) = \sigma(T) \setminus \{\lambda\}$.

(iv) $N(T - \lambda I) \subseteq R(E_\lambda)$.

(v) $E_\lambda T = T E_\lambda$.

Theorem 1.3.6. *([2, 57]) Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K})$ satisfying the following properties for every $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$.*

(i) $\phi(A^*) = \phi(A)^*,\ \phi(I_{\mathcal{H}}) = I_{\mathcal{K}},\ \phi(\alpha A + \beta B) = \alpha\phi(A) + \beta\phi(B),$
$\phi(AB) = \phi(A)\phi(B),\ \|\phi(A)\| = \|A\|,\ \phi(A) \leq \phi(B)$ *if* $A \leq B$.

(ii) $\phi(A) \geq 0$ *whenever* $A \geq 0$.

(iii) $\sigma_a(A) = \sigma_a(\phi(A)) = \sigma_p(\phi(A)) = \sigma(\phi(A))$.

(iv) $\sigma_{ja}(A) = \sigma_{jp}(\phi(A))$.

Let $\mathcal{L}$ denotes the set of all compact subsets of $\mathbb{C}$. The map $\sigma : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{L}$, which maps $T \in \mathcal{B}(\mathcal{H})$ to its spectrum is known as spectral map ([9]).

Theorem 1.3.7. *([9]) Let $C(i)$ denotes the collection of all operators $T \in \mathcal{B}(\mathcal{H})$ satisfying the following properties*

(i) If $\sigma(T) = \{0\}$, then T is nilpotent.

(ii) The matrix representation of $\phi(T)$ on $N(\phi(T) - \lambda) \oplus N(\phi(T) - \lambda)^{\perp}$ is

$$\phi(T) = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix},$$

where λ is a nonzero eigen value of $\phi(T)$. Also $\lambda \notin \sigma_p(B)$ and $\sigma(\phi(T)) = \{\lambda\} \cup \sigma(B)$.

Then the spectral map is continuous on $C(i)$.

Numerical range of $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a *finite operator* if $0 \in \overline{W(TX - XT)}$, for all $X \in \mathcal{B}(\mathcal{H})$ ([56]). For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, 2x_2, 3x_3, 3x_4, \cdots)$ is finite.

Theorem 1.3.8. *([56]) Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a finite operator if and only if $\|I - (TX - XT)\| \geq 1$, for all $X \in \mathcal{B}(\mathcal{H})$.*

Note that every hyponormal, paranormal and finite rank operators operators are finite ([35, 56]).

It is well known that if $\|A\| \leq \|A - (TX - XT)\|$ for all $X \in \mathcal{B}(\mathcal{H})$ and for all $A \in N(\delta_T)$, where $\delta_T(X) = TX - XT$, then $R(\delta_T)$ is orthogonal to $N(\delta_T)$ ([35]).

Theorem 1.3.9. *([34, 56]) Let $T \in \mathcal{B}(\mathcal{H})$. If $\sigma_{ja}(T) \neq \emptyset$, then T is a finite operator.*

Theorem 1.3.10. *(Putnam- Fuglede Theorem)([42])*

Let $A, B \in \mathcal{B}(\mathcal{H})$ be normal operators. If $AX = XB$ for some $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB^*$.

Theorem 1.3.11. *(Asymmetric Putnam- Fuglede Theorem)*

Let $A, B \in \mathcal{B}(\mathcal{H})$ be normal operators. If $AX = XB^*$ for some $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB$.

Now we give some basic definitions and results related to densely defined operators.

Consider a linear map $T : \mathcal{H} \supset D(T) \to \mathcal{H}$, where $D(T)$ denotes the domain of T. Let $\mathcal{L}(\mathcal{H})$ denotes the space of all linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *closed* if for any sequence (x_n) in $D(T)$ such that $x_n \to x$ and $Tx_n \to y$ implies $Tx = y$ for some $x \in D(T)$. Let $\mathcal{C}(\mathcal{H})$ denotes the space of all closed linear operators on $\mathcal{H}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be *invariant* under $T \in \mathcal{L}(\mathcal{H})$ if for any $x \in D(T) \cap \mathcal{M}$, then $Tx \in \mathcal{M}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be *reduces* under $T \in \mathcal{L}(\mathcal{H})$ if $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under T. A closed subspace $\mathcal{M}$ is said to be a *core* for $T \in \mathcal{L}(\mathcal{H})$ if $\mathcal{G}(T) \subseteq \overline{\mathcal{G}(T|_{\mathcal{M}})}$, where $\mathcal{G}(T)$ denotes the graph of T.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *densely defined* if $\overline{D(T)} = \mathcal{H}$ ([1]). For example, let $T : l^2(\mathbb{N}) \supset D(T) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (2x_1, 3x_2, 4x_3, \cdots)$ with $D(T) = \{(x_1, x_2, x_3, \cdots) \in l^2(\mathbb{N}) : \sum\limits_{j=1}^{\infty} |(j+1)x_j|^2 < \infty\}$ is densely defined.

Theorem 1.3.12. *([1]) If $T \in \mathcal{L}(\mathcal{H})$ is a densely defined operator, then there exist a unique operator $T^* \in \mathcal{L}(\mathcal{H})$ which satisfies*
$\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in D(T),\ y \in D(T^*)$,
where $D(T^*) = \{y \in \mathcal{H} : x \longrightarrow \langle Tx, y\rangle$ is continuous on $D(T)\}$.
The operator T^* is called the adjoint of T.

$T \in \mathcal{C}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a normal operator S on $\mathcal{K}$ such that $T \subseteq S$ i.e, $D(T) \subset D(S)$ and $Tx = Sx, \forall x \in D(T)$ ([52]). For example, let $T : D(T) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, 2x_2, 3x_3, 4x_4, \cdots)$ with $D(T) = \{(x_1, x_2, x_3, \cdots) \in l^2(\mathbb{N}) : \sum\limits_{j=1}^{\infty} |jx_j|^2 < \infty\}$ is subnormal.

A densely defined operator $T \in \mathcal{C}(\mathcal{H})$ is said to be normal if $D(T) = D(T^*)$ and $\|T^*x\| = \|Tx\|$ for all $x \in D(T)$ ([36]).

Theorem 1.3.13. *([1]) Let $T \in \mathcal{C}(\mathcal{H})$. Then*

(i) $N(T) = R(T^*)^{\perp}$.

(ii) $N(T^*) = R(T)^{\perp}$.

(iii) $N(T^*T) = N(T)$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be subspaces of $\mathcal{H}$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $P_{\mathcal{H}_i}$, denotes the projection onto $\mathcal{H}_i$, $i = 1, 2$. Then $T \in \mathcal{C}(\mathcal{H})$ has the block matrix representation on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{ij} : D(T) \cap \mathcal{H}_j \to \mathcal{H}_i$ is defined by $T_{ij} = P_{\mathcal{H}_i} T P_{\mathcal{H}_j}|_{D(T)\cap\mathcal{H}_j}$ for $j = 1, 2$ (see [54, Page No. 287]).
Note that if $\mathcal{H}_1$ is invariant under T then $T_{21} = 0$.

Theorem 1.3.14. *([7]) Let $T_1, T_2 \in \mathcal{C}(\mathcal{H})$. If $R(T_1) \subset R(T_2)$, then there exist an operator $T_3 \in \mathcal{C}(\mathcal{H})$ such that $T_1 = T_2T_3$.*

Theorem 1.3.15. *([43, 52]) Let $T \in \mathcal{C}(\mathcal{H})$ be normal. Then* $\bigcap\limits_{\lambda\in\sigma(T)} R(T - \lambda I) = \{0\}$.

Theorem 1.3.16. *([52]) Let $\mathcal{M}$ be a core for $T \in \mathcal{C}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator with $N(A) = \{0\}$. If $AT \subseteq TA$, then $A(\mathcal{M})$ is a core for T.*

Let X be a nonempty set and $\mathcal{A}$ be a σ-algebra of subsets of X. A spectral measure on a measure space $(X, \mathcal{A})$ is a mapping $E : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ such that

(i) $E(\Lambda)$ is an orthogonal projection for every $\Lambda \in \mathcal{A}$.

(ii) $E(\emptyset) = 0$ and $E(X) = 1$.

(iii) $E(\Lambda_1 \cap \Lambda_2) = E(\Lambda_1)E(\Lambda_2), \forall \Lambda_1, \Lambda_2 \in \mathcal{A}$.

(iv) $E(\bigcup\limits_k \Lambda_k) = \sum\limits_k E(\Lambda_k)$ whenever $\{\Lambda_k\}$ is a countable collection of pairwise sets in $\mathcal{A}$.

A spectral measure is said to be *regular* if $E(\Lambda) = \sup\{E(C) : C \subset \Lambda, C \text{ is compact}\}$, for all $\Lambda \in \mathcal{A}$. A spectral measure is said to be complex if $X = \mathbb{C}$. Note that complex spectral measure is regular.

Theorem 1.3.17. *([29, 52]) If $T \in \mathcal{C}(\mathcal{H})$ is normal, then there is a unique complex spectral measure $E : A_{\sigma(T)} \longrightarrow \mathcal{C}(\mathcal{H})$ such that $T = \int \lambda dE_\lambda$, where $A_{\sigma(T)}$ is the σ-algebra of Borel subsets of $\sigma(T)$.*

Theorem 1.3.18. *([29, 52]) Let $T = \int \lambda dE_\lambda$ be the spectral decomposition of the normal operator in $\mathcal{C}(\mathcal{H})$ and $S \in \mathcal{C}(\mathcal{H})$. Then the following are equivalent:*

(i) S commutes with T.

(ii) S commutes with T^*.

(iii) S commutes with every $E(\Lambda)$, where $\Lambda \in A_{\sigma(T)}$.

(iv) $R(E(\Lambda))$ reduces S for every $\Lambda \in A_{\sigma(T)}$.

Theorem 1.3.19. *([52]) Let E be the spectral measure of a normal operator N in $\mathcal{C}(\mathcal{H})$. If Ω is a Borel subset of $\mathbb{C}$ and $x \in \bigcap_{z\in\Omega} R(N - \lambda I)$, then $E(\Omega)x = 0$.*

1.4 Composition operators

Composition operators play an important role in classical mechanics and ergotic theory ([24]). In this section, we give some basic definition and results related with composition operators.

For a non empty set X, we denote $\mathcal{A}$ as the collection of all measurable subsets of X. Let $\mu : \mathcal{A} \to [0, \infty]$ be a measure on $\mathcal{A}$. The measure μ is said to be a σ *finite measure* if X can be covered with atmost countably many measurable subsets of X with finite measure.

A function $T : X \to X$ is said to be measurable if $T^{-1}(B) \in \mathcal{A}$, for every $B \in \mathcal{A}$. The measurable function T is said to be *nonsingular* if $\mu(T^{-1}(B)) = 0$ whenever $\mu(B) = 0$. Let μ and m be two measures on $\mathcal{A}$. The measure μ is said to be *absolutely continuous* with respect to m if $\mu(B) = 0$ whenever $m(B) = 0$, for every $B \in \mathcal{A}$.

Note that if T is a nonsingular measurable function, the measure μT^{-1} defined by $\mu T^{-1}(B) = \mu(T^{-1}(B))$, $B \in \mathcal{A}$ is absolutely continuous with respect to μ.

Theorem 1.4.1. *([49]) Let $(X, \mathcal{A}, \mu)$ be a σ finite measure space and m be a σ finite measure defined on $\mathcal{A}$ which is absolutely continuous with respect to μ. Then there exist a non negative function f on X which is measurable with respect to $\mathcal{A}$ and $m(B) = \int_B f d\mu$ for all $B \in \mathcal{A}$.*

From the above theorem, we get a non negative measurable function f_T such that $\mu T^{-1}(B) = \int_B f_T d\mu$, for all $B \in \mathcal{A}$. The measurable function f_T is called Radon-Nikodym derivative of μT^{-1} with respect to μ and is denoted by h. We denote the Radon-Nikodym derivative of $\mu(T^{-1})^k$ with respect to μ by h_k.

Definition 1.4.1. *([49]) Let $T : X \to X$ be a non singular measurable transformation. A composition operator C_T on $L^2(\mu)$ is defined by $C_T f = f \circ T,\ f \in L^2(\mu)$.*

More details on general properties of (measure based) composition operators can be found in ([39, 49]). In ([32]), A. Lambert studied hyponormal composition operators. Composition operators of class $\mathcal{Q}$ and (m, n)-paranormal composition operators were studied in ([27, 40]).

Let T be a measurable function defined on a nonempty set X and E be a function defined on the set of all non-negative functions $f \in L^p(\mu)$, $1 \leq p < \infty$. The conditional expectation operator E is uniquely determined by the following conditions:

(i) $E(f)$ is $T^{-1}(\mathcal{A})$ measurable.

(ii) $\int_B f d\mu = \int_B E(f) d\mu$, for any $T^{-1}(\mathcal{A})$ measurable set B.

Theorem 1.4.2. *([11, 22]) Let T be a measurable function defined on a nonempty set X. The conditional expectation operator E satisfies the following properties: For $f, g \in L^2(\mu)$,*

(i) $E(g) = g$ if and only if g is $T^{-1}(\mathcal{A})$ measurable.

(ii) If g is $T^{-1}(\mathcal{A})$ measurable, then $E(fg) = E(f)g$.

(iii) $E(fg \circ T) = (E(f))(g \circ T)$ and $E(E(f)g) = E(f)E(g)$.

(iv) $E(1) = 1$.

(v) *E is the identity operator on $L^2(\mu)$ if and only if $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

(vi) *E is a projection operator from $L^2(\mu)$ onto $\overline{R(C_T)}$.*

Theorem 1.4.3. *([3, 21]) Let T be a measurable function defined on X and P be a projection from $L^2(\mu)$ onto $\overline{R(C_T)}$. Then the following results holds for every $f \in L^2(\mu)$:*

(i) $C_T^* f = h.E(f) \circ T^{-1}$.

(ii) $C_T^k f = f \circ T^k$, $C_T^{*k} f = h_k E(f) \circ T^{-k}$, *for any* $k \in \mathbb{N}$.

(iii) $C_T C_T^* f = (h \circ T)Pf$, $C_T^* C_T f = hf$.

1.5 Weighted composition operators

Definition 1.5.1. *Let $(X, \mathcal{A}, \mu)$ be a σ- finite measure space. The weighted composition operator W on $L^2(\mu)$ induced by a measurable transformation T and a complex valued measurable function π is defined as $Wf = \pi.(f \circ T)$, for all $f \in L^2(\mu)$.*

Theorem 1.5.1. *([3]) Let W be the weighted composition operator induced by T and π. The following statements hold: For $f \in L^2(\mu)$ and $k \in \mathbb{N}$,*

(i) $W^k f = \pi_k . f \circ T^k$

(ii) $W^{*k} f = h_k E(\overline{\pi} f) \circ T^{-k}$

(iii) $W^* W(f) = hE(|\pi|^2) \circ T^{-1}(f)$

(iv) $WW^*(f) = \pi(h \circ T)E(\overline{\pi} f)$

(v) $W^{*k} W^k(f) = h_k E(|\pi_k|^2) \circ T^{-k}(f)$

(vi) $W^k W^{*k} f = \pi_k (h_k \circ T^k) E(\bar{\pi}_k f)$

where $\pi_k = \pi(\pi \circ T)(\pi \circ T^2) \ldots (\pi \circ T^{k-1})$.

Chapter 2

k-quasi (m, n)-paranormal and k-quasi $(m, n)^*$- paranormal operators

In this chapter, we introduce two new classes of operators namely, k-quasi $(m, n)^*$-paranormal and k-quasi (m, n)-paranormal operators, which contains known class of operators $(m, n)^*$-paranormal and (m, n)-paranormal operators respectively. We prove that the restriction of the operator to its closed subspace is the corresponding operator. Also we prove some characterizations and we give 2×2 matrix representation of these operators. We show that these two classes of operators are independent.

2.1 k-quasi $(m, n)^*$- paranormal operators

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. For $m \in \mathbb{R}^+$ and $n \in \mathbb{N}$, T is said to be $(m, n)^*$-*paranormal* if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$ ([5]). Now we define a new classes of operator namely k-quasi $(m, n)^*$-paranormal operator which contains the class of $(m, n)^*$-paranormal operators.

Definition 2.1.1. *Let $m \in \mathbb{R}^+, n \in \mathbb{N}$ and k be a non-negative integer. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a k-quasi $(m, n)^*$-paranormal if*

$$\|T^*T^kx\|^{n+1} \leq m\|T^{n+1}T^kx\|\|T^kx\|^n, \ \forall x \in \mathcal{H}.$$

If $k = 0$, then k-quasi $(m, n)^*$-paranormal operator becomes $(m, n)^*$-paranormal operator. That is, the class of all $(m,n)^*$-paranormal operators is a subclass of class of all k-quasi $(m,n)^*$-paranormal operators. In general, k-quasi $(m,n)^*$-paranormal operator need not be $(m,n)^*$-paranormal operator.

For example, let $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $T^2 = 0$. From the definition, it is clear that if $k \geq 2$ then T is a k-quasi $(m,n)^*$-paranormal operator for any m, n. Recall that T is $(m,n)^*$-paranormal if and only if

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n TT^* + m^{\frac{2}{n+1}} n \ a^{n+1} \ I \geq 0, \forall a > 0 \ ([5]).$$

For $m = 25$, $n = 3$ and for any $a > 0$, we have

$$\begin{aligned} m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n TT^* + m^{\frac{2}{n+1}} n \ a^{n+1} \ I &= -4a^3 TT^* + 15a^4 I \\ &= \begin{pmatrix} 15a^4 & 0 \\ 0 & -4a^3 + 15a^4 \end{pmatrix} \end{aligned}$$

If $a = \frac{1}{5}$, then $-4a^3 TT^* + 15a^4 I < 0$. Hence T is not $(25,3)^*$-paranormal.

Note that if $k = 0$ and $m = 1$, then k-quasi $(m,n)^*$-paranormal operator coincides with the class of n^*-paranormal operators introduced by M. H. M. Rashid ([45]). If $k = 0$ and $m = n = 1$, then k-quasi $(m,n)^*$-paranormal operators coincide with $*$-paranormal operators ([53]).

Now we give some characterizations of k-quasi $(m,n)^*$-paranormal operators.

Theorem 2.1.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a k-quasi $(m,n)^*$-paranormal operator if and only if*

$$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1) \ a^n T^{*k} TT^* T^k + m^{\frac{2}{n+1}} \ n \ a^{n+1} T^{*k} T^k \geq 0 \tag{2.1}$$

for all $a > 0$.

Proof. T is a k-quasi $(m,n)^*$-paranormal operator

$$\Leftrightarrow m\|T^{n+1}T^k x\| \|T^k x\|^n \geq \|T^* T^k x\|^{n+1}, \ \forall x \in \mathcal{H}.$$
$$\Leftrightarrow m^{\frac{2}{n+1}} \|T^{n+1}T^k x\|^{\frac{2}{n+1}} \|T^k x\|^{\frac{2n}{n+1}} \geq \|T^* T^k x\|^2, \ \forall x \in \mathcal{H}.$$

$$\Leftrightarrow m^{\frac{2}{n+1}}\langle T^{*n+1}T^{n+1}T^kx,T^kx\rangle^{\frac{1}{n+1}}\ \langle T^kx,T^kx\rangle^{\frac{n}{n+1}}\geq\langle TT^*T^kx,T^kx\rangle,\ \forall x\in\mathcal{H}.$$

$$\Leftrightarrow m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle^{\frac{1}{n+1}}\ \langle T^kx,T^kx\rangle^{\frac{n}{n+1}}\geq\langle|T^*|^2T^kx,T^kx\rangle,\ \forall x\in\mathcal{H}. \quad (2.2)$$

Hence T is a k-quasi $(m,n)^*$-paranormal operator if and only if T satisfies (2.2). For $a>0$ and $x\in\mathcal{H}$, using the weighted arithmetic mean-geometric mean inequality (1.1) and (2.2), it follows that

$$\begin{aligned}\tfrac{1}{n+1}\langle a^{-n}m^{\frac{2}{n+1}}|T^{n+1}|^2T^kx,T^kx\rangle+\tfrac{n}{n+1}\langle a\ m^{\frac{2}{n+1}}T^kx,T^kx\rangle\\ \geq\ \langle a^{-n}m^{\frac{2}{n+1}}|T^{n+1}|^2T^kx,T^kx\rangle^{\frac{1}{n+1}}\ \langle a\ m^{\frac{2}{n+1}}T^kx,T^kx\rangle^{\frac{n}{n+1}}\\ =\ m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle^{\frac{1}{n+1}}\ \langle T^kx,T^kx\rangle^{\frac{n}{n+1}}\\ \geq\ \langle|T^*|^2T^kx,T^kx\rangle.\end{aligned}$$

Thus,

$\frac{a^{-n}}{n+1}\ m^{\frac{2}{n+1}}\langle T^{*k}T^{*n+1}T^{n+1}T^kx,x\rangle+\frac{na}{n+1}\ m^{\frac{2}{n+1}}\langle T^{*k}T^kx,x\rangle-\langle T^{*k}TT^*T^kx,x\rangle\geq 0,$

for all $x\in\mathcal{H}$ and $a>0$.

Hence,

$m^{\frac{2}{n+1}}T^{*k}T^{*n+1}T^{n+1}T^k-(n+1)\ a^nT^{*k}TT^*T^k+m^{\frac{2}{n+1}}\ n\ a^{n+1}T^{*k}T^k\geq 0$, for all $a\geq 0$.

Conversely, suppose that (2.1) holds. Hence for every $x\in\mathcal{H}$,

$$m^{\frac{2}{n+1}}\langle T^{*k}T^{*n+1}T^{n+1}T^kx,x\rangle-(n+1)\ a^n\langle T^{*k}TT^*T^kx,x\rangle+m^{\frac{2}{n+1}}\ n\ a^{n+1}\langle T^{*k}T^kx,x\rangle\geq 0.$$

Hence,

$$m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle-(n+1)\ a^n\langle|T^*|^2T^kx,T^kx\rangle+m^{\frac{2}{n+1}}\ n\ a^{n+1}\langle T^kx,T^kx\rangle\geq 0. \quad (2.3)$$

Let $x\in\mathcal{H}$, with $\langle|T^{n+1}|^2T^kx,T^kx\rangle=0$. From (2.3) we get,

$$m^{\frac{2}{n+1}}\ n\ a\ \langle T^{*k}T^kx,x\rangle-(n+1)\langle|T^*|^2T^kx,T^kx\rangle\geq 0,\ \forall a\geq 0.$$

Letting $a \to 0$, we get $\langle |T^*|^2 T^k x, T^k x\rangle = 0$. Hence,

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}} \ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T^*|^2 T^k x, T^k x\rangle$$

is satisfied. Thus (2.2) holds.

Let $x \in \mathcal{H}$, with $\langle |T^{n+1}|^2 T^k x, T^k x\rangle > 0$. Hence, $\langle T^k x, T^k x\rangle \neq 0$.
By taking $a = \left(\frac{\langle |T^{n+1}|^2 T^k x, T^k x\rangle}{\langle T^k x, T^k x\rangle}\right)^{\frac{1}{n+1}}$ in (2.3), we get

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle + m^{\frac{2}{n+1}} \ n \ \frac{\langle |T^{n+1}|^2 T^k x, T^k x\rangle}{\langle T^k x, T^k x\rangle} \langle T^k x, T^k x\rangle$$

$$\geq (n+1)\langle |T^*|^2 T^k x, T^k x\rangle \left(\frac{\langle |T^{n+1}|^2 T^k x, T^k x\rangle}{\langle T^k x, T^k x\rangle}\right)^{\frac{n}{n+1}}$$

Hence,

$$m^{\frac{2}{n+1}}(1+n)\langle |T^{n+1}|^2 T^k x, T^k x\rangle \geq (n+1)\langle |T^*|^2 T^k x, T^k x\rangle \left(\frac{\langle |T^{n+1}|^2 T^k x, T^k x\rangle}{\langle T^k x, T^k x\rangle}\right)^{\frac{n}{n+1}}$$

Thus $m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{1-\frac{n}{n+1}} \ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T^*|^2 T^k x, T^k x\rangle$. Hence,

$$m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}} \ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T^*|^2 T^k x, T^k x\rangle.$$

Thus (2.2) holds for all $x \in \mathcal{H}$. Hence T is k-quasi $(m,n)^*$-paranormal. □

Theorem 2.1.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi $(m,n)^*$-paranormal operator and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. Then $T|_{\mathcal{M}}$ is a k-quasi $(m,n)^*$-paranormal operator.*

Proof. Let $B = T|_{\mathcal{M}}$ and P be an orthogonal projection onto $\mathcal{M}$. Then we can see that $B = TP = PTP$. Hence $B^{*j}B^j = PT^{*j}T^jP$, for all $j \in \mathbb{N}$.

Now for any $a > 0$, we have

$$\begin{aligned}
&m^{\frac{2}{n+1}} B^{*k} B^{*n+1} B^{n+1} B^k - (n+1)\ a^n B^{*k} B B^* B^k + m^{\frac{2}{n+1}}\, n\, a^{n+1} B^{*k} B^k \\
&= m^{\frac{2}{n+1}} P T^{*k+n+1} T^{k+n+1} P - (n+1) a^n P T^{*k} T P P T^* T^k P + m^{\frac{2}{n+1}}\, n\, a^{n+1} P T^{*k} T^k P \\
&= P T^{*k} (m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n T P T^* + m^{\frac{2}{n+1}} n a^{n+1} I) T^k P \\
&\geq P T^{*k} (m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n T T^* + m^{\frac{2}{n+1}} n a^{n+1} I) T^k P.
\end{aligned}$$

Since T is k-quasi $(m,n)^*$-paranormal operator, we get

$$m^{\frac{2}{n+1}} B^{*k} B^{*n+1} B^{n+1} B^k - (n+1)\ a^n B^{*k} B B^* B^k + m^{\frac{2}{n+1}}\ n\ a^{n+1} B^{*k} B^k \geq 0.$$

Hence $T|_{\mathcal{M}}$ is a k-quasi $(m,n)^*$-paranormal operator. □

Now we give a matrix representation for k-quasi $(m,n)^*$-paranormal operators.

Theorem 2.1.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\overline{R(T^k)} \neq \mathcal{H}$. If T is a k-quasi $(m,n)^*$-paranormal operator, then*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \ on\ \overline{R(T^k)} \oplus N(T^{*k}),$$

where A is a $(m,n)^*$-paranormal operator on $\overline{R(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Assume that T is a k-quasi $(m,n)^*$-paranormal operator. Then

$$\|T^* T^k x\|^{n+1} \leq m \|T^{n+1} T^k x\| \|T^k x\|^n, \text{ for all } x \in \mathcal{H}.$$

Let $T^k x = z$. Then from the above equation we have

$$\|T^* z\|^{n+1} \leq m \|T^{n+1} z\| \|z\|^n. \tag{2.4}$$

Since $R(T^k)$ is not dense in $\mathcal{H}$ and $\overline{R(T^k)}$ is invariant under T, we have $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{R(T^k)} \oplus N(T^{*k})$, where $A = T|_{\overline{R(T^k)}}$. Hence from (2.4), we get $\|A^* z\|^{n+1} \leq m \|A^{n+1} z\| \|z\|^n$ for all $z \in \overline{R(T^k)}$. Hence A is a $(m,n)^*$-paranormal operator on $\overline{R(T^k)}$.

Let $x \in N(T^{*k})$. Then,

$$T^k(x) = \begin{pmatrix} A^k & \sum\limits_{i=0}^{k-1} A^i B C^{k-1-i} \\ 0 & C^k \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

Thus $C^k x = T^k x - \sum\limits_{i=0}^{k-1} A^i B C^{k-1-i} x$. Since $A = T|_{\overline{R(T^k)}}$, we have $C^k x \in \overline{R(T^k)}$. Also $C^k x \in N(T^{*k})$ since $C = T|_{N(T^{*k})}$. Thus $C^k x \in \overline{R(T^k)} \cap N(T^{*k})$. Hence $C^k = 0$. Thus $\sigma(C) = \{0\}$. Hence $\sigma(A) \cap \sigma(C) = \sigma(A) \cap \{0\}$ has no interior point. From Theorem 1.3.4, we get $\sigma(T) = \sigma(A) \cup \{0\}$. □

2.2 k-quasi (m, n)-paranormal operators

In this section, we introduce a new class of operators, k-quasi (m, n)-paranormal operators which contains the class of all (m, n)-paranormal operators. Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be (m, n)*-paranormal* if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$ ([5]).

Definition 2.2.1. *Let $m \in \mathbb{R}^+, n \in \mathbb{N}$ and k be a non-negative integer. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi (m, n)-paranormal if*

$$\|TT^k x\|^{n+1} \leq m\|T^{n+1}T^k x\|\|T^k x\|^n, \text{ for all } x \in \mathcal{H}.$$

If $k = 0$, then k-quasi (m, n)-paranormal operator becomes (m, n)-paranormal operator. That is, the class of all (m, n)-paranormal operators forms a subclass of class of all k-quasi (m, n)-paranormal operators. In general, every k-quasi (m, n)-paranormal operator need not be (m, n)-paranormal operator.

For example, let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T^2 = 0$. Hence T is a k-quasi (m, n)-paranormal operator for $k \geq 2$ and for any m, n.

Recall that T is (m, n)-paranormal if and only if

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n T^* T + m^{\frac{2}{n+1}} n \, a^{n+1} \, I \geq 0, \, \forall a > 0.$$

If $m = 25$, $n = 3$ and for any $a > 0$, we have

$$m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)a^n T^*T + m^{\frac{2}{n+1}} n\ a^{n+1}\ I = -4a^3T^*T + 15a^4 I$$
$$= \begin{pmatrix} 15a^4 & 0 \\ 0 & -4a^3 + 15a^4 \end{pmatrix}.$$

In particular, if $a = \frac{1}{5}$, then $-4a^3TT^* + 15a^4 I < 0$. Hence T is not a $(25,3)$-paranormal operator.

Note that if $k = 0$ and $m = n = 1$, then k-quasi (m,n)-paranormal operators coincide with paranormal operators ([13]).

Now we give some characterizations of k-quasi (m,n)-paranormal operators.

Theorem 2.2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is k-quasi (m,n)-paranormal if and only if*

$$m^{\frac{2}{n+1}} T^{*k}T^{*n+1}T^{n+1}T^k - (n+1)\ a^n T^{*k}T^*TT^k + m^{\frac{2}{n+1}}\ n\ a^{n+1}T^{*k}T^k \geq 0, \quad (2.5)$$

for all $a > 0$.

Proof. T is a k-quasi (m,n)-paranormal operator

$$\Leftrightarrow m\|T^{n+1}T^k x\|\|T^k x\|^n \geq \|TT^k x\|^{n+1},\ \forall x \in \mathcal{H}.$$
$$\Leftrightarrow m^{\frac{2}{n+1}} \|T^{n+1}T^k x\|^{\frac{2}{n+1}} \|T^k x\|^{\frac{2n}{n+1}} \geq \|TT^k x\|^2,\ \forall x \in \mathcal{H}.$$
$$\Leftrightarrow m^{\frac{2}{n+1}} \langle T^{*n+1}T^{n+1}T^k x, T^k x\rangle^{\frac{1}{n+1}}\ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle T^*TT^k x, T^k x\rangle,\ \forall x \in \mathcal{H}.$$

$$\Leftrightarrow\ m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}}\ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \geq \langle |T|^2 T^k x, T^k x\rangle,\ \forall x \in \mathcal{H}. \quad (2.6)$$

Hence T is a k-quasi (m,n)-paranormal operator if and only if T satisfies (2.6).
For $a > 0$ and $x \in \mathcal{H}$, using the weighted arithmetic mean-geometric mean inequality (1.1) and (2.6), it follows that
$\frac{1}{n+1}\langle a^{-n} m^{\frac{2}{n+1}} |T^{n+1}|^2 T^k x, T^k x\rangle + \frac{n}{n+1}\langle a\ m^{\frac{2}{n+1}} T^k x, T^k x\rangle$

$$\begin{aligned} &\geq\ \langle a^{-n} m^{\frac{2}{n+1}} |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}}\ \langle a\ m^{\frac{2}{n+1}} T^k x, T^k x\rangle^{\frac{n}{n+1}} \\ &=\ m^{\frac{2}{n+1}} \langle |T^{n+1}|^2 T^k x, T^k x\rangle^{\frac{1}{n+1}}\ \langle T^k x, T^k x\rangle^{\frac{n}{n+1}} \\ &\geq\ \langle |T|^2 T^k x, T^k x\rangle. \end{aligned}$$

Thus,
$\frac{a^{-n}}{n+1}\ m^{\frac{2}{n+1}}\langle T^{*k}T^{*n+1}T^{n+1}T^kx,x\rangle+\frac{na}{n+1}\ m^{\frac{2}{n+1}}\langle T^{*k}T^kx,x\rangle-\langle T^{*k}T^*TT^kx,x\rangle\geq 0$, for all $x\in\mathcal{H}$ and $a>0$.
Hence,
$m^{\frac{2}{n+1}}T^{*k}T^{*n+1}T^{n+1}T^k-(n+1)\,a^nT^{*k}T^*TT^k+m^{\frac{2}{n+1}}\,n\,a^{n+1}T^{*k}T^k\geq 0$, for all $a\geq 0$.
Assume that (2.5) holds. Then for every $x\in\mathcal{H}$,

$$m^{\frac{2}{n+1}}\langle T^{*k}T^{*n+1}T^{n+1}T^kx,x\rangle-(n+1)\,a^n\langle T^{*k}T^*TT^kx,x\rangle+m^{\frac{2}{n+1}}\,n\,a^{n+1}\langle T^{*k}T^kx,x\rangle\geq 0.$$

Hence,

$$m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle-(n+1)\ a^n\langle|T|^2T^kx,T^kx\rangle+m^{\frac{2}{n+1}}\ n\ a^{n+1}\langle T^kx,T^kx\rangle\geq 0. \tag{2.7}$$

Let $x\in\mathcal{H}$ with $\langle|T^{n+1}|^2T^kx,T^kx\rangle=0$. From (2.7), we get

$$m^{\frac{2}{n+1}}\ n\ a\langle T^{*k}T^kx,x\rangle-(n+1)\langle|T|^2T^kx,T^kx\rangle\geq 0.$$

Letting $a\to 0$, we get $\langle|T|^2T^kx,T^kx\rangle=0$. Hence,

$$m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle^{\frac{1}{n+1}}\ \langle T^kx,T^kx\rangle^{\frac{n}{n+1}}\geq\langle|T|^2T^kx,T^kx\rangle.$$

Thus (2.6) is satisfied.
Let $x\in\mathcal{H}$ with $\langle|T^{n+1}|^2T^kx,T^kx\rangle>0$. Hence $\langle T^kx,T^kx\rangle\neq 0$.
By taking $a=\left(\frac{\langle|T^{n+1}|^2T^kx,T^kx\rangle}{\langle T^kx,T^kx\rangle}\right)^{\frac{1}{n+1}}$ in (2.7), we get

$$m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle+m^{\frac{2}{n+1}}\ n\ \frac{\langle|T^{n+1}|^2T^kx,T^kx\rangle}{\langle T^kx,T^kx\rangle}\langle T^kx,T^kx\rangle$$

$$\geq(n+1)\langle|T|^2T^kx,T^kx\rangle\left(\frac{\langle|T^{n+1}|^2T^kx,T^kx\rangle}{\langle T^kx,T^kx\rangle}\right)^{\frac{n}{n+1}}$$

$$m^{\frac{2}{n+1}}(1+n)\langle|T^{n+1}|^2T^kx,T^kx\rangle\geq(n+1)\langle|T|^2T^kx,T^kx\rangle\left(\frac{\langle|T^{n+1}|^2T^kx,T^kx\rangle}{\langle T^kx,T^kx\rangle}\right)^{\frac{n}{n+1}}$$

$$m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle^{1-\frac{n}{n+1}}\langle T^kx,T^kx\rangle^{\frac{n}{n+1}}\geq\langle|T|^2T^kx,T^kx\rangle$$

$$m^{\frac{2}{n+1}}\langle|T^{n+1}|^2T^kx,T^kx\rangle^{\frac{1}{n+1}}\ \langle T^kx,T^kx\rangle^{\frac{n}{n+1}}\geq\langle|T|^2T^kx,T^kx\rangle.$$

Thus (2.6) holds for all $x\in\mathcal{H}$. Hence T is k-quasi (m,n)-paranormal. □

Now we show that the newly defined classes of operators, k-quasi (m,n)-paranormal and k-quasi $(m,n)^*$-paranormal are independent.
For example, consider the operator T defined by the matrix

$$T=\begin{pmatrix}1&0\\1&0\end{pmatrix}.$$

Let $m=n=1$. Then for any $a>0$, we have
$m^{\frac{2}{n+1}}T^{*k}T^{*n+1}T^{n+1}T^k-(n+1)\ a^nT^{*k}T^*TT^k+m^{\frac{2}{n+1}}\ n\ a^{n+1}T^{*k}T^k$

$$\begin{aligned}&=T^{*k}(T^{*2}T^2-2aT^*T+a^2I)T^k\\&=\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}a^2-4a+2&0\\0&a^2\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix}\\&=\begin{pmatrix}2(a-1)^2&0\\0&0\end{pmatrix}\geq0.\end{aligned}$$

Hence T is k-quasi $(1,1)$-paranormal for any $k\geq1$.
But for $m=n=1$ and $a>0$, we have
$m^{\frac{2}{n+1}}T^{*k}T^{*n+1}T^{n+1}T^k-(n+1)a^nT^{*k}TT^*T^k+m^{\frac{2}{n+1}}n\ a^{n+1}\ T^{*k}T^k$

$$\begin{aligned}&=T^{*k}(T^{*2}T^2-2aTT^*+a^2I)T^k\\&=\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}a^2-2a+2&-2a\\-2a&a^2-2a\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix}\\&=\begin{pmatrix}2a^2-8a+2&0\\0&0\end{pmatrix}.\end{aligned}$$

In particular if $a = 1$, then

$$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1) a^n T^{*k} T T^* T^k + m^{\frac{2}{n+1}} n \ a^{n+1} \ T^{*k} T^k < 0.$$

Hence T is not k-quasi $(1,1)^*$-paranormal for any k.

Now we give an example of a k-quasi $(m,n)^*$-paranormal, which is not k-quasi (m,n)-paranormal. For this, we prove the following result.

Theorem 2.2.2. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_n\}, (n = 1, 2, ...)$, defined by $Te_n = \alpha_n \ e_{n+1}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is a k-quasi $(1,1)$-paranormal operator if and only if*

$$|\alpha_{n+k}|^2 |\alpha_{n+k+1}|^2 - 2 \ a \ |\alpha_{n+k}|^2 + a^2 \geq 0, \ \forall a > 0, \ \forall n \in \mathbb{N}. \tag{2.8}$$

Proof. Since $Te_n = \alpha_n \ e_{n+1}$, we have $T^* e_n = \overline{\alpha_{n-1}} \ e_{n-1}$. Hence, $T^*Te_n = |\alpha_n|^2 \ e_n$, and $T^{*2}T^2 e_n = |\alpha_n|^2 |\alpha_{n+1}|^2 e_n$. Therefore,

$$\begin{aligned} T^*(T^{*2}T^2)Te_n = & \ |\alpha_n|^2 |\alpha_{n+1}|^2 |\alpha_{n+2}|^2 e_n \\ T^{*2}(T^{*2}T^2)T^2 e_n = & \ |\alpha_n|^2 |\alpha_{n+1}|^2 |\alpha_{n+2}|^2 |\alpha_{n+3}|^2 e_n \end{aligned}$$

Hence we get,

$$T^{*l}(T^{*2}T^2)T^l e_n = |\alpha_n|^2 |\alpha_{n+1}|^2 \cdots |\alpha_{n+l-1}|^2 |\alpha_{n+l}|^2 |\alpha_{n+l+1}|^2 e_n, \ \text{for any } l \in \mathbb{N}.$$

Also,

$$\begin{aligned} T^*(T^*T)Te_n = & \ |\alpha_n|^2 |\alpha_{n+1}|^2 e_n \\ T^{*2}(T^*T)T^2 e_n = & \ |\alpha_n|^2 |\alpha_{n+1}|^2 |\alpha_{n+2}|^2 e_n \end{aligned}$$

Hence we get,

$$T^{*l}(T^*T)T^l e_n = |\alpha_n|^2 |\alpha_{n+1}|^2 \cdots |\alpha_{n+l-1}|^2 |\alpha_{n+l}|^2 e_n, \ \text{for any } l \in \mathbb{N}.$$

In the similar way we get,

$$T^{*l}T^{l}e_n = |\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+l-1}|^2e_n, \quad \text{for any } l \in \mathbb{N}.$$

Using the above equations and (2.5) we get, T is k-quasi $(1,1)$-paranormal

$\Leftrightarrow T^{*k}(T^{*2}T^2 - 2aT^*T + a^2I)T^k \geq 0,\ \forall a > 0.$

$\Leftrightarrow \big\langle (T^{*k}(T^{*2}T^2 - 2aT^*T + a^2I)T^k)e_n, e_n\big\rangle \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}.$

$\Leftrightarrow |\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+k-1}|^2(|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ a\ |\alpha_{n+k}|^2 + a^2) \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}.$

$\Leftrightarrow |\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ a\ |\alpha_{n+k}|^2 + a^2 \geq 0, \forall a > 0,\ \forall n \in \mathbb{N}.$

□

Theorem 2.2.3. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_n\}, (n = 1, 2, ...)$, defined by $Te_n = \alpha_n\, e_{n+1}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is k-quasi $(1,1)^*$-paranormal if and only if*

$$|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ a\ |\alpha_{n+k-1}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}. \tag{2.9}$$

Proof. Since $Te_n = \alpha_n\ e_{n+1}$, we have $T^*e_n = \overline{\alpha_{n-1}}\ e_{n-1}$. Hence, $T^*Te_n = |\alpha_n|^2\ e_n$, and $T^{*2}T^2e_n = |\alpha_n|^2|\alpha_{n+1}|^2e_n$. Therefore,

$$\begin{aligned} T^*(T^{*2}T^2)Te_n =&\ |\alpha_n|^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2e_n \\ T^{*2}(T^{*2}T^2)T^2e_n =&\ |\alpha_n|^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2|\alpha_{n+3}|^2e_n \end{aligned}$$

Hence we get,

$$T^{*l}(T^{*2}T^2)T^le_n = |\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+l-1}|^2|\alpha_{n+l}|^2|\alpha_{n+l+1}|^2e_n, \text{ for any } l \in \mathbb{N}.$$

Also,

$$\begin{aligned} T^*(TT^*)Te_n =&\ |\alpha_n|^2|\alpha_n|^2e_n \\ T^{*2}(TT^*)T^2 =&\ |\alpha_n|^2|\alpha_{n+1}|^2|\alpha_{n+1}|^2e_n \end{aligned}$$

Hence we get,

$$T^{*l}(TT^*)T^l e_n = |\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+l-1}|^2|\alpha_{n+l-1}|^2 e_n, \text{ for any } l \in \mathbb{N}.$$

In the similar way we get,

$$T^{*l}T^l e_n = |\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+l-1}|^2 e_n, \text{ for any } l \in \mathbb{N}.$$

Using above equations and (2.1), we get T is k-quasi $(1,1)^*$-paranormal

$\Leftrightarrow T^{*k}(T^{*2}T^2 - 2aTT^* + a^2I)T^k \geq 0,\ \forall a > 0.$

$\Leftrightarrow \langle (T^{*k}(T^{*2}T^2 - 2aTT^* + a^2I)T^k)e_n, e_n\rangle \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}.$

$\Leftrightarrow |\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+k-1}|^2(|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ a\ |\alpha_{n+k-1}|^2 + a^2) \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}.$

$\Leftrightarrow |\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ a\ |\alpha_{n+k-1}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}.$

□

Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, 2x_1, x_2, 2x_3, x_4, 6x_5, 6x_6, \cdots)$.
Here $\alpha_1 = 2\ \alpha_2 = 1, \alpha_3 = 2,\ \alpha_4 = 1,\ \alpha_n = 6\ \text{ for }\ n \geq 5$.
From (2.9), T is 2-quasi $(1,1)^*$-paranormal if and only if

$$|\alpha_{n+2}|^2|\alpha_{n+3}|^2 - 2\ a\ |\alpha_{n+1}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}. \tag{2.10}$$

If $n = 1$, $|\alpha_3|^2|\alpha_4|^2 - 2a\ |\alpha_2|^2 + a^2 = (a-1)^2 + 3 \geq 0, \forall a > 0$.
If $n = 2$, $|\alpha_4|^2|\alpha_5|^2 - 2a\ |\alpha_3|^2 + a^2 = (a-4)^2 + 20 \geq 0, \forall a > 0$.
If $n = 3$, $|\alpha_5|^2|\alpha_6|^2 - 2a\ |\alpha_4|^2 + a^2 = (a-1)^2 + 1295 \geq 0, \forall a > 0$.
If $n \geq 4$, $|\alpha_{n+2}|^2|\alpha_{n+3}|^2 - 2\ a\ |\alpha_{n+1}|^2 + a^2 = (a-36)^2 \geq 0, \forall a > 0$.
Thus, (2.10) holds for all $a > 0$, for all $n \in \mathbb{N}$.
Hence, T is 2-quasi $(1,1)^*$-paranormal.
From (2.8), T is 2-quasi $(1,1)$-paranormal if and only if

$$|\alpha_{n+2}|^2|\alpha_{n+3}|^2 - 2\ a\ |\alpha_{n+2}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}. \tag{2.11}$$

If $n = 1$, $|\alpha_3|^2|\alpha_4|^2 - 2a\,|\alpha_3|^2 + a^2 = (a-4)^2 - 12 < 0$, for $a = 4$.
Hence T is not 2-quasi $(1,1)$-paranormal.

Theorem 2.2.4. *Let $T \in \mathcal{B}(\mathcal{H})$.*

(i) If T is a $(k+1)$-quasi $(m,n)^*$-paranormal, then T is k-quasi $(m,n+1)$-paranormal.

(ii) If T is a k-quasi $(m,n)^*$-paranormal operator, then T is $(k+1)$-quasi $(m,n)^*$-paranormal operator.

Proof. (i) Suppose that T is a $(k+1)$-quasi $(m,n)^*$-paranormal operator. Then

$$\|T^*T^{k+1}x\|^{n+1} \le m\|T^{n+1}T^{k+1}x\|\|T^{k+1}x\|^n, \text{ for all } x \in \mathcal{H}.$$

Hence, for every $x \in \mathcal{H}$ we have

$$\begin{aligned}\|T^{k+1}x\|^{2n+2} &= \langle T^*T^{k+1}x, T^kx\rangle^{n+1}\\ &\le \|T^*T^{k+1}x\|^{n+1}\|T^kx\|^{n+1}\\ &\le m\|T^{n+1}T^{k+1}x\|\|T^{k+1}x\|^n\|T^kx\|^{n+1}.\end{aligned}$$

Thus, $\|TT^kx\|^{n+2} \le m\|T^{n+2}T^kx\|\|T^kx\|^{n+1}$, for all $x \in \mathcal{H}$.
Hence T is a k-quasi $(m,n+1)$-paranormal operator.

(ii) Assume that T is a k-quasi $(m,n)^*$-paranormal operator. Then

$$\|T^*T^kx\|^{n+1} \le m\|T^{n+1}T^kx\|\|T^kx\|^n, \text{ for all } x \in \mathcal{H}.$$

Replacing x by Tx, we get

$$\|T^*T^{k+1}x\|^{n+1} \le m\|T^{n+1}T^{k+1}x\|\|T^{k+1}x\|^n, \text{ for all } x \in \mathcal{H}.$$

Hence T is a $(k+1)$-quasi $(m,n)^*$-paranormal operator.

□

Theorem 2.2.5. *If $T \in \mathcal{B}(\mathcal{H})$ is a k-quasi (m,n)-paranormal operator, then T is a $(k+1)$-quasi (m,n)-paranormal operator.*

Proof. Assume that T is a k-quasi (m,n)-paranormal operator. Then

$$\|TT^k x\|^{n+1} \leq m\|T^{n+1}T^k x\|\|T^k x\|^n, \text{ for all } x \in \mathcal{H}.$$

Replacing x by Tx, we get

$$\|TT^{k+1} x\|^{n+1} \leq m\|T^{n+1}T^{k+1} x\|\|T^{k+1} x\|^n, \text{ for all } x \in \mathcal{H}.$$

Hence T is a $(k+1)$-quasi (m,n)-paranormal operator. □

Remark 2.2.1.

From Theorem 2.2.4 it is clear that for any non negative integer k,

$$k\text{-quasi } (m,n)^*\text{-paranormal} \subset (k+1)\text{-quasi } (m,n)^*\text{-paranormal}.$$

But the converse need not be true always. For, we give an example of a 2-quasi $(1,1)^*$-paranormal operator which is not 1-quasi $(1,1)^*$-paranormal.
Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 2x_1, x_2, 2x_3, x_4, 6x_5, 6x_6, \cdots).$$

In this section, we proved that T is a 2-quasi $(1,1)^*$-paranormal operator . From Theorem 2.2.3, T is 1-quasi $(1,1)^*$-paranormal operator if and only if

$$|\alpha_{n+1}|^2|\alpha_{n+2}|^2 - 2\ a\ |\alpha_n|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}. \tag{2.12}$$

If $n = 1$ and $a = 4$, (2.12) is not satisfied. Hence T is not 1-quasi $(1,1)^*$-paranormal operator.

Remark 2.2.2.

From Theorem 2.2.5 it is clear that for any non negative integer k,

$$k\text{-quasi } (m,n)\text{-paranormal} \subset (k+1)\text{-quasi } (m,n)\text{-paranormal}.$$

But the converse need not be true. Here we give an example of a 3-quasi $(1,1)$-paranormal operator, which is not 2-quasi $(1,1)$-paranormal.
Consider $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{2}x_2, \frac{1}{4}x_3, \frac{1}{5}x_4, \frac{1}{4}x_5, \frac{1}{4}x_6, \cdots).$$

From Theorem 2.2.2, T is 3-quasi $(1,1)$-paranormal. Using Theorem 2.2.2, T is 2-quasi $(1,1)$-paranormal if and only if

$$|\alpha_{n+2}|^2|\alpha_{n+3}|^2 - 2\ a\ |\alpha_{n+2}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}. \tag{2.13}$$

If $n = 1$ and $a = \frac{1}{16}$, (2.13) is not satisfied. Hence T is not 2-quasi $(1,1)$-paranormal.

Now we give a matrix representation for k-quasi (m,n)-paranormal operators.

Theorem 2.2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\overline{R(T^k)} \neq \mathcal{H}$. Then the following are equivalent:*

(i) T is a k-quasi (m,n)- paranormal operator.

(ii) $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\overline{R(T^k)} \oplus N(T^{*k})$, where A is a (m,n)-paranormal operator on $\overline{R(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Assume that T is a k-quasi (m,n)-paranormal operator. Since $R(T^k)$ is not dense in $\mathcal{H}$ and $\overline{R(T^k)}$ is invariant under T, we have $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{R(T^k)} \oplus N(T^{*k})$. Since $A = T|_{\overline{R(T^k)}}$, we have

$$\langle (T^{*k}(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\ a^nT^*T + m^{\frac{2}{n+1}}\ n\ a^{n+1}I)T^k)x, x\rangle$$

$$\begin{aligned}
&= \langle (m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\ a^nT^*T + m^{\frac{2}{n+1}}\ n\ a^{n+1}I)T^kx, T^kx\rangle,\ \forall x \in \mathcal{H}.\\
&= \langle (m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\ a^nT^*T + m^{\frac{2}{n+1}}\ n\ a^{n+1}I)y, y\rangle, \forall y \in \overline{R(T^k)}.\\
&= \langle (m^{\frac{2}{n+1}}A^{*n+1}A^{n+1} - (n+1)\ a^nA^*A + m^{\frac{2}{n+1}}\ n\ a^{n+1}I)y, y\rangle, \forall y \in \overline{R(T^k)}.
\end{aligned}$$

Since T is a k-quasi (m,n)- paranormal operator, we have $\langle (m^{\frac{2}{n+1}}A^{*n+1}A^{n+1} - (n+1)\ a^nA^*A + m^{\frac{2}{n+1}}\ n\ a^{n+1}I)y, y\rangle \geq 0$, for all $y \in \overline{R(T^k)}$. Hence A is a (m,n)-paranormal operator on $\overline{R(T^k)}$.

Let $x \in N(T^{*k})$. Then

$$T^k(x) = \begin{pmatrix} A^k & \sum\limits_{i=0}^{k-1} A^iBC^{k-1-i} \\ 0 & C^k \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

Thus $C^kx = T^kx - \sum\limits_{i=0}^{k-1} A^iBC^{k-1-i}x$. Since $A = T|_{\overline{R(T^k)}}$, we have $C^kx \in \overline{R(T^k)}$. From the matrix representation of T, we have $C^kx \in N(T^{*k})$. Thus $C^kx \in \overline{R(T^k)} \cap N(T^{*k})$. Hence $C^k = 0$. From Theorem 1.3.4, we get $\sigma(T) = \sigma(A) \cup \{0\}$.

Conversely, let $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathcal{H} = \overline{R(T^k)} \oplus N(T^{*k})$, where A is a (m,n)-paranormal operator on $\overline{R(T^k)}$ and $C^k = 0$. Thus

$$T^k = \begin{pmatrix} A^k & \sum\limits_{i=0}^{k-1} A^iBC^{k-1-i} \\ 0 & 0 \end{pmatrix}$$

and $T^kT^{*k} = \begin{pmatrix} A^kA^{*k} + \sum\limits_{i=0}^{k-1} A^iBC^{k-1-i}(\sum\limits_{i=0}^{k-1} A^iBC^{k-1-i})^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$, where $S = A^kA^{*k} + \sum\limits_{i=0}^{k-1} A^iBC^{k-1-i}(\sum\limits_{i=0}^{k-1} A^iBC^{k-1-i})^*$. Since, S is a positive operator on $\overline{R(T^k)}$ and $A = T|_{\overline{R(T^k)}}$ is a (m,n)-paranormal operator, we have

$$T^kT^{*k}(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)a^nT^*T + m^{\frac{2}{n+1}}na^{n+1}I)T^kT^{*k}$$
$$= \begin{pmatrix} S(m^{\frac{2}{n+1}}A^{*n+1}A^{n+1} - (n+1)a^nA^*A + m^{\frac{2}{n+1}}na^{n+1}I)S & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

Let $D = T^{*k}(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)a^nT^*T + m^{\frac{2}{n+1}}na^{n+1}I)T^k$. Then, $T^kDT^{*k} \geq 0$. Let $x \in \mathcal{H}$. Then $x = y + z$ where $y \in \overline{R(T^{*k})},\ z \in N(T^k)$. Since $y \in \overline{R(T^{*k})}$, there exists a sequence (x_n) in $\mathcal{H}$ such that $T^{*k}(x_n) \to y$. Since $z \in N(T^k)$, $Dz = 0$. Since $T^kDT^{*k} \geq 0$, we have

$$\begin{aligned} \langle Dx, x\rangle &= \langle D(lim\ T^{*k}(x_n) + z), lim\ T^{*k}(x_n) + z\rangle \\ &= \langle D\ lim\ T^{*k}(x_n), lim\ T^{*k}(x_n)\rangle + \langle D\ lim\ T^{*k}(x_n), z\rangle \\ &= lim\langle DT^{*k}(x_n), T^{*k}(x_n)\rangle + lim\langle DT^{*k}(x_n), z\rangle \\ &= lim\langle DT^{*k}(x_n), T^{*k}(x_n)\rangle + lim\langle T^{*k}(x_n), Dz\rangle \\ &\geq 0. \end{aligned}$$

Hence T is a k-quasi (m, n)-paranormal operator. □

Chapter 3

(m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ operators

In this chapter, we define two independent new classes of operators (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$, which contains some well known class of operators, (m,n)-paranormal and $(m,n)^*$-paranormal operators respectively. Here we prove that the restriction of the operator to its closed subspace is the corresponding operator. Also, we give characterizations for the weighted shift operators to become (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ operators. Finally we characterize the composition operators of these two classes on L^2 space.

3.1 (m,n)-class $\mathcal{Q}$ operators

Let $m \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Recall that $T \in \mathcal{B}(\mathcal{H})$ is (m,n)-paranormal if and only if

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) a^n T^* T + m^{\frac{2}{n+1}} n \, a^{n+1} \, I \geq 0,$$

for all $a > 0$ ([5]).
Let $\mathcal{Q}_{(m,n)} = m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) T^* T + m^{\frac{2}{n+1}} \, n \, I$.
Now we define (m,n)-class $\mathcal{Q}$ operator as follows:

Definition 3.1.1. *Let $m \in \mathbb{R}^+$, $n \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (m,n)-class $\mathcal{Q}$ if $\mathcal{Q}_{(m,n)} \geq 0$.*

For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$. Hence $T^*T = I$ and $T^{*n+1}T^{n+1} = I$. Therefore,

$$\begin{aligned}\mathcal{Q}_{(m,n)} &= m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I \\ &= (m^{\frac{2}{n+1}} - 1)(1+n)I \geq 0,\ \text{for all } m \geq 1 \text{ and } n \in \mathbb{N}.\end{aligned}$$

Hence if $m \geq 1$, then T is a (m,n)-class $\mathcal{Q}$ operator for every $n \in \mathbb{N}$.
Note that if $m = n = 1$, then (m,n)-class $\mathcal{Q}$ operator become class $\mathcal{Q}$ ([10]). That is, class $\mathcal{Q}$ operators form a subclass of the classes of all (m,n)-class $\mathcal{Q}$ operators. In general, every (m,n)-class $\mathcal{Q}$ operators need not be class $\mathcal{Q}$.
For example, let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T^*T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $T^2 = 0$.
If $m = 25,\ n = 3$, then

$$\begin{aligned}m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)\ T^*T + m^{\frac{2}{n+1}}\ n\ I &= -4T^*T + 15I \\ &= \begin{pmatrix} 15 & 0 \\ 0 & 11 \end{pmatrix} \geq 0.\end{aligned}$$

Hence T is a $(25,3)$- class $\mathcal{Q}$ operator.
By definition, T is class $\mathcal{Q}$ if and only if $T^{*2}T^2 - 2\ T^*T + I \geq 0$.

$$T^{*2}T^2 - 2\ T^*T + I = -2T^*T + I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} < 0.$$

Hence T is not a class $\mathcal{Q}$ operator.

Now we prove some characterizations of (m,n)-class $\mathcal{Q}$ operators.

Theorem 3.1.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is (m,n)-class $\mathcal{Q}$ if and only if*

$$\|Tx\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1} \left(\|T^{n+1}x\|^2 + n\|x\|^2\right),\ \forall x \in \mathcal{H}.$$

Proof. T is a (m,n)-class $\mathcal{Q}$ operator

$$\Leftrightarrow m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}} \ \langle T^{*n+1}T^{n+1}x, x\rangle - (n+1)\ \langle T^*Tx, x\rangle \ + m^{\frac{2}{n+1}}\ n\ \langle x, x\rangle \ \geq 0,\ \forall x \in \mathcal{H}.$$

$$\Leftrightarrow m^{\frac{2}{n+1}} \ \|T^{n+1}x\|^2 - (n+1)\|Tx\|^2 + m^{\frac{2}{n+1}}\ n\ \|x\|^2 \geq 0,\ \forall x \in \mathcal{H}.$$

$$\Leftrightarrow \|Tx\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right),\ \forall x \in \mathcal{H}.$$

□

Theorem 3.1.2. *Let $T \in \mathcal{B}(\mathcal{H})$. T is (m,n)-paranormal if and only if λT is (m,n)-class $\mathcal{Q}$ operator, for each $\lambda > 0$.*

Proof. λT is (m,n)-class $\mathcal{Q}$

$$\Leftrightarrow m^{\frac{2}{n+1}}|\lambda|^{2(n+1)}T^{*n+1}T^{n+1} - (n+1)\ |\lambda|^2\ T^*T + m^{\frac{2}{n+1}}\ n\ I \geq 0,\ \forall \lambda > 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\left(\frac{1}{\lambda^2}\right)^n\ T^*T + m^{\frac{2}{n+1}}\ n\left(\frac{1}{\lambda^2}\right)^{n+1} I \geq 0,\ \forall \lambda > 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\ a^n\ T^*T + m^{\frac{2}{n+1}}\ n\ a^{n+1}\ I \geq 0,\ a = \frac{1}{\lambda^2} > 0.$$

$\Leftrightarrow T$ is (m,n)-paranormal.

□

From the above theorem, it is clear that every (m,n)-paranormal operator is (m,n)-class $\mathcal{Q}$.

Theorem 3.1.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m,n)-class $\mathcal{Q}$ operator and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. Then $T|_{\mathcal{M}}$ is a (m,n)-class $\mathcal{Q}$ operator.*

Proof. Let $x \in \mathcal{M}$. Since T is (m,n)-class $\mathcal{Q}$, we have

$$\begin{aligned}\|T|_{\mathcal{M}}x\|^2 &= \|Tx\|^2 \\ &\leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right) \\ &= \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|(T|_{\mathcal{M}})^{n+1}x\|^2 + n\|x\|^2\right).\end{aligned}$$

Hence $T|_{\mathcal{M}}$ is a (m,n)-class $\mathcal{Q}$ operator. □

Theorem 3.1.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and $c = \dfrac{l}{m^{\frac{2}{n+1}}\ n}$, where $l \geq n+1$. If $\sqrt{c}\ T$ is a contraction, then T is a (m,n)-class $\mathcal{Q}$ operator.*

Proof. Since $\sqrt{c}\,T$ is a contraction, $(\sqrt{c}\,T)^*(\sqrt{c}\,T) \leq I$. Therefore, $-c\,T^*T + I \geq 0$. Since $\dfrac{T^{*n+1}T^{n+1}}{n}$ is a positive operator, we have

$$\frac{T^{*n+1}T^{n+1}}{n} - c\,T^*T + I \geq 0.$$

Hence,

$$\frac{T^{*n+1}T^{n+1}}{n} - \frac{l}{m^{\frac{2}{n+1}}\,n}\,T^*T + I \geq 0.$$

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - l\,T^*T + m^{\frac{2}{n+1}}\,n\,I \geq 0.$$

Therefore,

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\,n\,I \geq m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - l\,T^*T + m^{\frac{2}{n+1}}\,n\,I \geq 0.$$

Thus T is a (m,n)-class $\mathcal{Q}$ operator. □

From Theorem 3.1.2, every (m,n)-paranormal operator is (m,n)-class $\mathcal{Q}$. In general, the converse need not be true. For proving this, we use the following result.

Theorem 3.1.5. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_k\}, (k = 1, 2, ...)$, defined by $Te_k = \alpha_k\ e_{k+1}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is a (m,n)-class $\mathcal{Q}$ operator if and only if*

$$\frac{n+1}{m^{\frac{2}{n+1}}}\left(|\alpha_k|^2\right) \leq |\alpha_k|^2\ |\alpha_{k+1}|^2......|\alpha_{k+n}|^2 + n,\ \forall\ k \in \mathbb{N}.$$

Proof. Since $Te_k = \alpha_k\ e_{k+1}$, we have $T^{n+1}e_k = \alpha_k\ \alpha_{k+1}.....\alpha_{k+n}\ e_{k+n+1}$.
T is a (m,n)-class $\mathcal{Q}$

$$\Leftrightarrow \|Tx\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right),\ \forall\ x \in \mathcal{H}.$$

$$\Leftrightarrow \|Te_k\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}e_k\|^2 + n\|e_k\|^2\right),\ \forall\ k \in \mathbb{N}.$$

$$\Leftrightarrow \frac{n+1}{m^{\frac{2}{n+1}}}\ |\alpha_k|^2 \leq\ |\alpha_k|^2\ |\alpha_{k+1}|^2.....|\alpha_{k+n}|^2 + n,\ \forall\ k \in \mathbb{N}.$$

□

Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, ...) = (0, \frac{1}{2}x_1, \frac{1}{4}x_2, \frac{1}{4}x_3, ...).$$

Here $\alpha_1 = \frac{1}{2}$, $\alpha_k = \frac{1}{4}$ for $k \geq 2$. From Theorem 3.1.5, we get T is a $(\frac{1}{3}, 3)$-class $\mathcal{Q}$ operator. From Lemma 1.3.6, T is (m, n)-paranormal if and only if

$$|\alpha_k|^{n+1} \leq m\ |\alpha_k|\ |\alpha_{k+1}|......|\alpha_{k+n}|,\ \forall\ k \in \mathbb{N}. \tag{3.1}$$

If $m = \frac{1}{3}$, $n = 3$, $k = 4$, then (3.1) is not satisfied. Hence T is not a $(\frac{1}{3}, 3)$-paranormal operator.

Theorem 3.1.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m, n)-class $\mathcal{Q}$ operator and $A \in \mathcal{B}(\mathcal{H})$ be an isometric operator such that $AT = TA$. Then TA is a (m, n)-class $\mathcal{Q}$ operator.*

Proof. Let $S = TA$. Since $AT = TA$ and $A^*A = I$, we have

$m^{\frac{2}{n+1}} S^{*n+1} S^{n+1} - (n+1)S^*S + m^{\frac{2}{n+1}}\ n\ I$

$$\begin{aligned} &= m^{\frac{2}{n+1}} (A^*T^*)^{n+1} (TA)^{n+1} - (n+1)A^*T^*TA + m^{\frac{2}{n+1}}\ nI \\ &= m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I \end{aligned}$$

Since T is (m, n)-class $\mathcal{Q}$, $m^{\frac{2}{n+1}} S^{*n+1} S^{n+1} - (n+1)S^*S + m^{\frac{2}{n+1}}\ n\ I \geq 0$. Hence $S = TA$ is a (m, n)-class $\mathcal{Q}$ operator. □

Theorem 3.1.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m, n)-class $\mathcal{Q}$ operator and T be unitarily equivalent to an operator $B \in \mathcal{B}(\mathcal{H})$. Then B is a (m, n)-class $\mathcal{Q}$ operator.*

Proof. Since T is unitarily equivalent to B, there exist an unitary operator U such that $B = U^*TU$. Now,

$m^{\frac{2}{n+1}} B^{*n+1} B^{n+1} - (n+1)B^*B + m^{\frac{2}{n+1}}\ n\ I$

$$\begin{aligned} &= m^{\frac{2}{n+1}} U^*T^{*n+1}T^{n+1}U - (n+1)U^*T^*TU + m^{\frac{2}{n+1}}\ nI \\ &= U^* \left(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I \right) U \end{aligned}$$

Since T is (m,n)-class $\mathcal{Q}$, $U^*\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1}-(n+1)T^*T+m^{\frac{2}{n+1}}\ n\ I\right)U \geq 0$. Hence B is a (m,n)-class $\mathcal{Q}$ operator. □

3.2 (m,n)- class $\mathcal{Q}^*$ operators

Let $T \in \mathcal{B}(\mathcal{H})$. Let $m \in \mathbb{R}^+$, $n \in \mathbb{N}$. Recall that T is said to be $(m,n)^*$-paranormal if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$. Equivalently, T is $(m,n)^*$-paranormal if and only if

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1}-(n+1)a^nTT^*+m^{\frac{2}{n+1}}\ n\ a^{n+1}\ I \geq 0,$$

for each $a > 0$ ([5]). Now we define a new classes of operator namely (m,n)-class $\mathcal{Q}^*$ operators, which contains the classes of all $(m,n)^*$-paranormal operators.

Let $\mathcal{Q}_{(m,n)^*} = m^{\frac{2}{n+1}}T^{*n+1}T^{n+1}-(n+1)TT^*+m^{\frac{2}{n+1}}\ n\ I$.

Definition 3.2.1. *Let $m \in \mathbb{R}^+$, $n \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (m,n)-class $\mathcal{Q}^*$ if $\mathcal{Q}_{(m,n)^*} \geq 0$.*

For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1,x_2,x_3,\cdots) = (0,x_1,x_2,x_3,\cdots)$. Then $TT^*(x_1,x_2,x_3,......) = (0,x_2,x_3,......)$ and $T^{*n+1}T^{n+1} = I$, for all $n \in \mathbb{N}$. Hence, for every $x = (x_1,x_2,x_3,......) \in l^2(\mathbb{N})$

$$\begin{aligned}\langle \mathcal{Q}_{(m,n)^*}x,x\rangle &= \left\langle \left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1}-(n+1)TT^*+m^{\frac{2}{n+1}}\ n\ I\right)x,x\right\rangle \\ &= \left\langle (1+n)\left(m^{\frac{2}{n+1}}I-TT^*\right)x,x\right\rangle \\ &= (1+n)\left(m^{\frac{2}{n+1}}|x_1|^2+(m^{\frac{2}{n+1}}-1)|x_2|^2+\cdots\right) \geq 0,\end{aligned}$$

for every $n \in \mathbb{N}$ and $m \geq 1$. Hence T is a (m,n)-class $\mathcal{Q}^*$ operator, for $m \geq 1$ and $n \in \mathbb{N}$.

Note that if $m = n = 1$, then (m,n)-class $\mathcal{Q}^*$ is class $\mathcal{Q}^*$. That is, class $\mathcal{Q}^*$ operators form a subclass of the classes of all (m,n)-class $\mathcal{Q}^*$ operators. But, every (m,n)-class $\mathcal{Q}^*$ operator need not be a class $\mathcal{Q}^*$ operator.

For example, let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T^2 = 0$.

If $m = 25,\ n = 3$, then

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)\ TT^* + m^{\frac{2}{n+1}} n\ I = -4TT^* + 15I$$
$$= \begin{pmatrix} 11 & 0 \\ 0 & 15 \end{pmatrix} \geq 0.$$

Hence T is a $(25,3)$-class $\mathcal{Q}^*$ operator.

By definition, T is class $\mathcal{Q}^*$ if and only if $T^{*2}T^2 - 2\ TT^* + I \geq 0$. Here

$$T^{*2}T^2 - 2\ TT^* + I = -2TT^* + I$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} < 0.$$

Hence T is not a class $\mathcal{Q}^*$ operator.

That is, every (m,n)-class $\mathcal{Q}^*$ need not be class $\mathcal{Q}^*$.

Now we prove some characterizations of (m,n)-class $\mathcal{Q}^*$ operators.

Theorem 3.2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is (m,n)-class $\mathcal{Q}^*$ if and only if*

$$\|T^*x\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1} \left(\|T^{n+1}x\|^2 + n\|x\|^2 \right), \forall x \in \mathcal{H}.$$

Proof. T is a (m,n)-class $\mathcal{Q}^*$ operator

$$\Leftrightarrow m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$
$$\Leftrightarrow m^{\frac{2}{n+1}}\ \langle T^{*n+1}T^{n+1}x, x\rangle - (n+1)\ \langle TT^*x, x\rangle\ + m^{\frac{2}{n+1}}\ n\ \langle x, x\rangle\ \geq 0, \forall x \in \mathcal{H}.$$
$$\Leftrightarrow m^{\frac{2}{n+1}}\ \|T^{n+1}x\|^2 - (n+1)\|T^*x\|^2 + m^{\frac{2}{n+1}}\ n\ \|x\|^2 \geq 0,\ \forall x \in \mathcal{H}.$$
$$\Leftrightarrow \|T^*x\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1} \left(\|T^{n+1}x\|^2 + n\|x\|^2 \right), \forall x \in \mathcal{H}.$$

□

Now we give a characterization for the weighted shift operator become (m,n)-class $\mathcal{Q}^*$.

Theorem 3.2.2. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_k\}, (k = 1, 2, ...)$, defined by $Te_k = \alpha_k\ e_{k+1}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is a (m,n)-class $\mathcal{Q}^*$ operator if and only if*

$$\frac{n+1}{m^{\frac{2}{n+1}}}\left(|\alpha_k|^2\right) \leq |\alpha_{k+1}|^2\ |\alpha_{k+2}|^2 \cdots |\alpha_{k+n+1}|^2 + n, \forall\ k \in \mathbb{N}.$$

Proof. Since $Te_k = \alpha_k\ e_{k+1}$, we have $T^{n+1}e_k = \alpha_k\ \alpha_{k+1}.....\alpha_{k+n}\ e_{k+n+1},\ \forall\ k \in \mathbb{N}$. and $T^*e_k = \overline{\alpha_{k-1}}\ e_{k-1},\ \forall\ k \geq 2$. Now,
T is (m,n)-class $\mathcal{Q}^*$

$$\Leftrightarrow \|T^*x\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right), \forall\, x \in \mathcal{H}.$$
$$\Leftrightarrow \|T^*e_k\|^2 \leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}e_k\|^2 + n\|e_k\|^2\right), \forall\ k \in \mathbb{N}.$$
$$\Leftrightarrow \frac{n+1}{m^{\frac{2}{n+1}}}\ |\alpha_k|^2 \leq\ |\alpha_{k+1}|^2\ |\alpha_{k+2}|^2.....|\alpha_{k+n+1}|^2 + n, \forall\ k \in \mathbb{N}.$$

□

Theorem 3.2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $c = \dfrac{l}{m^{\frac{2}{n+1}}\ n}$, where $l \geq n+1$. If $\sqrt{c}\ T^*$ is a contraction, then T is a (m,n)-class $\mathcal{Q}^*$ operator.*

Proof. Since $\sqrt{c}\ T^*$ is a contraction, we have $-c\ TT^* + I \geq 0$. Hence

$$\frac{T^{*n+1}T^{n+1}}{n} - c\ TT^* + I \geq 0.$$

$$\frac{T^{*n+1}T^{n+1}}{n} - \frac{l}{m^{\frac{2}{n+1}}\ n}\ TT^* + I \geq 0.$$

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - l\,TT^* + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

$$m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I \geq m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - l\,TT^* + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

Thus T is a (m,n)-class $\mathcal{Q}^*$ operator. □

Now we show that the class of all $(m,n)^*$-paranormal operators is contained in the class of all (m,n)-class $\mathcal{Q}^*$ operators.

Theorem 3.2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is $(m,n)^*$-paranormal if and only if λT is (m,n)-class $\mathcal{Q}^*$, for each $\lambda > 0$.*

Proof. We have λT is (m,n)-class $\mathcal{Q}^*$

$$\Leftrightarrow m^{\frac{2}{n+1}}|\lambda|^{2(n+1)}T^{*n+1}T^{n+1} - (n+1)\,|\lambda|^2\,TT^* + m^{\frac{2}{n+1}}\,n\,I \geq 0,\ \forall \lambda > 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\left(\frac{1}{\lambda^2}\right)^n TT^* + m^{\frac{2}{n+1}}\,n\left(\frac{1}{\lambda^2}\right)^{n+1} I \geq 0, \forall \lambda > 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\,a^n\,TT^* + m^{\frac{2}{n+1}}\,n\,a^{n+1}\,I \geq 0,\ a = \frac{1}{\lambda^2} > 0.$$

$$\Leftrightarrow T \text{ is } (m,n)^*\text{-paranormal.}$$

□

By the above theorem, every $(m,n)^*$-paranormal operators is (m,n)-class $\mathcal{Q}^*$ operator. In general, the converse need not be true.

For example, let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T^2 = 0$.

If $m = 25,\ n = 3$, then

$$\begin{aligned} m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\,TT^* + m^{\frac{2}{n+1}}n\,I &= -4TT^* + 15I \\ &= \begin{pmatrix} 11 & 0 \\ 0 & 15 \end{pmatrix} \geq 0. \end{aligned}$$

Hence T is a $(25,3)$-class $\mathcal{Q}^*$ operator.

If $m = 25,\ n = 3$, then

$$\begin{aligned} m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\,a^n\,TT^* + m^{\frac{2}{n+1}}n\,a^{n+1}I &= -4a^3TT^* + 15a^4I \\ &= \begin{pmatrix} -4a^3 + 15a^4 & 0 \\ 0 & 15a^4 \end{pmatrix} \end{aligned}$$

If $a = \frac{1}{5}$, then $-4a^3TT^* + 15a^4I < 0$. Hence T is not $(25,3)^*$-paranormal.

Theorem 3.2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m,n)-class $\mathcal{Q}^*$ operator and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. Then $T|_{\mathcal{M}}$ is a (m,n)-class $\mathcal{Q}^*$ operator.*

Proof. Let $x \in \mathcal{M}$. Then

$$\begin{aligned}\|(T|_{\mathcal{M}})^* x\|^2 &= \|T^* x\|^2 \\ &\leq \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right) \\ &= \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|(T|_{\mathcal{M}})^{n+1}x\|^2 + n\|x\|^2\right).\end{aligned}$$

Hence $T|_{\mathcal{M}}$ is a (m,n)-class $\mathcal{Q}^*$ operator. □

Theorem 3.2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (m,n)-class $\mathcal{Q}^*$ operator and T be unitarily equivalent to an operator $B \in \mathcal{B}(\mathcal{H})$. Then B is a (m,n)-class $\mathcal{Q}^*$ operator.*

Proof. Since T is unitarily equivalent to an operator B, there exist an unitary operator U such that $B = U^*TU$.

$$m^{\frac{2}{n+1}} B^{*n+1}B^{n+1} - (n+1)BB^* + m^{\frac{2}{n+1}}\ n\ I$$

$$\begin{aligned}&= m^{\frac{2}{n+1}} U^*T^{*n+1}T^{n+1}U - (n+1)U^*TT^*U + m^{\frac{2}{n+1}}\ n\ I. \\ &= U^*\left(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I\right)U.\end{aligned}$$

Since T is (m,n)-class $\mathcal{Q}^*$, $U^*\left(m^{\frac{2}{n+1}} T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I\right)U \geq 0$. Hence B is a (m,n)-class $\mathcal{Q}^*$ operator. □

Now we show that the newly defined classes of operators, (m,n)-class $\mathcal{Q}$ and class of (m,n)-class $\mathcal{Q}^*$ are independent. For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{4}x_2, \frac{1}{4}x_3, \cdots).$$

Here $\alpha_1 = \frac{1}{2}$, $\alpha_k = \frac{1}{4}$ for $k \geq 2$. If $m = \dfrac{1}{9.00033}$, $n = 3$, using Theorem 3.1.5, T is a (m,n)-class $\mathcal{Q}$ operator.

From Theorem 3.2.2, we have T is a (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$\frac{n+1}{m^{\frac{2}{n+1}}}\left(|\alpha_k|^2\right) \leq |\alpha_{k+1}|^2\ |\alpha_{k+2}|^2 \cdots |\alpha_{k+n+1}|^2 + n, \forall\ k \in \mathbb{N}. \tag{3.2}$$

If $m = \dfrac{1}{9.00033}$, $n = 3$ and $k = 1$, (3.2) is not satisfied. Hence T is not a (m,n)-class $\mathcal{Q}^*$ operator.

Next we consider an operator $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 2x_1, 4x_2, 3x_3, 4x_4, 6x_5, 6x_6, \cdots).$$

Here $\alpha_1 = 2$ $\alpha_2 = 4, \alpha_3 = 3,$ $\alpha_4 = 4,$ $\alpha_n = 6$ for $n \geq 5$.

If $m = 2 \times 10^{-7}$, $n = 3$, using Theorem 3.2.2, T is a (m,n)-class $\mathcal{Q}^*$ operator.

From Theorem 3.1.5, T is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$\frac{n+1}{m^{\frac{2}{n+1}}} \left(|\alpha_k|^2\right) \leq |\alpha_k|^2 |\alpha_{k+1}|^2 ......|\alpha_{k+n}|^2 + n, \forall\ k \in \mathbb{N}. \tag{3.3}$$

If $m = 2 \times 10^{-7}$, $n = 3$ and $k = 2$, (3.3) is not satisfied.

Hence T is not a (m,n)-class $\mathcal{Q}$ operator.

That is, the classes of operators (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ are independent.

3.3 (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ Composition operators

In this section, we give some characterizations of (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ composition operators defined on L^2 space. For a non singular measurable function T on X, let C_T denotes the composition operator of T on $L^2(\mu)$ and h_n denotes the Radon-Nikodym derivative of $\mu(T^{-1})^n$ with respect to μ. We denote h_1 by h and adjoint of C_T by C_T^*.

Theorem 3.3.1. *C_T is a (m,n)-class $\mathcal{Q}$ operator if and only if* $m^{\frac{2}{n+1}}(h_{n+1} + nI) \geq (n+1)h$

Proof. By definition, we have C_T is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} - (n+1) C_T^* C_T + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

From Theorem 1.4.3, for any $f \in L^2(\mu)$, we have $(n+1)C_T^*C_Tf = (n+1)hf$ and

$$\begin{aligned}
m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} f &= m^{\frac{2}{n+1}} C_T^{*n+1}(f \circ T^{n+1}). \\
&= m^{\frac{2}{n+1}} h_{n+1} E(f \circ T^{n+1}) \circ T^{-(n+1)}. \\
&= m^{\frac{2}{n+1}} h_{n+1} E(1.f \circ T^{n+1}) \circ T^{-(n+1)}. \\
&= m^{\frac{2}{n+1}} h_{n+1} E(1) f. \\
&= m^{\frac{2}{n+1}} h_{n+1} f.
\end{aligned}$$

Hence, C_T is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$\langle m^{\frac{2}{n+1}} h_{n+1} f - (n+1)hf + m^{\frac{2}{n+1}} nf, f \rangle \geq 0, \text{ for every } f \in L^2(\mu).$$

That is, C_T is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}}(h_{n+1} + nI) \geq (n+1)h.$$

□

Example 3.3.1. *Let $X = \mathbb{N} \cup \{0\}$, $\mathcal{A} = P(X)$ and μ be the measure defined by*

$$\mu(A) = \sum_{k \in A} m_k,$$

where

$$m_k = \begin{cases} 1 & \text{if } k = 0. \\ \dfrac{1}{4^{k-1}} & \text{if } k \geq 1. \end{cases}$$

Let $T : X \to X$ be defined by

$$T(k) = \begin{cases} 0 & \text{if } k = 0, 1. \\ k-1 & \text{if } k \geq 2. \end{cases}$$

Then for $q > 1$, we have

$$T^q(k) = \begin{cases} 0 & \text{if } k = 0, 1, 2, \ldots, q. \\ k - q & \text{if } k \geq q + 1. \end{cases}$$

Hence, $\quad h(k) = \frac{\mu T^{-1}(\{k\})}{\mu\{k\}} = \begin{cases} 2 & \text{if } k = 0 \\ \frac{1}{4} & \text{if } k \geq 1. \end{cases}$

For $q > 1$, $h_q(k) = \frac{\mu T^{-q}(\{k\})}{\mu\{k\}} = \begin{cases} 2 + \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^{q-1}} & \text{if } k = 0. \\ \frac{1}{4^q} & \text{if } k \geq 1. \end{cases}$

By computation, it can be seen that if $m \geq 2$ and $n = 1$, then $m(h_2 + I) \geq 2h$. Hence C_T is a (m, n)-class Ω operator for $m \geq 2$ and $n = 1$.

Theorem 3.3.2. *C_T^* is a (m, n)-class Ω operator if and only if $m^{\frac{2}{n+1}}(h_{n+1} \circ T^{n+1} + nI) \geq (n+1) h \circ T$.*

Proof. We have, C_T^* is a (m, n)-class Ω operator if and only if

$$m^{\frac{2}{n+1}} C_T^{n+1} C_T^{*n+1} - (n+1) C_T C_T^* + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

Let $f \in L^2(\mu)$. From Theorem 1.4.3, we have

$$(n+1) C_T C_T^* f = (n+1)(h \circ T) P f = (n+1)(h \circ T) f,$$

where P is a projection onto $\overline{R(C_T)}$. Also from Theorem 1.4.3, for any $f \in L^2(\mu)$ we get

$$\begin{aligned} m^{\frac{2}{n+1}} C_T^{n+1} C_T^{*n+1} f &= m^{\frac{2}{n+1}} C_T^{n+1} (h_{n+1}.E(f) \circ T^{-(n+1)}). \\ &= m^{\frac{2}{n+1}} (h_{n+1}.E(f) \circ T^{-(n+1)}) \circ T^{n+1}. \\ &= m^{\frac{2}{n+1}} \left((h_{n+1} \circ T^{n+1}).(E(f) \circ T^{-(n+1)} \circ T^{n+1}) \right). \\ &= m^{\frac{2}{n+1}} h_{n+1} \circ T^{n+1} E(f). \\ &= m^{\frac{2}{n+1}} h_{n+1} \circ T^{n+1} f. \end{aligned}$$

Thus C_T^* is a (m,n)-class $\mathcal{Q}$ operator

$$\Leftrightarrow \langle m^{\frac{2}{n+1}} h_{n+1} \circ T^{n+1} f - (n+1)(h \circ T) f + m^{\frac{2}{n+1}} n f, f\rangle \geq 0, \forall f \in L^2(\mu).$$
$$\Leftrightarrow m^{\frac{2}{n+1}} (h_{n+1} \circ T^{n+1} + nI) \geq (n+1) h \circ T.$$

□

Example 3.3.2. *In example 3.3.1, if we choose $m \geq 2$ then C_T^* is a $(m,1)$-class $\mathcal{Q}$ operator.*

Theorem 3.3.3. *C_T is a (m,n)-class $\mathcal{Q}^*$ operator if and only if $m^{\frac{2}{n+1}}(h_{n+1} + nI) \geq (n+1) h \circ T$.*

Proof. We have, C_T is a (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} - (n+1) C_T C_T^* + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

From Theorem 1.4.3, we get
$(n+1) C_T C_T^* f = (n+1)(h \circ T) P f = (n+1)(h \circ T) f,\ \forall f \in L^2(\mu)$.
Also, for every $f \in L^2(\mu)$, we have

$$\begin{aligned} m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} f &= m^{\frac{2}{n+1}} C_T^{*n+1} (f \circ T^{n+1}). \\ &= m^{\frac{2}{n+1}} h_{n+1} E(f \circ T^{n+1}) \circ T^{-(n+1)}. \\ &= m^{\frac{2}{n+1}} h_{n+1} E(1.f \circ T^{n+1}) \circ T^{-(n+1)}. \\ &= m^{\frac{2}{n+1}} h_{n+1} E(1) f. \\ &= m^{\frac{2}{n+1}} h_{n+1} f. \end{aligned}$$

Hence C_T is a (m,n)-class $\mathcal{Q}^*$ operator

$$\Leftrightarrow \left\langle m^{\frac{2}{n+1}} h_{n+1} f - (n+1)(h \circ T) f + m^{\frac{2}{n+1}} n f, f \right\rangle \geq 0, \forall f \in L^2(\mu).$$
$$\Leftrightarrow m^{\frac{2}{n+1}} (h_{n+1} + nI) \geq (n+1) h \circ T.$$

□

Example 3.3.3. *In example 3.3.1, if we choose $m \geq 4$ and $n = 1$ then C_T is a (m,n)-class $\mathcal{Q}^*$ operator.*

Theorem 3.3.4. *C_T^* is (m,n)-class $\mathcal{Q}^*$ if and only if $m^{\frac{2}{n+1}}(h_{n+1}\circ T^{n+1}+nI) \geq (n+1)h$.*

Proof. By definition, C_T^* is a (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}} C_T^{n+1} C_T^{*n+1} - (n+1)C_T^* C_T + m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

From Theorem 1.4.3, we get $(n+1)C_T^*C_Tf = (n+1)hf$, $\forall f \in L^2(\mu)$. Also,

$$\begin{aligned} m^{\frac{2}{n+1}} C_T^{n+1} C_T^{*n+1} f &= m^{\frac{2}{n+1}} C_T^{n+1} (h_{n+1}.E(f) \circ T^{-(n+1)}). \\ &= m^{\frac{2}{n+1}} (h_{n+1}.E(f) \circ T^{-(n+1)}) \circ T^{n+1}. \\ &= m^{\frac{2}{n+1}} \left((h_{n+1} \circ T^{n+1}).(E(f) \circ T^{-(n+1)} \circ T^{n+1})\right). \\ &= m^{\frac{2}{n+1}} h_{n+1} \circ T^{n+1} E(f). \\ &= m^{\frac{2}{n+1}} h_{n+1} \circ T^{n+1} f,\ \forall f \in L^2(\mu). \end{aligned}$$

Hence C_T^* is a (m,n)-class $\mathcal{Q}^*$ operator

$$\begin{aligned} &\Leftrightarrow \left\langle m^{\frac{2}{n+1}} h_{n+1} \circ T^{n+1} f - (n+1)hf + m^{\frac{2}{n+1}} nf, f \right\rangle \geq 0,\ \forall f \in L^2(\mu). \\ &\Leftrightarrow m^{\frac{2}{n+1}} (h_{n+1} \circ T^{n+1} + nI) \geq (n+1)h. \end{aligned}$$

□

Example 3.3.4. *In example 3.3.1, if we choose $m \geq 2$ then C_T^* is a $(m,1)$-class $\mathcal{Q}^*$ operator.*

Now we prove some characterizations of (m,n)-class $\mathcal{Q}$ and (m,n)-class $\mathcal{Q}^*$ weighted composition operators. For a non empty set X, let $T : X \to X$ be a measurable function and π be a complex valued measurable function defined on X. Let W denotes the weighted composition operator on $L^2(\mu)$, induced by T and π.
For $f \in L^2(\mu)$ and $k \in \mathbb{N}$, let $J_kf = h_kE(|\pi_k|^2)\circ T^{-k}f$ and $L_kf = \pi_k(h_k\circ T^k)E(\bar{\pi_k}f)$, where $\pi_k = \pi(\pi \circ T)(\pi \circ T^2)\dots(\pi \circ T^{k-1})$.

Theorem 3.3.5. *W is a (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(J_{n+1}+nI) \geq (n+1)J_1.$$

Proof. We have, W is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}}W^{*n+1}W^{n+1}-(n+1)W^*W+m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

By Theorem 1.5.1, for any $f \in L^2(\mu)$, we have

$$\begin{aligned}(n+1)W^*Wf &= (n+1)hE(|\pi|^2)\circ T^{-1}f.\\ &= (n+1)J_1 f.\\ m^{\frac{2}{n+1}}W^{*n+1}W^{n+1}f &= m^{\frac{2}{n+1}}\ h_{n+1}\ E(|\pi_{n+1}|^2)\circ T^{-(n+1)}f.\\ &= m^{\frac{2}{n+1}}J_{n+1}f.\end{aligned}$$

Hence, W is a (m,n)-class $\mathcal{Q}$ operator

$$\begin{aligned}&\Leftrightarrow\ \left\langle m^{\frac{2}{n+1}}J_{n+1}f-(n+1)J_1f+m^{\frac{2}{n+1}}nf,f\right\rangle \geq 0, \forall f\in L^2(\mu).\\ &\Leftrightarrow\ m^{\frac{2}{n+1}}(J_{n+1}+nI)\geq (n+1)J_1.\end{aligned}$$

□

Theorem 3.3.6. *W is a (m,n)-class $\mathcal{Q}^*$ operator if and only if*

$$m^{\frac{2}{n+1}}(J_{n+1}+nI) \geq (n+1)L_1.$$

Proof. We have, W is a (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}}W^{*n+1}W^{n+1}-(n+1)WW^*+m^{\frac{2}{n+1}}\ n\ I \geq 0.$$

By Theorem 1.5.1, for any $f \in L^2(\mu)$, we have

$$\begin{aligned}(n+1)WW^*f &= (n+1)(h\circ T)E(\overline{\pi} f).\\ &= (n+1)L_1 f.\\ m^{\frac{2}{n+1}}W^{*n+1}W^{n+1}f &= m^{\frac{2}{n+1}}\ h_{n+1}\ E(|\pi_{n+1}|^2)\circ T^{-(n+1)}f.\\ &= m^{\frac{2}{n+1}}J_{n+1}f.\end{aligned}$$

Hence, W is a (m,n)-class $\mathcal{Q}^*$ operator

$$\begin{aligned}&\Leftrightarrow\ \left\langle m^{\frac{2}{n+1}}J_{n+1}f-(n+1)L_1 f+m^{\frac{2}{n+1}}nf, f\right\rangle \geq 0, \forall f\in L^2(\mu).\\ &\Leftrightarrow\ m^{\frac{2}{n+1}}(J_{n+1}+nI)\geq (n+1)L_1.\end{aligned}$$

□

Theorem 3.3.7. *W^* is a (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(L_{n+1}+nI)\geq ((n+1)L_1.$$

Proof. By definition, W^* is a (m,n)-class $\mathcal{Q}$ operator if and only if

$$m^{\frac{2}{n+1}}W^{n+1}W^{*n+1}-(n+1)WW^*+m^{\frac{2}{n+1}}\ n\ I\geq 0.$$

Let $f \in L^2(\mu)$. From Theorem 1.5.1, we have

$$\begin{aligned}(n+1)WW^*f &= (n+1)(h\circ T)E(\overline{\pi} f).\\ &= (n+1)L_1 f.\\ m^{\frac{2}{n+1}}W^{n+1}W^{*n+1}f &= m^{\frac{2}{n+1}}\ \pi_{n+1}(h_{n+1}\ \circ T^{n+1})E(\overline{\pi_{n+1}} f).\\ &= m^{\frac{2}{n+1}}\ L_{n+1}f.\end{aligned}$$

Hence, W^* is a (m,n)-class $\mathcal{Q}$ operator

$$\Leftrightarrow \left\langle m^{\frac{2}{n+1}} \ L_{n+1}f - (n+1)L_1 f + m^{\frac{2}{n+1}} nf, f \right\rangle \geq 0, \forall f \in L^2(\mu).$$
$$\Leftrightarrow m^{\frac{2}{n+1}}(L_{n+1} + nI) \geq ((n+1)L_1.$$

□

Theorem 3.3.8. *W^* is a (m,n)-class $\mathcal{Q}^*$ operator if and only if*

$$m^{\frac{2}{n+1}}(L_{n+1} + nI) \geq (n+1)J_1.$$

Proof. W^* is a (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}} W^{n+1} W^{*n+1} - (n+1)W^*W + m^{\frac{2}{n+1}} \ n \ I \geq 0.$$

From Theorem 1.5.1, for any $f \in L^2(\mu)$, we have

$$\begin{aligned} (n+1)W^*Wf &= (n+1)hE(|\pi|^2) \circ T^{-1} f. \\ &= (n+1)J_1 f. \\ m^{\frac{2}{n+1}} W^{n+1} W^{*n+1} f &= m^{\frac{2}{n+1}} \ \pi_{n+1}(h_{n+1} \ \circ T^{n+1})E(\bar{\pi}_{n+1} f). \\ &= m^{\frac{2}{n+1}} \ L_{n+1} f. \end{aligned}$$

Hence, W^* is a (m,n)-class $\mathcal{Q}^*$ operator

$$\Leftrightarrow \left\langle m^{\frac{2}{n+1}} L_{n+1}f - (n+1)J_1 f + m^{\frac{2}{n+1}} nf, f \right\rangle \geq 0, \forall f \in L^2(\mu).$$
$$\Leftrightarrow m^{\frac{2}{n+1}}(L_{n+1} + nI) \geq (n+1)J_1.$$

□

Chapter 4

k-quasi (m, n)-class $\mathcal{Q}$ and k-quasi (m, n)-class $\mathcal{Q}^*$ operators

In this chapter, we define two independent new classes of operators namely k-quasi (m, n)-class $\mathcal{Q}$ and k-quasi (m, n)-class $\mathcal{Q}^*$ operators which contains some well known classes of operators, k-quasi (m, n)-paranormal and k-quasi $(m, n)^*$-paranormal operators respectively. Here we give some characterizations, examples and a 2×2 matrix representation for these classes of operators. Finally we characterize the composition operators of these classes on L^2 space.

4.1 k-quasi (m, n)-class $\mathcal{Q}$ operators

Let $T \in \mathcal{B}(\mathcal{H})$ and $m \in \mathbb{R}^+$, $n \in \mathbb{N}$. Recall that T is said to be (m, n)-class $\mathcal{Q}$ if

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) T^* T + m^{\frac{2}{n+1}} \ n \ I \geq 0,$$

(Definition 3.1.1). Also, T is k-quasi (m, n)-paranormal if and only if

$$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1) \ a^n T^{*k} T^* T T^k + m^{\frac{2}{n+1}} \ n \ a^{n+1} T^{*k} T^k \geq 0, \forall a > 0,$$

(Theorem 2.2.1).

Definition 4.1.1. *([19]) Let k be a non-negative integer. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi class $\mathcal{Q}$ operator if* $T^{*k}\left(T^{*2}T^2 - 2T^*T + I\right)T^k \geq 0$.

Now we define k-quasi (m,n)-class $\mathcal{Q}$ operators which includes k-quasi (m,n)-paranormal and (m,n)-class $\mathcal{Q}$ operators.

Definition 4.1.2. *Let $m \in \mathbb{R}^+$, $n \in \mathbb{N}$ and k be a non-negative integer. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi (m,n)-class $\mathcal{Q}$ operator if*

$$T^{*k}\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^k \geq 0.$$

Note that 1-quasi (m,n)-class $\mathcal{Q}$ operator is said to be quasi (m,n)-class $\mathcal{Q}$ operator.

For example, let $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $T^2 = 0$. From the definition, it is clear that if $k \geq 2$, then T is a k-quasi (m,n)-class $\mathcal{Q}$ operator for any m, n.

If $m = n = 1$, then k-quasi (m,n)-class $\mathcal{Q}$ operators coincides with k-quasi class $\mathcal{Q}$ operators ([19]).

Note that if $k = 0$, then k-quasi (m,n)-class $\mathcal{Q}$ operator becomes (m,n)-class $\mathcal{Q}$ operator introduced in chapter 3 (refer definition 3.1.1). That is, the class of all (m,n)-class $\mathcal{Q}$ operator forms a subclass of the classes all k-quasi (m,n)-class $\mathcal{Q}$ operators. In general, every k-quasi (m,n)-class $\mathcal{Q}$ operator need not be (m,n)-class $\mathcal{Q}$ operator.

For example, the operator T defined by the matrix $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

If $m = n = 1$ and for any integer $k \geq 1$, we have

$m^{\frac{2}{n+1}}T^{*k}T^{*n+1}T^{n+1}T^k - (n+1)\ T^{*k}T^*TT^k + m^{\frac{2}{n+1}}\ n\ T^{*k}T^k$

$$\begin{aligned} &= T^{*k}(T^{*2}T^2 - 2T^*T + I)T^k \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence T is a k-quasi $(1,1)$-class $\mathcal{Q}$ operator for any $k \geq 1$.

Now for $m = n = 1$, we have

$$\begin{aligned} m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) T^*T + m^{\frac{2}{n+1}} n\ I &= T^{*2}T^2 - 2T^*T + I \\ &= T^*T - 2T^*T + I \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} < 0. \end{aligned}$$

Hence T is not a $(1,1)$-class $\mathcal{Q}$ operator.

That is, T is a k-quasi $(1,1)$-class $\mathcal{Q}$ operator for any $k \geq 1$ but T is not $(1,1)$-class $\mathcal{Q}$ operator.

Now we give some characterizations of k-quasi (m,n)-class $\mathcal{Q}$ operators.

Theorem 4.1.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if* $\dfrac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{k+n+1}x\|^2 + n\|T^k x\|^2\right) \geq \|T^{k+1}x\|^2$ *for all* $x \in \mathcal{H}$

Proof. Let $x \in \mathcal{H}$. T is a k-quasi (m,n)-class $\mathcal{Q}$ operator

$$\Leftrightarrow \left\langle \left(T^{*k} \left(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) T^*T + m^{\frac{2}{n+1}}\ n\ I \right) T^k \right) x, x \right\rangle \geq 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}} \ \langle T^{*k+n+1} T^{k+n+1} x, x\rangle - (n+1)\ \langle T^{*k+1} T^{k+1} x, x\rangle \ + m^{\frac{2}{n+1}}\ n\ \langle T^{*k} T^k x, x\rangle \ \geq 0.$$

$$\Leftrightarrow m^{\frac{2}{n+1}}\ \|T^{k+n+1}x\|^2 - (n+1)\|T^{k+1}x\|^2 + m^{\frac{2}{n+1}}\ n\ \|T^k x\|^2 \geq 0.$$

$$\Leftrightarrow \frac{m^{\frac{2}{n+1}}}{n+1}\left(\|T^{k+n+1}x\|^2 + n\|T^k x\|^2\right) \geq \|T^{k+1}x\|^2.$$

□

Theorem 4.1.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\lambda^{\frac{-m}{n+1}} T$ is k-quasi (m,n)-class $\mathcal{Q}$ for all $\lambda > 0$ if and only if T is k-quasi (m,n)-paranormal.*

Proof. Let $\lambda > 0$. $\lambda^{\frac{-m}{n+1}} T$ is a k-quasi (m,n)-class $\mathcal{Q}$ operator

$$\Leftrightarrow (\lambda^{\frac{-m}{n+1}} T)^{*k} \left[m^{\frac{2}{n+1}} (\lambda^{\frac{-m}{n+1}} T)^{*n+1} (\lambda^{\frac{-m}{n+1}} T)^{n+1} - (n+1) \lambda^{\frac{-2m}{n+1}} T^*T + m^{\frac{2}{n+1}} n\ I \right] (\lambda^{\frac{-m}{n+1}} T)^k \geq 0.$$

$$\Leftrightarrow (\lambda^{\frac{-2mk}{n+1}})T^{*k}\left[m^{\frac{2}{n+1}}\lambda^{-2m}T^{*n+1}T^{n+1}-(n+1)\lambda^{\frac{-2m}{n+1}}T^{*}T+m^{\frac{2}{n+1}}nI\right]T^{k}\geq 0.$$

$$\Leftrightarrow (\lambda^{\frac{-2mk}{n+1}})\lambda^{-2m}T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1}-(n+1)\lambda^{\frac{2mn}{n+1}}T^{*}T+m^{\frac{2}{n+1}}n\,\lambda^{2m}I\right]T^{k}\geq 0.$$

$$\Leftrightarrow T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1}-(n+1)(\lambda^{\frac{2m}{n+1}})^{n}T^{*}T+m^{\frac{2}{n+1}}\,n\,(\lambda^{\frac{2m}{n+1}})^{n+1}I\right]T^{k}\geq 0.$$

$$\Leftrightarrow T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1}-(n+1)a^{n}T^{*}T+m^{\frac{2}{n+1}}\,n\,a^{n+1}I\right]T^{k}\geq 0, a>0.$$

$\Leftrightarrow T$ is a k-quasi (m,n)-paranormal operator.

□

From the above theorem it is clear that every k-quasi (m,n)-paranormal operator is k-quasi (m,n)-class $\mathcal{Q}$. In general, the converse need not be true. For proving this, we use the following theorem.

Theorem 4.1.3. *Let $T\in\mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_n\}, (n=1,2,....)$, defined by $Te_n=\alpha_n\ e_{n+1}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is k-quasi $(1,1)$-class $\mathcal{Q}$ if and only if*

$$|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2-2\ |\alpha_{n+k}|^2+1\geq 0,\ \forall n\in\mathbb{N}. \tag{4.1}$$

Proof. From the proof of Theorem 2.2.2, we get

$$T^{*k}(T^{*2}T^2)T^ke_n=|\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+k-1}|^2|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2e_n,\ \text{for any } k\in\mathbb{N}.$$

$$T^{*k}(T^{*}T)T^ke_n=|\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+k-1}|^2|\alpha_{n+k}|^2e_n,\ \text{for any } k\in\mathbb{N}.$$

and

$$T^{*k}T^ke_n=|\alpha_n|^2|\alpha_{n+1}|^2\cdots|\alpha_{n+k-1}|^2e_n,\ \text{for any } k\in\mathbb{N}.$$

Now, T is k-quasi $(1,1)$-class $\mathcal{Q}$

$$\Leftrightarrow T^{*k}(T^{*2}T^2-2T^*T+I)T^k\geq 0.$$

$$\Leftrightarrow\langle(T^{*k}(T^{*2}T^2-2T^*T+I)T^k)e_n,e_n\rangle\geq 0,\ \forall n\in\mathbb{N}.$$

$$\Leftrightarrow |\alpha_n|^2|\alpha_{n+1}|^2 \cdots |\alpha_{n+k-1}|^2(|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ |\alpha_{n+k}|^2 + 1) \geq 0,\ \forall n \in \mathbb{N}.$$
$$\Leftrightarrow |\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ |\alpha_{n+k}|^2 + 1 \geq 0,\ \forall n \in \mathbb{N}.$$

□

Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{2}x_2, \frac{1}{4}x_3, \frac{1}{5}x_4, \frac{1}{4}x_5, \frac{1}{4}x_6, \cdots).$$

Here $\alpha_1 = \frac{1}{2},\ \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{1}{4},\ \alpha_4 = \frac{1}{5}$ and $\alpha_n = \frac{1}{4}$ for $n \geq 5$.
From Theorem 4.1.3, it can be seen that T is a 2-quasi $(1,1)$-class $\mathcal{Q}$ operator.
Also, from Theorem 2.2.2, T is k-quasi $(1,1)$- paranormal if and only if

$$|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ a\ |\alpha_{n+k}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall n \in \mathbb{N}. \quad (4.2)$$

If $k = 2, n = 1$ and $a = \frac{1}{16}$, we get $|\alpha_3|^2|\alpha_4|^2 - 2\ a\ |\alpha_3|^2 + a^2 < 0$. Hence, from (4.2), T is not 2-quasi $(1,1)$- paranormal.
That is, every k-quasi (m,n)-class $\mathcal{Q}$ operators need not be k-quasi (m,n)- paranormal.
Now, we give some characterization of k-quasi (m,n)-class $\mathcal{Q}$ operators.

Theorem 4.1.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m,n)-class $\mathcal{Q}$ operator and $A \in \mathcal{B}(\mathcal{H})$ be an isometric operator such that $AT = TA$. Then TA is a k-quasi (m,n)-class $\mathcal{Q}$ operator.*

Proof. Let $S = TA$. Since $AT = TA$, and $A^*A = I$, we have

$$m^{\frac{2}{n+1}}S^{*k+n+1}S^{k+n+1} - (n+1)S^{*k+1}S^{k+1} + m^{\frac{2}{n+1}}\ n\ S^{*k}S^k$$
$$= m^{\frac{2}{n+2}}(A^*T^*)^{k+n+1}(TA)^{k+n+1} - (n+1)(A^*T^*)^{k+1}(TA)^{k+1} + m^{\frac{2}{n+1}}n(A^*T^*)^k(TA)^k$$
$$= m^{\frac{2}{n+1}}T^{*k+n+1}T^{k+n+1} - (n+1)T^{*k+1}T^{k+1} + m^{\frac{2}{n+1}}\ n\ T^{*k}T^k$$

Since T is a k-quasi (m,n)-class $\mathcal{Q}$ operator, we get

$$m^{\frac{2}{n+1}}S^{*k+n+1}S^{k+n+1} - (n+1)S^{*k+1}S^{k+1} + m^{\frac{2}{n+1}}\ n\ S^{*k}S^k \geq 0.$$

Hence $S = TA$ is a k-quasi (m, n)- class $\mathcal{Q}$ operator. □

Theorem 4.1.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m, n)-class $\mathcal{Q}$ operator and T be unitarily equivalent to an operator $B \in \mathcal{B}(\mathcal{H})$. Then B is a k-quasi (m,n)-class $\mathcal{Q}$ operator.*

Proof. Since T is unitarily equivalent to B, there exist an unitary operator U such that $B = U^*TU$.

Now, for any non-negative integer k we have,

$$B^{*k}\left(m^{\frac{2}{n+1}}B^{*n+1}B^{n+1} - (n+1)B^*B + m^{\frac{2}{n+1}}\ n\ I\right)B^k$$

$$=U^*T^{*k}U\left[U^*\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ nI\right)U\right]U^*T^kU$$
$$=U^*T^{*k}\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^kU$$

Since T is a k-quasi (m, n)-class $\mathcal{Q}$ operator, we get

$$U^*\left(T^{*k}\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^k\right)U \geq 0.$$

Thus, $B^{*k}\left(m^{\frac{2}{n+1}}B^{*n+1}B^{n+1} - (n+1)B^*B + m^{\frac{2}{n+1}}\ n\ I\right)B^k \geq 0$. Hence B is a k-quasi (m, n)-class $\mathcal{Q}$ operator. □

Theorem 4.1.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m,n)-class $\mathcal{Q}$ operator. If $\overline{R(T^k)} = \mathcal{H}$, then T is a (m, n)-class $\mathcal{Q}$ operator.*

Proof. Let $y \in \mathcal{H}$. Since $\overline{R(T^k)} = \mathcal{H}$, there exist a sequence (x_i) in $\mathcal{H}$ such that $\left(T^k(x_i)\right)$ converges to y in $\mathcal{H}$. Since T is a k-quasi (m, n)-class $\mathcal{Q}$ operator, we have

$$\left\langle\left(T^{*k}\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^k\right)x_i, x_i\right\rangle \geq 0, \forall i.$$

Hence, $\left\langle\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I\right)T^kx_i, T^kx_i\right\rangle \geq 0, \forall i.$

Since $\left(T^k(x_i)\right) \to y$, we get

$$\left\langle(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)T^*T + m^{\frac{2}{n+1}}\ n\ I)y, y\right\rangle \geq 0.$$

Hence T is a (m, n)-class $\mathcal{Q}$ operator. □

Theorem 4.1.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m, n)-class $\mathcal{Q}$ operator and $\overline{R(T^k)} \neq \mathcal{H}$. If*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } \overline{R(T^k)} \oplus N(T^{*k}),$$

then A is a (m, n)-class $\mathcal{Q}$ operator on $\overline{R(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Since T is a k-quasi (m, n)-class $\mathcal{Q}$ operator, we have

$$m^{\frac{2}{n+1}} \left(\|T^{k+n+1}y\|^2 + n\|T^k y\|^2 \right) \geq (n+1)\|T^{k+1}y\|^2, \ \forall y \in \mathcal{H}.$$

Let $z = T^k y$, then we get

$$m^{\frac{2}{n+1}} \left(\|T^{n+1}z\|^2 + n\|z\|^2 \right) \geq (n+1)\|Tz\|^2$$

Since $A = T|_{\overline{R(T^k)}}$, we get

$$m^{\frac{2}{n+1}} \left(\|A^{n+1}z\|^2 + n\|z\|^2 \right) \geq (n+1)\|Az\|^2, \forall z \in \overline{R(T^k)}.$$

Hence A is a (m, n)-class $\mathcal{Q}$ operator on $\overline{R(T^k)}$.
Let $x \in N(T^{*k})$. Then

$$T^k(x) = \begin{pmatrix} A^k & \sum_{i=0}^{k} A^i B C^{k-1-i} \\ 0 & C^k \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Hence, $C^k x = T^k x - \sum_{i=0}^{k-1} A^i B C^{k-1-i} x$. Since $A = T|_{\overline{R(T^k)}}$, we get $C^k x \in \overline{R(T^k)}$. We have $C = T|_{N(T^{*k})}$. Hence, $C^k x \in N(T^{*k})$. Thus $C^k x \in \overline{R(T^k)} \cap N(T^{*k})$. Hence $C^k = 0$. Therefore, $\sigma(C) = \{0\}$. Now $\sigma(A) \cap \sigma(C) = \sigma(A) \cap \{0\}$ and hence it has no interior point. Using Theorem 1.3.4, we get $\sigma(T) = \sigma(A) \cup \{0\}$. □

4.2 k-quasi (m, n)-class $\mathcal{Q}^*$ operators

In this section we define a new class of operator, k-quasi (m, n)-class $\mathcal{Q}^*$ operator, which contains the classes of k-quasi $(m, n)^*$-paranormal and (m, n)-class $\mathcal{Q}^*$. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is k-quasi $(m,n)^*$-paranormal if and only if

$$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1)\ a^n T^{*k} T T^* T^k + m^{\frac{2}{n+1}}\ n\ a^{n+1} T^{*k} T^k \geq 0, \forall a > 0,$$

(Theorem 2.1.1). $T \in \mathcal{B}(\mathcal{H})$ is said to be (m,n)-class $\mathcal{Q}^*$ operator if

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) T T^* + m^{\frac{2}{n+1}}\ n\ I \geq 0,$$

(Definition 3.2.1).

Definition 4.2.1. *Let $m \in \mathbb{R}^+$, $n \in \mathbb{N}$ and k be a non-negative integer. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if*

$$T^{*k} \left(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) T T^* + m^{\frac{2}{n+1}}\ n\ I \right) T^k \geq 0.$$

For example, let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T^2 = 0$. Hence T is k-quasi (m,n)-class $\mathcal{Q}^*$ operator for $k \geq 2$ and for any m, n.

Note that if $k = 1$, then T is said to be a quasi (m,n)-class $\mathcal{Q}^*$ operator. Also, every (m,n)-class $\mathcal{Q}^*$ operators are 0-quasi (m,n)-class $\mathcal{Q}^*$ operators. It can be proved that every k-quasi (m,n)-class $\mathcal{Q}^*$ operators need not be (m,n)-class $\mathcal{Q}^*$ operators. For proving this, we use the following theorem.

Theorem 4.2.1. *Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_n\}, (n = 1, 2, ...)$, defined by $T e_n = \alpha_n\ e_{n+1}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. Then T is k-quasi $(1,1)$-class $\mathcal{Q}^*$ if and only if*

$$|\alpha_{n+k}|^2 |\alpha_{n+k+1}|^2 - 2\ |\alpha_{n+k-1}|^2 + 1 \geq 0,\ \forall n \in \mathbb{N}. \tag{4.3}$$

Proof. From the proof of Theorem 2.2.3, we get

$$T^{*k} (T^{*2} T^2) T^k e_n = |\alpha_n|^2 |\alpha_{n+1}|^2 \cdots |\alpha_{n+k-1}|^2 |\alpha_{n+k}|^2 |\alpha_{n+k+1}|^2 e_n,\ \text{for any } k \in \mathbb{N}.$$

$$T^{*k} (T T^*) T^k e_n = |\alpha_n|^2 |\alpha_{n+1}|^2 \cdots |\alpha_{n+k-1}|^2 |\alpha_{n+k-1}|^2 e_n,\ \text{for any } k \in \mathbb{N}.$$

and

$$T^{*k}T^k e_n = |\alpha_n|^2|\alpha_{n+1}|^2 \cdots |\alpha_{n+k-1}|^2 e_n, \text{ for any } k \in \mathbb{N}.$$

Thus, T is k-quasi $(1, 1)$-class $\mathcal{Q}^*$

$$\begin{aligned}
&\Leftrightarrow T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k \geq 0.\\
&\Leftrightarrow \langle (T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k)e_n, e_n\rangle \geq 0, \ \forall n \in \mathbb{N}.\\
&\Leftrightarrow |\alpha_n|^2|\alpha_{n+1}|^2 \cdots |\alpha_{n+k-1}|^2(|\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ |\alpha_{n+k-1}|^2 + 1) \geq 0, \ \forall n \in \mathbb{N}.\\
&\Leftrightarrow |\alpha_{n+k}|^2|\alpha_{n+k+1}|^2 - 2\ |\alpha_{n+k-1}|^2 + 1 \geq 0, \ \forall n \in \mathbb{N}.
\end{aligned}$$

□

Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, 2x_1, x_2, 2x_3, x_4, 6x_5, 6x_6, \cdots)$ Here $\alpha_1 = 2\ \alpha_2 = 1, \alpha_3 = 2,\ \alpha_4 = 1$ and $\alpha_n = 6$ for $n \geq 5$. From Theorem 4.2.1, we get T is a 2-quasi $(1, 1)$-class $\mathcal{Q}^*$ operator.
From Theorem 3.2.2, T is a (m, n)- class $\mathcal{Q}^*$ operator if and only if

$$\frac{n+1}{m^{\frac{2}{n+1}}}\left(|\alpha_l|^2\right) \leq |\alpha_{l+1}|^2\ |\alpha_{l+2}|^2 \cdots |\alpha_{l+n+1}|^2 + n, \forall\ l \in \mathbb{N}. \tag{4.4}$$

If $m = n = l = 1$, (4.4) is not satisfied. Hence, T is not $(1, 1)$- class $\mathcal{Q}^*$.
That is, every k-quasi (m, n)-class $\mathcal{Q}^*$ operators need not be (m, n)-class $\mathcal{Q}^*$.
Now we show that the classes of operators k-quasi (m, n)-class $\mathcal{Q}$ and k-quasi (m, n)-class $\mathcal{Q}^*$ are independent.

For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 2x_1, x_2, 2x_3, x_4, 6x_5, 6x_6, \cdots).$$

From Theorem 4.2.1, we get T is 2-quasi $(1, 1)$-class $\mathcal{Q}^*$.
Also from (4.1), T is 2-quasi $(1, 1)$-class $\mathcal{Q}$ if and only if

$$|\alpha_{n+2}|^2|\alpha_{n+3}|^2 - 2\ |\alpha_{n+2}|^2 + 1 \geq 0, \ \forall n \in \mathbb{N}. \tag{4.5}$$

If $n = 1$, the above relation does not holds. Hence T is not 2-quasi $(1, 1)$-class $\mathcal{Q}$.

Next we consider the operator T defined by the matrix

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

In section 4.1, we proved that T is k-quasi $(1, 1)$-class $\mathcal{Q}$ for any $k \geq 1$.

But, for $m = n = 1$,

$m^{\frac{2}{n+1}} T^{*k} T^{*n+1} T^{n+1} T^k - (n+1) T^{*k} T T^* T^k + m^{\frac{2}{n+1}} n \; T^{*k} T^k$

$$\begin{aligned} &= T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix} < 0, \text{ for any } k. \end{aligned}$$

Hence T is not k-quasi $(1, 1)$-class $\mathcal{Q}^*$ for any k.

Now we give some characterizations for k-quasi (m, n)-class $\mathcal{Q}^*$ operators.

Theorem 4.2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator if and only if* $\dfrac{m^{\frac{2}{n+1}}}{n+1} \left(\|T^{k+n+1}x\|^2 + n\|T^k x\|^2 \right) \geq \|T^* T^k x\|^2, \ \forall x \in \mathcal{H}.$

Proof. Let $x \in \mathcal{H}$. T is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator

$$\begin{aligned} &\Leftrightarrow \left\langle \left(T^{*k} \left(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) T T^* + m^{\frac{2}{n+1}} \; n \; I \right) T^k \right) x, x \right\rangle \geq 0. \\ &\Leftrightarrow m^{\frac{2}{n+1}} \left\langle T^{*k+n+1} T^{k+n+1} x, x \right\rangle - (n+1) \left\langle T^{*k} T T^* T^k x, x \right\rangle + m^{\frac{2}{n+1}} \; n \left\langle T^{*k} T^k x, x \right\rangle \geq 0. \\ &\Leftrightarrow m^{\frac{2}{n+1}} \|T^{k+n+1} x\|^2 - (n+1) \|T^* T^k x\|^2 + m^{\frac{2}{n+1}} \; n \; \|T^k x\|^2 \geq 0. \\ &\Leftrightarrow \frac{m^{\frac{2}{n+1}}}{n+1} \left(\|T^{k+n+1} x\|^2 + n \|T^k x\|^2 \right) \geq \|T^* T^k x\|^2. \end{aligned}$$

□

Theorem 4.2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\lambda^{\frac{-m}{n+1}} T$ is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator for all $\lambda > 0$ if and only if T is a k-quasi $(m, n)^*$-paranormal operator.*

Proof. Let $\lambda > 0$. $\lambda^{\frac{-m}{n+1}}T$ is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator

$$\Leftrightarrow (\lambda^{\frac{-m}{n+1}}T)^{*k}\left[m^{\frac{2}{n+1}}(\lambda^{\frac{-m}{n+1}}T)^{*n+1}(\lambda^{\frac{-m}{n+1}}T)^{n+1} - (n+1)\lambda^{\frac{-2m}{n+1}}TT^* + m^{\frac{2}{n+1}}n\ I\right](\lambda^{\frac{-m}{n+1}}T)^k \geq 0.$$
$$\Leftrightarrow (\lambda^{\frac{-2mk}{n+1}})T^{*k}\left[m^{\frac{2}{n+1}}\lambda^{-2m}T^{*n+1}T^{n+1} - (n+1)\lambda^{\frac{-2m}{n+1}}TT^* + m^{\frac{2}{n+1}}nI\right]T^k \geq 0.$$
$$\Leftrightarrow (\lambda^{\frac{-2mk}{n+1}})\lambda^{-2m}T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)\lambda^{\frac{2mn}{n+1}}TT^* + m^{\frac{2}{n+1}}n\lambda^{2m}I\right]T^k \geq 0.$$
$$\Leftrightarrow T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)(\lambda^{\frac{2m}{n+1}})^nTT^* + m^{\frac{2}{n+1}}n(\lambda^{\frac{2m}{n+1}})^{n+1}I\right]T^k \geq 0.$$
$$\Leftrightarrow T^{*k}\left[m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)a^nTT^* + m^{\frac{2}{n+1}}na^{n+1}I\right]T^k \geq 0, a > 0.$$
$\Leftrightarrow T$ is k-quasi $(m,n)^*$-paranormal .

□

From the above theorem, it is clear that every k-quasi $(m,n)^*$-paranormal operator is k-quasi (m,n)-class $\mathcal{Q}^*$ operator. But the converse need not be true.
For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, ...) = (0, \frac{1}{2}x_1, \frac{1}{4}x_2, \frac{1}{4}x_3, ...).$$

From Theorem 4.2.1, T is k-quasi $(1,1)$-class $\mathcal{Q}^*$ if and only if

$$|\alpha_{l+k}|^2|\alpha_{l+k+1}|^2 - 2\ |\alpha_{l+k-1}|^2 + 1 \geq 0,\ \forall\, l \in \mathbb{N}. \tag{4.6}$$

If $k = 1$, (4.6) holds. Hence T is 1-quasi $(1,1)$-class $\mathcal{Q}^*$.
From Theorem 2.2.3, we have T is k-quasi $(1,1)^*$-paranormal if and only if

$$|\alpha_{l+k}|^2|\alpha_{l+k+1}|^2 - 2\ a\ |\alpha_{l+k-1}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall\, l \in \mathbb{N}. \tag{4.7}$$

If $k = 1,\ l = 1$ and $a = \frac{1}{4}$, (4.7) is not satisfied. Hence T is not 1-quasi $(1,1)^*$-paranormal.
That is, every k-quasi (m,n)-class $\mathcal{Q}^*$ operators need not be k-quasi $(m,n)^*$-paranormal operators.

Theorem 4.2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi (m,n)-class $\mathcal{Q}^*$ operator and $A \in \mathcal{B}(\mathcal{H})$ be an isometric operator such that $AT = TA$. Then TA is quasi (m,n)-class $\mathcal{Q}^*$.*

Proof. Let $S = TA$. Since $AT = TA$ and $A^*A = I$, we have
$m^{\frac{2}{n+1}} S^{*n+2} S^{n+2} - (n+1) S^*SS^*S + m^{\frac{2}{n+1}}\ n\ S^*S$

$$= m^{\frac{2}{n+2}} (A^*T^*)^{n+2} (TA)^{n+2} - (n+1) A^*T^*TAA^*T^*TA + m^{\frac{2}{n+1}}\ n\ A^*T^*TA$$
$$= m^{\frac{2}{n+1}} T^{*n+2} T^{n+2} - (n+1) T^*TT^*T + m^{\frac{2}{n+1}}\ n\ T^*T$$

Since T is a quasi (m,n)-class $\mathcal{Q}^*$ operator, we get

$$m^{\frac{2}{n+1}} S^{*n+2} S^{n+2} - (n+1) S^*SS^*S + m^{\frac{2}{n+1}}\ n\ S^*S \geq 0.$$

Hence $S = TA$ is a quasi (m,n)-class $\mathcal{Q}^*$ operator. □

Theorem 4.2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m,n)-class $\mathcal{Q}^*$ operator and is unitarily equivalent to an operator $B \in \mathcal{B}(\mathcal{H})$. Then B is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator.*

Proof. Since T is unitarily equivalent to B, there exist an unitary operator U such that $B = U^*TU$. Now,
$B^{*k} \left(m^{\frac{2}{n+1}} B^{*n+1} B^{n+1} - (n+1) BB^* + m^{\frac{2}{n+1}}\ n\ I \right) B^k$

$$= U^*T^{*k}U \left[U^* \left(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) TT^* + m^{\frac{2}{n+1}}\ nI \right) U \right] U^*T^kU$$
$$= U^*T^{*k} \left(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) TT^* + m^{\frac{2}{n+1}}\ n\ I \right) T^kU$$

Since T is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator, we get

$$U^* \left(T^{*k} \left(m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1) TT^* + m^{\frac{2}{n+1}}\ n\ I \right) T^k \right) U \geq 0.$$

Thus $B^{*k} \left(m^{\frac{2}{n+1}} B^{*n+1} B^{n+1} - (n+1) BB^* + m^{\frac{2}{n+1}}\ n\ I \right) B^k \geq 0$. Hence B is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator. □

Theorem 4.2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m,n)-class $\mathcal{Q}^*$ operator. If $\overline{R(T^k)} = \mathcal{H}$, then T is a (m,n)-class $\mathcal{Q}^*$ operator.*

Proof. Let $y \in \mathcal{H}$. Since $\overline{R(T^k)} = \mathcal{H}$, there exist a sequence (x_i) in $\mathcal{H}$ such that $\big(T^k(x_i)\big)$ converges to y in $\mathcal{H}$. Since T is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator,

$$\left\langle \left[T^{*k}\left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I\right)T^k\right]x_i, x_i\right\rangle \geq 0, \forall i.$$

Then, $\left\langle \left(m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I\right)T^k x_i, T^k x_i\right\rangle \geq 0, \forall i.$
Since $\big(T^k(x_i)\big)$ converges to y, we get

$$\left\langle (m^{\frac{2}{n+1}}T^{*n+1}T^{n+1} - (n+1)TT^* + m^{\frac{2}{n+1}}\ n\ I)y, y\right\rangle \geq 0.$$

Hence T is a (m,n)-class $\mathcal{Q}^*$ operator. □

Theorem 4.2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi (m,n)-class $\mathcal{Q}^*$ operator and $\overline{R(T^k)} \neq \mathcal{H}$. If*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } \overline{R(T^k)} \oplus N(T^{*k}),$$

then A is a (m,n)-class $\mathcal{Q}^*$ operator on $\overline{R(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Since T is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator, we have

$$m^{\frac{2}{n+1}}\left(\|T^{k+n+1}y\|^2 + n\|T^k y\|^2\right) \geq (n+1)\|T^*T^k y\|^2,\ \forall y \in \mathcal{H}.$$

Let $z = T^k y$. Then we get

$$m^{\frac{2}{n+1}}\left(\|T^{n+1}z\|^2 + n\|z\|^2\right) \geq (n+1)\|T^*z\|^2.$$

Hence, $m^{\frac{2}{n+1}}(\|A^{n+1}z\|^2 + n\|z\|^2) \geq (n+1)\|A^*z\|^2$, $\forall z \in \overline{R(T^k)}$. Thus A is a (m,n)-class $\mathcal{Q}^*$ operator on $\overline{R(T^k)}$.
Let $x \in N(T^{*k})$. Then

$$T^k(x) = \begin{pmatrix} A^k & \sum_{i=0}^{k} A^i B C^{k-1-i} \\ 0 & C^k \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}$$

Hence, $C^k x = T^k x - \sum_{i=0}^{k-1} A^i B C^{k-1-i}x$. Since $A = T|_{\overline{R(T^k)}}$, we have $C^k x \in \overline{R(T^k)}$.

Also, $C^k x \in N(T^{*k})$. Thus $C^k x \in \overline{R(T^k)} \cap N(T^{*k})$. Hence $C^k = 0$. Therefore $\sigma(C) = \{0\}$. From Theorem 1.3.4, we get $\sigma(T) = \sigma(A) \cup \{0\}$. □

Remark 4.2.1.

(i) From the earlier sections, it can be seen that

(m,n)-paranormal $\subset$ k-quasi (m,n)-paranormal $\subset$ k-quasi (m,n)-class $\mathcal{Q}$ and

$(m,n)^*$-paranormal $\subset$ k-quasi $(m,n)^*$-paranormal $\subset$ k-quasi (m,n)-class $\mathcal{Q}^*$

Also, (m,n)-class $\mathcal{Q} \subset$ k-quasi (m,n)-class $\mathcal{Q}$ and

(m,n)-class $\mathcal{Q}^* \subset$ k-quasi (m,n)-class $\mathcal{Q}^*$

(ii) The classes of k-quasi (m,n)-paranormal and (m,n)-class $\mathcal{Q}$ are independent. For example, consider the operator T defined by the matrix $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

In section 2.2, we proved that T is k-quasi $(1,1)$-paranormal for any $k \geq 1$. We know that T is $(1,1)$-class $\mathcal{Q}$ if and only if $T^{*2}T^2 - 2T^*T + I \geq 0$. Now,

$$\begin{aligned} T^{*2}T^2 - 2T^*T + I &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} - 2 \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} < 0. \end{aligned}$$

Hence T is not $(1,1)$-class $\mathcal{Q}$.

Next let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{2}x_2, \frac{1}{4}x_3, \frac{1}{5}x_4, \frac{1}{4}x_5, \frac{1}{4}x_6, \cdots).$$

Here $\alpha_1 = \frac{1}{2}$ $\alpha_2 = \frac{1}{2}, \alpha_3 = \frac{1}{4},\ \alpha_4 = \frac{1}{5},\ \alpha_n = \frac{1}{4}$ for $n \geq 5$.

From Theorem 3.1.5, we get T is a $(1,1)$-class $\mathcal{Q}$ operator. From Theorem 2.2.2, T is k-quasi $(1,1)$- paranormal if and only if

$$|\alpha_{l+k}|^2 |\alpha_{l+k+1}|^2 - 2\ a\ |\alpha_{l+k}|^2 + a^2 \geq 0,\ \forall a > 0,\ \forall\, l \in \mathbb{N}. \tag{4.8}$$

If $a = \frac{1}{16}$, $l = 1$, then (4.8) is not satisfied for $k = 2$. Hence T is not 2-quasi $(1, 1)$- paranormal.

(iii) The classes of operators, k-quasi $(m, n)^*$-paranormal and (m, n)-class $\mathcal{Q}^*$ are independent.

Consider $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 2x_1, x_2, 2x_3, x_4, 6x_5, 6x_6, \cdots).$$

In section 2.2, we proved that T is 2-quasi $(1, 1)^*$-paranormal.

From Theorem 3.2.2, T is a (m, n)-class $\mathcal{Q}^*$ operator if and only if

$$\frac{n+1}{m^{\frac{2}{n+1}}} \left(|\alpha_k|^2\right) \leq |\alpha_{k+1}|^2 \ |\alpha_{k+2}|^2 \cdots |\alpha_{k+n+1}|^2 + n, \forall \ k \in \mathbb{N}. \tag{4.9}$$

If $m = n = k = 1$, (4.9) is not satisfied. Hence, T is not $(1, 1)$-class $\mathcal{Q}^*$.

Let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 2x_1, 4x_2, 3x_3, 4x_4, 6x_5, 6x_6, \cdots).$$

Here $\alpha_1 = 2 \ \alpha_2 = 4, \alpha_3 = 3, \ \alpha_4 = 4, \ \alpha_n = 6$ for $n \geq 5$. Using Theorem 3.2.2, T is a $(1, 1)$-class $\mathcal{Q}^*$ operator.

From Theorem 2.2.3, T is 1-quasi $(1, 1)^*$-paranormal if and only if

$$|\alpha_{l+1}|^2 |\alpha_{l+2}|^2 - 2 \ a \ |\alpha_l|^2 + a^2 \geq 0, \ \forall a > 0, \ \forall l \in \mathbb{N}. \tag{4.10}$$

If $a = 16$, (4.10) is not satisfied for $l = 2$. Hence T is not 1-quasi $(1, 1)^*$-paranormal.

4.3 k-quasi (m, n)-class $\mathcal{Q}$ and k-quasi (m, n)-class $\mathcal{Q}^*$ Composition operators

In this section, we give some characterizations for k-quasi (m, n)-class $\mathcal{Q}$ and k-quasi (m, n)-class $\mathcal{Q}^*$ composition operators on L^2-spaces. For a non singular measurable function T on X, let C_T denotes the composition operator of T on $L^2(\mu)$ and C_T^* be the adjoint of C_T. Let h_n denotes the Radon-Nikodym derivative of $\mu(T^{-1})^n$ with respect to μ. We denote h_1 by h.

Theorem 4.3.1. *C_T is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(h_{k+n+1} + n\ h_k) \geq (n+1)h_{k+1} \tag{4.11}$$

Proof. C_T is a k-quasi (m,n)-class $\mathcal{Q}$ if and only if

$$\left\langle \left(m^{\frac{2}{n+1}} C_T^{*k+n+1} C_T^{k+n+1} - (n+1) C_T^{*k+1} C_T^{k+1} + m^{\frac{2}{n+1}}\ n\ C_T^{*k} C_T^{k} \right) f, f \right\rangle \geq 0, \forall f \in L^2(\mu).$$

For any $f \in L^2(\mu)$, using Theorem 1.4.3 we get,

$$\begin{aligned} C_T^{*k+n+1} C_T^{k+n+1} f &= C_T^{*k+n+1}(f \circ T^{k+n+1}) \\ &= h_{k+n+1} E(f \circ T^{k+n+1}) \circ T^{-(k+n+1)} \\ &= h_{k+n+1} E(1.f \circ T^{k+n+1}) \circ T^{-(k+n+1)} \\ &= h_{k+n+1} E(1)(f \circ T^{k+n+1} \circ T^{-(k+n+1)}) \\ &= h_{k+n+1} f. \\ C_T^{*k+1} C_T^{k+1} f &= h_{k+1} f. \\ C_T^{*k} C_T^{k} f &= h_k f. \end{aligned}$$

Hence, C_T is k-quasi (m,n)-class $\mathcal{Q}$

$$\Leftrightarrow \left\langle m^{\frac{2}{n+1}} h_{k+n+1} f - (n+1) h_{k+1} f + m^{\frac{2}{n+1}}\ n\ h_k f, f \right\rangle \geq 0, \forall f \in L^2(\mu).$$
$$\Leftrightarrow m^{\frac{2}{n+1}}(h_{k+n+1} + n\ h_k) \geq (n+1)h_{k+1}.$$

□

Example 4.3.1. *Let $X = \mathbb{N} \cup \{0\}$, $\mathcal{A} = P(X)$ and μ be the measure defined by*

$$\mu(A) = \sum_{k \in A} m_k,$$

where

$$m_k = \begin{cases} 1 & \text{if } k = 0. \\ \dfrac{1}{4^{k-1}} & \text{if } k \geq 1. \end{cases}$$

Let $T : X \to X$ *be defined by*

$$T(k) = \begin{cases} 0 & \text{if } k = 0, 1. \\ k-1 & \text{if } k \geq 2. \end{cases}$$

Then for $q > 1$, *we have*

$$T^q(k) = \begin{cases} 0 & \text{if } k = 0, 1, 2, \ldots, q. \\ k-q & \text{if } k \geq q+1. \end{cases}$$

Hence, $\quad h(k) = \frac{\mu T^{-1}(\{k\})}{\mu\{k\}} = \begin{cases} 2 & \text{if } k = 0. \\ \frac{1}{4} & \text{if } k \geq 1. \end{cases}$

For $q > 1$, $h_q(k) = \frac{\mu T^{-q}(\{k\})}{\mu\{k\}} = \begin{cases} 2 + \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^{q-1}} & \text{if } k = 0. \\ \frac{1}{4^q} & \text{if } k \geq 1. \end{cases}$

If $m \geq 2$ *and* $n = 3$, *(4.11) is satisfied for* $k = 2$. *Hence* C_T *is a 2-quasi* $(m, 3)$*-class* $\mathcal{Q}$ *operator for* $m \geq 2$.

Theorem 4.3.2. C_T^* *is a* k*-quasi* (m,n)*-class* $\mathcal{Q}$ *operator if and only if*

$$m^{\frac{2}{n+1}} \left(h_{k+n+1} \circ T^{k+n+1} + n \left(h_k \circ T^k\right)\right) \geq (n+1) h_{k+1} \circ T^{k+1}.$$

Proof. By definition, C_T^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}} C_T^{k+n+1} C_T^{*k+n+1} - (n+1) C_T^{k+1} C_T^{*k+1} + m^{\frac{2}{n+1}}\ n\ C_T^k C_T^{*k}) f, f \right\rangle \geq 0, \forall f \in L^2(\mu).$$

For $f \in L^2(\mu)$, using Theorem 1.4.3 we get,

$$\begin{aligned} C_T^{k+n+1} C_T^{*k+n+1} f &= C_T^{k+n+1} \left(h_{k+n+1} E(f) \circ T^{-(k+n+1)}\right) \\ &= \left(h_{k+n+1}.\ E(f) \circ T^{-(k+n+1)}\right) \circ T^{k+n+1} \end{aligned}$$

$$
\begin{aligned}
&= \left((h_{k+n+1}\circ T^{k+n+1}).(E(f)\circ T^{-(k+n+1)}\circ T^{k+n+1})\right)\\
&= h_{k+n+1}\circ T^{k+n+1}E(f)\\
&= h_{k+n+1}\circ T^{k+n+1}f\\
C_T^{k+1}C_T^{*k+1}f &= h_{k+1}\circ T^{k+1}f\\
C_T^{k}C_T^{*k}f &= h_k\circ T^k f.
\end{aligned}
$$

Hence, C_T^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator

$$\Leftrightarrow \left\langle m^{\frac{2}{n+1}}h_{k+n+1}\circ T^{k+n+1}f-(n+1)h_{k+1}\circ T^{k+1}f+m^{\frac{2}{n+1}}\ n\ (h_k\circ T^k)f,f\right\rangle\geq 0,$$

for every $f\in L^2(\mu)$.

$$\Leftrightarrow m^{\frac{2}{n+1}}\left(h_{k+n+1}\circ T^{k+n+1}+n\ (h_k\circ T^k)\right)\geq (n+1)h_{k+1}\circ T^{k+1}.$$

□

Example 4.3.2. *In example 4.3.1, if $m\geq 14$ and $n=3$ then C_T^* is 2-quasi $(m,3)$-class $\mathcal{Q}$ operator.*

Theorem 4.3.3. *Let P be a projection from $L^2(\mu)$ onto $\overline{R(C_T)}$. Then*

(i) C_T is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}}(h_{k+n+1}+n\ h_k)\geq (n+1)h_k.h\circ T^{1-k}.$$

(ii) C_T^* is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$m^{\frac{2}{n+1}}\left(h_{k+n+1}\circ T^{k+n+1}+n\ (h_k\circ T^k)\right)\geq (n+1)(h\circ T^k)(h_k\circ T^k).$$

Proof. (i) C_T is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if

$$\left\langle\left(m^{\frac{2}{n+1}}C_T^{*k+n+1}C_T^{k+n+1}-(n+1)C_T^{*k}C_TC_T^*C_T^k+m^{\frac{2}{n+1}}\ n\ C_T^{*k}C_T^k\right)f,f\right\rangle\geq 0,\forall f\in L^2(\mu).$$

From Theorem 1.4.3, for any $f \in L^2(\mu)$ we have

$$\begin{aligned} C_T^{*k+n+1}C_T^{k+n+1}f &= C_T^{*k+n+1}(f \circ T^{k+n+1}) \\ &= h_{k+n+1}E(f \circ T^{k+n+1}) \circ T^{-(k+n+1)} \\ &= h_{k+n+1}f. \\ C_T^{*k}C_T^{k}f &= h_k f. \end{aligned}$$

Also, from Theorem 1.4.3 and for any $f \in L^2(\mu)$, we have

$$\begin{aligned} C_T^{*k}C_TC_T^{*}C_T^{k}f &= C_T^{*k}C_TC_T^{*}(f \circ T^k) \\ &= C_T^{*k}(h \circ T)P(f \circ T^k) \\ &= h_k.E\left((h \circ T)P(f \circ T^k)\right) \circ T^{-k} \\ &= h_k.\left(E(h \circ T)P(f \circ T^k)\right) \circ T^{-k} \\ &= h_k.E(h \circ T) \circ T^{-k}f \\ &= h_k.h \circ T^{1-k}f. \end{aligned}$$

Hence C_T is k-quasi (m,n)-class $\mathcal{Q}^*$

$$\Leftrightarrow \left\langle m^{\frac{2}{n+1}}h_{k+n+1}f - (n+1)h_k.h \circ T^{1-k}f + m^{\frac{2}{n+1}}\ n\ h_kf, f\right\rangle \geq 0, \forall f \in L^2(\mu).$$

$$\Leftrightarrow m^{\frac{2}{n+1}}(h_{k+n+1} + n\ h_k) \geq (n+1)h_k.h \circ T^{1-k}.$$

(ii) C_T^* is a k-quasi (m,n)-class $\mathcal{Q}^*$

$$\Leftrightarrow m^{\frac{2}{n+1}}C_T^{k+n+1}C_T^{*k+n+1} - (n+1)C_T^kC_T^*C_TC_T^{*k} + m^{\frac{2}{n+1}}\ n\ C_T^kC_T^{*k} \geq 0.$$

$$\Leftrightarrow \left\langle \left(m^{\frac{2}{n+1}}C_T^{k+n+1}C_T^{*k+n+1} - (n+1)C_T^kC_T^*C_TC_T^{*k} + m^{\frac{2}{n+1}}\ n\ C_T^kC_T^{*k}\right)f, f\right\rangle \geq 0,$$

for every $f \in L^2(\mu)$.

From Theorem 1.4.3, for any $f \in L^2(\mu)$ we have

$$\begin{aligned} C_T^{k+n+1} C_T^{*k+n+1} f &= C_T^{k+n+1} \left(h_{k+n+1} E(f) \circ T^{-(k+n+1)}\right) \\ &= \left(h_{k+n+1}\ E(f) \circ T^{-(k+n+1)}\right) \circ T^{k+n+1} \\ &= h_{k+n+1} \circ T^{k+n+1} f. \\ C_T^k C_T^{*k} f &= h_k \circ T^k f. \end{aligned}$$

Also, for every $f \in L^2(\mu)$ and from Theorem 1.4.3, we have

$$\begin{aligned} C_T^k C_T^* C_T C_T^{*k} f &= C_T^k C_T^* C_T \left(h_k E(f) \circ T^k\right) \\ &= C_T^k\ h. \left(h_k E(f) \circ T^{-k}\right) \\ &= \left[h. \left(h_k E(f) \circ T^{-k}\right)\right] \circ T^k \\ &= (h \circ T^k)\ \left(h_k . E(f) \circ T^{-k}\right) \circ T^k \\ &= (h \circ T^k)(h_k \circ T^k) E(f) \\ &= (h \circ T^k)(h_k \circ T^k) f. \end{aligned}$$

Hence C_T^* is k-quasi (m,n)-class $\mathcal{Q}^*$

$$\Leftrightarrow \left\langle m^{\frac{2}{n+1}} h_{k+n+1} \circ T^{k+n+1} f - (n+1)(h \circ T^k)(h_k \circ T^k) f + m^{\frac{2}{n+1}}\ n\ (h_k \circ T^k) f, f \right\rangle \geq 0,$$

for every $f \in L^2(\mu)$.

$$\Leftrightarrow m^{\frac{2}{n+1}} \left(h_{k+n+1} \circ T^{k+n+1} + n\ (h_k \circ T^k)\right) \geq (n+1)(h \circ T^k)(h_k \circ T^k).$$

□

Example 4.3.3. *Consider the example 4.3.1. It can be seen that if* $m \geq 4$ *and* $n = 3$, C_T *is a* 1- *quasi* (m,n)*-class* $\mathcal{Q}^*$ *operator.*
Also if $m \geq 4$ *and* $n = 3$, C_T^* *is a* 2- *quasi* (m,n)*-class* $\mathcal{Q}^*$ *operator.*

Now we give some characterizations for k-quasi (m,n)-class $\mathcal{Q}$ weighted composition operators on $L^2(\mu)$.

Let $T : X \to X$ be a measurable transformation on X and π be complex valued measurable transformation on X. Let $\pi_k = \pi(\pi \circ T)(\pi \circ T^2) \cdots (\pi \circ T^{k-1})$ and W be the weighted composition operator induced by T and π. For $f \in L^2(\mu)$, we denote $h_k E(|\pi_k|^2) \circ T^{-k}(f)$ by $J_k f$ and $\pi_k(h_k \circ T^k)E(\bar{\pi}_k f)$ by $L_k f$.

Theorem 4.3.4. *W is a k-quasi (m, n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(J_{k+n+1} + nJ_k) \geq (n+1)J_{k+1}.$$

Proof. We have, W is a k-quasi (m, n)-class $\mathcal{Q}$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}} W^{*k+n+1}W^{k+n+1} - (n+1)W^{*k+1}W^{k+1} + m^{\frac{2}{n+1}}\ n\ W^{*k}W^k)f, f \right\rangle \geq 0,$$

$\forall f \in L^2(\mu)$. Let $f \in L^2(\mu)$. Using Theorem 1.5.1, we get

$$\begin{aligned} W^{*k+n+1}W^{k+n+1}f &= h_{k+n+1}\ E(|\pi_{k+n+1}|^2) \circ T^{-(k+n+1)}f \\ &= J_{k+n+1}f \\ W^{*k+1}W^{k+1}f &= h_{k+1}E(|\pi_{k+1}|^2) \circ T^{-(k+1)}f \\ &= J_{k+1}f \\ W^{*k}W^k f &= h_k\ E(|\pi_k|^2) \circ T^{-k}f \\ &= J_k f. \end{aligned}$$

Hence W is a k-quasi (m, n)-class $\mathcal{Q}$ operator

$$\Leftrightarrow \left\langle (m^{\frac{2}{n+1}} J_{k+n+1}f - (n+1)J_{k+1}f + m^{\frac{2}{n+1}}\ n\ J_k f, f \right\rangle \geq 0, \forall f \in L^2(\mu).$$
$$\Leftrightarrow m^{\frac{2}{n+1}}(J_{k+n+1} + nJ_k) \geq (n+1)J_{k+1}.$$

□

Theorem 4.3.5. *W^* is a k-quasi (m, n)-class $\mathcal{Q}$ operator if and only if*

$$m^{\frac{2}{n+1}}(L_{k+n+1} + nL_k) \geq (n+1)L_{k+1}.$$

Proof. We have W^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}} W^{k+n+1} W^{*k+n+1} - (n+1) W^{k+1} W^{*k+1} + m^{\frac{2}{n+1}}\ n\ W^k W^{*k}) f, f \right\rangle \geq 0,$$

for every $f \in L^2(\mu)$. From Theorem 1.5.1, for any $f \in L^2(\mu)$, we have

$$\begin{aligned} W^{k+n+1} W^{*k+n+1} f &= \pi_{k+n+1}(h_{k+n+1} \circ T^{k+n+1}) E(\bar{\pi}_{k+n+1} f) \\ &= L_{k+n+1} f \\ W^{k+1} W^{*k+1} f &= \pi_{k+1}(h_{k+1} \circ T^{k+1}) E(\bar{\pi}_{k+1} f) \\ &= L_{k+1} f \\ W^k W^{*k} f &= \pi_k (h_k \circ T^k) E(\bar{\pi}_k f) \\ &= L_k f \end{aligned}$$

Hence W^* is a k-quasi (m,n)-class $\mathcal{Q}$ operator

$$\begin{aligned} &\Leftrightarrow \left\langle m^{\frac{2}{n+1}} L_{k+n+1} f - (n+1) L_{k+1} f + m^{\frac{2}{n+1}}\ n\ L_k f, f \right\rangle \geq 0, \forall f \in L^2(\mu). \\ &\Leftrightarrow m^{\frac{2}{n+1}} (L_{k+n+1} + n L_k) \geq (n+1) L_{k+1}. \end{aligned}$$

□

Theorem 4.3.6. *Let W be the weighted composition operator induced by T on $L^2(\mu)$. Then*

(i) W is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if

$$\left\langle (m^{\frac{2}{n+1}} J_{k+n+1} f - (n+1) h_k |E(\bar{\pi}_k \pi)|^2 \circ T^{-k} . h \circ T^{1-k} f + m^{\frac{2}{n+1}}\ n\ J_k f, f \right\rangle \geq 0,$$

for every $f \in L^2(\mu)$.

(ii) W^* is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if

$$\left\langle (m^{\frac{2}{n+1}} L_{k+n+1} f - (n+1) \pi_k (J_1 \circ T^k)(h_k \circ T^k) E(\bar{\pi}_k f) + m^{\frac{2}{n+1}}\ n\ L_k f, f \right\rangle \geq 0,$$

for every $f \in L^2(\mu)$.

Proof. (i) W is a k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if

$$\left\langle (m^{\frac{2}{n+1}} W^{*k+n+1}W^{k+n+1} - (n+1)W^{*k}WW^*W^k + m^{\frac{2}{n+1}}\ n\ W^{*k}W^k)f, f \right\rangle \geq 0,$$

$\forall f \in L^2(\mu)$. From Theorem 1.5.1, for any $f \in L^2(\mu)$, we have

$$\begin{aligned}
W^{*k+n+1}W^{k+n+1}f &= h_{k+n+1}\ E(|\pi_{k+n+1}|^2) \circ T^{-(k+n+1)}f \\
&= J_{k+n+1}f. \\
W^{*k}W^k f &= h_k\ E(|\pi_k|^2) \circ T^{-k} f \\
&= J_k f. \\
W^{*k}WW^*W^k f &= W^{*k}WW^*(\pi_k.f \circ T^k) \\
&= W^{*k}\left(\pi(h \circ T)E(\bar{\pi}\pi_k.f \circ T^k)\right) \\
&= h_k E\left[\left(\bar{\pi_k}\pi(h \circ T)E(\bar{\pi}\pi_k.f \circ T^k)\right)\right] \circ T^{-k} \\
&= h_k\left[E\left(\bar{\pi_k}\pi(h \circ T)\right)E(\bar{\pi}\pi_k.f \circ T^k)\right] \circ T^{-k} \\
&= h_k\left[|E(\bar{\pi_k}\pi)|^2(h \circ T).(f \circ T^k)\right] \circ T^{-k} \\
&= h_k\ |E(\bar{\pi_k}\pi)|^2 \circ T^{-k}.h \circ T^{1-k} f
\end{aligned}$$

Hence W is a k-quasi (m,n)-class $\mathcal{Q}^*$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}} J_{k+n+1}f - (n+1)h_k\ |E(\bar{\pi_k}\pi)|^2 \circ T^{-k}.h \circ T^{1-k}f + m^{\frac{2}{n+1}}\ n\ J_k f, f \right\rangle \geq 0,$$

for every $f \in L^2(\mu)$.

(ii) W^* is k-quasi (m,n)-class $\mathcal{Q}^*$ if and only if

$$\left\langle (m^{\frac{2}{n+1}} W^{k+n+1}W^{*k+n+1} - (n+1)W^kW^*WW^{*k} + m^{\frac{2}{n+1}}\ n\ W^kW^{*k})f, f \right\rangle \geq 0,$$

for every $f \in L^2(\mu)$. Let $f \in L^2(\mu)$. From Theorem 1.5.1, we have

$$\begin{aligned}
W^{k+n+1}W^{*k+n+1}f &= \pi_{k+n+1}(h_{k+n+1} \circ T^{k+n+1})E(\bar{\pi}_{k+n+1}f) \\
&= L_{k+n+1}f.
\end{aligned}$$

$$\begin{aligned}
W^k W^{*k} f &= \pi_k (h_k \circ T^k) E(\bar{\pi}_k f) \\
&= L_k f. \\
W^k W^* W W^{*k} f &= W^k W^* W \left(h_k E(\bar{\pi}_k f) \circ T^{-k} \right) \\
&= W^k \left(J_1 h_k E(\bar{\pi}_k f) \circ T^{-k} \right) \\
&= \pi_k \left[\left(J_1 h_k E(\bar{\pi}_k f) \circ T^{-k} \right) \right] \circ T^k \\
&= \pi_k (J_1 \circ T^k)(h_k \circ T^k) E(\bar{\pi}_k f)
\end{aligned}$$

Hence W^* is a k-quasi (m, n)-class $\mathcal{Q}^*$ operator if and only if

$$\left\langle (m^{\frac{2}{n+1}} L_{k+n+1} f - (n+1) \pi_k (J_1 \circ T^k)(h_k \circ T^k) E(\bar{\pi}_k f) + m^{\frac{2}{n+1}} \; n \; L_k f, f \right\rangle \geq 0,$$

for every $f \in L^2(\mu)$.

□

Chapter 5

Totally $(m, n)^*$-paranormal operators

In this chapter, we introduce a new class of operators namely totally $(m, n)^*$-paranormal operators which is contained in the class of $(m, n)^*$- paranormal operators. Here we show that this class of operators has certain nice properties like translation invariant, finiteness, spectral continuity and range kernel orthogonality. Moreover, we define another class of operators, k-quasi totally $(m, n)^*$-paranormal, which contains the class of totally $(m, n)^*$-paranormal operators. Also we give a 2×2 matrix representation for k-quasi totally $(m, n)^*$-paranormal operators.

5.1 Totally $(m, n)^*$-paranormal operators

Let $T \in \mathcal{B}(\mathcal{H})$. T is said to be $(m, n)^*$- *paranormal* if $\|T^*x\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$ ([5]). Recall that, T is said to be $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|\|x\|$, for all $x \in \mathcal{H}$. In ([25]), E. Ko, H. Nam and Y. Yang defined the class of totally $*$-paranormal operators which is a sub class of $*$-paranormal operators. T is said to be totally $*$-paranormal if $\|(T-\lambda I)^*x\|^2 \leq \|(T-\lambda I)^2x\|\|x\|$, for all $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Now we define, totally $(m, n)^*$-paranormal operators.

Definition 5.1.1. *Let $m \in \mathbb{R}^+$, $n \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be totally $(m, n)^*$-paranormal if*

$$\|(T-\lambda I)^*x\|^{n+1} \leq m\|(T-\lambda I)^{n+1}x\|\|x\|^n, \textit{ for all } x \in \mathcal{H} \textit{ and } \lambda \in \mathbb{C}.$$

If $m = 1$, then totally $(m,n)^*$-paranormal operator is totally n^*-paranormal ([46]). If $\lambda = 0$, then totally $(m,n)^*$-paranormal operator is $(m,n)^*$-paranormal. If $\lambda = 0$, $m = n = 1$, then totally $(m,n)^*$-paranormal operator is $*$-paranormal.

Lemma 5.1.1. *([25]) Let $T \in \mathcal{B}(l^2(\mathbb{N}))$ be a weighted shift operator with non zero weights $\{\alpha_n\}, (n = 1, 2, ...)$, defined by $Te_n = \alpha_n \ e_{n+1}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $l^2(\mathbb{N})$. If $2|\alpha_k|^2 < |\alpha_{k-1}|^2$ for some $k \in \mathbb{N}$, then T is not totally $*$-paranormal.*

We know that every totally $(m,n)^*$-paranormal operator is $(m,n)^*$-paranormal operator. But the converse need not be true.

For example, let $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, x_3, x_4, x_5, x_6, \cdots).$$

From Lemma 1.3.2, T is a $(1,1)^*$-paranormal operator. We can see that the inequality $2|\alpha_k|^2 < |\alpha_{k-1}|^2$ holds for $k = 2$. Hence from Lemma 5.1.1, T is not totally $*$-paranormal. Therefore, T is not a totally $(1,1)^*$-paranormal operator.
Next we give some characterizations for totally $(m,n)^*$-paranormal operators.

Theorem 5.1.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is totally $(m,n)^*$-paranormal if and only if*

$$m^{\frac{2}{n+1}}(T-\lambda I)^{*n+1}(T-\lambda I)^{n+1} - (n+1)\ a^n (T-\lambda I)(T-\lambda I)^* + m^{\frac{2}{n+1}}\ n\ a^{n+1} I \geq 0 \quad (5.1)$$

for all $a \geq 0$ and $\lambda \in \mathbb{C}$.

Proof. T is a totally $(m,n)^*$-paranormal operator

$$\Leftrightarrow \|(T-\lambda I)^* x\|^{n+1} \leq m\|(T-\lambda I)^{n+1}x\|\|x\|^n, \forall x \in \mathcal{H}, \forall \lambda \in \mathbb{C}.$$
$$\Leftrightarrow \|(T-\lambda I)^* x\|^2 \leq m^{\frac{2}{n+1}}\|(T-\lambda I)^{n+1}x\|^{\frac{2}{n+1}}\|x\|^{\frac{2n}{n+1}}, \forall x \in \mathcal{H}, \forall \lambda \in \mathbb{C}.$$
$$\Leftrightarrow \langle (T-\lambda I)(T-\lambda I)^* x, x\rangle \leq m^{\frac{2}{n+1}}\langle (T-\lambda I)^{*n+1}(T-\lambda I)^{n+1}x, x\rangle^{\frac{1}{n+1}}\langle x, x\rangle^{\frac{n}{n+1}},$$

for every $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

$$\Leftrightarrow \langle |(T-\lambda I)^*|^2 x, x\rangle \leq m^{\frac{2}{n+1}} \langle |(T-\lambda I)^{n+1}|^2 x, x\rangle^{\frac{1}{n+1}} \langle x, x\rangle^{\frac{n}{n+1}}, \forall x \in \mathcal{H}, \forall \lambda \in \mathbb{C}. \tag{5.2}$$

Thus, T is a totally $(m,n)^*$-paranormal operator if and only if T satisfies (5.2). For any $a > 0$, by weighted arithmetic mean-geometric mean inequality (1.1) and (5.2), we have

$$\frac{1}{n+1}\langle a^{-n} m^{\frac{2}{n+1}} |(T-\lambda I)^{n+1}|^2 x, x\rangle + \frac{n}{n+1}\langle a\, m^{\frac{2}{n+1}} x, x\rangle$$

$$\begin{aligned} &\geq \langle a^{-n} m^{\frac{2}{n+1}} |(T-\lambda I)^{n+1}|^2 x, x\rangle^{\frac{1}{n+1}} \ \langle a\, m^{\frac{2}{n+1}} x, x\rangle^{\frac{n}{n+1}} \\ &= m^{\frac{2}{n+1}} \langle |(T-\lambda I)^{n+1}|^2 x, x\rangle^{\frac{1}{n+1}} \ \langle x, x\rangle^{\frac{n}{n+1}} \\ &\geq \langle |(T-\lambda I)^*|^2 x, x\rangle, \forall x \in \mathcal{H},\ \forall \lambda \in \mathbb{C}. \end{aligned}$$

Thus,

$$\frac{a^{-n}}{n+1}\ m^{\frac{2}{n+1}} \langle |(T-\lambda I)^{n+1}|^2 x, x\rangle + \frac{na}{n+1}\ m^{\frac{2}{n+1}} \langle x, x\rangle - \langle (T-\lambda I)(T-\lambda I)^* x, x\rangle \geq 0,$$

$\forall x \in \mathcal{H}, \forall a > 0, \forall \lambda \in \mathbb{C}$.

Hence,

$$m^{\frac{2}{n+1}} (T-\lambda I)^{*n+1} (T-\lambda I)^{n+1} - (n+1)\ a^n (T-\lambda I)(T-\lambda I)^* + m^{\frac{2}{n+1}}\ n\ a^{n+1} I \geq 0,$$

for all $a \geq 0, \forall \lambda \in \mathbb{C}$.

Conversely, suppose that (5.1) holds. Then

$$m^{\frac{2}{n+1}} \langle (T-\lambda I)^{*n+1} (T-\lambda I)^{n+1} x, x\rangle - (n+1)\ a^n \langle (T-\lambda I)(T-\lambda I)^* x, x\rangle + m^{\frac{2}{n+1}}\ n\ a^{n+1} \langle x, x\rangle \geq 0, \tag{5.3}$$

for every $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

Let $x \in \mathcal{H}$ be such that $\langle |(T-\lambda I)^{n+1}|^2 x, x\rangle = 0$. From (5.3), we get

$$m^{\frac{2}{n+1}}\ n\ a\langle x, x\rangle - (n+1)\langle (T-\lambda I)(T-\lambda I)^* x, x\rangle \geq 0$$

Letting $a \to 0$, we get $\langle (T-\lambda I)(T-\lambda I)^* x, x\rangle = 0$. Hence

$$\langle |(T-\lambda I)^*|^2 x, x\rangle \leq m^{\frac{2}{n+1}} \langle |(T-\lambda I)^{n+1}|^2 x, x\rangle^{\frac{1}{n+1}} \ \langle x, x\rangle^{\frac{n}{n+1}}.$$

Thus (5.2) is satisfied.

Let $x \in \mathcal{H}$ be such that $\langle |(T-\lambda I)^{n+1}|^2 x, x\rangle > 0$. Hence $\langle x,x\rangle > 0$.
By taking $a = \left(\frac{\langle |(T-\lambda I)^{n+1}|^2 x,x\rangle}{\langle x,x\rangle}\right)^{\frac{1}{n+1}}$ in (5.3), we get

$$m^{\frac{2}{n+1}}\langle (T-\lambda I)^{*n+1}(T-\lambda I)^{n+1}x,x\rangle + m^{\frac{2}{n+1}}\ n\ \frac{\langle |(T-\lambda I)^{n+1}|^2 x,x\rangle}{\langle x,x\rangle}\langle x,x\rangle$$

$$\geq (n+1)\left(\frac{\langle |(T-\lambda I)^{n+1}|^2 x,x\rangle}{\langle x,x\rangle}\right)^{\frac{n}{n+1}} \langle (T-\lambda I)(T-\lambda I)^* x,x\rangle$$

$$m^{\frac{2}{n+1}}\langle |(T-\lambda)^{n+1}|^2 x,x\rangle + m^{\frac{2}{n+1}}\ n\ \langle |(T-\lambda)^{n+1}|^2 x,x\rangle$$

$$\geq (n+1)\left(\frac{\langle |(T-\lambda)^{n+1}|^2 x,x\rangle}{\langle x,x\rangle}\right)^{\frac{n}{n+1}} \langle (T-\lambda)(T-\lambda)^* x,x\rangle.$$

Hence, $m^{\frac{2}{n+1}}(1+n)\langle |(T-\lambda I)^{n+1}|^2 x,x\rangle$

$$\geq (n+1)\ \left(\frac{\langle |(T-\lambda I)^{n+1}|^2 x,x\rangle}{\langle x,x\rangle}\right)^{\frac{n}{n+1}} \langle (T-\lambda I)(T-\lambda I)^* x,x\rangle.$$

Therefore, $m^{\frac{2}{n+1}}\langle |(T-\lambda I)^{n+1}|^2 x,x\rangle^{1-\frac{n}{n+1}}\ \langle x,x\rangle^{\frac{n}{n+1}} \geq \langle (T-\lambda I)(T-\lambda I)^* x,x\rangle$. Hence,

$$m^{\frac{2}{n+1}}\langle |(T-\lambda I)^{n+1}|^2 x,x\rangle^{\frac{1}{n+1}}\ \langle x,x\rangle^{\frac{n}{n+1}} \geq \langle |(T-\lambda I)^*|^2 x,x\rangle$$

Thus (5.2) is satisfied. Therefore, T is totally $(m,n)^*$-paranormal. □

Theorem 5.1.2. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a totally $(m,n)^*$-paranormal operator and $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which is invariant under T. Then $T|_{\mathcal{M}}$ is a totally $(m,n)^*$-paranormal operator.*

Proof. Let $T_1 = T|_{\mathcal{M}}$ and P be an orthogonal projection onto $\mathcal{M}$. Since $\mathcal{M}$ is invariant under T, we have $(T|_{\mathcal{M}})^* = PT^*|_{\mathcal{M}}$. Let $x \in \mathcal{M}$.

$$\begin{aligned}
\|(T_1-\lambda I)^* x\|^{n+1} &= \|PT^*x - \overline{\lambda}x\|^{n+1}\\
&= \|P(T^*-\overline{\lambda}I)Px\|^{n+1}\\
&\leq \|(T^*-\overline{\lambda}I)x\|^{n+1}\\
&\leq m\|(T-\lambda I)^{n+1}x\|\|x\|^{n}
\end{aligned}$$

$$= m\|(T_1 - \lambda I)^{n+1}x\|\|x\|^n$$

Hence $T|_{\mathcal{M}}$ is a totally $(m,n)^*$-paranormal operator. □

Theorem 5.1.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. Then $T - \alpha I$ and αT are totally $(m,n)^*$-paranormal operator for every $\alpha \in \mathbb{C}$.*

Proof. Suppose that T is a totally $(m,n)^*$-paranormal operator. Let $x \in \mathcal{H}$ and λ, $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
\|[(T-\alpha I) - \lambda I]^* x\|^{n+1} &= \|[T - (\alpha+\lambda)I]^* x\|^{n+1} \\
&\leq m\|[T - (\alpha+\lambda)I]^{n+1}x\|\|x\|^n \\
&= m\|[(T-\alpha I) - \lambda I]^{n+1}x\|\|x\|^n
\end{aligned}$$

Thus $T - \alpha I$ is totally $(m,n)^*$-paranormal.
If $\alpha = 0$, then αT is totally $(m,n)^*$-paranormal.
Let $\alpha \neq 0$. For $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\|(\alpha T - \lambda I)^* x\|^{n+1} &= \left\|\bar{\alpha}\left(T - \frac{\lambda}{\alpha} I\right)^* x\right\|^{n+1} \\
&\leq m|\alpha|^{n+1}\left\|\left(T - \frac{\lambda}{\alpha} I\right)^{n+1} x\right\|\|x\|^n \\
&= m\|(\alpha T - \lambda I)^{n+1}x\|\|x\|^n
\end{aligned}$$

Hence αT is totally $(m,n)^*$-paranormal. □

Theorem 5.1.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. Then $N(T - \lambda I) \subset N(T - \lambda I)^*$, for all $\lambda \in \mathbb{C}$.*

Proof. Since T is a totally $(m,n)^*$-paranormal operator, we have

$$\|(T - \lambda I)^* x\|^{n+1} \leq m\|(T - \lambda I)^{n+1}x\|\|x\|^n, \text{ for all } x \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}. \tag{5.4}$$

Let $x \in N(T - \lambda I)$. Then $(T - \lambda I)^{n+1}x = 0$. From (5.4), we get $(T - \lambda I)^* x = 0$. Hence $x \in N(T - \lambda I)^*$. Thus $N(T - \lambda I) \subset N(T - \lambda I)^*$, for all $\lambda \in \mathbb{C}$. □

Lemma 5.1.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. If $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

Proof. Since T is totally $(m,n)^*$-paranormal, using Theorem 5.1.3 we have $T - \lambda I$ is totally $(m,n)^*$-paranormal. Also, $\sigma(T - \lambda I) = \{0.\}$ But every quasinilpotent totally $(m,n)^*$-paranormal operator is a zero operator. Hence $T = \lambda I$. □

Let E_λ denotes the Riesz projection of T with respect to an isolated spectral value λ. In ([46]), M. H. M. Rashid proved that if $T \in \mathcal{B}(\mathcal{H})$ is a totally n^*-paranormal operator, then $N(T - \lambda I) = R(E_\lambda)$. Now we prove that this result holds for totally $(m,n)^*$-paranormal operators also.

Theorem 5.1.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator and λ be an isolated point of $\sigma(T)$. Then $N(T - \lambda I) = R(E_\lambda)$,*

Proof. From Theorem 1.3.5, we have $N(T - \lambda I) \subseteq R(E_\lambda)$ and $R(E_\lambda)$ is invariant under T. Since T is totally $(m,n)^*$-paranormal operator, from Theorem 5.1.2, $T|_{R(E_\lambda)}$ is totally $(m,n)^*$-paranormal. Hence from Theorem 1.3.5, $\sigma(T|_{R(E_\lambda)}) = \{\lambda\}$.
If $\lambda = 0$, then $\sigma(T|_{R(E_\lambda)}) = \{0\}$. Hence from Lemma 5.1.2, $T|_{R(E_\lambda)} = 0$. Therefore, $R(E_\lambda) \subseteq N(T)$.
If $\lambda \neq 0$, then $\sigma(T|_{R(E_\lambda)} - \lambda I|_{R(E_\lambda)}) = \{0\}$. From Lemma 5.1.2, $(T - \lambda I)|_{R(E_\lambda)} = 0$. Hence $R(E_\lambda) \subseteq N(T - \lambda I)$. □

5.2 Spectral properties of totally $(m,n)^*$-paranormal operators

In this section, we prove some properties of different kinds of spectra for totally $(m,n)^*$- paranormal operators.

Theorem 5.2.1. *If $T \in \mathcal{B}(\mathcal{H})$ is a totally $(m,n)^*$-paranormal operator then $\sigma_{ja}(T) = \sigma_a(T)$.*

Proof. Let $\lambda \in \sigma_a(T)$. Then there exist a sequence (x_n) in $\mathcal{H}$ with $\|x_n\| = 1$ such that $\|(T - \lambda I)x_n\| \longrightarrow 0$ as $n \to \infty$. Since T is a totally $(m,n)^*$-paranormal operator, we have

$$\|(T-\lambda I)^* x_n\|^{n+1} \leq m\|(T-\lambda I)^{n+1} x_n\|\|x_n\|^n \leq m\|(T-\lambda I)^n\|\|(T-\lambda I)x_n\|$$

Hence, $\|(T-\lambda I)^* x_n\| \longrightarrow 0$ as $n \to \infty$. Therefore, $\bar{\lambda} \in \sigma_a(T^*)$.
Hence $\sigma_a(T) \subset \sigma_{ja}(T)$. Also, we have $\sigma_{ja}(T) \subset \sigma_a(T)$. Hence $\sigma_{ja}(T) = \sigma_a(T)$. □

Theorem 5.2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. Then $\{\bar{\lambda} : \lambda \in \sigma_a(T^*)\} = \sigma(T)$.*

Proof. Let $S_a(T) = \{\bar{\lambda} : \lambda \in \sigma_a(T^*)\}$ and $S_{ja}(T) = \{\bar{\lambda} : \lambda \in \sigma_{ja}(T^*)\}$. Since T is a totally $(m,n)^*$-paranormal operator, we have $\sigma_{ja}(T) = \sigma_a(T)$. Obviously, $\sigma_{ja}(T) = S_{ja}(T) \subset S_a(T)$. Therefore, $\sigma_a(T) \subset S_a(T)$. From Theorem 1.3.1, we have $\sigma(T) = \sigma_a(T) \cup S_a(T)$. Hence $\sigma(T) \subset S_a(T)$. Also, $S_a(T) \subset \sigma(T)$. Hence $S_a(T) = \sigma(T)$. □

Let $\mathcal{K}$ be a Hilbert space containing $\mathcal{H}$ and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a linear transformation satisfies certain properties defined in Theorem 1.3.6.

Theorem 5.2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. Then $\phi(T)$ is a totally $(m,n)^*$-paranormal operator.*

Proof. Since T is totally $(m,n)^*$-paranormal, we have

$$m^{\frac{2}{n+1}}(T-\lambda I)^{*n+1}(T-\lambda I)^{n+1} - (n+1)\, a^n (T-\lambda I)(T-\lambda I)^* + m^{\frac{2}{n+1}}\, n\, a^{n+1} I \geq 0 \quad (5.5)$$

for all $a > 0$. From Theorem 1.3.6, we have
$m^{\frac{2}{n+1}}(\phi(T)-\lambda I)^{*n+1}(\phi(T)-\lambda I)^{n+1} - (n+1)\, a^n (\phi(T)-\lambda I)(\phi(T)-\lambda I)^* + m^{\frac{2}{n+1}}\, n\, a^{n+1} I$

$$= m^{\frac{2}{n+1}}(\phi(T-\lambda I))^{*n+1}(\phi(T-\lambda I))^{n+1} - (n+1)\, a^n \phi(T-\lambda I)(\phi(T-\lambda I))^* + m^{\frac{2}{n+1}} n a^{n+1} I$$
$$= m^{\frac{2}{n+1}}\phi\left((T-\lambda I)^{*n+1}\right)\phi\left((T-\lambda I)^{n+1}\right) - (n+1)\, a^n \phi(T-\lambda I)(\phi(T-\lambda I)^*) + m^{\frac{2}{n+1}} n a^{n+1} I$$
$$= \phi\left(m^{\frac{2}{n+1}}(T-\lambda I)^{*n+1}(T-\lambda I)^{n+1} - (n+1)\; a^n (T-\lambda I)(T-\lambda I)^* + m^{\frac{2}{n+1}}\; n\; a^{n+1} I\right)$$

From (5.5) and Theorem 1.3.6, we have

$$\phi\left(m^{\frac{2}{n+1}}(T-\lambda I)^{*n+1}(T-\lambda I)^{n+1} - (n+1)\; a^n (T-\lambda I)(T-\lambda I)^* + m^{\frac{2}{n+1}}\; n\; a^{n+1} I\right) \geq 0.$$

Hence,

$$m^{\frac{2}{n+1}}(\phi(T)-\lambda I)^{*n+1}(\phi(T)-\lambda I)^{n+1}-(n+1)\,a^n(\phi(T)-\lambda I)(\phi(T)-\lambda I)^*+m^{\frac{2}{n+1}}\,n\,a^{n+1}I \geq 0.$$

Thus $\phi(T)$ is a totally $(m,n)^*$-paranormal operator. □

Theorem 5.2.4. *Assume that $T \in \mathcal{B}(\mathcal{H})$ is a totally $(m,n)^*$-paranormal operator. Then the following holds:*

(i) If $\sigma(T) = \{0\}$, then T is nilpotent.

(ii) The matrix representation of T on $\mathcal{H} = N(T-\lambda I) \oplus N(T-\lambda I)^{\perp}$ is given by

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix},$$

where λ is a nonzero eigen value of T. Also $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator.

(i) Assume that $\sigma(T) = \{0\}$. Hence by Lemma 5.1.2, we have $T = 0$. Thus T is nilpotent.

(ii) Let λ be a nonzero eigen value of T. Since T is totally $(m,n)^*$-paranormal, from Theorem 5.1.4, we get $N(T-\lambda I) \subset N(T-\lambda I)^*$. Hence, $N(T-\lambda I)^{\perp}$ is invariant under T. Thus, $N(T-\lambda I)$ reduces T. Hence, the matrix representation of T on $\mathcal{H} = N(T-\lambda I) \oplus N(T-\lambda I)^{\perp}$ is given by

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix},$$

where $B = T|_{N(T-\lambda I)^{\perp}}$. Let $x \in N(B-\lambda I)$. Then

$$(T-\lambda I)\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ (B-\lambda I)x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $x \in N(T-\lambda I)$. Since $B = T|_{N(T-\lambda I)^{\perp}}$, we have $x \in N(T-\lambda I)^{\perp}$. Thus, $x = 0$. Hence, $N(B-\lambda I) = \{0\}$. i.e, $\lambda \notin \sigma_p(B)$. Since $T = \lambda I \oplus B$, we have $\sigma(T) = \{\lambda\} \cup \sigma(B)$.

□

Let $\mathcal{L}$ denotes the set of all compact subsets of $\mathbb{C}$. Recall that the spectral map σ is the function from $\mathcal{B}(\mathcal{H})$ to $\mathcal{L}$, which maps $T \in \mathcal{B}(\mathcal{H})$ to its spectrum.

Theorem 5.2.5. *The spectral map, σ on the class of totally $(m,n)^*$-paranormal operators is continuous.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. If $\sigma(T) = \{0\}$, then from Theorem 5.2.4, we have T is nilpotent. Also from Theorem 5.2.3, we have $\phi(T)$ is totally $(m,n)^*$-paranormal. Hence from Theorem 5.2.4 and Theorem 1.3.7 we have, the spectral map σ is continuous on the set of all totally $(m,n)^*$-paranormal operators. □

Theorem 5.2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. Then $\sigma_r(T^*)$, residual spectrum of T^* is empty and $\sigma_a(T^*) = \sigma(T^*)$.*

Proof. Assume that T has no eigen value. Then $N(T-\lambda I) = \{0\}$, for all $\lambda \in \mathbb{C}$. Therefore $\overline{R(T^* - \overline{\lambda} I)} = \mathcal{H}$, for all $\lambda \in \mathbb{C}$. Thus $\sigma_r(T^*) = \emptyset$.
Suppose that T has eigen value. Since T is totally $(m,n)^*$-paranormal operator, from Theorem 5.1.4, we have $N(T-\lambda I) \subset N(T^* - \bar{\lambda} I)$. Therefore, $(T^* - \bar{\lambda} I)$ is not one-one for every $\lambda \in \mathbb{C}$. Hence, $\sigma_r(T^*) = \emptyset$. Since $\sigma(T^*) = \sigma_r(T^*) \cup \sigma_a(T^*)$ and $\sigma_r(T^*) = \emptyset$, we have $\sigma_a(T^*) = \sigma(T^*)$. □

5.3 Finite operator

In this section, we prove that every totally $(m,n)^*$-paranormal operators are finite operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is *finite* if and only if $\|I - (TX - XT)\| \geq 1$ for all $X \in \mathcal{B}(\mathcal{H})$.

Theorem 5.3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator. Then T is a finite operator.*

Proof. Assume that $T \in \mathcal{B}(\mathcal{H})$ is a totally $(m,n)^*$-paranormal operator. From Theorem 5.2.1, we have $\sigma_{ja}(T) = \sigma_a(T)$. Also from Theorem 1.3.2, $\partial\sigma(T) \subset \sigma_a(T)$. Hence $\sigma_{ja}(T) \neq \emptyset$. Now using Theorem 1.3.9, we have T is a finite operator. □

Let $T \in \mathcal{B}(\mathcal{H})$. For $X \in \mathcal{B}(\mathcal{H})$, let $\delta_T(X) = TX - XT$. Now we show that $R(\delta_T)$ is orthogonal to $N(\delta_T)$ for totally $(m,n)^*$-paranormal operators. We use the following lemma for proving the result.

Lemma 5.3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally $(m,n)^*$-paranormal operator and $A \in \mathcal{B}(\mathcal{H})$ be a normal operator with $AT = TA$. Then for all $\lambda \in \sigma_p(A)$ and for all $X \in \mathcal{B}(\mathcal{H})$,*

$$|\lambda| \leq \|A - (TX - XT)\|$$

Proof. Let $\lambda \in \sigma_p(A)$. If $\lambda = 0$, then the result holds trivially.
Let $\lambda \neq 0$. Since A is normal and $AT = TA$, from Fuglede-Putnam Theorem 1.3.10 we get $A^*T = TA^*$. Hence $N(A - \lambda I)$ reduces T and A. Thus the matrix representation of T and A on $\mathcal{H} = N(A - \lambda I) \oplus N(A - \lambda I)^{\perp}$ is given by

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda I & 0 \\ 0 & A_2 \end{pmatrix}$$

Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

Hence $A - (TX - XT) = \begin{pmatrix} \lambda I - (T_1X_1 - X_1T_1) & C \\ D & E \end{pmatrix}$, where $C, D, E \in \mathcal{B}(\mathcal{H})$. Now

$$\begin{aligned} \|A - (TX - XT)\| &\geq \|\lambda I - (T_1X_1 - X_1T_1)\| \\ &= |\lambda| \; \|I - \left(T_1\frac{X_1}{\lambda} - \frac{X_1}{\lambda}T_1\right)\| \end{aligned}$$

Since $N(A - \lambda I)$ is invariant under T and $T_1 = T|_{N(A-\lambda I)}$, from Theorem 5.1.2 we have T_1 is totally $(m,n)^*$-paranormal. From Theorem 5.3.1, T_1 is a finite operator. Hence $\|A - (TX - XT)\| \geq |\lambda|$. □

Theorem 5.3.2. *If $T \in \mathcal{B}(\mathcal{H})$ is a totally $(m,n)^*$-paranormal operator and $A \in \mathcal{B}(\mathcal{H})$ is a normal operator with $AT = TA$, then $R(\delta_T)$ is orthogonal to $N(\delta_T)$.*

Proof. Since A is normal, we have $\phi(A)$ is normal. From Theorem 5.2.3, we have $\phi(T)$ is totally $(m,n)^*$-paranormal. Hence $\phi(T)$ is finite. Since $AT = TA$, we have $\phi(A)\phi(T) = \phi(T)\phi(A)$. Let $\lambda \in \sigma_p(\phi(A))$. From Lemma 5.3.1, we have

$$\begin{aligned} |\lambda| &\leq \|\phi(A) - (\phi(T)\phi(X) - \phi(X)\phi(T))\| \\ &= \|\phi(A - (TX - XT))\| \\ &= \|A - (TX - XT)\| \end{aligned}$$

Hence

$$|\lambda| \leq \|A - (TX - XT)\|, \ \forall X \in \mathcal{B}(\mathcal{H}), \ \lambda \in \sigma_p(\phi(A)). \tag{5.6}$$

Since $\phi(A)$ is normal, we have $\|\phi(A)\| = \sup\limits_{\lambda\in\sigma(\phi(A))} |\lambda|$

From Theorem 1.3.6, we have $\sigma_p(\phi(A)) = \sigma(\phi(A))$. Thus from (5.6) , we get

$$\|\phi(A)\| \leq \|A - (TX - XT)\|, \text{ for all } X \in \mathcal{B}(\mathcal{H}).$$

Thus, $\|A\| \leq \|A - (TX - XT)\|,$ for all $X \in \mathcal{B}(\mathcal{H})$. Hence $R(\delta_T)$ is orthogonal to $N(\delta_T)$. □

5.4 k-quasi totally $(m,n)^*$-paranormal operators

In this section, we introduce a new classes of operators namely k-quasi totally $(m,n)^*$-paranormal operators which includes the class of totally $(m,n)^*$-paranormal operators.

Definition 5.4.1. *Let $m \in \mathbb{R}^+, n \in \mathbb{N}$ and k be a non-negative integer. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k-quasi totally $(m,n)^*$-paranormal operator if*

$$\|(T-\lambda I)^*T^kx\|^{n+1} \leq m\|(T-\lambda I)^{n+1}T^kx\|\|T^kx\|^n, \ \textit{for all } x \in \mathcal{H} \textit{ and } \lambda \in \mathbb{C}.$$

Equivalently by proceeding in the similar way as in Theorem 5.1.1, it can be proved that T is a k-quasi totally $(m,n)^*$-paranormal operator if and only if

$$T^{*k}\left(m^{\frac{2}{n+1}}(T-\lambda I)^{*n+1}(T-\lambda I)^{n+1}-(n+1)\ a^n(T-\lambda I)(T-\lambda I)^*+m^{\frac{2}{n+1}}\ n\ a^{n+1}I\right)T^k\geq 0$$

for all $a>0$ and $\lambda\in\mathbb{C}$.

If $k=0$, then k-quasi totally $(m,n)^*$-paranormal operator is totally $(m,n)^*$-paranormal operator. In particular, if $k=0,\ m=n=1$, then T is totally $*$-paranormal ([25]).
For example, let $T: l^2(\mathbb{N})\to l^2(\mathbb{N})$ be defined by $T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,x_3,\cdots)$ is 2-quasi totally $(1,1)^*$-paranormal operator.

Theorem 5.4.1. *Let $T\in\mathcal{B}(\mathcal{H})$ and $\overline{R(T^k)}\neq\mathcal{H}$. Then T is a k-quasi totally $(m,n)^*$-paranormal operator if and only if $T=\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}$ on $\overline{R(T^k)}\oplus N(T^{*k})$, where*

$$m^{\frac{2}{n+1}}(A-\lambda I)^{*n+1}(A-\lambda I)^{n+1}-(n+1)\ a^n(A-\lambda I)(A-\lambda I)^*+m^{\frac{2}{n+1}}\ n\ a^{n+1}I\geq (n+1)a^nBB^*,$$

for all $a>0$ and $\lambda\in\mathbb{C}$. Also $C^k=0$ and $\sigma(T)=\sigma(A)\cup\{0\}$.

Proof. Assume that T is a k-quasi totally $(m,n)^*$-paranormal operator.
Let $T=\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}$ on $\overline{R(T^k)}\oplus N(T^{*k})$ and P be the orthogonal projection onto $\overline{R(T^k)}$. Since T is a k-quasi totally $(m,n)^*$-paranormal operator, we have

$$P\left(m^{\frac{2}{n+1}}(T-\lambda I)^{*n+1}(T-\lambda I)^{n+1}-(n+1)\ a^n(T-\lambda I)(T-\lambda I)^*+m^{\frac{2}{n+1}}\ n\ a^{n+1}I\right)P\geq 0,$$

for all $a>0$ and $\lambda\in\mathbb{C}$. Since $A=T|_{\overline{R(T^k)}}$, from the above relation we get

$$m^{\frac{2}{n+1}}(A-\lambda I)^{*n+1}(A-\lambda I)^{n+1}-(n+1)\ a^n(A-\lambda I)(A-\lambda I)^*+m^{\frac{2}{n+1}}\ n\ a^{n+1}I\geq (n+1)a^nBB^*,$$

for all $a>0$ and $\lambda\in\mathbb{C}$.

Let $x \in N(T^{*k})$. we have

$$T^k(x) = \begin{pmatrix} A^k & \sum_{i=0}^{k-1} A^iBC^{k-1-i} \\ 0 & C^k \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Hence $C^kx = T^kx - \sum_{i=0}^{k-1} A^iBC^{k-1-i}x$. Since $A = T|_{\overline{R(T^k)}}$, we have $C^kx \in \overline{R(T^k)}$. Also $C^kx \in N(T^{*k})$. Thus $C^kx \in \overline{R(T^k)} \cap N(T^{*k})$. Hence $C^k = 0$. Therefore, $\sigma(C) = \{0\}$. Hence $\sigma(A) \cap \sigma(C) = \sigma(A) \cap \{0\}$, has no interior point. From Theorem 1.3.4, we get $\sigma(T) = \sigma(A) \cup \{0\}$.

Conversely assume that $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\overline{R(T^k)} \oplus N(T^{*k})$, where

$$m^{\frac{2}{n+1}}(A-\lambda I)^{*n+1}(A-\lambda I)^{n+1} - (n+1)\, a^n(A-\lambda I)(A-\lambda I)^* + m^{\frac{2}{n+1}}\, n\, a^{n+1}I \geq (n+1)a^nBB^*,$$

for all $a > 0$, for all $\lambda \in \mathbb{C}$ and $C^k = 0$. Thus

$$T^k = \begin{pmatrix} A^k & \sum_{i=0}^{k-1} A^iBC^{k-1-i} \\ 0 & 0 \end{pmatrix}$$

and $T^kT^{*k} = \begin{pmatrix} A^kA^{*k} + \sum_{i=0}^{k-1} A^iBC^{k-1-i}(\sum_{i=0}^{k-1} A^iBC^{k-1-i})^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$,

where $S = A^kA^{*k} + \sum_{i=0}^{k-1} A^iBC^{k-1-i}(\sum_{i=0}^{k-1} A^iBC^{k-1-i})^*$. Let

$$T_\lambda = T^{*k}\left(m^{\frac{2}{n+1}}(T-\lambda)^{*n+1}(T-\lambda)^{n+1} - (n+1)\, a^n(T-\lambda)(T-\lambda)^* + m^{\frac{2}{n+1}}\, n\, a^{n+1}I\right)T^k.$$

Then for any $a > 0$ and $\lambda \in \mathbb{C}$,

$$T^kT_\lambda T^{*k} = \begin{pmatrix} SDS & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$D = m^{\frac{2}{n+1}}(A-\lambda I)^{*n+1}(A-\lambda I)^{n+1} - (n+1)\, a^n[(A-\lambda I)(A-\lambda I)^* + BB^*] + m^{\frac{2}{n+1}}\, n\, a^{n+1}I.$$

Since $D \geq 0$, we have $T^k T_\lambda T^{*k} \geq 0$.
Let $x \in \mathcal{H}$. Then there exist $y \in \overline{R(T^{*k})}$, $z \in N(T^k)$ such that $x = y + z$. Since $y \in \overline{R(T^{*k})}$, there exists a sequence (x_n) in $\mathcal{H}$ such that $T^{*k}(x_n) \to y$ as $n \to \infty$. Also $T_\lambda z = 0$, since $z \in N(T^k)$. Hence, $\langle T_\lambda x, x\rangle = \langle T_\lambda y, y\rangle \geq 0$. Thus T is a k-quasi totally $(m,n)^*$-paranormal operator. □

Corollary 5.4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\overline{R(T^k)} \neq \mathcal{H}$. If T is a k-quasi totally $(m,n)^*$-paranormal operator, then*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } \overline{R(T^k)} \oplus N(T^{*k}),$$

where A is a totally $(m,n)^*$-paranormal operator on $\overline{R(T^k)}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Assume that T is a k-quasi totally $(m,n)^*$-paranormal operator. Then

$$\|(T-\lambda)^* T^k x\|^{n+1} \leq m\|(T-\lambda)^{n+1} T^k x\| \|T^k x\|^n, \text{for all } x \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}.$$

Let $x \in \mathcal{H}$. Let $z = T^k x$ in the above equation we get

$$\|(T-\lambda)^* z\|^{n+1} \leq m\|(T-\lambda)^{n+1} z\| \|z\|^n.$$

Since $A = T|_{\overline{R(T^k)}}$, we have $\|(A-\lambda)^* z\|^{n+1} \leq m\|(A-\lambda)^{n+1} z\|\|z\|^n$, for all $z \in \overline{R(T^k)}$. Hence A is a totally $(m,n)^*$-paranormal operator on $\overline{R(T^k)}$.
Let $x \in N(T^{*k})$. Then

$$T^k(x) = \begin{pmatrix} A^k & \sum_{i=0}^{k-1} A^i B C^{k-1-i} \\ 0 & C^k \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Therefore, $C^k x = T^k x - \sum_{i=0}^{k-1} A^i B C^{k-1-i} x$. Thus $C^k x \in \overline{R(T^k)} \cap N(T^{*k})$. Hence $C^k = 0$. Thus $\sigma(C) = \{0\}$. From Theorem 1.3.4, we get $\sigma(T) = \sigma(A) \cup \{0\}$. □

Chapter 6

Totally P-posinormal operators

In this chapter, we concentrate on studying the class of totally P-posinormal operators. Here we show that the restriction of totally P-posinormal to its closed subspace is again totally P-posinormal. Also we show that these operators are finite and spectral continuous. Finally we study about range kernel orthogonality and Riesz projection for this operator.

6.1 Totally P-posinormal operators

Definition 6.1.1. *([38]) An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be totally P-posinormal if $\|(P(T - zI))^* x\| \leq M(z)\|(T - zI)x\|$ for all $x \in \mathcal{H}$ and $z \in \mathbb{C}$, where $P(z)$ is a polynomial with zero constant term and $M(z)$ is bounded on compact sets of $\mathbb{C}$.*

If $P(z) = z$ and $M(z) = M$ a constant, then totally P-posinormal operator is M-hyponormal. If $P(z) = z$, $M(z) = C$, a constant and $z = 0$, then totally P-posinormal operator is posinormal. If $M(z) = C$ and $z = 0$, then totally P-posinormal operator is polynomially P-posinormal.

For example, let $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x_1, x_2, \cdots, x_n) = (x_1, 2x_2, 2x_3 \cdots, 2x_n)$ is totally P-posinormal ([26]).

Let T be a nilpotent operator. Hence $T^n = 0$, for some $n \in \mathbb{N}$. Then T is totally z^{2n-1}-posinormal ([26]).

Now, we prove that restriction of a totally P-posinormal operator on a closed subspace is again a totally P-posinormal operator.

Theorem 6.1.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. If T is a totally P-posinormal operator, then $T|_{\mathcal{M}}$ is totally P-posinormal.*

Proof. Let $x, y \in \mathcal{M}$ and Q be an orthogonal projection onto $\mathcal{M}$.
Since $\mathcal{M}$ is invariant under T,

$$\begin{aligned} \langle (T|_{\mathcal{M}})^* x, y\rangle &= \langle x, T|_{\mathcal{M}}\, y\rangle \\ &= \langle x, Ty\rangle \\ &= \langle x, TQy\rangle \\ &= \langle QT^* x, y\rangle \\ &= \langle QT^*|_{\mathcal{M}}\, x, y\rangle \end{aligned}$$

Hence $QT^*|_{\mathcal{M}} = (T|_{\mathcal{M}})^*$. Therefore, $(T|_{\mathcal{M}} - zI)^* x = Q(T - zI)^* x$.
Also since $\mathcal{M}$ is invariant under T, we have $Q(T^*)^2|_{\mathcal{M}} = (T^2|_{\mathcal{M}})^*$.
Thus $\left((T|_{\mathcal{M}} - zI)^2\right)^* x = Q(T - zI)^{*2} x$. Hence $\left((T|_{\mathcal{M}} - zI)^n\right)^* x = Q(T - zI)^{*n} x$, for all $n \in \mathbb{N}$. Thus $\left(P(T|_{\mathcal{M}} - zI)\right)^* x = Q\left(P(T - zI)\right)^* x$. Since T is totally P-posinormal, we have

$$\begin{aligned} \|\left(P(T|_{\mathcal{M}} - zI)\right)^* x\| &= \|Q\left(P(T - zI)\right)^* x\| \\ &\le M(z)\|(T - zI)x\| \\ &= M(z)\|(T|_{\mathcal{M}} - zI)x\|. \end{aligned}$$

Hence $T|_{\mathcal{M}}$ is totally P-posinormal. □

Let $\mathcal{PB}$ denotes the collection of all totally P-posinormal operators, where $P(z) = z^n + \sum_{j=1}^{n-1} c_j z^j, c_1 > 0$.

Theorem 6.1.2. *If $T \in \mathcal{PB}$, then $N(T - zI) \subseteq N(T - zI)^*$.*

Proof. Since T is totally P-posinormal operator, we have

$$\left(P(T - zI)\right)\left(P(T - zI)\right)^* \le M(z)^2 (T - zI)^*(T - zI), \ \forall z \in \mathbb{C} \qquad (6.1)$$

Let $x \in N(T - zI)$. From above equation, we get

$$(P(T - zI))\,(P(T - zI))^*\, x = 0.$$

Therefore, $\|\,(P(T - zI))^*\, x\|^2 = 0$. Hence $x \in N((P(T - zI))^*)$.
Thus, $\bar{c_1}(T - zI)^* x = -(T - zI)^{*n} x + \sum_{j=2}^{n-1} -\bar{c_j}(T - zI)^{*j} x$.
Hence,

$$\begin{aligned}\|\overline{c_1}(T - zI)^* x\| &\leq \|\,(P(T - zI))^*\, x\| \\ &\leq M(z)\|(T - zI)x\|.\end{aligned}$$

Since $x \in N(T - zI)$, we have $\overline{c_1}(T - zI)^* x = 0$. As $c_1 > 0$, we have $(T - zI)^* x = 0$. Hence $N(T - zI) \subseteq N(T - zI)^*$. □

For $T \in \mathcal{B}(\mathcal{H})$, define $H_0(T) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}$. Let $q \in \mathbb{N}$. T is said to satisfy the property $H(q)$, if $H_0(T - \lambda I) = N(T - \lambda I)^q$ for all $\lambda \in \mathbb{C}$. In ([8]), B. P. Duggal proved that totally P-posinormal operators satisfy the property $H(q)$. It is known that if T satisfies the property $H(q)$ with $\sigma(T) = \{\lambda\}$, then $T = \lambda I$ ([8]). Hence the following theorem holds for totally P-posinormal operators.

Theorem 6.1.3. *([8]) If $T \in \mathcal{B}(\mathcal{H})$ is a totally P-posinormal operator and $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

In ([4]), M. Cho and Y. M. Han proved that if $T \in \mathcal{B}(\mathcal{H})$ is a M-hyponormal operator, then $N(E_\lambda) = R(T - \lambda I)$, where E_λ is the Reisz projection of T with respect to an isolated spectral value λ. Now we prove this result holds for totally P-posinormal operators also.

Theorem 6.1.4. *Suppose T is a totally P-posinormal operator and λ is an isolated point of $\sigma(T)$. Then $N(T - \lambda I) = R(E_\lambda)$.*

Proof. From Theorem 1.3.5, we have $N(T - \lambda I) \subseteq R(E_\lambda)$.
From Theorem 6.1.1, $T|_{R(E_\lambda)}$ is totally P-posinormal. Since λ is an isolated point of $\sigma(T)$, from Theorem 1.3.5 we have $\sigma(T|_{R(E_\lambda)}) = \{\lambda\}$.

If $\lambda = 0$, then $\sigma(T|_{R(E_\lambda)}) = \{0\}$. Hence from Theorem 6.1.3, we have $T|_{R(E_\lambda)} = 0$. Therefore, $R(E_\lambda) \subseteq N(T)$.
If $\lambda \neq 0$, then $\sigma(T|_{R(E_\lambda)} - \lambda I|_{R(E_\lambda)}) = \{0\}$. Now from Theorem 6.1.3, we have $(T - \lambda I)|_{R(E_\lambda)} = 0$. Hence $R(E_\lambda) \subseteq N(T - \lambda I)$. □

Theorem 6.1.5. *If $T \in \mathcal{PB}$, then $\sigma_a(T) = \sigma_{ja}(T)$.*

Proof. Since T is totally P-posinormal,

$$M(z)^2(T - zI)^*(T - zI) - (P(T - zI))\,(P(T - zI))^* \geq 0,\ \forall z \in \mathbb{C}.$$

Let $z \in \mathbb{C}$. Hence from Theorem 1.3.6, we have
$M(z)^2(\phi(T) - zI)^*(\phi(T) - zI) - (P(\phi(T) - zI))\,(P(\phi(T) - zI))^*$

$$\begin{aligned} &= M(z)^2\phi((T - zI)^*)\phi(T - zI) - \phi\,(P(T - zI))\,\phi\,(P(T - zI)^*) \\ &= \phi\left(M(z)^2(T - zI)^*(T - zI) - (P(T - zI))\,(P(T - zI))^*\right) \geq 0. \end{aligned}$$

Hence, $\phi(T)$ is totally P-posinormal. Now, from Theorem 1.3.6, $\sigma_a(T) = \sigma_p(\phi(T))$. Also, from Theorem 6.1.2 $N(\phi(T) - zI) \subset N(\phi(T) - zI)^*$. Hence, $\sigma_p(\phi(T)) = \sigma_{jp}(\phi(T))$. Also from Theorem 1.3.6, $\sigma_{jp}(\phi(T)) = \sigma_{ja}(T)$. Hence $\sigma_a(T) = \sigma_{ja}(T)$. □

Theorem 6.1.6. *If $T \in \mathcal{PB}$, then the following holds:*

(i) If $\sigma(T) = \{0\}$, then T is nilpotent.

(ii) Let λ be a nonzero eigen value of T. Then the matrix representation of T on $\mathcal{H} = N(T - \lambda I) \oplus (N(T - \lambda I))^\perp$ is given by

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix}.$$

Also $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.

Proof. Since $\sigma(T) = \{0\}$, it follows from Theorem 6.1.3 that $T = 0$. Hence T is nilpotent.

Let λ be a nonzero eigen value of T. Since $T \in \mathcal{PB}$, from Theorem 6.1.2, we have $N(T-\lambda I) \subseteq N(T-\lambda I)^*$. Therefore, $N(T-\lambda I)^\perp$ is invariant under T. Hence $N(T-\lambda I)$ reduces T. Thus,

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & B \end{pmatrix} \text{ on } \mathcal{H} = N(T-\lambda I) \oplus N(T-\lambda I)^\perp,$$

where $B = T|_{N(T-\lambda I)^\perp}$. Let $x \in N(B-\lambda I)$. Then

$$(T-\lambda I)\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ (B-\lambda I)x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $x \in N(T-\lambda I)$. Since $B = T|_{N(T-\lambda I)^\perp}$ and $x \in N(B-\lambda I)$, $x \in N(T-\lambda I)^\perp$. Thus, $x = 0$. Hence $N(B-\lambda I) = 0$. i.e, $\lambda \notin \sigma_p(B)$. Since $T = \lambda I \oplus B$, we have $\sigma(T) = \{\lambda\} \cup \sigma(B)$. □

Let σ be the spectral map which sends $T \in \mathcal{B}(\mathcal{H})$ to its spectrum. Now we discuss the continuity of spectral map on the set of all totally P-posinormal operators.

Theorem 6.1.7. *The spectral map σ is continuous on the class of all $\mathcal{PB}$ operators.*

Proof. Let $T \in \mathcal{PB}$. Then from Theorem 6.1.6, if $\sigma(T) = \{0\}$, then T is nilpotent. Also we have, $\phi(T)$ is totally P-posinormal. From Theorem 6.1.6 and Theorem 1.3.7, we have the spectral map σ is continuous on $\mathcal{PB}$. □

6.2 Finite operator

Recall that, $T \in \mathcal{B}(\mathcal{H})$ is *finite* if and only if $\|I - (TX - XT)\| \geq 1, \ \forall X \in \mathcal{B}(\mathcal{H})$. Now we show that every $\mathcal{PB}$ operators are finite.

Theorem 6.2.1. *If $T \in \mathcal{PB}$, then T is a finite operator.*

Proof. First we show that $\sigma_{ja}(T) \neq \emptyset$. Let $z \in \sigma_a(T)$. Then there exist a sequence (x_n) in $\mathcal{H}$ with $\|x_n\| = 1$ such that $\|(T - zI)x_n\| \to 0$ as $n \to \infty$. Since T is totally P-posinormal, we have

$$\|(P(T - zI))^* x_n\| \leq M(z)\|(T - zI)x_n\|.$$

Hence $\|(P(T-zI))^*x_n\| \to 0$ as $n \to \infty$. We have,

$$(P(T-zI))^*x_n = (T-zI)^{*n}x_n + \sum_{j=1}^{n-1} \overline{c_j}(T-zI)^{*j}x_n, \forall n \in \mathbb{N}.$$

Hence, $\overline{c_1}(T-zI)^*x_n = (P(T-zI))^*x_n - (T-zI)^{*n}x_n + \sum_{j=2}^{n-1} -\overline{c_j}(T-zI)^{*j}x_n$. Therefore,

$$\begin{aligned}\|\overline{c_1}(T-zI)^*x_n\| &\leq \|\left(P(T-zI)\right)^* x_n\| + \|(T-zI)^{*n}x_n + \sum_{j=2}^{n-1} \overline{c_j}(T-zI)^{*j}x_n\| \\ &\leq 2\|\left(P(T-zI)\right)^* x_n\|.\end{aligned}$$

Since $\|(P(T-zI))^*x_n\| \to 0$ as $n \to \infty$, we have $\|\overline{c_1}(T-zI)^*x_n\| \to 0$ as $n \to \infty$. Hence $\|(T-zI)^*x_n\| \to 0$ as $n \to \infty$. Therefore, $\overline{z} \in \sigma_a(T^*)$. Thus, $z \in \sigma_{ja}(T)$. Hence $\sigma_a(T) = \sigma_{ja}(T)$. From Theorem 1.3.2, $\partial\sigma(T) \subset \sigma_a(T)$. Hence $\sigma_{ja}(T) \neq \emptyset$. Now by using Theorem 1.3.9, we have T is a finite operator. □

Next we show that if T is a $\mathcal{PB}$ operator, then $R(\delta_T)$ is orthogonal to $N(\delta_T)$. For proving the result we use the following lemma.

Lemma 6.2.1. *If $T \in \mathcal{PB}$ and $A \in \mathcal{B}(\mathcal{H})$ is a normal operator with $AT = TA$. Then*

$$|\lambda| \leq \|A - (TX - XT)\|$$

for all $\lambda \in \sigma_p(A)$ and $X \in \mathcal{B}(\mathcal{H})$.

Proof. Let $\lambda \in \sigma_p(A)$. If $\lambda = 0$, the result trivially holds.
If $\lambda \neq 0$. Let $D_\lambda = N(A - \lambda I)$. Since A is a normal operator with $AT = TA$ and by Fuglede-Putnam theorem 1.3.10, we have $A^*T = TA^*$. Hence D_λ reduces T and A. Thus the matrix representation of T and A on $D_\lambda \oplus D_\lambda^\perp$ is

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda I & 0 \\ 0 & A_2 \end{pmatrix}$$

Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

Hence $A - (TX - XT) = \begin{pmatrix} \lambda I - (T_1X_1 - X_1T_1) & B \\ R & S \end{pmatrix}$,
where $B, R, S \in \mathcal{B}(\mathcal{H})$. Then

$$\begin{aligned} \|A - (TX - XT)\| &\geq \|\lambda I - (T_1X_1 - X_1T_1)\| \\ &= |\lambda| \ \|I - \left(T_1\frac{X_1}{\lambda} - \frac{X_1}{\lambda}T_1\right)\| \end{aligned}$$

Since D_λ is invariant under T and $T_1 = T|_{D_\lambda}$, we have T_1 is a totally P-posinormal operator from Theorem 6.1.1. Also from Theorem 6.2.1, T_1 is a finite operator. Therefore, $\|A - (TX - XT)\| \geq |\lambda|$. □

Theorem 6.2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{PB}$ and $A \in \mathcal{B}(\mathcal{H})$ is a normal operator with $AT = TA$, then $R(\delta_T)$ is orthogonal to $N(\delta_T)$.*

Proof. Let ϕ be the function as mentioned in Theorem 1.3.6. Since A is normal, $\phi(A)$ is normal. Since T is totally P-posinormal and from the proof of Theorem 6.1.5, we have $\phi(T)$ is totally P-posinormal. Also from Theorem 6.2.1, $\phi(T)$ is a finite operator. Since $AT = TA$, $\phi(A)\phi(T) = \phi(T)\phi(A)$. Let $\lambda \in \sigma_p(\phi(A))$. From Theorem 6.2.1, we have

$$|\lambda| \leq \|\phi(A) - (\phi(T)\phi(X) - \phi(X)\phi(T))\| = \|A - (TX - XT)\|, \qquad (6.2)$$

for all $X \in \mathcal{B}(\mathcal{H})$. Since $\phi(A)$ is normal, we have

$$\|\phi(A)\| = \sup_{\lambda \in \sigma(\phi(A))} |\lambda|$$

From Theorem 1.3.6, we have $\sigma_p(\phi(A)) = \sigma(\phi(A))$. Hence from (6.2), we have

$$\|\phi(A)\| \leq \|A - (TX - XT)\|, \text{ for all } X \in \mathcal{B}(\mathcal{H}).$$

Thus,

$$\|A\| \leq \|A - (TX - XT)\|, \text{ for all } X \in \mathcal{B}(\mathcal{H}).$$

□

Chapter 7

Closed densely defined M-hyponormal operators

In this chapter, we define closed densely defined M-hyponormal operator which contains some well known classes of operators namely, closed densely defined hyponormal operators. In this chapter we mainly focus on proving asymmetric Fuglede - Putnam theorem for this classes of operators.

7.1 Closed densely defined M-hyponormal operators

Let $\mathcal{H}$ be a Hilbert Space. We denote the classes of all linear operators and closed linear operatos on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ respectively. Recall that a closed operator $T \in \mathcal{L}(\mathcal{H})$ is said to be densely defined if $\overline{D(T)} = \mathcal{H}$. Recall that a densely defined operator $T \in \mathcal{C}(\mathcal{H})$ is said to be hyponormal if $D(T) \subset D(T^*)$ and $\|T^*x\| \leq \|Tx\|$ for all $x \in D(T)$.

Now, we define a new classes of operators, closed densely defined M-hyponormal operators which contains the class of all closed densely defined hyponormal operators.

Definition 7.1.1. *A densely defined operator $T \in \mathcal{C}(\mathcal{H})$ is said to be M-hyponormal if $D(T) \subset D(T^*)$ and $\| (T - zI)^*x \| \leq M \| (T - zI)x \|$ for all $z \in \mathbb{C}$ and $x \in D(T)$, for some constant $M > 0$.*

In particular, if $z = 0$ and $M = 1$ then the M-hyponormal operator are hyponormal. In general, the converse need not be true.
For example, let $T : l^2(\mathbb{N}) \longrightarrow l^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1, 2x_2, x_3, 4x_4, 5x_5, \cdots).$$

Here the weights are given by $\alpha_j = \begin{cases} 1 & \text{if} \quad j = 1, 3 \\ j & \text{if} \quad j = 2 \text{ and } j \geq 4 \end{cases}$
Let $D(T) = \{(x_1, x_2 \cdots) \in l^2(\mathbb{N}) : \sum_{j=1}^{\infty} |\alpha_j x_j|^2 < \infty\}$. Since $C_{00} \subseteq D(T)$, and C_{00} is dense in $l^2(\mathbb{N})$, $D(T)$ is dense in $l^2(\mathbb{N})$. Since (α_n) is eventually increasing, T is M-hyponormal ([18]). The adjoint of T, T^* is given by

$$T^*(x_1, x_2, x_3, \cdots) = (x_2, 2x_3, 3x_4, 4x_5, 5x_6, \cdots).$$

Let $e_i = (0, 0, \cdots, 1, 0, 0, \cdots)$, where 1 occurs in the i^{th} place. Then

$$Te_1 = e_2, Te_2 = 2e_3, Te_3 = e_4, Te_i = ie_{i+1} \text{ for } i \geqslant 4.$$

$$T^*e_1 = 0, T^*e_2 = e_1, T^*e_3 = 2e_2, T^*e_4 = e_3, T^*e_i = (i-1)e_{i-1} \text{ for } i \geqslant 5.$$

Since $\|T^*e_3\| = 2$ and $\|Te_3\| = 1$, it follows that T is not hyponormal.

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. We define $T|_{\mathcal{M}}$ as an operator on $\mathcal{M}$ with domain

$$D(T|_{\mathcal{M}}) = \{x \in D(T) \cap \mathcal{M} : Tx \in \mathcal{M}\} \text{ and } T|_{\mathcal{M}}x = Tx, \ x \in D(T|_{\mathcal{M}}).$$

Let $B = T|_{\mathcal{M}}$, then we say that $\mathcal{M}$ reduces T to an operator B.
Now we show that restriction of a closed densely defined M- hyponormal operator is again M-hyponormal.

Lemma 7.1.1. *Let $T \in \mathcal{C}(\mathcal{H})$ be a densely defined M-hyponormal operator and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T. Then $T|_{\mathcal{M}}$ is a closed M-hyponormal operator.*

Proof. Let $x \in D(T|_{\mathcal{M}})$ and P be an orthogonal projection onto $\mathcal{M}$. Since T is M-hyponormal, we have

$$\begin{aligned} \| (T|_{\mathcal{M}} - \lambda I)^* x \| &= \| P(T-\lambda I)^* x \| \\ &\leq M \| (T-\lambda I)x \| \\ &= M \| (T|_{\mathcal{M}} - \lambda I)x \| . \end{aligned}$$

Hence $T|_{\mathcal{M}}$ is a closed M-hyponormal operator. □

Now we prove a characterization for closed densely defined M-hyponormal operators.

Lemma 7.1.2. *Let $T \in \mathcal{C}(\mathcal{H})$ be a densely defined M-hyponormal operator. Then there exist a contraction $C_\lambda \in \mathcal{B}(\mathcal{H})$ such that $\frac{1}{M}(T-\lambda I) \subseteq (T-\lambda I)^* C_\lambda$ for every $\lambda \in \mathbb{C}$.*

Proof. Define $K : R(T-\lambda I) \to R(T^* - \overline{\lambda} I)$ by

$$K\left((T-\lambda I)x\right) = \frac{1}{M}(T^* - \overline{\lambda} I)x, \quad \text{for all } x \in D(T).$$

Since T is M-hyponormal, K is a contraction with $K(T-\lambda I) \subseteq \frac{1}{M}(T^* - \overline{\lambda} I)$. Now we extend K to $K' \in \mathcal{B}\left(\overline{R(T-\lambda I)}, \overline{R(T^* - \overline{\lambda} I)}\right)$ such that $K'(T-\lambda I) \subseteq \frac{1}{M}(T^* - \overline{\lambda} I)$. Let $A \in \mathcal{B}(\mathcal{H})$ be defined by $Ax = \begin{cases} K'x & \text{if } x \in \overline{R(T-\lambda I)} \\ 0 & \text{if } x \in \overline{R(T-\lambda I)}^{\perp} \end{cases}$

It is clear that A ia a contraction. Also

$$A(T-\lambda I) \subseteq \frac{1}{M}(T^* - \overline{\lambda} I).$$

Hence,

$$\frac{1}{M}(T-\lambda I) \subseteq (T-\lambda I)^* A^*.$$

Let $A^* = C_\lambda$, then $\frac{1}{M}(T-\lambda I) \subseteq (T-\lambda I)^* C_\lambda$, where C_λ is a contraction. □

Stochel ([52]) proved if $T \in \mathcal{C}(\mathcal{H})$ is hyponormal operator and $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which is invariant under T with $T|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T.

Now we prove this result holds for closed densely defined M-hyponormal operators.

Theorem 7.1.1. *Let $T \in \mathcal{C}(\mathcal{H})$ be a densely defined M-hyponormal operator. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ which is invariant under T and $T|_{\mathcal{M}}$ is normal. Then $\mathcal{M}$ reduces T.*

Proof. Let $\mathcal{H}_1 = \mathcal{M}$ and $\mathcal{H}_2 = \mathcal{M}^{\perp}$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $P_{\mathcal{H}_i}$ denotes the orthogonal projection onto $\mathcal{H}_i$, for $i = 1, 2$. Then T has a block matrix representation,

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{ij} : D(T) \cap \mathcal{H}_j \to \mathcal{H}_i$ is defined by $T_{ij} = P_{\mathcal{H}_i} T P_{\mathcal{H}_j}|_{D(T)\cap\mathcal{H}_j}$ for $j = 1, 2$. Since $\mathcal{M}$ is invariant under T, we have

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}.$$

Let $y \in D(T) \cap \mathcal{M}^{\perp}$. By Lemma 7.1.2, we have

$$\frac{1}{M}(T - \lambda I) \subseteq (T - \lambda I)^* C_\lambda$$

for every $\lambda \in \mathbb{C}$, where C_λ is a contraction. Thus, $R(T - \lambda I) \subseteq R(T - \lambda I)^*$ for every $\lambda \in \mathbb{C}$. From Theorem 1.3.14, there exist a densely defined operator B such that $(T - \lambda I) = (T - \lambda I)^* B$. Then on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ we have

$$\begin{pmatrix} T_{11} - \lambda I & T_{12} \\ 0 & T_{22} - \lambda I \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} (T_{11} - \lambda I)^* & 0 \\ T_{12}^* & (T_{22} - \lambda I)^* \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix},$$

where $B_{ij} : D(T) \cap \mathcal{H}_j \to \mathcal{H}_i$ is defined by $B_{ij} = P_{\mathcal{H}_i} T P_{\mathcal{H}_j}|_{D(T)\cap\mathcal{H}_j}$ for $j = 1, 2$. Hence $T_{12}(y) = (T_{11} - \lambda I)^* B_{12} y$. Then $T_{12}(y) = (T_{11} - \lambda I)^* u$, where $u = B_{12} y \in \mathcal{M}$. Since $T_{11} = T|_{\mathcal{M}}$ is normal, we have $N(T_{11} - \lambda I)^* = N(T_{11} - \lambda I)$. From Theorem 1.3.13, we have $R(T_{11} - \lambda I)^* = R(T_{11} - \lambda I)$. Thus we can choose $v \in D(T) \cap \mathcal{M}$ such that $(T_{11} - \lambda I)^* u = (T_{11} - \lambda I) v$. Therefore, $T_{12}(y) = (T_{11} - \lambda I) v$ for every $\lambda \in \mathbb{C}$.

Hence,

$$T_{12}(y) \in \bigcap_{\lambda \in \mathbb{C}} R(T_{11} - \lambda I).$$

From Theorem 1.3.15, we have $T_{12}(y) = 0$. Hence $T_{12} = 0$. □

Now we prove some properties of closed densely defined M-hyponormal operators.

Theorem 7.1.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Assume that $S \in \mathcal{C}(\mathcal{H})$ is normal and $T \in \mathcal{C}(\mathcal{K})$ is M-hyponormal. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $AS \subseteq TA$. Then the following holds:*

(i) $|A|S \subseteq S|A|$

(ii) If $A \geq 0, N(A) = \{0\}$ *and* $\mathcal{K} = \mathcal{H}$*, then* $S = T$.

Proof. (i) Suppose T is M-hyponormal. Then by Lemma 7.1.2, there exist a contraction C_λ such that

$$\frac{1}{M}(T - \lambda I) \subseteq (T - \lambda I)^* C_\lambda \text{ for all } \lambda \in \mathbb{C}. \tag{7.1}$$

Let E be a spectral measure of S. Let Ω be a compact subset of $\mathbb{C}$ and $x \in R(E(\Omega))$. We have E is regular. Hence from Theorem 1.3.18, it is sufficient to prove that $|A|E(\Omega) = E(\Omega)|A|$ for every compact set Ω of $\mathbb{C}$. Since S is normal, from Theorem 1.3.18, we have $R(E(\Omega))$ reduces $S - \lambda$ and $R(E(\Omega)) \subset D(S - \lambda)$ for every $\lambda \in \mathbb{C}$. Define the function $\psi : \mathbb{C} \setminus \Omega \to R(E(\Omega))$ by

$$\psi(\lambda) = \int_\Omega \frac{1}{(z - \lambda I)} E(dz)x = (S|_{R(E(\Omega))} - \lambda I)^{-1}x, \, \lambda \notin \Omega$$

([52]). Then $x = (S - \lambda I)\psi(\lambda)$, $\lambda \notin \Omega$. Thus, $Ax = A(S - \lambda I)\psi(\lambda)$, for $\lambda \notin \Omega$. Since $AS \subseteq TA$, we have $Ax = (T - \lambda)A\psi(\lambda)$, for $\lambda \notin \Omega$. Hence from (7.1), we get $A^*Ax = MA^*(T - \lambda)^* C_\lambda A\psi(\lambda)$. Since $A(S - \lambda) \subseteq (T - \lambda)A$, for $\lambda \in \mathbb{C}$, $A^*Ax = M(S - \lambda)^* A^* C_\lambda A\psi(\lambda)$, for $\lambda \notin \Omega$. Hence $A^*Ax \in \bigcap_{z \in \mathbb{C} \setminus \Omega^*} R(S^* - z)$, where $\Omega^* = \{z : \overline{z} \in \Omega\}$. From Theorem 1.3.19, we have $E(\mathbb{C} \setminus \Omega)A^*Ax = 0$.

Therefore, $A^*Ax = E(\Omega)A^*Ax$. Since $x \in R(E(\Omega))$ is arbitrary,

$$A^*A(R(E(\Omega))) \subseteq R(E(\Omega)).$$

Since A^*A is selfadjoint, $R(E(\Omega))$ reduces A^*A. From Theorem 1.3.18, we have $A^*AE(\Omega) = E(\Omega)A^*A$, for every compact set Ω of $\mathbb{C}$.

(ii) Since $A \geq 0$, from (i) we have $SAx = ASx = TAx$ for $x \in D(S)$. Thus, $S|_{AD(S)} \subseteq T$. Since $D(S)$ is a core for S, $AD(S)$ is a core for S from Theorem 1.3.16. Hence,

$$\mathcal{G}(S) \subseteq \overline{\mathcal{G}(S|_{AD(S)})} \subseteq \overline{\mathcal{G}(T)} = \mathcal{G}(T),$$

where $\mathcal{G}(S)$ is the graph of S. Therefore, $S \subseteq T$. Thus $D(S) \subset D(T)$. Since S is normal, we have $D(S^*) = D(S)$. Hence $D(T) \subseteq D(T^*) \subseteq D(S)$. Thus, $S = T$.

□

Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we denote $\overline{R(A^*)}$ by $\mathcal{R}(A^*)$ and $\overline{R(A)}$ by $\mathcal{R}(A)$. It is known that $\overline{R(A^*)} = \overline{R(|A|)}$, where $|A| = (A^*A)^{\frac{1}{2}}$ ([16]). It is known that the polar decomposition of A is given by $A = U|A|$ where $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a partial isometry with initial space $\mathcal{R}(A^*)$ and final space $\mathcal{R}(A)$ ([16]). Also $N(U) = N(A)$ and $U|_{\mathcal{R}(A^*)}, A|_{\mathcal{R}(A^*)}$ are in $\mathcal{B}(\mathcal{R}(A^*), \mathcal{R}(A))$. Also $U|_{\mathcal{R}(A^*)}$ is a bounded unique unitary isomorphism from $\mathcal{R}(A^*)$ into $\mathcal{R}(A)$ with

$$U|_{\mathcal{R}(A^*)}\,|A|\,x = A\,x, x \in \mathcal{H} \text{ ([52]).}$$

Lemma 7.1.3. *([52]) For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The following holds:*

(i) $N\left(A|_{\mathcal{R}(A^*)}\right) = \{0\}$.

(ii) $R(A) = R\left(A|_{\mathcal{R}(A^*)}\right)$.

(iii) $A^*|_{\mathcal{R}(A)} = \left(A|_{\mathcal{R}(A^*)}\right)^*$.

(iv) $(|A|)|_{\mathcal{R}(|A|^*)} = \left|A|_{\mathcal{R}(A^*)}\right|$.

Lemma 7.1.4. *([52]) Let T and B be closed densely defined operators in $\mathcal{H}$ and $\mathcal{K}$ respectively and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that $AT^* \subseteq BA$.*
(i) If $\mathcal{R}(A^*)$ reduces T, then $B|_{\mathcal{R}(A)}$ is a closed densely defined operator in $\mathcal{R}(A)$ and

$$A|_{\mathcal{R}(A^*)} \quad (T|_{\mathcal{R}(A^*)})^* \subseteq B|_{\mathcal{R}(A)} \quad A|_{\mathcal{R}(A^*)}.$$

(ii) If $\mathcal{R}(A^*)$ reduces T and $\mathcal{R}(A)$ reduces B to normal operators, then

$$AT \subseteq B^*A, \quad |A|T \subseteq T|A|, \quad |A^*|B \subseteq B|A^*| \text{ and}$$

$$(T|_{\mathcal{R}(A^*)})^* = (U|_{\mathcal{R}(A^*)})^* B|_{\mathcal{R}(A)} \quad U|_{\mathcal{R}(A^*)}.$$

Now we prove the asymmetric Fuglede-Putnam theorem for closed densely defined M-hyponormal and normal operators.

Theorem 7.1.3. *Suppose $S \in \mathcal{C}(\mathcal{H})$ is normal and $T \in \mathcal{C}(\mathcal{K})$ is M-hyponormal. If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $AS \subseteq TA$. Then $\mathcal{R}(A^*)$ reduces S, $\mathcal{R}(A)$ reduces T and $T|_{\mathcal{R}(A)}$, $S|_{\mathcal{R}(A^*)}$ are unitarily equivalent normal operators.*

Proof. Let Ω be a Borel subset of $\mathbb{C}$ and let E be the spectral measure of S. We have $E(\Omega)$ is an orthogonal projection. To prove $\mathcal{R}(A^*)$ reduces S, it is sufficient to prove that $\mathcal{R}(A^*) := \overline{R(|A|)}$ is invariant under $E(\Omega)$.

Let $y \in \overline{R(|A|)}$. Then there exist a sequence $(y_n) \in R(|A|)$ such that y_n converges to y. Since $E(\Omega)$ is bounded, $E(\Omega)y_n$ converges to $E(\Omega)y$. Since $y_n \in R(|A|)$, there exist $x_n \in D(|A|)$ such that $y_n = |A|x_n$. Therefore, $E(\Omega)|A|x_n$ converges to $E(\Omega)y$. From Theorem 7.1.2 (i), $|A|E(\Omega) = E(\Omega)|A|$. Hence, $|A|E(\Omega)x_n$ converges to $E(\Omega)y$. Thus $\mathcal{R}(A^*)$ is invariant under $E(\Omega)$.

Since $AS \subseteq TA$ and $\mathcal{R}(A^*)$ reduces S, from Lemma 7.1.1, we have $T|_{\mathcal{R}(A)}$ is a closed densely defined operator in $\mathcal{R}(A)$ and

$$A|_{\mathcal{R}(A^*)} \quad S|_{\mathcal{R}(A^*)} \subseteq T|_{\mathcal{R}(A)} \quad A|_{\mathcal{R}(A^*)}. \tag{7.2}$$

Since T is M-hyponormal, $T|_{\mathcal{R}(A)}$ is a closed M-hyponormal operator in $\mathcal{R}(A)$.

Since $U|_{\mathcal{R}(A^*)}$ is unitary, we have

$$| A|_{\mathcal{R}(A^*)} | \ \ S|_{\mathcal{R}(A^*)} \subseteq (U|_{\mathcal{R}(A^*)})^* \ \ T|_{\mathcal{R}(A)} \ \ U|_{\mathcal{R}(A^*)} \ \ |A|_{\mathcal{R}(A^*)}|. \tag{7.3}$$

Let $W = U|_{\mathcal{R}(A^*)}$ and $V = W^* \ T|_{\mathcal{R}(A)} \ W$. Also we have W is unitary isomorphism and $T|_{\mathcal{R}(A)}$ is M-hyponormal. Then for $x \in \mathcal{R}(A^*)$,

$$\begin{aligned} \| (V - zI)^* x \| &= \| W^*(T|_{\mathcal{R}(A)} \ - zI)^* \ Wx \| \\ &= \| \ (T|_{\mathcal{R}(A)} \ - zI)^* \ Wx \| \\ &\leq M \ \| \ (T|_{\mathcal{R}(A)} \ - zI) \, Wx \| \\ &= M \ \| \ W^*(T|_{\mathcal{R}(A)} \ - zI) Wx \| \\ &= M \ \| \ (V - zI)x \| \ . \end{aligned}$$

Hence $(U|_{\mathcal{R}(A^*)})^* \ \ T|_{\mathcal{R}(A)} \ \ U|_{\mathcal{R}(A^*)}$ is a closed M-hyponormal operator.
Also, we have $N(| A|_{\mathcal{R}(A^*)} |) = N(A|_{\mathcal{R}(A^*)})$. From Lemma 7.1.3, $N(A|_{\mathcal{R}(A^*)}) = \{0\}$. Thus $N(| A|_{\mathcal{R}(A^*)} |) = \{0\}$. From (7.3) and Theorem 7.1.2 (ii), we get $S|_{\mathcal{R}(A^*)} = (U|_{\mathcal{R}(A^*)})^* \ T|_{\mathcal{R}(A)} \ U|_{\mathcal{R}(A^*)}$. Thus, $T|_{\mathcal{R}(A)}, \ S|_{\mathcal{R}(A^*)}$ are unitarily equivalent normal operators. From Theorem 7.1.1, $\mathcal{R}(A)$ reduces T.

□

Stochel ([52]) proved the following result for closed hyponormal and closed subnormal operators. Now we extend the result to closed M-hyponormal and closed subnormal operators.

Theorem 7.1.4. *Let $B \in \mathcal{C}(\mathcal{H})$ be subnormal (resp. a closed M-hyponormal operator in $\mathcal{H}$), $T \in \mathcal{C}(\mathcal{K})$ be M-hyponormal (resp. a closed subnormal operator in $\mathcal{K}$) and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $AB^* \subseteq TA$. Then*

(i) $AB \subseteq T^*A$.

(ii) $\mathcal{R}(A^*)$ *reduces* B *to the normal operator* $B|_{\mathcal{R}(A^*)}$.

(iii) $\mathcal{R}(A)$ *reduces* T *to the normal operator* $T|_{\mathcal{R}(A)}$.

Proof. Assume that $B \in \mathcal{C}(\mathcal{H})$ is subnormal and $T \in \mathcal{C}(\mathcal{K})$ is M-hyponormal. Since B is subnormal, there exist a normal extension S on the Hilbert Space $\mathcal{L} \supseteq \mathcal{H}$. Define $Y \in \mathcal{B}(\mathcal{K}, \mathcal{L})$ by $Yx = A^*x, x \in \mathcal{K}$.
Let $J \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ defined by $Jx = x, x \in \mathcal{H}$. Then for $x \in \mathcal{K}$, $JA^*x = A^*x = Yx$. Thus, $Y = JA^*$. Hence, $Y^* = AP$, where P is an orthogonal projection of $\mathcal{L}$ on to $\mathcal{H}$. Hence, $\mathcal{R}(Y^*) = \mathcal{R}(A)$. Since $AB^* \subseteq TA$, we have

$$A^*T^* \subseteq BA^*. \tag{7.4}$$

Since from equation (7.4) and $Yx = A^*x$, we have

$$\begin{aligned} YT^*x &= A^*T^*x \\ &= BA^*x \\ &= SA^*x \\ &= SYx, \; x \in \mathcal{D}(T^*). \end{aligned}$$

Thus $YT^* \subseteq SY$. Hence $Y^*S^* \subseteq TY^*$. Since $\mathcal{R}(Y^*) = \mathcal{R}(A)$, we have $T|_{\mathcal{R}(A)}$ and $S|_{\mathcal{R}(A^*)}$ are unitarily equivalent normal operators by Theorem 7.1.3. Thus, $\mathcal{R}(A)$ reduces T to the normal operator $T|_{\mathcal{R}(A)}$.
Since $A^*T^* \subseteq BA^*$ and $\mathcal{R}(A)$ reduces T, we have $B|_{\mathcal{R}(A^*)}$ is closed densely defined in $\mathcal{R}(A^*)$ and

$$(A|_{\mathcal{R}(A^*)})^* \;\; (T|_{\mathcal{R}(A)})^* \subseteq B|_{\mathcal{R}(A^*)} \;\; (A|_{\mathcal{R}(A^*)})^* \tag{7.5}$$

by Lemma 7.1.4 (i) and Lemma 7.1.3. Since $B|_{\mathcal{R}(A^*)} \subseteq B \subseteq S$, $B|_{\mathcal{R}(A^*)}$ is subnormal. Then $B|_{\mathcal{R}(A^*)}$ is M-hyponormal. Thus, $\mathcal{R}(A|_{\mathcal{R}(A^*)})^* = \mathcal{R}(A^*)$ reduces $B|_{\mathcal{R}(A^*)}$ to the normal operator from equation (7.5) and Theorem 7.1.3. Hence by Theorem 7.1.1, $\mathcal{R}(A^*)$ reduces B to the normal operator $B|_{\mathcal{R}(A^*)}$. The result (i) follows from Lemma 7.1.4 (ii).
Next we assume that $B \in \mathcal{C}(\mathcal{H})$ is M-hyponormal and $T \in \mathcal{C}(\mathcal{K})$ is subnormal. Since $A^*T^* \subseteq BA^*$, $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ are reducing subspace for T and B respectively. Also the part $T|_{\mathcal{R}(A)}$ and $B|_{\mathcal{R}(A^*)}$ are normal. By Proposition 7.1.4 (ii), $AB \subseteq T^*A$. □

Corollary 7.1.1. *Let $B \in \mathcal{C}(\mathcal{H})$ be subnormal (resp. a closed M-hyponormal operator in $\mathcal{H}$), $T \in \mathcal{C}(\mathcal{K})$ be M-hyponormal (resp. a closed subnormal operator in $\mathcal{K}$) and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $AB^* \subseteq TA$. Then*

(i) If $N(A) = \{0\}$, then B is normal.

(ii) If $N(A^*) = \{0\}$, then T is normal.

Proof. Assume that $N(A) = \{0\}$. From Theorem 1.3.13, we have $R(A^*)^{\perp} = \{0\}$. Thus $\overline{R(A^*)} = \mathcal{H}$. That is, $\mathcal{R}(A^*) = \mathcal{H}$. From Theorem 7.1.4 (ii), B is normal.
Assume that $N(A^*) = \{0\}$. From Theorem 1.3.13, we have $R(A)^{\perp} = \{0\}$. Thus, $\overline{R(A)} = \mathcal{K}$. That is, $\mathcal{R}(A) = \mathcal{K}$. From Theorem 7.1.4 (iii), T is normal. □

Conclusion

In this thesis, we introduced some new classes of bounded operators namely k-quasi (m, n)-paranormal, (m, n)-class $\mathcal{Q}$, k-quasi (m, n)-class $\mathcal{Q}$ operators which are the extensions of (m, n)-paranormal operators. Also we introduced k-quasi $(m, n)^*$-paranormal, (m, n)-class $\mathcal{Q}^*$ and k-quasi (m, n)-class $\mathcal{Q}^*$ operators which contains $(m, n)^*$-paranormal operators. Here we obtained some characterizations and matrix representations of these classes of operators with proper illustrations. Next we defined totally $(m, n)^*$-paranormal, which is having nice characteristics, like translation invariance and finiteness. Also, we studied some properties of polynomially P-posinormal operators namely, finiteness, spectral continuity etc. Finally, we introduced a closed densely defined M- hyponormal operator and proved asymmetric Fuglede- Putnam theorem for this class of operators.

Recommendations

In this thesis, we defined some new classes of operators which are the extensions of well defined classes of operators such as (m, n)-paranormal, $(m, n)^*$-paranormal and closed densely defined hyponormal operators. In future, we are aim to introduce some new classes of operators which are extensions of newly introduced class of operators, which are having nice properties like finiteness, spectral continuity, matrix representation etc. Also we try to introduce a closed densely defined dominant operator, which contains closed densely defined M- hyponormal operator and prove asymmetric Fuglede-Putnam theorem for this operators.

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List of publications

- Shine Lal Enose, Ramya Perumal and Prasad Thankarajan, *Some classes of operators related to* (m,n)*-paranormal and* $(m,n)*$*-paranormal operators,* Commun. Korean Math. Soc. Vol. 38(4), (2023), 1075–1090. https://doi.org/10.4134/CKMS.c220355

- P. Ramya, T. Prasad and E. Shine Lal, *On* (m,n)*-class* Q *and* (m,n)*-class* Q^* *operators,* Palesteine J. Math. Vol. 12 (2), (2023), 353–360.

- E Shine Lal, T Prasad and P Ramya *On the class of totally polynomially posinormal operators,* Aust. J. Math. Anal. Appl. Vol. 20 (2023), No. 1, Art. 9, 7 pp.

- T Prasad, E Shine Lal and P Ramya *Asymmetric Fuglede-Putnam theorem for unbounded M-hyponormal operators,* Advances in Operator Theory, 8, Article no 4, 2023. https://doi.org/10.1007/s43036-022-00231-z

List of presentations

- Presented a paper titled " Some Properties of Unbounded *-Paranormal Operators" in the International Conference on Research Trends in Mathematics (ICRTM 2020) , organized by the Division of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai, India held during August 25-26, 2020, through online platform.

- Presented a paper titled "Tikhonov Type Regularization for Unbounded Operators" in the International Conference on Mathematical Analysis and Application in online mode during November 02-04, 2020, organized by the Department of Mathematics, National Institute of Technology, Jamshedpur, India.

- Presented a paper titled"Riesz Projection for Closed M-hyponormal Operators" in the International Research Conclave on Advances in Science and Technology-2023, organized by Arunai International Research Foundation, Tiruvannamalai, India held during February 18-19, 2023, through online platform.