MOMENTUM MAPS ON SYMPLECTIC MANIFOLDS

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by

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CERTIFICATE

I hereby certify that the work presented in this thesis entitled **MOMENTUM MAPS ON SYMPLECTIC MANIFOLDS** is a bonafide work carried out by Mrs. FAZEELA.K.(KAITHACKAL), under my guidance for the award of the Degree of Ph.D in Mathematics, of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Calicut University **Dr. K.S.SUBRAMANIAN MOOSATH**

Date : **Supervising Guide**

DECLARATION

I hereby declare that the work presented in this thesis entitled **MOMENTUM MAPS ON SYMPLECTIC MANIFOLDS** is based on the original work done by me under the guidance of Dr. K.S.SUBRAMANIAN MOOSATH, Reader, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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CERTIFICATE

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MOMENTUM MAPS ON SYMPLECTIC MANIFOLDS

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Contents

Chapter 0

Introduction and Summary

Symmetries are used to study the dynamics of a physical system. In classical mechanics symmetries are usually induced by point transformations, that is, they come exclusively from symmetries of the configuration space. The symmetry based techniques are implemented using integrals of motion which are quantities that are conserved along the flow of that system. This idea can be generalized to many symmetries of the entire phase space of the dynamical system. This is done by associating a map from the phase space to the dual of the Lie algebra of the Lie group which is acting on the phase space encoding the symmetry. This map, whose level sets are preserved by the dynamics of any symmetric system is referred as the Momentum map (Standard Momentum map) of the symmetry. Momentum maps are at the centre of many geometrical facts that are useful in variety of fields of both pure and applied Mathematics. Also these maps are very much useful in Physics and Engineering applications.

This thesis grew out of our study on Momentum maps. In this thesis we present the existence results, elementary properties, convexity properties and certain generalizations of the standard momentum maps. The thesis contains four chapters.

In the first chapter main focus is on the standard momentum map associated to Lie group action on a symplectic manifold. The concept of momentum map and some existence results are given. Also some elementary properties of momentum map are discussed. This chapter contains five sections.

In the first section we have given the basic ideas required. Many of the standard results are recalled. First we give the definitions of Lie group action, proper action, Lie algebra action, symplectic manifold, symplectomorphism, Lagrange submanifold. Also we state some basic theorems on symplectic manifolds. Then the definitions of Hamiltonian vector field, Hamiltonian functions, Poisson manifold, Poisson tensor, canonical mappings, Hamiltonian and Poisson dynamical systems are given. The canonical Lie group and Lie algebra actions, almost Hamiltonian actions and Hamiltonian actions are also discussed.

In section 2 the notion of Noether momentum map on a symplectic manifold is introduced.

Definition 1.2.1 Let $(M, \{.,.\})$ be a Poisson manifold and G (respectively $\mathcal{G}$) a Lie group (respectively Lie algebra) acting canonically on it. Let S be a set and $J : M \rightarrow S$ a map. We say that J is a *Noether momentum map* for the G-action (respectively $\mathcal{G}$-action) on $(M, \{.,.\})$ when the flow F_t of any Hamiltonian vector field associated to any G-invariant (respectively $\mathcal{G}$-invariant) Hamiltonian function $h \in C^\infty(M)$ preserves the fibers of J. That is,

$$J \circ F_t = J\mid_{Dom(F_t)} .$$

Then given Chu momentum map whose definition makes essential use of the symplectic structure and some properties are also proved in section 3.

Definition 1.3.1 Let (M, ω) be a symplectic manifold and $\mathcal{G}$ be a Lie algebra acting canonically on it. The *Chu map* is defined as the map $\Psi : M \to Z^2(\mathcal{G})$ given by

$$\Psi(m)(\xi, \eta) \quad = \quad \omega(m)(\xi_M(m), \eta_M(m)),$$

for every $\xi, \eta \in \mathcal{G}$. The fact that Ψ maps into $Z^2(\mathcal{G})$ is a consequence of the closedness of the symplectic form ω and the canonical character of the $\mathcal{G}$- action.

Apart from its intrinsic interest as a Noether momentum map, this construction will be extremely important in the statement and proof of a symplectic version of slice theorem, presented in chapter 3.

In section 4 the standard momentum map, whose values are in the dual of Lie algebra of symmetries, is given.

Definition 1.4.1 Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold (M, ω). Suppose that for any $\xi \in \mathcal{G}$, the vector field ξ_M, (infinitesimal generator) is globally Hamiltonian with Hamiltonian function $J^\xi \in C^\infty(M)$. The map $J : M \to \mathcal{G}^*$ defined by the relation

$$< J(z), \xi >= J^\xi(z),$$

for all $\xi \in \mathcal{G}$ and $z \in M$, is called a *standard momentum map* or simply a momentum map of the $\mathcal{G}$-action.

After giving examples of such momentum maps, we consider the problem of existence of momentum maps. Its existence is guaranteed when the infinitesimal generators of this action are Hamiltonian vector fields. In other words, if the Lie algebra $\mathcal{G}$ acts canonically on the Poisson manifold $(M, \{.,.\})$, then for each $\xi \in \mathcal{G}$, we require the existence of a globally defined function $J^\xi \in C^\infty(M)$ such that $\xi_M = X_{J^\xi}$. In general this is not guaranteed even if there is a canonical Lie algebra action.

Then various situations on the existence of such momentum maps is given. The following result characterizes the existence of momentum maps in the symplectic case.

Proposition 1.4.8 Let (M, ω) be a symplectic manifold and $\mathcal{G}$ a Lie algebra acting canonically on it . There exists a momentum map associated to this action if and only if the linear map $\rho : \frac{\mathcal{G}}{[\mathcal{G},\mathcal{G}]} \to H^1(M, \Re)$, by $\rho([\xi]) = [i_{\xi_M}\omega]$ is identically zero.

As a consequence of this proposition we have the following theorem.

Theorem 1.4.9 Let (M, ω) be a symplectic manifold and $\mathcal{G}$ be a Lie algebra acting canonically on it . There exists a momentum map associated to this action if and only if one of the following is true.

(i) $H^1(M, \Re) = 0$.

(ii) $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$.

(iii) $H^1(\mathcal{G}, \Re) = 0$.

(iv) $\mathcal{G}$ is semisimple.

Then we look at the coadjoint or G-equivariant momentum maps. Existence results of such momentum maps are given using fixed points of the action, Lie algebra cohomology, G-invariant 1-form on M and compact Lie group action. The existence of coadjoint equivariant moment maps for the action of semidirect product $G_1 \times_\sigma G_2$ is also given using conditions on G_1. We have proved two theorems on the existence of coadjoint equivariant momentum maps on the product manifold.

Theorem 1.4.19 Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds and let G be a Lie group acting canonically on both M_1 and M_2 . Suppose the above actions admit coadjoint equivariant momentum maps. Then G has a coadjoint equivariant momentum map on $(M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$ where π_1 and π_2 are projections on M_1 and M_2 respectively.

Theorem 1.4.20 Let G_1 and G_2 be Lie groups acting canonically on connected symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with G_1 is connected and $H^1(\mathcal{G}_1, \Re) = 0$. Suppose the above actions admits coadjoint equivariant momentum maps. If $G = G_1 \times_\sigma G_2$ then G has a coadjoint equivariant momentum map on $(M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$ where π_1 and π_2 are projections on M_1 and M_2 respectively.

Infinitesimally equivariant momentum map is defined for the Lie algebra action and the results on existence using the extension of Lie algebra is also given.

In general it is not possible to choose a coadjoint equivariant momentum map, but we could ask whether one can define another action on this space with respect to which we have equivariance.

Definition Let G be a Lie group acting canonically on the connected symplectic manifold (M, ω) with associated momentum map $J : M \to \mathcal{G}^*$. If $\sigma : G \longrightarrow \mathcal{G}^*$ is the non-equivariance one cocycle of J, we define the *affine action* of G on $\mathcal{G}^*$ with cocycle σ by

$$\Theta : G \times \mathcal{G}^* \longrightarrow \mathcal{G}^*, \text{ given by}$$
$$\Theta(g, \mu) = Ad^*_{g^{-1}}\mu + \sigma(g).$$

Proposition 1.4.34 : The affine action Θ of G on $\mathcal{G}^*$ determines a left action. The momentum map $J : M \to \mathcal{G}^*$ is equivariant with respect to the symplectic action ϕ on M and the affine action Θ on $\mathcal{G}^*$.

In the last section we discuss certain properties of momentum maps. First we prove that the momentum map J is a submersion on the open dense subset of principal orbits in M. Then the Noether's theorem, that is, they are constant on the dynamics of any symmetric Hamiltonian vector field is given. An equivalent condition for the moment map to be constant on the orbits is given. We establishes a link between the symmetry of a point and the rank of the momentum map at the point, called bifurcation lemma. Also proved that the zero level set of the moment map is locally arc wise connected.

In chapter 2 we consider the action of a torus T^n on a symplectic manifold (M, ω). Hamiltonian actions of tori of maximal dimension are a special case of integrable systems. More than that they are the local form of all integrable systems with compact level sets. Convexity property of momentum map for the torus action using Morse theory is also discussed. This chapter contains 2 sections. In the first section we define Hamiltonian torus action and give examples of it. Then we prove a Hamiltonian circle action on a compact symplectic manifold has fixed point.

One of the most striking aspects of momentum maps is the convexity properties of its image. In section 2 we discuss the convexity properties of momentum map. We discuss the developments in this area starting from the first convexity result by Atiyah, Guillemin and Sternberg. They have proved the convexity theorem for compact M on which a torus acts in a Hamiltonian fashion. Then Guillemin and Sternberg conjectured and partially proved the convexity theorem to actions of non-abelian compact groups on compact manifolds. This was completely proved by Kirwan .

First we strengthen the Poincare Lemma to deal with invariant forms. Then Darboux theorem for momentum maps is given using the G-relative Darboux theorem.

Theorem 2.2.9 If J is the momentum map for a Hamiltonian action of a torus T on the symplectic manifold (M, ω) and m a T-fixed point then there is an invariant neighborhood U of m in M and a neighborhood U' of $J(m)$ in $\mathcal{T}^*$ such that $J(U)$ is $U' \cap (J(m) + C(\alpha_1, \alpha_2, ..., \alpha_n))$ where $\mathcal{T}$ is the Lie algebra of T and $C(\alpha_1, \alpha_2, ..., \alpha_n))$ is the positive cone spanned by the weights $\alpha_1, \alpha_2, ..., \alpha_n$ of the action of T on M.

Now to improve it again, that is, for M compact $J(M)$ is a compact convex polytope, Morse theory is used. Then Atiyah - Guillemin -Sternberg convexity theorem is proved.

Theorem 2.2.27 Let (M, ω) be a compact connected symplectic manifold, and let T be a torus acts in a Hamiltonian fashion with associated invariant momentum map $J : M \to \mathcal{T}^*$. Here $\mathcal{T}$ denotes the Lie algebra of T and $\mathcal{T}^*$ its dual. Then the image $J(M)$ of J is a compact convex polytope, called the T-momentum polytope. Moreover, it is equal to the convex hull of the image of the fixed point set of the T-action. The fibers of J are connected.

As a corollary of this convexity theorem, if the T -action is effective, then there must be at least $m + 1$ fixed points and $dimM \geq 2m$ where $2m$ is the dimension of the torus.

Then prove the convexity theorem to actions of non-abelian compact groups on compact manifolds.

Theorem 2.2.30 Let M be a compact connected symplectic manifold on which the compact connected Lie group G acts in a Hamiltonian fashion with associated equivariant momentum map $J : M \to \mathcal{G}^*$. Here $\mathcal{G}$ denotes the Lie algebra of G and $\mathcal{G}^*$ is its dual. Let T be a maximal torus of G, $\mathcal{T}$ its Lie algebra, $\mathcal{T}^*$ its dual, and $\mathcal{T}^*_+$ the positive Weyl chamber relative to a fixed ordering of the roots. Then $J(M) \cap \mathcal{T}^*_+$ is a compact convex polytope, called the G-momentum polytope. The fibers of J are connected.

In general Morse theory is not sufficient to study convexity properties of the image of the momentum map. The case of compact symplectic manifolds is rich but quite particular. For noncompact manifolds the results in the previous chapter no longer hold. Convexity results to compact group actions on noncompact manifolds with proper momentum maps were given by Condevaux, Dazord, and Molino and later by Hilgert, Neeb and Plank. The Lokal-global-prinzip is the main tool in these works. Yael Karshon And Christina Marshall gave a generalization of Lokal-global-prinzip for a proper map. But Petre Birtea, Juan-Pablo Ortega and Tudor S.Ratiu gave a generalization of Lokal-global-prinzip for a closed map. Using this, many stronger results in convexity are obtained.

In chapter 3 we discuss the convexity property for a general Lie group action using topological properties. This chapter contains 3 sections. The essential attributes underlying the convexity theorems for momentum maps are the openness of the map onto its image and the local convexity data. The classical convexity theorems given in Chapter 2 are also satisfy these conditions. In this chapter more general theorems on convexity are given using the topological ingredients.

To do convexity results using topological properties we need normal form for the momentum map which we have discussed in section 1. Most of the technical behavior of proper Lie group action is a direct consequence of the existence of slices and tubes; they provide a privileged system of semiglobal coordinates in which the group action takes on a particularly simple form. Proper symplectic Lie group actions turnout to behave similarly: the tubular chart can be constructed in such a way that the expression of the symplectic form is very natural and, moreover, if there is a momentum map associated to this canonical action, this construction provides a normal form for it. We start with the Witt-Artin decomposition of the tangent space. Then the construction of a symplectic tube (Y_r, ω_{Y_r}) at a point m of a symplectic manifold (M, ω) is given. Then the symplectic slice theorem is given.

Theorem 3.1.7 Let (M, ω) be a symplectic manifold and G be a Lie group acting properly and canonically on it. Let $m \in M$, and let (Y_r, ω_{Y_r}) be the G-symplectic tube at that point constructed in proposition 3.1.6. Then there is a G- invariant neighborhood U of m in M and a G-equivariant symplectomorphism $\phi : U \longrightarrow Y_r$ satisfying $\phi(m) = [e, 0, 0]$.

Tubewise Hamiltonian action is defined and sufficient conditions for the action to be tubewise Hamiltonian also given. Then the expression of the momentum map in the slice coordinate, which is usually referred to as the Marle-Guillemin-Sternberg normal form is given.

Theorem 3.1.12 Let (M, ω) be a connected symplectic manifold and G be a Lie group acting properly and canonically on it. Suppose that this action has an associated momentum map $J : M \longrightarrow \mathcal{G}^*$ with non equivariance cocycle $\sigma : G \longrightarrow \mathcal{G}^*$. Let $m \in M$, and (Y_r, ω_{Y_r}) be the symplectic tube at m that models a G-invariant open neighborhood U of the orbit $G.m$ via the G-equivariant symplectomorphism $\phi : (U, \omega_U) \longrightarrow (Y_r, \omega_{Y_r})$. Then the canonical left G-action on (Y_r, ω_{Y_r}) admits a momentum map $J_{Y_r} : Y_r \longrightarrow \mathcal{G}^*$ given by the expression

$$J_{Y_r} : Y_r = G \times_{G_m} (\mathcal{M}_r)^* \times V_r \longrightarrow \mathcal{G}^* \text{given by}$$
$$J_{Y_r}([g, \rho, \upsilon]) = Ad^*_{g^{-1}}(J(m) + \rho + J_V(\upsilon)) + \sigma(g).$$

The map $J_{Y_r} \times \phi$ is a momentum map for the canonical G-action on (U, ω_U). Moreover, if the group G is connected, this momentum map satisfies $J \mid_U = J_{Y_r} \times \phi$.

In section 2 we discuss the convexity properties of the image of the momentum map. We give the statement of Lokal-global-prinzip and a generalization of it for a closed map using some topological vector space results.

Theorem 3.2.14 Let $f : X \to V$ be a closed map with values in a finite dimensional Euclidean vector space V and X a connected, locally connected, first countable, and normal topological space. Assume that f has local convexity data

and is locally fiber connected. Then
(i) All the fibers of f are connected.
(ii) f is open on to its image.
(iii) The image $f(X)$ is a closed convex set.

Using this we obtained that the convexity is rooted on the map being open onto its image and having local convexity data. Next we look at the convexity for momentum maps. Then a generalization of Atiyah-Guillemin-Sternberg Convexity theorem for non compact manifolds is given.
Theorem 3.2.21 Let M be a paracompact connected symplectic manifold on which a torus T acts in a Hamiltonian fashion. Let $J_T : M \to \mathcal{T}^*$ be an associated momentum map which we suppose is closed. Then the image $J_T(m)$ is a closed convex locally polyhedral subset in $\mathcal{T}^*$. The fibers of J_T are connected and J_T is open on to its image.

Further generalization of convexity results are obtained in two cases: when the momentum map has connected fibers and the case when the momentum map has only the locally fiber connectedness property. In the first case we have the theorem :
Theorem 3.2.28 Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action which has connected fibers. Then J_T is open on to its image if and only if $\mathbf{C}J_T(M^{reg}) := J_T(M)\backslash J_T(M^{reg})$ (where M^{reg} denotes the union of all regular orbits) does not disconnect any region in $J_T(M)$. Moreover, the image of the momentum map is locally convex and locally polyhedral.

In the second case suppose that M_{J_T} is a Hausdorff space. Then we have the following theorem.
Theorem 3.2.34 Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action of a connected symplectic manifold (M, ω). Suppose that M_{J_T} is a Hausdorff space. Then J_T is open on to its image if and only if $J_T(M)$ is locally compact,

$\mathbf{C}J_T(M^{reg})$ does not disconnect any region in $J_T(M)$, and J_T satisfies the connected component fiber condition . Moreover under these hypothesis the image of the momentum map is locally convex and locally polyhedral.

Next a generalization of Kirwan's convexity result for a paracompact connected symplectic manifold (M, ω) is given.

Theorem 3.2.39 Let M be a paracompact connected Hamiltonian G-manifold with G a compact connected Lie group. If the momentum map J_G is closed then $J_G(M) \cap \mathcal{T}_+^*$ is a closed convex locally polyhedral set. Moreover, J_G is G-open on to its image and all its fibers are connected.

Also we prove non-abelian analogues of the Theorems 3.2.28 and 3.2.34.

Pull backs by J of smooth functions on $\mathcal{G}^*$ are called collective functions. A collective function is clearly constant on the level sets of the momentum map. The converse need not be true. A momentum map has the division property if any smooth function on M that is locally constant on the level set of Φ is a collective function. In section 3 we generalize a result on division property of momentum map by replacing the compactness of the Lie group with proper and effective action.

Theorem 3.3.13 Let G be a connected abelian Lie group acting properly and effectively on a connected symplectic manifold (M, ω). Let $J : M \longrightarrow \mathcal{G}^*$ be a proper momentum map associated to this action. Then J has the division property if and only if every smooth function on M that is locally constant on the level sets of J is a formal pull back with respect to J.

Then we prove that Torus action has division property if J_T is closed and semi-proper.

Theorem 3.3.17 Let M be a paracompact connected symplectic manifold on which a torus $\mathcal{T}$ acts in a Hamiltonian fashion. If the associated momentum map

$J_{\mathcal{T}}$ is closed and semi proper as a map into some open subset of $\mathcal{T}^*$, then J has the division property.

Then considered the general case. Let G be a compact connected Lie group acting on a compact connected symplectic manifold M in a Hamiltonian fashion with a momentum map $J : M \longrightarrow \mathcal{G}^*$. Put a G-invariant metric on $\mathcal{G}^*$, and use it to identify $\mathcal{G}^*$ with $\mathcal{G}$. Let $\mathcal{G}_{reg}$ be the elements of $\mathcal{G}$ whose stabilizers under the coadjoint action of G are tori, that is, if,

$$\mathcal{G}_{reg} = \{\xi \in \mathcal{G} : \text{stabilizer of } \xi \text{ is a torus }\}.$$

Then we prove the following theorem.

Theorem 3.3.21 Let M be a paracompact connected symplectic Hamiltonian G-manifold with G a compact connected Lie group. If the associated momentum map J is closed and semi proper as a map into some open subset of $\mathcal{G}^*$, then J has the division property if the image $J(M)$ is contained the $\mathcal{G}^*_{reg}$.

In chapter 4 we discuss certain generalizations of standard momentum map. This chapter contains 3 sections. The first section is on cylinder valued momentum maps, which has the important property of being always defined, unlike the standard momentum map. To introduce cylinder valued momentum maps, we need connections on a principal fiber bundle. Then we define holonomy bundle and some properties are discussed. The definition of cylinder valued momentum map is given as a generalization of the standard momentum map.

Definition 4.1.12 For $(z, \mu) \in M \times \mathcal{G}^*$, let $M \times \mathcal{G}^*(z, \mu)$ be the holonomy bundle through (z, μ) and let $\hbar(z, \mu)$ be the holonomy group of α with reference point (z, μ). The reduction theorem guarantees that $(M\times\mathcal{G}^*(z, \mu), M, \pi/_{M\times\mathcal{G}^*(z,\mu)}, \hbar(z, \mu))$ is a reduction of $(M \times \mathcal{G}^*, M, \pi, \mathcal{G}^*)$ For simplicity we use $(\widetilde{M}, M, \widetilde{P}, \hbar)$ instead of $(M \times \mathcal{G}^*(z, \mu), M, \pi/_{M\times\mathcal{G}^*(z,\mu)}, \hbar(z, \mu))$. Let $\widetilde{K} : \widetilde{M} \subset M \times \mathcal{G}^* \to \mathcal{G}^*$ be the projection into the $\mathcal{G}^*$-factor.

Consider now the closure $\overline{\hbar}$ of $\hbar$ in $\mathcal{G}^*$. Since $\overline{\hbar}$ is a closed subgroup of $(\mathcal{G}^*, +)$, the quotient $D := \frac{\mathcal{G}^*}{\overline{\hbar}}$ is a cylinder, that is, it is isomorphic to the abelian Lie group $\Re^a \times T^b$ for some $a, b \in \aleph$. Let $\pi_D : \mathcal{G}^* \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ be the projection. Define $K : M \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{K}} & \mathcal{G}^* \\ \widetilde{P} \downarrow & & \downarrow \pi_D \\ M & \xrightarrow{K} & \dfrac{\mathcal{G}^*}{\overline{\hbar}}. \end{array}$$

In other words, K is defined by $K(m) = \pi_D(v)$, where $v \in \mathcal{G}^*$ is any element such that$(m, v) \in C$. This is well defined because if we have two points $(m, v), (m, v^{'}) \in \widetilde{M}$, then $(m, v), (m, v^{'}) \in \widetilde{P}^{-1}(m)$, that is, there exists $\rho \in \hbar$ such that $v' = v + \rho$. So $\pi_D(v) = \pi_D(v')$.

Then the map $K : M \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ is referred as a *cylinder valued momentum map* associated to the canonical $\mathcal{G}$ action on (M, ω). The definition of K depends on the choice of the holonomy bundle, that is, if we let $\widetilde{M_1}$ and $\widetilde{M_2}$ are two holonomy bundles of $(M \times \mathcal{G}^*, M, \pi, \mathcal{G}^*)$. Then

$$K_{\widetilde{M_1}} = K_{\widetilde{M_2}} + \pi_D(\tau)$$

where $\tau \in \mathcal{G}^*$.

We look at certain properties of Cylinder valued momentum maps. Cylinder valued momentum maps are genuine generalizations of the standard ones in the sense that whenever a Lie algebra action admits a standard momentum map, there is a cylinder valued momentum map that coincides with it.

Proposition 4.1.16 Let (M, ω) be a connected paracompact symplectic manifold and $\mathcal{G}$ a Lie algebra acting canonically on it. Let $K : M \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ be a cylinder valued momentum map . Then there exists a standard momentum map if and only if

$\hbar = \{0\}$. In this case K is a standard momentum map.

In section 2 we discuss Lie group valued momentum maps. We define Lie group valued momentum maps and then show that it is a Noether Momentum Map.

Definition 4.2.1 Let G be an Abelian Lie Group whose Lie algebra $\mathcal{G}$ acts canonically on a symplectic manifold (M, ω). Let $(.,.)$ be some bilinear symmetric non-degenerate form on the lie algebra $\mathcal{G}$. The map $J : M \to G$ is called a G-valued momentum map for the $\mathcal{G}$ action on M whenever

$$i_{\xi_M}\omega(m).v_m \;\;=\;\; (T_m(L_{J(m)^{-1}} \circ J)(v_m), \xi),$$

for any $\xi \in \mathcal{G}$, $m \in M$, and $v_m \in T_mM$.

For abelian symmetries, cylinder valued momentum maps are closely related to the so- called Lie group valued momentum maps. This relation ship is discussed in detail.

Theorem 4.2.4 Let (M, ω) be a connected paracompact symplectic manifold and $\mathcal{G}$ Abelian Lie algebra acting canonically on it. Let $\hbar \subset \mathcal{G}^*$ be the holonomy group associated to the connection α associated to the $\mathcal{G}$-action and let $(.,.) : \mathcal{G} \times \mathcal{G} \to \Re$ be a bilinear symmetric non degenerate form on $\mathcal{G}$. Let $f : \mathcal{G} \to \mathcal{G}^*$, $\widetilde{f} : \frac{\mathcal{G}}{\mathcal{T}} \to \frac{\mathcal{G}^*}{\hbar}$ and let $\mathcal{T} := f^{-1}(\hbar)$ be as in the statement of above proposition. Let G be a connected Abelian Lie group whose Lie algebra is $\mathcal{G}$ and suppose that there exists a G-valued momentum map $A : M \to G$ associated to the $\mathcal{G}$-action whose definition uses the form $(.,.)$

(i) If $exp : \mathcal{G} \to G$ is the exponential map , then $\hbar \subset f(Ker\ exp)$.

(ii) $\hbar$ is closed in $\mathcal{G}^*$.

Let $J := \widetilde{f}^{-1} \circ K : M \to \frac{\mathcal{G}}{\mathcal{T}}$, where K is a cylinder valued momentum map for the $\mathcal{G}$-action on (M, ω). If $f(Ker\ exp) \subset \hbar$, then $J : M \to \frac{\mathcal{G}}{\mathcal{T}} = \frac{\mathcal{G}}{Ker\ exp} \simeq G$ is a G-valued momentum map that differs from A by a constant in G. Conversely, if

$\hbar = f(Ker\ exp)$, then $J : M \to \frac{\mathcal{G}}{Ker\ exp} \simeq G$ is a G-valued momentum map.

In section 3 we discuss a generalization of the standard momentum map not involving the group action. The classical notion of momentum map from Weinstein's point of view is given first. To do this we recall some ideas related to the symplectic category. Then we look at the standard momentum map in a more general set up as a map $\widetilde{J} : M \times G \to \mathcal{G}^*$. In this case we have shown that $\widetilde{J}$ is a momentum map. Then introduce the notion of generalization of the momentum map, where the group action is replaced by a family of symplectomorphisms.

Let (M, ω) be a symplectic manifold, S an arbitrary manifold and $f_s, s \in S$, a family of symplectomorphisms of M depending smoothly on s. For $p \in M$ and $s_o \in S$, let $g_{s_o,p} : S \to M$ be the map, $g_{s_o,p}(s) = f_s \circ f_{s_o}^{-1}(p)$. Then the derivative at s_o is given by

$$(dg_{s_o,p})_{s_o} : T_{s_o}S \to T_pM.$$

From this we get the linear map

$$\widetilde{(dg_{s_o,p})}_{s_o} : T_{s_o}S \to T_p^*M.$$

Now, let J be the map of $M \times S$ into T^*S which is compatible with the projection, $M \times S \to S$ in the sense

$$\begin{array}{ccc} M \times S & \xrightarrow{J} & T^*S \\ & \searrow & \downarrow \\ & & S \end{array}$$

commutes; and for $s_o \in S$ let $J_{s_o} : M \to T_{s_o}^*S$ be the restriction of J to $M \times \{s_o\}$.

Definition 4.3.14 J is a *momentum map* if, for all s_o and p,

$$(dJ_{s_o})_p : T_pM \to T^*_{s_o}S$$

is the transpose of the map $\widetilde{(dg_{s_o,p})}_{s_o}$.

Then a sufficient condition for the existence of momentum map can be done in a more general set up. We do it in a more general set up which does not involve the group action. After giving a sufficient condition for the existence of momentum map, we have recaptured a generalization of standard momentum map by family of symplectomorphisms and the momentum map associated to Hamiltonian group action.

Let (M, ω) be a symplectic manifold . Let Z, X and S be manifolds and suppose that $\pi : Z \to S$ is a fibration with fibers diffeomorphic to X. Let $G : Z \to M$ be a smooth map and let

$$g_s : Z_s \to M, Z_s := \pi^{-1}s$$

denote the restriction of G to Z_s. We assume that g_s is a Lagrangian embedding and let $\Lambda_s := g_s(Z_s)$ denote the image of g_s. Thus, for each $s \in S$, G imbeds the fiber, $Z_s = \pi^{-1}s$, into M as the Lagrangian submanifold, Λ_s. Let $s \in S$ and $\xi \in T_sS$. For $z \in Z_s$ and $w \in T_zZ_s$ tangent to the fiber Z_s,

$$dG_zw = (dg_s)_zw \in T_{G(z)}\Lambda_s.$$

So, dG_z induces a map, which by abuse of language, we will continue to denote by dG_z

$$dG_z : \frac{T_zZ}{T_zZ_s} \to \frac{T_mM}{T_m\Lambda_s}, \qquad m = G(z).$$

But $d\pi_z$ induces an identification $\frac{T_zZ}{T_zZ_s} = T_sS$.

Furthermore, we have an identification $\frac{T_mM}{T_m\Lambda_s} = T_m^*\Lambda_s$.

Using the identifications, we have $dG_z : T_sS \longrightarrow T_s^*Z_s$. Now, let $J : Z \to T^*S$ be a lifting of $\pi : Z \to S$, so that

$$\begin{array}{ccc} Z & \xrightarrow{J} & T^*S \\ \pi & \searrow & \downarrow \\ & & S \end{array}$$

commutes, and for $s \in S$, let $J_s : Z_s \to T_s^*S$ be the restriction of J to Z_s.

Definition 4.3.16 J is a momentum map if, for all s and all $z \in Z_s$,

$$(dJ_s)_z : T_zZ_s \to T_s^*S$$

is the transpose of dG_z.

We have an embedding $(G, J) : Z \to M \times T^*S$. from the momentum map $J : Z \to T^*S$. Then we prove a theorem on the existence of momentum maps.

Theorem 4.3.17 Let (M, ω) be a symplectic manifold . Let Z, X and S be manifolds and suppose that $\pi : Z \to S$ is a fibration with fibers diffeomorphic to X. Let $G : Z \to M$ be a smooth map and J is a momentum map. The pull back by (G, J) of the symplectic form on $M \times T^*S$ is the pull back by π of a closed two form ρ on S. If $[\rho] = 0$, there exists a momentum map, J, for which the imbedding (G, J) is Lagrangian.

Theorem 4.3.18 Let J be a map of Z into T^*S lifting the map, π, of Z into S. Then, if the imbedding (G, J) is Lagrangian, J is a momentum map.

Chapter 1

Existence and Certain Properties of the Standard Momentum Map

The concept of momentum map is a generalization of that of a Hamiltonian function. Its importance is given by the fact that it is able to describe some of the conservation laws associated to a symmetry of the system. The notion of the momentum map associated to a group action on a symplectic manifold formalizes the Noether principle, which states that to every symmetry, such as a group action in a mechanical system there corresponds a conserved quantity. In this chapter main focus is on the standard momentum map associated to Lie group action on a symplectic manifold.

In the first section we have given the basic ideas required. Many of the standard results are recalled. First we give the definitions of Lie group action, proper action, Lie algebra action, symplectic manifold, symplectomorphism, Lagrange submanifold. Also we state some basic theorems on symplectic manifolds. Then the definitions of Hamiltonian vector field, Hamiltonian functions, Poisson manifold, Poisson tensor, canonical mappings, Hamiltonian and Poisson dynamical systems are given. The canonical Lie group and Lie algebra actions, almost Hamiltonian actions and Hamiltonian actions are also discussed.

In section 2 the notion of Noether momentum map on a symplectic manifold is introduced. Then Chu momentum map is introduced and some properties are proved in section 3. In section 4 the standard momentum map, whose values are in the dual of Lie algebra of symmetries, is given. After giving examples of such momentum maps, a thorough discussion on the existence of such momentum maps is given. Existence results of momentum maps are given using fixed points of the action, Lie algebra cohomology, G-invariant 1-form on M and compact Lie group action. The existence of coadjoint equivariant momentum maps for the action of semidirect product $G_1 \times_\sigma G_2$ is also given using conditions on G_1. Using this, we prove two results on existence of momentum maps for $M_1 \times M_2$. Then the results on existence using the extension of Lie algebra is also given. In general it is not possible to choose a coadjoint equivariant momentum map, but we define another action with respect to which we have equivariance.

In section 5 we discuss certain properties of the momentum map. First we prove J is a submersion on the open dense subset of principal orbits in M. Then Noether's theorem, that is, they are constant on the dynamics of any symmetric Hamiltonian vector field is given. An equivalent condition for the momentum map to be constant on the orbits is given. We establishes a link between the symmetry of a point and the rank of the momentum map at the point, called bifurcation lemma. Also proved that the level sets of the momentum map is locally arcwise connected.

1.1 Preleminaries

In this section we have given the basic ideas required. Many of the standard results are recalled. First we give the definitions of Lie group action, proper action, Lie algebra action, symplectic manifold, symplectomorphism, Lagrange submanifold. Also we state some basic theorems on symplectic manifolds. Then the definitions of Hamiltonian vector field, Hamiltonian functions, Poisson manifold, Poisson tensor, canonical mappings, Hamiltonian and Poisson dynamical systems are given. The canonical Lie group and Lie algebra actions, almost Hamiltonian actions and Hamiltonian actions are also discussed. [32], [33], [23], [22], [40], [3], [37], [2], [35], [34], [29], [28] .

Definition 1.1.1. Let M be a manifold and G a Lie group. A *left action* of G on M is a smooth mapping $\phi : G \times M \to M$ such that
(i) $\phi(e, m) = m$, for all $m \in M$ and
(ii) $\phi(g, \phi(h, m)) = \phi(gh, m)$ for all $g, h \in G$ and $m \in M$.
We sometimes write $g.m := \phi(g, m) := \phi_g(m) := \phi^m(g)$. For any $g \in G$, the map $\phi_g : M \to M$ is a diffeomorphism of M with inverse $\phi_{g^{-1}}$. We will denote by A_G, the subgroup of diffeomorphism group $Diff(M)$ of M associated to the G-action on M, that is, $A_G = \{\phi_g | g \in G\}$. The triple (M, G, ϕ) is called a *G-space or a G-manifold.*

Similarly we can define *right action* of G on M.

Example 1.1.2. Translation and Conjugation : The *left translation* on G defined by $g \in G$, that is, the map $L_g : G \to G$, given by $L_g(h) = gh$, induces a left action of G on itself. *Right translation*, $R_g : G \to G$, given by $R_g(h) = hg$, defines a right action of G on itself. The inner automorphism $AD_g \equiv I_g : G \to G$, given by $I_g = R_{g^{-1}} \circ L_g$ defines a left action of G on itself called *conjugation.*

Example 1.1.3. Adjoint and Coadjoint actions : The differential T_eI_g at the identity of the conjugation mapping defines a linear left action of G on the Lie algebra $\mathcal{G}$ of G called the *adjoint representation* Ad_g of G on $\mathcal{G}$. That is $Ad_g := T_e I_g : \mathcal{G} \to \mathcal{G}$.

If $Ad_g^* : \mathcal{G}^* \to \mathcal{G}^*$ is the dual of Ad_g, then the map

$$\phi : G \times \mathcal{G}^* \to \mathcal{G}^* \quad by \quad \phi((g, v)) = Ad^*_{g^{-1}}(v)$$

defines also a linear left action of G on the dual space $\mathcal{G}^*$ of $\mathcal{G}$ called the *coadjoint representation* of G on $\mathcal{G}^*$.

Definition 1.1.4. Let M be a manifold and G a Lie group with Lie algebra $\mathcal{G}$. Given a (left) action $\phi : G \times M \to M$, the *infinitesimal generator* $\xi_M \in \mathcal{X}(M)$, the collection of vector fields on M, associated to $\xi \in \mathcal{G}$ is the vector field on M defined by $\xi_M(m) = \frac{d}{dt} \mid_{t=0} \phi_{expt\xi}(m) = T_e\phi^m.\xi$. Thus we can define a map $\dot{\phi} : \mathcal{G} \to \mathcal{X}(M)$, by $\dot{\phi}(\xi) = \xi_M$.

The infinitesimal generators are complete vector fields. Moreover the map $\dot{\phi}$ is a Lie algebra antihomomorphism.

Definition 1.1.5. Let $\mathcal{G}$ be a Lie algebra and M a smooth manifold. A right(left) *Lie algebra action* of $\mathcal{G}$ on M is a Lie algebra (anti) homomorphism $\xi \in \mathcal{G} \to \xi_M \in \mathcal{X}(M)$ such that the mapping $(m, \xi) \in M \times \mathcal{G} \to \xi_M(m) \in TM$ is smooth. Given a Lie group action, we will refer to the Lie algebra action induced by its infinitesimal generators as the associated Lie algebra action.

Definition 1.1.6. The *isotropy subgroup or stabilizer* of an element m in the manifold M acted upon by the Lie group G is the closed subgroup $G_m := \{g \in G \mid \phi_g(m) = m\}$ whose Lie algebra $\mathcal{G}_m$ equals $\mathcal{G}_m = \{\xi \in \mathcal{G} \mid \xi_M(m) = 0\}$.

In particular if one considers the coadjoint representation, the stabilizer is called the *coadjoint isotropy subgroup* of G. Its Lie algebra is the *coadjoint isotropy* *subalgebra* $\mathcal{G}_\mu = \{\xi \in \mathcal{G} \mid ad^*_\xi \mu = 0\}$, where $ad^*_\xi : \mathcal{G}^* \to \mathcal{G}^*$ (coadjoint representation of $\mathcal{G}$ on $\mathcal{G}^*$) the dual of $ad_\xi : \mathcal{G} \to \mathcal{G}$ (the adjoint representation of $\mathcal{G}$ on itself) defined by $ad_\xi \eta = [\xi, \eta], \xi, \eta \in \mathcal{G}$.

The *orbit* O_m of the element $m \in M$ under the group action ϕ is the set $O_m \equiv G.m := \{\phi_g(m) \mid g \in G\}$. In particular, in the case of the adjoint and coadjoint representations, the orbits will be called the *adjoint and coadjoint orbits* respectively.

The group action on M is said to be *transitive* if there is only one orbit, *free* if the isotropy subgroup of every element in M consists only the identity element, and *effective or faithful* if $\phi_g = id_M$ implies that $g = e.$. A Lie algebra action on M is *locally free* when $\mathcal{G}_m = 0$, for any $m \in M$.

Definition 1.1.7. A subset $S \subset M$ of the G-space (M, G, ϕ) is called *G-invariant* or *G-saturated* if $\phi_g(S) = S$, for all $g \in G$ or, equivalently, if it is a union of G-orbits. Functions in $C^\infty(M)$ which satisfy $f \circ \phi_g = f$, for all $g \in G$ are called smooth G-invariant functions on the G-space M denoted by $C^\infty(M)^G$.

Definition 1.1.8. A mapping $\phi : M_1 \to M_2$, between two G-spaces M_1 and M_2 is said to be *G-equivariant* provided that for any $g \in G$ and $z \in M_1$, the mapping ϕ satisfies the identity $\phi(g.z) = g.\phi(z)$.

Definition 1.1.9. Let $\mathcal{G}$ be a Lie algebra acting on the manifold M. We say that a function $f \in C^\infty(M)$ is *$\mathcal{G}$-invariant* if $\xi_M[f] = 0$, for any $\xi \in \mathcal{G}$. We will denote by $C^\infty(M)^{\mathcal{G}}$ the set of $\mathcal{G}$-invariant functions on M. Let M and N be two manifolds acted upon by the same Lie algebra $\mathcal{G}$. A smooth map $F : M \to N$ is said to be *$\mathcal{G}$-equivariant* if $\xi_N(F(m)) = T_m F.\xi_M(m)$ for any $m \in M$ and $\xi \in \mathcal{G}$.

Definition 1.1.10. A continuous map between two topological spaces with Hausdorff domain is said to be *proper* if it is closed and all its fibers are compact subsets of its domain.

Definition 1.1.11. Let G be a Lie group acting on the manifold M via the map $\phi : G \times M \to M$. We say that ϕ is a *proper action* whenever the map $\Phi : G \times M \to M \times M$ defined by $\Phi(g, z) = (z, \phi(g, z))$ is proper. The properness of the action is equivalent to the following condition : for any two convergent sequences $\{m_n\}$ and $\{g_n.m_n\}$in M, there exists a convergent subsequence $\{g_{n_k}\}$ in G. We say that the action ϕ is proper at the point z in M when for any two convergent sequences $\{m_n\}$ and $\{g_n.m_n\}$in M such that $m_n \to z$ and $g_n.m_n \to z$, there exists a convergent subsequence $\{g_{n_k}\}$ in G.

Compact group actions are proper.

Theorem 1.1.12. *Let M be a manifold and G be a Lie group acting on M via the map $\phi : G \times M \to M$. If the action is free and proper then $\frac{M}{G}$, space of orbits, is a smooth manifold and has a principal G-bundle structure $M \xrightarrow{\pi} \frac{M}{G}$.*

Definition 1.1.13. Let G be a Lie group and $H \subset G$ a subgroup. Suppose that H acts on the left on the manifold A. The *twisted action* of H on the product $G \times A$ is defined by

$$h.(g, a) = (gh, h^{-1}a), h \in H, g \in G \quad and \quad a \in A.$$

Note that this action is free and proper. The twisted product $G \times_H A$ is defined as the orbit space $\frac{(G \times A)}{H}$ corresponding to the twisted action. The elements of $G \times_H A$ will be denoted by $[g, a], g \in G, a \in A$.

Definition 1.1.14. Let M be a manifold and G be a Lie group acting properly on M. Let $m \in M$ and denote $H = G_m$. A *tube* around the orbit $G.m$ is a G-equivariant diffeomorphism $\phi : G \times_H A \to U$, where U is a G-invariant neighborhood of $G.m$ and A is some manifold on which H acts.

Definition 1.1.15. Let M be a manifold and G be a Lie group acting properly on M. Let $m \in M$ and denote $H = G_m$. Let S be a submanifold of M such that

$m \in S$ and $H.S = S$. We say that S is a *slice* at m if the G-equivariant map $\phi : G \times_H S \to U$ is a tube about $G.m$ for some G-invariant open neighborhood U of $G.m$. Notice that if S is a slice at m, then $\phi_g(S)$ is a slice at the point $\phi_g(m)$.

Lemma 1.1.16. Let H be a compact Lie group that acts on the manifold M. Assume that $m \in M$ is a fixed point of the action, that is, $H.m = \{m\}$. Then any open neighborhood of m contains an H-invariant open neighborhood of m.

Proof : Let $\phi : H \times M \longrightarrow M$ be the group action and U an arbitrary neighborhood of m. The set $\phi^{-1}(U)$ is clearly open and contains $H \times \{m\}$. Now, for any $g \in H$, there exists neighborhoods W_g of g in H and V_g of m in M such that $W_g \times V_g \subset \phi^{-1}(U)$. Since by hypothesis H is compact, the open cover $\{W_g \mid g \in H\}$ of H has a finite subcover $W_{g_1}, W_{g_2}, ...W_{g_n}$. Let $V := \cap_{i=1}^{n} V_{g_i}$. Then the H-invariant set $W : \phi(H, V) \subset U$ is clearly open since $W = \cup_{g \in H} \phi_g(V)$, and by construction, contains the point m. •

Lemma 1.1.17. Let H be a compact Lie group that acts on the manifold M and let $m \in M$ be such that $H.m = \{m\}$. Then there exists an H-invariant Riemannian metric defined on some H-invariant neighborhood of m.

Proof : Let $\varphi : U \longrightarrow \varphi(U) \subset \Re^n$ be a local chart of M around the point m. By the previous lemma, the open neighborhood U can be chosen to be H-invariant. The pull back of the Euclidean metric on $\varphi(U) \subset \Re^n$ gives a Riemannian metric g on U. Let now g' be the averaged metric on U given by

$$g'(z)(u, v) := \int_H g(h.z)(T_z\phi_h.u, T_z\phi_h.v)dh$$

where the integral is defined using the normalized Haar measure on H. By construction, g' is an H-invariant Riemannian metric on U. •

Theorem 1.1.18. (Tube theorem) Let M be a manifold and G be a Lie group acting properly on M at the point $m \in M$, $H := G_m$. There exists a tube

$\phi : G \times_H B \to U$ about $G.m$. B is an open H-invariant neighborhood of 0 in a vector space H-equivariantly isomorphic to $T_m M / T_m(G.m)$ on which H acts linearly by

$$h.(v + T_m(G.m)) := T_m \phi_h.v + T_m(G.m).$$

Proof : Since the G-action is proper, the isotropy subgroup $H := G_m$ is compact. Let g be an H-invariant Riemannian metric defined on some H-invariant neighborhood of m, whose existence is guaranteed by the lemmas 1.1.16 and 1.1.17. Let Exp be the corresponding Riemannian exponential and N_m the orthogonal complement to $\mathcal{G}.m$ in $T_m M$ with respect to the inner product induced by $g(m)$. The vector subspace N_m is clearly H-invariant and equivariantly isomorphic to $T_m M / T_m(G.m)$ via the map $v \longmapsto v + \mathcal{G}.m$; recall that $\mathcal{G}.m = T_m(G.m)$. Now, an elementary fact in Riemannian geometry guarantees the existence of a neighborhood W of the origin in $T_m M$ such that the restriction of Riemannian exponential map Exp_m to W is a diffeomorphism onto its image. By lemma 1.1.16, W can be chosen to be H-invariant. Let $V := W \cap N_m$. The open set V is H-invariant by construction and hence the twisted product $G \times_H V$ is well-defined. Let now τ be the map defined by

$$\begin{aligned} \tau : G \times_H V &\longrightarrow M \text{ given by,} \\ \tau([g, v]) &= g.Exp_m v. \end{aligned}$$

The map τ is well defined due to the H-equivariance of Exp and is obviously G-equivariant. Then we can show that $T_{[e,0]}\tau$ is an isomorphism.

The Local Diffeomorphism Theorem implies the existence of a neighborhood W' of $[e, 0]$ in $G \times_H V$ such that the restriction of τ to W' is a diffeomorphism onto its image. In particular, there exists an open neighborhood V' of 0 in V

such that for any $\upsilon \in V'$, the point $[e, \upsilon]$ is contained in W'. Since $\tau\ |_{W'}$ is a diffeomorphism, it follows thus that its tangent map $T_{[e,\upsilon]}\tau$ is an isomorphism. The equivariance of τ implies that $T_{[g,\upsilon]}\tau$ is an isomorphism for any $g \in G$ and $n \in V'$. Hence, τ restricted to $G \times_H V'$ is a local diffeomorphism. Then using the properness condition on the G-action there exists an open H-invariant subset B of V' containing 0 such that the restriction of τ to $G \times_H B$ is injective.

The restriction of τ to $G \times_H B$ is an injective local G-equivariant diffeomorphism onto its image and therefore a diffeomorphism onto an open G-invariant neighborhood of $m \in M$. This is the map φ needed in the statement of the theorem. •

Theorem 1.1.19. (Slice theorem)Let M be a manifold and G be a Lie group acting properly on M at the point $m \in M$. Then there exists a slice at m.

Corollary 1.1.20. Let M be a manifold and G be a Lie group acting properly on M at the point $m \in M$. The orbit is a closed embedded submanifold of M.

Definition 1.1.21. A *Symplectic manifold* is a pair (M, ω) where M is a manifold and $\omega \in \Omega^2(M)$ is a closed nondegenerate 2-form on M, that is, $d\omega = 0$ and, for every $m \in M$, the map $v \in T_mM \to \omega(m)(v, .) \in T^*_mM$ is a linear isomorphism between the tangent space T_mM to M at m and the cotangent space T^*_mM.

Definition 1.1.22. If ω is allowed to be degenerate, (M, ω) is called a *presymplectic manifold.*

Definition 1.1.23. A submanifold W of a symplectic manifold (M, ω) is called
(i)*isotropic* if $T_xW \subset (T_xW)^\perp$,
(ii) *coisotropic* if $(T_xW)^\perp \subset T_xW$,
(iii) *Lagrangian* if it is a maximal isotropic submanifold of M,

(iv)*symplectic* in (M, ω) if $T_xW \cap (T_xW)^{\perp} = 0$, for each $x \in W$.
Here $(T_xW)^{\perp}$ is the symplectic orthogonal complement of T_xW.

Let (M, ω) be a $2n$-dimensional symplectic manifold. A submanifold W of M is a Lagrangian submanifold if, at each $p \in W$, T_pW is a Lagrangian subspace of T_pM, that is, $\omega_p \mid_{T_pW} = 0$ and $dimT_pW = \frac{1}{2}dimT_pM$. Equivalently, if $i : W \to M$ is the inclusion map, then W is Lagrangian if and only if $i^*\omega = 0$ and $dimW = \frac{1}{2}dimM$.

Definition 1.1.24. Let M be a manifold and G be a Lie group acting on M. Let H a closed subgroup of G. Then the set $M_H = \{z \in M/H = G_z\}$ is called the isotropy type submanifold.

Theorem 1.1.25. Let (M, ω) be a symplectic manifold. Let G be a Lie group acting on the manifold M via the map $\phi : G \times M \to M$. Let H a closed subgroup of G. If the action is proper then the isotropy type submanifold M_H is symplectic.

Definition 1.1.26. Let M a proper G-space and $z \in M$ such that $H := G_z$, the local orbit type manifold through the point z is the subset $M^{l_z}_{(H)}$ of M containing all the points $x \in M$ for which there exists a G-invariant open neighborhood U_x around x and a G-equivariant diffeomorphism from U_x onto a G-invariant neighborhood of z.

Theorem 1.1.27. Let M a proper G-space and $z \in M$ such that $H := G_z$. If G is connected and M/G connected , then there is a unique orbit type (H) in M for which $M_{(H)}$ is an open dense subset of M. Moreover $M_{(H)}/G$ is a connected manifold.

Definition 1.1.28. Let M a proper G-space and $z \in M$ such that $H := G_z$. The G-orbit $G.z$ is called principal if its corresponding local orbit type manifold through the point z is the subset $M^{l_z}_{(H)}$ is open in M.

Definition 1.1.29. Let $(M_1, \omega_1), (M_2, \omega_2)$ be $2n$-dimensional symplectic manifolds, let $\phi : M_1 \to M_2$ be a diffeomorphism. Then ϕ is a *symplectomorphism* if $\phi^*\omega_2 = \omega_1$. If $M_1 = M_2$, then ϕ is called a *canonical transformation.*

Definition 1.1.30. Let (M, ω) be a $2n$-dimensional symplectic manifold. A vector field X on M is called a *symplectic vector field* if its flow preserves the symplectic form.

On the symplectic 2-Torus $(T^2, d\theta_1 \wedge d\theta_2)$ vector fields $X_1 = \frac{\partial}{\partial\theta_1}, X_2 = \frac{\partial}{\partial\theta_2}$ are symplectic vector fields.

Proposition 1.1.31. The following assertions are equivalent.
(i) X is a symplectic vector field.
(ii) The Lie derivative $L_X\omega = 0$.
(iii) $i_X\omega = df$ locally for some function $f \in C^\infty(M)$.

Definition 1.1.32. Let (M, ω) be a symplectic manifold. A vector field $X \in \mathcal{X}$ (M) , the space of vector fields on M, is called a *Hamiltonian vector field* if $i_X\omega$ is exact, that is, there exists $f \in C^\infty(M)$ such that $i_X\omega = df$.

Then X is denoted by X_f and f is called the Hamiltonian function. The set of Hamiltonian vector fields on a symplectic manifold (M, ω) will be denoted by $\mathcal{X}_{\mathcal{H}}(M, \omega)$. Thus we can assign a map $j : C^\infty(M) \to \mathcal{X}_{\mathcal{H}}(M, \omega)$ by $j(f) = X_f$.

If the one form $i_X\omega$ is only closed, then we say that X is a *locally Hamiltonian vector field*. Using Cartan's formula and the closedness of symplectic form ω, we get X is a locally Hamiltonian vector field if and only if $L_X\omega = 0$. The set of locally Hamiltonian vector fields on a symplectic manifold (M, ω) will be denoted by $\mathcal{X}_{\mathcal{LH}}(M, \omega)$.

The Lie bracket $[X, Y]$ of two locally Hamiltonian vector fields is Hamiltonian with $\omega(Y, X)$ as the associated Hamiltonian function. Then the quotient $\mathcal{X}_{\mathcal{LH}}(M, \omega)/\mathcal{X}_{\mathcal{H}}(M, \omega)$ gets an induced Lie algebra structure. So we have an exact sequence of Lie algebras

$$0 \to \mathcal{X}_{\mathcal{H}} \quad (M, \omega) \to \mathcal{X}_{\mathcal{LH}} \quad (M, \omega) \to H^1(M, R) \to 0.$$

The map $j : C^{\infty}(M) \to \mathcal{X}_{\mathcal{H}}(M, \omega)$ is a homomorphism of Lie algebras. Thus we have another exact sequence of Lie algebras

$$0 \to \Re \xrightarrow{i} C^{\infty}(M) \xrightarrow{j} \mathcal{X}_{\mathcal{H}} \quad (M, \omega) \to 0,$$

where i denotes the inclusion map.

Proposition 1.1.33. Let (M, ω) be a symplectic manifold .
(i) A smooth vector field on M is symplectic if and only if it is locally Hamiltonian.
(ii) Every locally Hamiltonian vector field on M is globally Hamiltonian if and only if $H^1_{dR}(M) = 0$.

Definition 1.1.34. Let (M, ω) be a symplectic manifold . A *Hamiltonian dynamical system* is a triple (M, ω, h) where $h \in C^{\infty}(M)$ is called the Hamiltonian function of the system.

Definition 1.1.35. Let (M, ω) be a $2n$-dimensional symplectic manifold. The *Poisson bracket* of two functions $f, g \in C^{\infty}(M)$ is the function $\{f, g\} := \omega(X_f, X_g)$.

Definition 1.1.36. A *Poisson Manifold* is a pair $(M, \{., .\})$ where M is a manifold and $\{., .\}$ is a bilinear operation on $C^{\infty}(M)$ such that $(C^{\infty}(M), \{., .\})$ is a Lie algebra and $\{., .\}$ is a derivation in each argument. The pair $(C^{\infty}(M), \{., .\})$ is also called a *Poisson algebra.* The functions in the center of the Lie algebra $(C^{\infty}(M), \{., .\})$ are called *Casimir functions* . The set of all casimir functions is denoted by $\mathcal{C}(M)$.

Definition 1.1.37. Let $(M, \{., .\})$ be a Poisson Manifold . Then any $h \in C^{\infty}(M)$ induces a vector field on M via the expression $X_h = \{., h\}$, called the Hamiltonian vector field associated to the Hamiltonian function h. The triplet $(M, \{., .\}, h)$ is called a *Poisson dynamical system.*

The Lie algebra mapping $(C^{\infty}(M), \{., .\}) \to (\mathcal{X}(\mathrm{M}),[.,.])$ that assigns to each function $f \in C^\infty(M)$ the associated Hamiltonian vector field $X_f \in \mathcal{X}$(M) is a Lie algebra antihomomorphism, that is, $X_{\{f,g\}} = -[X_f, X_g]$ for all $f, g \in C^\infty(M)$

Any Hamiltonian system on a symplectic manifold is a Poisson dynamical system relative to the Poisson bracket induced by the symplectic structure.

Definition 1.1.38. Let (M, ω, h) be a Hamiltonian dynamical system. Then we define a contravariant antisymmetric two-tensor $B \in \Lambda^2(T^*M)$ by $B(z)(\alpha_z, \beta_z) = \{f, g\}(z)$ where $df(z) = \alpha_z \in T_z^*M$ and $dg(z) = \beta_z \in T_z^*M$. This tensor is called *Poisson tensor* of M. The vector bundle map $B^\# : T^*M \to TM$ naturally associated to B is defined by $B(z)(\alpha_z, \beta_z) = (\alpha_z, B^\#(\beta_z))$. Its range $D := B^\#(T^*M) \subset TM$ is called the *characteristic distribution.*

Definition 1.1.39. A smooth mapping $\phi : M_1 \to M_2$ between two Poisson manifolds $(M_1, \{.,.\}_1)$ and $(M_2, \{.,.\}_2)$ is called *canonical* or *Poisson* if for all $f, g \in C^\infty(M_2)$ we have $\phi^*\{f, g\}_2 = \{\phi^* g, \phi^* f\}_1$.

Proposition 1.1.40. Let (M, ω) be a symplectic manifold and $B \in \Lambda^2(T^*M)$ be the associated Poisson tensor. Then for any $m \in M$ and any vector space $V \subset T_m M$, $V^\omega = B^\#(m)(V^o)$, where V^ω is the ω-orthogonal complement of V in $T_m M$.

Proposition 1.1.41. Let $\phi : M_1 \to M_2$ be a smooth map between two Poisson manifolds $(M_1, \{.,.\}_1)$ and $(M_2, \{.,.\}_2)$. Then ϕ is a Poisson map if and only if $T\phi \circ X_{h \circ \phi} = X_h \circ \phi, \forall h \in C^\infty(M_2)$. In particular if F_t^2 is the flow of X_h and F_t^1 is the flow of $X_{h\circ\phi}$, then $F_t^2 \circ \phi = \phi \circ F_t^1$.

Definition 1.1.42. Let (M, ω) be a symplectic manifold . Let G be a Lie group acting smoothly on M through the map $\phi : G \times M \to M$. We say that the action ϕ is *canonical or symplectic* if ϕ acts by canonical transformations,that is, for any $g \in G$ we have $\phi_g^*\omega = \omega$.

The infinitesimal analogue of this concept is the canonical action of a Lie algebra $\mathcal{G}$. We say that an action of the Lie algebra $\mathcal{G}$ is *canonical* if $\mathcal{L}_{\xi_M}\omega = 0$ where $\xi_M \in \mathcal{X}(\mathrm{M})$, $\xi \in \mathcal{G}$.

Note 1.1.43. Let G be a Lie group and ϕ be a symplectic action of G on (M, ω). Then for $\xi \in \mathcal{G}^*$, $\dot{\phi}(\xi) \in \mathcal{X}_{\mathcal{LH}}(M, \omega)$. Thus we have the following picture.

$$
\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
0 \to \Re \xrightarrow{i} C^\infty(M) \xrightarrow{j} & \mathcal{X}_{\mathcal{H}} & (M, \omega) \quad \to 0 \\
 & & \downarrow \\
\mathcal{G} \xrightarrow{\dot{\phi}} & \mathcal{X}_{\mathcal{LH}} & (M, \omega) \\
 & & \downarrow \\
 & & H^1(M, R) \\
 & & \downarrow \\
 & & 0
\end{array}
$$

Proposition 1.1.44. Let (M, ω) be a symplectic manifold and $B \in \Lambda(T^*M)$ be the associated Poisson tensor. Let G be a Lie group acting canonically on M. Then, for any $m \in M$ and any vector subspace $V \subset T_m^*M$
(i) $B^{\sharp(m)} : T_m^*M \to T_mM$ is G_m-equivariant.
(ii) If the Poisson bracket $\{.,.\}$ is induced by ω, then $B^{\sharp(m)}(V^{G_m}) = (B^{\sharp(m)}(V))^{G_m}$.

Note 1.1.45. Let (M, ω) be a symplectic manifold and G be a Lie group acting canonically on M. For a G-invariant system (M, ω, h) the associated Hamiltonian vector field X_h is G-equivariant and consequently the associated flow F_t is G-equivariant. The converse implication is only true up to casimir functions.

Let (M, ω, h) be a G-invariant Hamiltonian system. The G-equivariance of X_h guarantees that its flow F_t leaves the connected components of the isotropy type manifolds corresponding to the G-action invariant. We will refer to this fact as *law of conservation of the isotropy.*

Definition 1.1.46. Let ϕ be a symplectic action of a Lie group G on a symplectic manifold (M, ω) . We say that the action is *almost Hamiltonian* if the vector field ξ_M for any $\xi \in \mathcal{G}$ is a Hamiltonian vector field.

Proposition 1.1.47. Let ϕ be a symplectic action of a Lie group G on a symplectic manifold (M, ω) . If either $H^1_{dR}(M) = 0$, or $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, then the action of G is almost Hamiltonian.

Suppose $\mathcal{G}$ is finite dimensional. Then the map $j : C^\infty(M) \to \mathcal{X}_{\mathcal{H}}(M, \omega)$ defined by $j(f) = X_f$ in Definition 1.1.32 is onto. Then we define;

Definition 1.1.48. We say that an action ϕ of G on (M, ω) is *Hamiltonian* if there is a homomorphism $\lambda : \mathcal{G} \to C^\infty(M)$ with $j \circ \lambda(\xi) = \xi_M$.

But for an almost Hamiltonian action we can choose a linear map $\lambda : \mathcal{G} \to C^\infty(M)$ with $j \circ \lambda(\xi) = \xi_M$. Then we have ;

Proposition 1.1.49. An almost Hamiltonian action ϕ is a Hamiltonian action if the linear map $\lambda : \mathcal{G} \to C^\infty(M)$, $f \to X_f$ is a homomorphism.

Theorem 1.1.50. To each almost Hamiltonian action ϕ of G on a symplectic manifold (M, ω) there is a cohomology class in $H^2\mathcal{G}$ which vanishes if and only if there is a homomorphism $\lambda : \mathcal{G} \to C^\infty(M)$ making the action Hamiltonian.

Proof: We have the exact sequence

$$0 \to \Re \xrightarrow{i} C^\infty(M) \xrightarrow{j} \mathcal{X}_{\mathcal{H}} \quad (M, \omega) \to 0,$$

and a map $\phi : \mathcal{G} \to \mathcal{X}_{\mathcal{H}}(M, \omega)$ with $\dot{\phi}(\xi) = \xi_M$.

Given the action is almost Hamiltonian, therefore we can choose $\lambda : \mathcal{G} \to$

$C^\infty(M)$ with $j \circ \lambda(\xi) = \xi_M$. Suppose λ is not a homomorphism, define c as

$$c(\xi, \eta) = \{\lambda(\xi), \lambda(\eta)\} - \lambda([\xi, \eta]).$$

From the antihomomorphism property of infinitesimal generators, we have c is actually a constant and so defines a 2-cochain with values in $\Re$ viewed as trivial coefficients. But the Jacobi identity implies that c is infact a 2-cocycle. If we choose another linear map λ', then we get another 2-cochain c'. From these we get a constant function b as $b(\xi) = \lambda(\xi) - \lambda'(\xi)$, and so a 1-cochain. Also we have

$$c'(\xi, \eta) = c(\xi, \eta) - db(\xi, \eta).$$

Thus c only changes by a coboundary when we choose another linear map. So we have associated canonically to the action ϕ a cohomology class c_ϕ in $H^2\mathcal{G}$.

Conversely, suppose that $c_\phi = 0$, and we calculate c_ϕ using a linear map $\mu : \mathcal{G} \to C^\infty(M)$ then we have some 1-cochain b with

$$\{\mu(\xi), \mu(\eta)\} - \mu([\xi, \eta]) = -b([\xi, \eta]).$$

Define $\lambda : \mathcal{G} \to C^\infty(M)$ as $\lambda = \mu - b$, then λ is a homomorphism. •

Corollary 1.1.51. If $H^2\mathcal{G} = 0$ then every almost Hamiltonian action is Hamiltonian for some homomorphism λ.

Remark 1.1.52. Given a symplectic manifold (M, ω), the set $C^\infty(M)$ is endowed not only with a commutative ring structure relative to pointwise multiplication, but also with a real Lie algebra structure relative to the Poisson bracket. These two algebraic structures on $C^\infty(M)$ are linked with the property that the Poisson bracket is a ring derivation in each of its arguments, that is, the Leibinitz rule holds. Therefore if (M, ω) is a $2n$-dimensional symplectic manifold, then

$(C^\infty(M), \{.,.\})$ is a Poisson algebra. Thus every symplectic manifold is a Poisson manifold.

The converse of this statement is provided by the Symplectic Foliation Theorem. We can prove that the maximal integral manifolds of the characteristic distribution mentioned in Definition 1.1.38 are symplectic with the unique symplectic form that makes the inclusion into the Poisson manifold is a Poisson map. These maximal integral manifolds of the characteristic distribution are called symplectic leaves of $(M, \{.,.\})$.

1.2 Noether Momentum Map

In this section we introduce the concept of Noether momentum map [32].

Definition 1.2.1. Let $(M, \{.,.\})$ be a Poisson manifold and G (respectively $\mathcal{G}$) a Lie group (respectively Lie algebra) acting canonically on it. Let S be a set and $J : M \to S$ a map. We say that J is a *Noether momentum map* for the G-action (respectively $\mathcal{G}$-action) on $(M, \{.,.\})$ when the flow F_t of any Hamiltonian vector field associated to any G-invariant (respectively $\mathcal{G}$-invariant) Hamiltonian function $h \in C^\infty(M)$ preserves the fibers of J. That is,

$$J \circ F_t = J \mid_{Dom(F_t)} . \tag{1.1}$$

Condition 1.1is known as *Noether's condition.*

Note 1.2.2. In most cases , the set S in the above definition has additional structures. S is sometimes a Poisson manifold itself and Noether momentum map J in that case is a Poisson map. In other situations, S is a G-space and J is G-equivariant.

Remark 1.2.3. If G is a Lie group acting canonically on $(M, \{.,.\})$ and $J : M \to S$ is a Noether momentum map for the associated canonical Lie algebra $\mathcal{G}$-action, then J is also a Noether momentum map for the G-action.

1.3 Chu Momentum Map

We present a Noether momentum map whose definition makes essential use of the symplectic structure. Apart from its intrinsic interest as a Noether momentum map, this construction will be extremely important in the statement and proof of a symplectic version of slice theorem, presented in chapter 3 [32], [25], [26].

Definition 1.3.1. Let (M, ω) be a symplectic manifold and $\mathcal{G}$ be a Lie algebra acting canonically on it. The *Chu map* is defined as the map $\Psi : M \to Z^2(\mathcal{G})$ given by

$$\Psi(m)(\xi, \eta) \quad = \quad \omega(m)(\xi_M(m), \eta_M(m)), \tag{1.2}$$

for every $\xi, \eta \in \mathcal{G}$. The fact that Ψ maps into $Z^2(\mathcal{G})$ is a consequence of the closedness of the symplectic form ω and the canonical character of the $\mathcal{G}$- action.

The main properties of the Chu map are summarized in the following proposition.

Proposition 1.3.2. Let (M, ω) be a symplectic manifold and $\mathcal{G}$ a Lie algebra acting canonically on it. Then the corresponding Chu map $\Psi : M \to Z^2(\mathcal{G})$ satisfying the following properties.

(i) $T_m\Psi(v_m)(\xi, \eta) = \omega(m)([\xi, \eta]_M(m), v_m)$ for any $m \in M$, $v_m \in T_mM$ and $\xi, \eta \in \mathcal{G}$.

(ii) $KerT_m\Psi = ([\mathcal{G}, \mathcal{G}]_M(m))^{\perp}$.

(iii) Ψ is a Noether momentum map for the $\mathcal{G}$-action on M.

Proof : (i) For any $v_m \in T_mM$

$$\begin{aligned} T_m\Psi(v_m)(\xi,\eta) &= d(\omega(\xi_M,\eta_M))(m).v_m \\ &= i_{[\xi_M,\eta_M]}\omega(v_m) \\ &= -i_{[\xi,\eta]_M}\omega(v_m) \\ &= \omega(m)([\xi,\eta]_M(m),v_m). \end{aligned}$$

(ii) It directly follows from (i)

(iii) Let $h \in C^\infty(M)^{\mathcal{G}}$ and let X_h be the corresponding Hamiltonian vector field with flow F_t. Then for any $m \in M$ and $\xi,\eta \in \mathcal{G}$, we have

$$\begin{aligned} \{h,\Psi(.)(\xi,\eta)\}(m) &= -d[\Psi(.)(\xi,\eta)](m).X_h(m) \\ &= -\omega(m)([\xi,\eta]_M(m),X_h(m)) \\ &= dh(m).[\xi,\eta]_M(m) \\ &= [\xi,\eta]_M[h](m) = 0. \end{aligned}$$

where we have used (i) and the $\mathcal{G}$-invariance of h. This computation shows that the function $\Psi(.)(\xi,\eta)$ is constant along the flow F_t of X_h. As $\xi,\eta \in \mathcal{G}$ are arbitrary, we have that $\Psi \circ F_t = \Psi/_{Dom(F_t)}$, as required. •

Proposition 1.3.3. Let G be a Lie group acting canonically on (M,ω) via the map $\phi : G \times M \to M$ and whose Lie algebra is $\mathcal{G}$. Suppose that the canonical Lie algebra action associated to this group action.

(i) Let ϕ^m be the orbit map through m, that is, $\phi^m : G \to M$ by $\phi^m(g) = g.m$, then $\Psi(m) = (\phi^m)^*\omega$.

(ii) The Chu map is G-equivariant. That is , for any $m \in M$ and any $g \in G$ we have that $\Psi(g.m) = Ad^*_{g^{-1}}\Psi(m)$.

Proof : (i) For every $\xi, \eta \in \mathcal{G}$,

$$\begin{aligned}(\phi^m)^*\omega(\xi,\eta) &= \omega(m)(T_e\phi^m\xi, T_e\phi^m\eta)\\ &= \omega(m)(\xi_M(m), \eta_M(m))\\ &= \Psi(m)(\xi,\eta).\end{aligned}$$

(ii) Using (i), for every $g \in G$ and $m \in M$,

$$\begin{aligned}\Psi(gm) &= (\phi^{gm})^*\omega\\ &= (\phi^m r_g)^*\omega\\ &= r_g^*(\phi^m)^*\omega\\ &= r_g^*\Psi(m)\\ &= (Ad_{g_{-1}})^*\Psi(m).\end{aligned}$$

•

1.4 Standard Momentum Map and Its Existence

The Chu map is always well defined as soon as we have a canonical action on a symplectic manifold. However, it has the disadvantage of not reproducing well known conservation laws and of giving trivial results in some relevant situations .

Some of these drawbacks are overcome with the use of standard momentum map, which we will refer as the momentum map. The momentum map is a generalization of the standard linear and angular momentum in classical mechanics. The definition of momentum map only requires a canonical Lie algebra action. Existence is not guaranteed always.

In this section the standard momentum map, whose values are in the dual of Lie algebra of symmetries, is presented. After giving examples of such momentum maps, a thorough discussion on the existence of such momentum maps is given. Existence results of such momentum maps are given using fixed points of the action, Lie algebra cohomology, G-invariant 1-form on M and compact Lie group action. The existence of coadjoint equivariant momentum maps for the action of semidirect product $G_1 \times_\sigma G_2$ is also given using conditions on G_1. Using this, we prove two results on existence of momentum maps for $M_1 \times M_2$. Then the results on existence using the extension of Lie algebra is also given. In general it is not possible to choose a coadjoint equivariant momentum map, but we define another action with respect to which we have equivariance. [10], [18], [34], [32], [25], [26], [2].

Definition 1.4.1. Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold (M, ω). Suppose that for any $\xi \in \mathcal{G}$, the vector field ξ_M is globally Hamiltonian with Hamiltonian function $J^\xi \in C^\infty(M)$. The map $J : M \to \mathcal{G}^*$ defined by the relation

$$< J(z), \xi >= J^\xi(z), \tag{1.3}$$

for all $\xi \in \mathcal{G}$ and $z \in M$, is called a *standard momentum map* or simply a momentum map of the $\mathcal{G}$-action. (For Poisson manifolds also the same definition holds.)

Let G be a Lie group acting canonically on the symplectic manifold (M, ω). Suppose that for any $\xi \in \mathcal{G}$, $J^\xi \in C^\infty(M)$ is the Hamiltonian function corresponding to the globally Hamiltonian vector field ξ_M. Therefore $\xi_M = X_{J^\xi}$. Hence

$$i_{\xi_M}\omega = i_{X_{J^\xi}}\omega = dJ^\xi = d < J, \xi > .$$

Hence the above expression 1.3 is equivalent to

$$i_{\xi_M}\omega = d < J, \xi > \forall \xi \in \mathcal{G}. \tag{1.4}$$

Proposition 1.4.2. Let G be a Lie group acting canonically on the symplectic manifold (M, ω). Then the momentum maps for the associated Lie algebra $\mathcal{G}$-action are not uniquely determined. In fact,

(i) J_1 and J_2 are momentum maps for the same canonical action if and only if for any $\xi \in \mathcal{G}$, $J_1^\xi - J_2^\xi \in C(M)$, where $C(M)$ is the set of all casimir functions.

(ii) If M is connected, then momentum maps are determined up to a constant in $\mathcal{G}^*$.

Proof : (i) J_1 and J_2 are momentum maps for the same canonical action if and only if for any $\xi \in \mathcal{G}$,

$$\begin{aligned}
\xi_M &= X_{J_1^\xi} = X_{J_2^\xi} \\
\Leftrightarrow X_{J_1^\xi}(f) &= X_{J_2^\xi}(f), \forall f \in C^\infty(M) \\
\Leftrightarrow \{f, J_1^\xi\} &= \{f, J_2^\xi\}, \forall f \in C^\infty(M) \\
\Leftrightarrow \{f, J_1^\xi - J_2^\xi\} &= 0, \forall f \in C^\infty(M) \\
\Leftrightarrow X_{J_1^\xi - J_2^\xi} &= 0 \\
\Leftrightarrow J_1^\xi - J_2^\xi &\in C(M).
\end{aligned}$$

(ii)Suppose M is connected. Let J_1 and J_2 are momentum maps for the same canonical action. Let $J = J_1 - J_2$ and fix $m \in M$. Then for every $\xi \in \mathcal{G}$ and every

$v_m \in T_m M$ we have

$$\begin{aligned}
(T_m J)(v_m)\xi &= dJ^\xi(m)v_m \\
&= dJ_1^\xi(m)v_m - dJ_2^\xi(m)v_m \\
&= \omega(m)(X_{J_1^\xi}(m) - X_{J_2^\xi}(m), v_m) \\
&= \omega(m)(\xi_M(m) - \xi_M(m), v_m) \\
&= 0.
\end{aligned}$$

Hence $T_m J$ vanishes for every $m \in M$. Since M is connected we get $J = \mu_o$ for some $\mu_o \in \mathcal{G}^*$. •

Example 1.4.3. (i) Linear momentum : Take the phase space of the N-particle system, that is, $T^*\Re^{3N}$. The additive group $\Re^3$ acts on it by spacial translation on each factor : $v.(q_i, p^i) = (q_i + v), p^i)$, with $i = 1, 2, ...N$. This action is canonical and has an associated momentum map that coincides with the classical linear momentum $J : T^*\Re^{3N} \to \Re^3$, $(q_i, p^i) \to \sum_{i=1}^N p_i$.

(ii) Angular momentum : Let SO(3) act on $\Re^3$ and then, by lift, on $T^*\Re^3$, that is, $A.(q,p) = (Aq, Ap)$. This action is canonical and has an associated momentum map $J : T^*\Re^3 \to \Re^3$, $(q,p) \to q \times p$.

(iii) Lifted actions on cotangent bundles : Let G be a Lie group acting on the manifold M and then by lift on its cotangent bundle T^*M. Any such lifted action is canonical with respect to the canonical symplectic form on T^*M and has an associated momentum map $J : T^*M \to \mathcal{G}^*$ given by $< J(\alpha_m), \xi >=< \alpha_m, \xi_M(m) >$ for any $\alpha_m \in T^*M$ and $\xi \in \mathcal{G}$.

Proposition 1.4.4. Let $\mathcal{G}$ be a Lie algebra acting canonically on the Poisson manifold $(M, \{.,.\})$, and $J : M \to \mathcal{G}^*$ an associated momentum map. Let $\mathcal{L}$ be a symplectic leaf of $(M, \{.,.\})$. Then,
(i) The $\mathcal{G}$-action on $(M, \{.,.\})$ restricts to a canonical $\mathcal{G}$-action on $(\mathcal{L}, \omega_{\mathcal{L}})$.

(ii) $J_{\mathcal{L}} := J\,|_{\mathcal{L}}: \mathcal{L} \longrightarrow \mathcal{G}^*$ is a momentum map for this action.

Proof : (i) Let $m \in \mathcal{L}$ and $\xi \in \mathcal{G}$ arbitrary. Then $\xi_M(m) = X_{J^\xi}(m) \in T_m\mathcal{L}$, hence the $\mathcal{G}$-action leaves $\mathcal{L}$-invariant and thus $\mathcal{G}$ acts canonically on $\mathcal{L}$ since the inclusion $\mathcal{L} \hookrightarrow M$ is a Poisson map.

(ii) follows from the canonical character of the inclusion $\mathcal{L} \hookrightarrow M$. •

Note 1.4.5. We consider the problem of existence of momentum maps. Its existence is guaranteed when the infinitesimal generators of this action are Hamiltonian vector fields. In other words, if the Lie algebra $\mathcal{G}$ acts canonically on the Poisson manifold $(M, \{.,.\})$, then for each $\xi \in \mathcal{G}$, we require the existence of a globally defined function $J^\xi \in C^\infty(M)$ such that $\xi_M = X_{J^\xi}$. In general this is not guaranteed even if there is a canonical Lie algebra action, as the next example shows.

Example 1.4.6. Let $T^2 = \{(e^{i\theta_1}, e^{i\theta_2})\}$ be the two-torus considered as a symplectic manifold with the area form $\omega := d\theta_1 \wedge d\theta_2$. The circle $S^1 = \{e^{i\phi}\}$ acts canonically on T^2 by

$$e^{i\phi}.(e^{i\theta_1}, e^{i\theta_2}) := (e^{i(\theta_1+\phi}, e^{i\theta_2}).$$

Then the infinitesimal generators generated by this action cannot be integrated to a Hamiltonian vector field.

The following results characterizes the existence of momentum maps.

Proposition 1.4.7. Let (M, ω) be a symplectic manifold and $\mathcal{G}$ be a Lie algebra acting canonically on it . Then there exists a momentum map if and only if the action is almost Hamiltonian.

Proof : From the definition of momentum map, the existence of it is guaranteed when the infinitesimal generators of the action are Hamiltonian vector fields. In other words, if the Lie algebra $\mathcal{G}$ acts canonically on the symplectic manifold (M, ω), then, for each $\xi \in \mathcal{G}$, we require the existence of a globally defined function$J^\xi \in C^\infty(M)$, such that $\xi_M = X_{J^\xi}$. But this is same as the action is almost Hamiltonian. •

Proposition 1.4.8. Let (M, ω) be a symplectic manifold and $\mathcal{G}$ a Lie algebra acting canonically on it . There exists a momentum map associated to this action if and only if the linear map $\rho : \frac{\mathcal{G}}{[\mathcal{G},\mathcal{G}]} \to H^1(M, \Re)$, by $\rho([\xi]) = [i_{\xi_M}\omega]$ is identically zero.

Proof : We start by showing that the map ρ is well defined. It suffices to show that if $\eta \in [\mathcal{G}, \mathcal{G}]$, then $i_{\eta_M}\omega$ is exact. The element η can be written as a sum of brackets of elements in $\mathcal{G}$. For simplicity suppose that $\eta = [\xi, \zeta]$. Now, as the Lie algebra action is canonical we have that

$$\mathcal{L}_{\xi_M}\omega = \mathcal{L}_{\zeta_M}\omega = 0,$$

that is , ξ_M and ζ_M are locally Hamiltonian vector fields and consequently their Lie bracket $[\xi_M, \zeta_M]$ is Hamiltonian with associated Hamiltonian function $\omega(\zeta_M, \xi_M)$. Therefore

$$i_{\eta_M}\omega = i_{[\xi,\zeta]_M}\omega = -i_{[\xi_M,\zeta_M]}\omega = -d(\omega(\zeta_M, \xi_M)),$$

which is an exact form, as required. The map ρ is clearly linear.
Now, notice that there exist a momentum map $J : M \to \mathcal{G}^*$ if and only if for any $\xi \in \mathcal{G}$ we can write $i_{\xi_M}\omega = dJ^\xi$. This is equivalent to $[i_{\xi_M}\omega] = 0 \in H^1(M, \Re)$, which in turn can be written as $\rho[\xi] = 0$, for any $\xi \in \mathcal{G}$. •

As a consequence of this proposition we have the following theorem.

Theorem 1.4.9. Let (M, ω) be a symplectic manifold and $\mathcal{G}$ be a Lie algebra acting canonically on it . There exists a momentum map associated to this action if and only if one of the following is true.

(i) $H^1(M, \Re) = 0$.

(ii) $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$.

(iii) $H^1(\mathcal{G}, \Re) = 0$.

(iv) $\mathcal{G}$ is semisimple.

Proof : The cases (i) , (ii) and (iii) trivially imply that $\rho \equiv 0$. In case (iv), if $\mathcal{G}$ is semisimple, the first Whitehead lemma implies that $H^1(\mathcal{G}, \Re) = 0$ and therefore a momentum map always exists. •

Theorem 1.4.10. Let (M, ω) be a symplectic manifold and G a Lie group acting canonically on it . There exists a momentum map associated to this action if and only if $H^1_{deRham}(G) = 0$.

Proof : We have $H^1_{deRham}(G) = H^1(\mathcal{G}, R)$. Using the above theorem we have there exist a momentum map associated to this action if and only if
$H^1_{deRham}(G) = 0$. •

Theorem 1.4.11. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds and let $\mathcal{G}$ be a Lie algebra acting canonically on both M_1 and M_2. Let $\phi : M_1 \to M_2$ be a canonical $\mathcal{G}$-equivariant diffeomorphism. Then the existence of momentum map for the $\mathcal{G}$-action on M_1 is a necessary and sufficient condition for the existence of momentum map for the $\mathcal{G}$-action on M_2.

Proof : Suppose there exist a momentum map $J_1 : M_1 \to \mathcal{G}^*$ for the $\mathcal{G}$-action on M_1. Then $J_2 : M_2 \to \mathcal{G}^*$ defined by $J_2 = J_1 \circ \phi^{-1}$ is a momentum map for the $\mathcal{G}$-action on M_2. Conversely, if $J_2 : M_2 \to \mathcal{G}^*$ is a momentum map for the $\mathcal{G}$-action on M_2 and ϕ is a $\mathcal{G}$-equivariant immersion, then $J_1 : M_1 \to \mathcal{G}^*$ defined by $J_1 = J_2 \circ \phi$ is a momentum map for the $\mathcal{G}$-action on M_1. •

The momentum map J maps the G-manifold M into the G-manifold $\mathcal{G}^*$. Now we look at the equivariance of momentum maps.

Definition 1.4.12. Let G be a Lie group acting canonically on the symplectic manifold (M, ω). Then the momentum map J for the Lie algebra $\mathcal{G}$-action associated to the Lie group action is called *coadjoint equivariant* if and only if J satisfies

$$Ad^*_{g^{-1}} \circ J = J \circ \Phi_g, \qquad \forall g \in G. \tag{1.5}$$

Proposition 1.4.13. Let (M, ω) be a connected symplectic manifold and let ϕ be a symplectic action by a Lie group G with the momentum map J. If the action has a fixed point, then it has a coadjoint equivariant momentum map.

Proof : Momentum map exists, so the action is almost Hamiltonian. Thus there exists a linear map $\widehat{J} : \mathcal{G} \to C^\infty(M)$ satisfying $j \circ \widehat{J}(\xi) = \xi_M$. Then for every $g \in G$,

$$\begin{aligned} j((\phi_g)^* \widehat{J}(\xi)) &= (\phi_g)_* j \widehat{J}(\xi) \\ &= (\phi_g)_* \xi_M \\ &= (Ad_{g^{-1}} \xi)_M \\ &= j \circ \widehat{J}(Ad_{g^{-1}} \xi). \end{aligned}$$

That is, $(\phi_g)^* \widehat{J}(\xi) - \widehat{J}(Ad_{g^{-1}}\xi)$ is a constant function on M and so

$$(\phi_g)^* \widehat{J}(\xi) - \widehat{J}(Ad_{g^{-1}}\xi) = (\phi_g)^* \widehat{J}(\xi)(m_o) - \widehat{J}(Ad_{g^{-1}}\xi)(m_o),$$

for any given point $m_o \in M$.

Let m_o be a fixed point for the symplectic action ϕ, that is , $\phi_g(m_o) = m_o$ for each $g \in G$. Then we have

$$(\phi_g)^* \widehat{J}(\xi) - \widehat{J}(Ad_{g^{-1}}\xi) = \widehat{J}(\xi)(m_o) - \widehat{J}(Ad_{g^{-1}}\xi)(m_o),$$

for every $g \in G$.

Define $\lambda : \mathcal{G} \to C^\infty(M)$ by $\lambda(\xi) = \widehat{J}(\xi) - \widehat{J}(\xi)(m_o)$. Then λ is linear and satisfies $j \circ \lambda(\xi) = \xi_M$.
Moreover we have

$$\begin{aligned}(\phi_g)^*\lambda(\xi) &= (\phi_g)^*\widehat{J}(\xi) - \widehat{J}(\xi)(m_o) \\ &= \widehat{J}(Ad_{g^{-1}}\xi) - \widehat{J}(Ad_{g^{-1}}\xi)(m_o) \\ &= \lambda(Ad_{g^{-1}}\xi).\end{aligned}$$

Thus λ defines a coadjoint equivariant momentum map for the symplectic action ϕ. Hence the proposition. •

Theorem 1.4.14. Let G be a Lie group acting canonically on a connected symplectic manifold (M, ω). If $H^1(\mathcal{G}, \Re) = H^2(\mathcal{G}, \Re) = 0$, then there exist a coadjoint equivariant momentum map for the associated Lie algebra action.

Proof : Since $H^1(\mathcal{G}, \Re) = H^2(\mathcal{G}, \Re) = 0$, we have the coboundary operator $\delta : \mathcal{G}^* \to Z^2(\mathcal{G})$ is bijective. Then we can define $J : M \to \mathcal{G}^*$ by $J(m) = \delta^{-1}(\phi^m)^*\omega$. Since

$$\begin{aligned}d < J, [\xi, \eta] > &= -d < \delta J, \xi \wedge \eta > \\ &= -d < (\phi^m)^*\omega, \xi \wedge \eta > \\ &= \omega([\xi, \eta]_M, .)\end{aligned}$$

and $[\xi, \eta]$ generates $\mathcal{G}$. Therefore J is a momentum map.

Also

$$
\begin{aligned}
J(gm) &= \delta^{-1}(\phi^{gm})^*\omega \\
&= \delta^{-1}(\phi^m r_g)^*\omega \\
&= \delta^{-1}r_g^*(\phi^m)^*\omega \\
&= \delta^{-1}Ad(g^{-1})^*(\phi^m)^*\omega \\
&= Ad(g^{-1})^*\delta^{-1}(\phi^m)^*\omega \\
&= Ad(g^{-1})^*J(m)
\end{aligned}
$$

that is, J is coadjoint equivariant. •

Theorem 1.4.15. Let G be a Lie group acting canonically on a connected symplectic manifold (M,ω). If $\omega = d\theta$, where θ a G-invariant 1-form, then there exist a coadjoint equivariant momentum map for the associated Lie algebra action.

Proof : Since $\omega = d\theta$, define $J : M \to \mathcal{G}^*$ as $J(m) = (\phi^m)^*\theta$. Then $\forall X \in \mathcal{X}(\mathrm{M})$,

$$
\begin{aligned}
d < J,\xi > (X) = d < (\phi^m)^*\theta, \xi > (X) &= d((\phi^m)^*\theta(\xi))(X) \\
&= i_X d((\phi^m)^*\theta(\xi)) = \mathcal{L}_X((\phi^m)^*\theta(\xi)) \\
&= \mathcal{L}_X(\theta(\phi^m_*)_e(\xi)) = \mathcal{L}_X(\theta(\xi_M)) \\
&= i_X d(\theta(\xi_M)) = d\theta(\xi_M)(X) \\
&= d\theta(\xi_M, X) = \omega(\xi_M, X).
\end{aligned}
$$

Hence J is a momentum map.To prove the coadjoint equivariancy

$$\begin{aligned} J(gm) &= (\phi^{gm})^*\theta \\ &= (\phi^m r_g)^*\theta \\ &= r_g^*(\phi^m)^*\theta \\ &= r_g^* J(m) \\ &= (Ad_{g-1})^* J(m) \end{aligned}$$

Thus J is a coadjoint equivariant momentum map. •

Theorem 1.4.16. Let G be a compact Lie group acting canonically on a symplectic manifold (M, ω) with an associated momentum map $J : M \to \mathcal{G}^*$. Then there exists a momentum map that is equivariant.

Proof : For each $g \in G, m \in M$ we define

$$J_g(m) = Ad^*_{g-1} J(g^{-1}.m).$$

or, equivalently

$$J_g^\xi = J^{Ad_{g^{-1}}\xi} \circ \Phi_{g^{-1}}.$$

Then J_g is also a momentum map for the G-action on M. Indeed, if $m \in M, \xi \in \mathcal{G}$, and $f \in C^\infty(M)$ we have

$$
\begin{aligned}
\{f, J_g^\xi\}(m) &= -dJ_g^\xi(m).X_f(m) \\
&= -dJ^{Ad_{g^{-1}}\xi}(g^{-1}.m).T_m\Phi_{g^{-1}}.X_f(m) \\
&= -dJ^{Ad_{g^{-1}}\xi}(g^{-1}.m).(\Phi_g^* X_f)(g^{-1}.m) \\
&= -dJ^{Ad_{g^{-1}}\xi}(g^{-1}.m).X_{\Phi_g^* f}(g^{-1}.m) \\
&= \{\Phi_g^* f, J^{Ad_{g^{-1}}\xi}\}(g^{-1}.m) \\
&= (Ad_{g^{-1}}\xi)_M[\Phi_g^* f](g^{-1}.m) \\
&= (\Phi_g^* \xi_M)[\Phi_g^* f](g^{-1}.m) \\
&= df(m).\xi_M(m) \\
&= \{f, J^\xi(m)\}.
\end{aligned}
$$

Therefore, $\{f, J_g^\xi - J^\xi\} = 0$ for every $f \in C^\infty(M)$, that is, $J_g^\xi - J^\xi$ is a casimir function on M for every $g \in G$ and every $\xi \in \mathcal{G}$.

Now define ,$< J >= \int_G J_g dg$, where dg denotes the Haar measure on G normalized such that the total volume of G is one. Equivalently, this definition states that $< J >^\xi = \int_G J_g^\xi dg$, for every $\xi \in \mathcal{G}$.

By the linearity of the Poisson bracket in each factor, it follows that

$$
\begin{aligned}
\{f, < J >^\xi\} &= \int_G \{f, J_g^\xi\} dg \\
&= \int_G \{f, J^\xi\} dg = \{f, J^\xi\}.
\end{aligned}
$$

Thus $< J >$ is also a momentum map for the G-action on M and $< J >^\xi - J^\xi$ is a casimir function on M for every $\xi \in \mathcal{G}$.

The momentum map $< J >$ is equivariant. Noting that

$$J_g(h.m) = Ad^*_{h_{-1}} J_{h^{-1}g}(m)$$

and using invariance of the Haar measure on G under translations, for any $h \in G$, we indeed have,

$$\begin{aligned} < J > (h.m) &= \int_G Ad^*_{h_{-1}} J_{h^{-1}g}(m)dg \\ &= Ad^*_{h_{-1}} \int_G J_{h^{-1}g}(m)dg \\ &= Ad^*_{h_{-1}} \int_G J_k(m)dk, \text{ where } k = h^{-1}g \\ &= Ad^*_{h_{-1}} < J > (m). \end{aligned}$$

•

Definition 1.4.17. Let G_1 and G_2 be Lie groups. The *semi direct product* $G_1 \times_\sigma G_2$ of G_1 and G_2 is the set $G_1 \times G_2$ relative to the multiplication $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1(g_1.h_2))$.

Theorem 1.4.18. Let G_1 and G_2 be Lie groups acting canonically on a connected symplectic manifold (M, ω) with G_1 is connected and $H^1(\mathcal{G}_1, \Re) = 0$. Suppose the above actions admits coadjoint equivariant momentum maps. If $G = G_1 \times_\sigma G_2$ is the semi direct product, then G has a coadjoint equivariant momentum map.

Proof : Let J_1 and J_2 be the coadjoint equivariant momentum maps corresponding to the Lie group actions by G_1 and G_2 respectively. Then $J = J_1 + J_2$ is a momentum map for the action by the Lie group G. To prove coadjoint equivariance, it is enough to prove $< J(gm) - Ad^*_{g^{-1}} J(m), \eta >= 0$ for two cases (i) $g \in G_2$ and $\eta \in \mathcal{G}_1$ or (ii) $g \in G_1$ and $\eta \in \mathcal{G}_2$.

For the case (i), since $H^1(\mathcal{G}_1, \Re) = 0$, we may assume $\eta = [\eta_1, \eta_2]$ where η_1 and η_2

are in $\mathcal{G}_1$. Then we have

$$< J(gm) - Ad^*_{g^{-1}} J(m), \eta > = < J(gm) - Ad^*_{g^{-1}} J(m), [\eta_1, \eta_2] > = 0.$$

Hence the coadjoint equivariancy of J.
For the case (ii), consider the map $G \ni g \to < J(gm) - Ad^*_{g^{-1}} J(m), \eta >$. Differentiating this map to the direction $\xi \in \mathcal{G}_1$, we have

$$i_{\xi_M} d < J, \eta > - < J, ad(\xi)\eta > = \omega(\xi_M, \eta_M) - < J, [\xi, \eta] > = 0.$$

Thus $< J(gm) - Ad^*_{g^{-1}} J(m), \eta >$ is constant on G_1. If G_1 is connected , $< J(gm) - Ad^*_{g^{-1}} J(m), \eta > = 0$ identically. Hence J is coadjoint equivariant. •

Theorem 1.4.19. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds and let G be a Lie group acting canonically on both M_1 and M_2 . Suppose the above actions admit coadjoint equivariant momentum maps. Then G has a coadjoint equivariant momentum map on $(M_1 \times M_2, \pi_1^* \omega_1 - \pi_2^* \omega_2)$ where π_1 and π_2 are projections on M_1 and M_2 respectively.
Proof : Suppose J_1 and J_2 are the coadjoint equivariant momentum maps corresponding to the Lie group actions by G on M_1 and M_2 respectively. Then $J = J_1 \circ \pi_1 - J_2 \circ \pi_2$ is a momentum map for the G-action on $(M_1 \times M_2, \pi_1^* \omega_1 - \pi_2^* \omega_2)$. To prove this it is enough to prove $i_{\xi_M} \omega = d < J, \xi >$, for any $\xi \in \mathcal{G}$. Then for

every $\xi \in \mathcal{G}$ and $m \in M$,

$$\begin{aligned}
i_{\xi_M}\omega(m) &= i_{\xi_M}(\pi_1^*\omega_1 - \pi_2^*\omega_2)(m) \\
&= i_{\xi_M}(\pi_1^*\omega_1(m)) - i_{\xi_M}(\pi_2^*\omega_2(m)) \\
&= \pi_1^* i_{\xi_M}\omega_1(m) - \pi_2^* i_{\xi_M}\omega_2(m) \\
&= i_{d\pi_1(m)\xi_M}\omega_1(\pi_1(m)) - i_{d\pi_2(m)\xi_M}\omega_2(\pi_2(m)) \\
&= i_{\xi_{M_1}}\omega_1(\pi_1(m)) - i_{\xi_{M_2}}\omega_2(\pi_2(m)) \\
&= d < J_1, \xi > (\pi_1(m)) - d < J_2, \xi > (\pi_2(m)) \\
&= d < J_1 \circ \pi_1 - J_2 \circ \pi_2, \xi > (m) \\
&= d < J, \xi > (m).
\end{aligned}$$

Thus J is a momentum map. Also

$$\begin{aligned}
J(gm) &= (J_1 \circ \pi_1 - J_2 \circ \pi_2)(gm) \\
&= J_1(gm_1) - J_2(gm_2) \\
&= Ad^*_{g^{-1}} J_1(m_1) - Ad^*_{g^{-1}} J_2(m_2) \\
&= Ad^*_{g^{-1}}(J_1 \circ \pi_1(m) - J_2 \circ \pi_2(m)) = Ad^*_{g^{-1}} J(m), \forall g \in G.
\end{aligned}$$

Thus J is a coadjoint equivariant momentum map. •

Theorem 1.4.20. Let G_1 and G_2 be Lie groups acting canonically on connected symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with G_1 is connected and $H^1(\mathcal{G}_1, \Re) = 0$. Suppose the above actions admits coadjoint equivariant momentum maps. If $G = G_1 \times_\sigma G_2$ then G has a coadjoint equivariant momentum map on $(M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$ where π_1 and π_2 are projections on M_1 and M_2 respectively.

Proof : Given G_1 and G_2 are Lie groups acting canonically on a connected symplectic manifold (M_1, ω_1) with G_1 is connected and

$H^1(\mathcal{G}_1, \Re) = 0$. Suppose the above actions admits coadjoint equivariant momentum maps J_1 and J_2 respectively. Then from the theorem 1.4.18 , $J^{'} = J_1 + J_2$ is a coadjoint equivariant momentum map for the G-action on M_1. Similarly, if J_3 and J_4 are coadjoint equivariant momentum maps corresponding to the G_1 and G_2 actions on (M_2, ω_2) respectively, then $J^{"} = J_3 + J_4$ is a coadjoint equivariant momentum map for the G-action on M_2. Then using theorem 1.4.19, $J = J^{'} \circ \pi_1 - J^{"} \circ \pi_2$ is a coadjoint equivariant momentum map for the G-action on $M_1 \times M_2$. •

For the Lie group action we look at the G-equivariance of momentum maps. For the Lie algebra action we look at the case when the map $\mathcal{G} \to C^\infty(M)$ is a Lie algebra homomorphism, which is an infinitesimal version of the G-equivariance.

Definition 1.4.21. Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold (M, ω). Then the momentum map $J : M \to \mathcal{G}^*$ for this action is called infinitesimally equivariant if and only if J satisfies

$$J^{[\xi,\eta]} = \{J^\xi, J^\eta\}, \qquad \xi, \eta \in \mathcal{G}. \tag{1.6}$$

Theorem 1.4.22. Let G be a Lie group acting canonically on the symplectic manifold (M, ω). Then there exists an infinitesimally equivariant momentum map for the associated Lie algebra action if and only if the action is Hamiltonian.

Proof : Let $J : M \to \mathcal{G}^*$ be the infinitesimally equivariant momentum map . That is, J satisfies

$$J^{[\xi,\eta]} = \{J^\xi, J^\eta\}, \qquad \xi, \eta \in \mathcal{G}.$$

That is if and only if there exist a map $\tau : (\mathcal{G}, [.,.]) \to (C^\infty(M), \{.,.\})$ defined by $\xi \to J^\xi, \xi \in \mathcal{G}$, is a Lie algebra homomorphism. Also for $j : C^\infty(M) \to \mathcal{X}_{\mathcal{H}}(M, \omega)$

defined by $j(f) = X_f$, in 1.1.32, we have

$$j \circ \tau(\xi) = j(J^{\xi}) = \xi_M$$

from the definition of J^{ξ}. That is if and only if the action is Hamiltonian. . •

When G is connected Guillemin and Sternberg proved that the G-equivariance and $\mathcal{G}$-equivariance are equivalent.

Proposition 1.4.23. Let J be the momentum map for the Hamiltonian action of the connected group G on the symplectic manifold (M, ω). Then J is an equivariant momentum map.

Proof : It is sufficient to prove the infinitesimal version. We consider an element ξ of $\mathcal{G}$ and we prove $T_m J.\xi_M(m) = -ad^*_{\xi} J(m)$.

For any $\zeta \in \mathcal{G}$,

$$\begin{aligned}
< T_m J.\xi_M(m), \zeta > &= < \xi_M(m), T_m J^t(\zeta) > \\
&= < \xi_M(m), (i_{\zeta_M}\omega)_m > \\
&= \omega_m(\zeta_M, \xi_M) \\
&= -\{J^{\xi}, J^{\zeta}\}(m) \\
&= -J^{[\xi,\zeta]}(m) \\
&= - < J(m), [\xi, \zeta] > \\
&= < -ad^*_{\xi} J(m), \zeta > .
\end{aligned}$$

•

Definition 1.4.24. Given a Lie algebra $\mathcal{G}$, an exact sequence $0 \rightarrow \mathcal{Z} \xrightarrow{i} \mathcal{E} \xrightarrow{j} \mathcal{G} \rightarrow 0$ of the Lie algebras is called an *extension of* $\mathcal{G}$ *by* $\mathcal{Z}$. If $i(\mathcal{Z})$ is contained in the center of $\mathcal{E}$ we call it a *central extension.*

Example 1.4.25. $0 \rightarrow \Re \rightarrow C^{\infty}(M) \rightarrow \mathcal{X}_{\mathcal{H}}(M, \omega) \rightarrow 0$ is a central extension of

$\mathcal{X}_{\mathcal{H}}(M,\omega)$ by $\Re$.

Theorem 1.4.26. Let G be a Lie group acting canonically on the symplectic manifold (M,ω) via the action ϕ. Suppose there exist a momentum map for the associated Lie algebra action. Then there is a central extension $G(\phi)$ of G by $\Re$ such that the action of $G(\phi)$ has an infinitesimally equivariant momentum map.

Proof: Given that there exist a momentum map, so the action is almost Hamiltonian. Therefore we have a sequence

$$0 \to \Re \xrightarrow{i} C^\infty(M) \xrightarrow{j} \mathcal{X}_{\mathcal{H}} \quad (M,\omega) \to 0.$$

Let ϕ be an almost Hamiltonian action of G on (M,ω). This gives a central extension $\widehat{\mathcal{G}}$ of $\mathcal{G}$ by $\Re$ which we denote by $\mathcal{G}(\phi)$ to indicates its dependence on the action. Explicitly

$$\mathcal{G}(\phi) = \{(\xi, f) \in \mathcal{G} \times C^\infty(M)/\dot{\phi}(\xi) = X_f\}, \quad and$$
$$i(z) = (0, z), j(\xi, f) = \xi.$$

The homomorphism j exponentiates to a homomorphism of Lie groups $G(\phi) \to G$ where $G(\phi)$ is the simply connected Lie group with Lie algebra $\mathcal{G}(\phi)$, and allows us to get an action ϕ_1 of $G(\phi)$ on M which is symplectic. It differentiates to give $\dot{\phi} \circ j$, so is almost Hamiltonian,with $\dot{\phi}_1(\xi, f) = X_f$. It follows that if we define $\lambda(\xi, f) = f$ then λ is Hamiltonian for this action. •

Definition 1.4.27. If we choose $\mathcal{Z} = H^2(\mathcal{G})^*$, then the corresponding central extension $\mathcal{E}_o$ is called the Universal central extension of $\mathcal{G}$.

Theorem 1.4.28. Let G be a Lie group acting canonically on the symplectic manifold (M,ω). Then there exists an infinitesimally equivariant momentum map if the action of G passing to the universal central extension of $\mathcal{G}$ by $H^2(\mathcal{G})^*$.

Proof: It directly follows from the theorem 1.4.26. •

Note 1.4.29. In general it is not possible to choose a coadjoint equivariant momentum map, but we could ask whether one can define another action on this space with respect to which we have equivariance.

Definition 1.4.30. Let (M, ω) be a connected symplectic manifold and G a Lie group acting on M in a canonical fashion with an associated momentum map $J : M \to \mathcal{G}^*$. Define the *non-equivariance one cocycle* associated to J as the map

$$\sigma : G \longrightarrow \mathcal{G}^*, \text{ given by}$$
$$\sigma(g) = J(\phi_g(z)) - Ad^*_{g^{-1}}(J(z)).$$

Proposition 1.4.31. Let (M, ω) be a connected symplectic manifold and G a Lie group acting on M in a canonical fashion with an associated momentum map $J : M \to \mathcal{G}^*$ and non-equivariance one cocycle σ. Then :

(i) The definition of σ does not depend on the choice of $z \in M$.

(ii) The mapping σ is a $\mathcal{G}^*$-valued one-cocycle on G with respect to the coadjoint representation of G on $\mathcal{G}^*$.

(iii) If J' is another momentum map for the same canonical action of G on M, then its non-equivariance one-cocycle σ' is in the same Lie group cohomology class as σ; that is , $\sigma - \sigma'$ is a one-boundary.

Proof: (i) Let $g \in G$ be fixed and $\tau_g : M \longrightarrow \mathcal{G}^*$ the mapping defined by $\tau_g(z) = J(\phi_g(z)) - Ad^*_{g^{-1}}(J(z)), z \in M$. Now, for any $\xi \in \mathcal{G}$ and $v_z \in T_zM$ we have that

$$\begin{aligned}
< T_z\tau_g.v_z, \xi > &= dJ^{\xi}(g.z).T_z\phi_g.v_z - dJ^{Ad_{g^{-1}}\xi}(z).v_z \\
&= \omega(g.z)(\xi_M(g.z), T_z\phi_g.v_z) - \omega(z)((Ad_{g^{-1}}\xi)_M(z), v_z) \\
&= \omega(g.z)(T_z\phi_g.(Ad_{g^{-1}}\xi)_M(z), T_z\phi_g.v_z) - \omega(z)((Ad_{g^{-1}}\xi)_M(z), v_z) \\
&= (\phi_g^*\omega)(z)((Ad_{g^{-1}}\xi)_M(z), v_z) - \omega(z)((Ad_{g^{-1}}\xi)_M(z), v_z) \\
&= \omega(z)((Ad_{g^{-1}}\xi)_M(z), v_z) - \omega(z)((Ad_{g^{-1}}\xi)_M(z), v_z) = 0.
\end{aligned}$$

Since v_z and ξ are arbitrary, this shows that $T\tau_g = 0$ and hence, as M is connected, the function τ_g is constant. This proves that the definition of σ does not depend on the choice of $z \in M$.

(ii) On the one hand we have that for any $g, h \in G$

$$\sigma(gh) = J(gh.z) - Ad^*_{(gh)^{-1}}J(z). \tag{1.7}$$

Using the independence of the definition of σ on the choice of the point in the manifold , we take the point $h.z$ and we write $\sigma(g) = J(gh.z) - Ad^*_{g^{-1}}J(h.z)$. We now take the point $z \in M$ and write $\sigma(h) = J(h.z) - Ad^*_{h^{-1}}J(z)$. Hence $\sigma(g) + Ad^*_{g^{-1}}\sigma(h) = J(gh.z) - Ad^*_{(gh)^{-1}}J(z)$ which, by 1.7, coincides with $\sigma(gh)$ establishing the cocycle identity.

(iii) The defining property of a momentum map implies that for any $\xi \in \mathcal{G}$, $d(J^{\xi} - J'^{\xi}) = i_{\xi_M}\omega - i_{\xi_M}\omega = 0$. The connectedness of M implies that $J - J'$ is a constant function. Now, using this fact, we have that for any $g \in G$:

$$\begin{aligned}
\sigma(g) - \sigma'(g) &= J(g.z) - J'(g.z) - Ad^*_{g^{-1}}(J(z) - J'(z)) \\
&= J(z) - J'(z) - Ad^*_{g^{-1}}(J(z) - J'(z)).
\end{aligned}$$

Hence, if we set $\mu = J(z) - J'(z)$ we have that $\sigma(g) - \sigma'(g) = \mu - Ad^*_{g^{-1}}\mu$, which is a coboundary. •

Remark 1.4.32. This proposition identifies the cohomology class $[\sigma]$ in the first group cohomology as the obstruction to the equivariance of a momentum map. If the Lie group G is semisimple, Whitehead's Lemma for groups implies that any non-equivariance one-cocycle is actually a coboundary. Therefore, by part(iii) of the proposition, any momentum map can be modified in this case to be G-equivariant.

Using the non-equivariance one-cocycle we can define a new action of G on $\mathcal{G}^*$, with respect to which a given momentum map J is equivariant.

Definition 1.4.33. Let G be a Lie group acting canonically on the connected symplectic manifold (M, ω) with associated momentum map $J : M \to \mathcal{G}^*$. If $\sigma : G \longrightarrow \mathcal{G}^*$ is the non-equivariance one cocycle of J, we define the *affine action* of G on $\mathcal{G}^*$ with cocycle σ by

$$\Theta : G \times \mathcal{G}^* \longrightarrow \mathcal{G}^*, \text{ given by}$$
$$\Theta(g, \mu) = Ad^*_{g^{-1}}\mu + \sigma(g).$$

Proposition 1.4.34. The affine action Θ of G on $\mathcal{G}^*$ determines a left action. The momentum map $J : M \to \mathcal{G}^*$ is equivariant with respect to the symplectic action ϕ on M and the affine action Θ on $\mathcal{G}^*$.

Proof: Since σ satisfying the cocycle identity, Θ of G on $\mathcal{G}^*$ determines a left action. The equivariance of J with respect to with respect to the affine action is clear from the Definition 1.4.33. •

Note 1.4.35. Next we show that the mathematical object that measures the lack of infinitesimal equivariance is a Lie algebra two-cocycle that, in the presence of a canonical group action, can be obtained as the derivative of the non-equivariance group cocycle.

Definition 1.4.36. Let $\mathcal{G}$ be a Lie algebra acting canonically on the connected symplectic manifold (M, ω) with associated momentum map $J : M \to \mathcal{G}^*$. Define the *infinitesimal non-equivariance two-cocycle* associated to J as the element $\Sigma \in \Lambda^2(\mathcal{G})$ given by

$$\Sigma(\xi, \eta) = J^{[\xi,\eta]}(z) - \{J^\xi, J^\eta\}(z), z \in M, \xi, \eta \in \mathcal{G}$$

Theorem 1.4.37. Let $\mathcal{G}$ be a Lie algebra acting canonically on the connected symplectic manifold (M, ω) with associated momentum map J and the infinitesimal non-equivariance two-cocycle associated to J is Σ. Then:
(i) The definition of Σ does not depend on the choice of $z \in M$.
(ii) For any $\xi, \eta \in \mathcal{G}$ we have $X_{\{J^\xi, J^\eta\}} = X_{J^{[\xi,\eta]}}$.
(iii) $\Sigma \in Z^2(\mathcal{G}, \Re)$, that is, Σ is a Lie algebra two-cocycle, that is, it satisfies the *two-cocycle identity*

$$\Sigma([\xi, \eta], \zeta) + \Sigma([\eta, \zeta], \xi) + \Sigma([\zeta, \xi], \eta) = 0, \forall \xi, \eta, \zeta \in \mathcal{G}. \tag{1.8}$$

(iv) For arbitrary $z \in M$ and $\eta \in \mathcal{G}$, we have

$$T_z J.\eta_M(z) = -ad^*_\eta J(z) + \Sigma(\eta, .). \tag{1.9}$$

(v) If $J : M \to \mathcal{G}^*$ is a momentum map associated to the canonical action of Lie group G that has $\sigma : G \to \mathcal{G}^*$ as non-equivariance cocycle, then $\Sigma \in Z^2(\mathcal{G}, \Re)$ is given by

$$\begin{aligned} \Sigma : \mathcal{G} \times \mathcal{G} \longrightarrow \Re, \text{ given by },& \\ \Sigma(\xi, \eta) = d\widehat{\sigma}_\eta(e).\xi,& \end{aligned} \tag{1.10}$$

where $\widehat{\sigma}_\eta : G \longrightarrow \Re$ is defined by $\widehat{\sigma}_\eta(g) =< \sigma(g), \eta >$.

Proof: (i) Define the function $\tau_{\xi,\eta} \in C^\infty(M)$ by $\tau_{\xi,\eta}(z) = J^{[\xi,\eta]}(z) - \{J^\xi, J^\eta\}(z)$. We now compute its derivative:

$$\begin{aligned} d\tau_{\xi,\eta} &= dJ^{[\xi,\eta]} - d(\{J^\xi, J^\eta\}) \\ &= i_{[\xi,\eta]_M}\omega - i_{X_{\{J^\xi, J^\eta\}}}\omega \\ &= -i_{[\xi_M,\eta_M]}\omega + i_{[X_{J^\xi}, X_{J^\eta}]}\omega \\ &= -i_{[\xi_M,\eta_M]}\omega + i_{[\xi_M,\eta_M]}\omega = 0. \end{aligned}$$

The connectedness of M implies that $\tau_{\xi,\eta}$ is constant and that therefore the definition of Σ does not depend on the point $z \in M$ used in its definition.

(ii) By point (i) the functions $\{J^\xi, J^\eta\}$ and $J^{[\xi,\eta]}$ differ by a constant, and hence $X_{\{J^\xi, J^\eta\}} = X_{J^{[\xi,\eta]}}$.

(iii) It is a straightforward consequence of the Jacobi identities satisfied by the brackets $[.,.]$ and $\{.,.\}$ as well as of point (ii).

(iv) For any $z \in M$ and $\xi, \eta \in \mathcal{G}$, we have that

$$\begin{aligned} < T_zJ.\eta_M(z), \xi > &= dJ^\xi(z).\eta_M(z) = \{J^\xi, J^\eta\}(z) = J^{[\xi,\eta]}(z) - \Sigma(\xi,\eta) \\ &= - < J(z), ad_\eta\xi > + \Sigma(\eta,\xi) = - < -ad_\eta^* J(z), \xi > + \Sigma(\eta,\xi). \end{aligned}$$

Since the element ξ is arbitrary, the relation follows.

(v) Using the relation in the previous point, as well as the definition of the non-equivariance cocycle, we can write, for any $\xi, \eta \in \mathcal{G}$

$$\begin{aligned}
\Sigma(\xi,\eta) &= < T_z J.\eta_M(z), \eta > + < ad^*_\xi J(z), \eta > \\
&= \frac{d}{dt}\mid_{t=0} < J(expt\xi.z), \eta > + < ad^*_\xi J(z), \eta > \\
&= \frac{d}{dt}\mid_{t=0} (< \sigma(expt\xi), \eta > + < Ad^*_{exp(-t\xi)} J(z), \eta >) + < ad^*_\xi J(z), \eta > \\
&= \frac{d}{dt}\mid_{t=0} < \sigma(expt\xi), \eta > = d\widehat{\sigma}_\eta(e).\xi.
\end{aligned}$$

•

Remark 1.4.38. In general the existence of momentum map is not guaranteed even if there is a canonical Lie algebra action. But there is a generalization of the standard momentum map, namely cylinder valued momentum maps, which has the important property of being always defined, unlike the standard momentum map. Cylinder valued momentum maps are genuine generalizations of the standard ones in the sense that whenever a Lie algebra action admits a standard momentum map, there is a cylinder valued momentum map that coincides with it. For Abelian symmetries, cylinder valued momentum maps are closely related to the so called Lie group valued momentum maps. We discuss them in the fourth chapter.

1.5 Properties of the Standard Momentum Map.

In this section we discuss certain properties of the momentum map. First we prove J is a submersion on the open dense subset of principal orbits in M. Then Noether's theorem, that is, they are constant on the dynamics of any symmetric Hamiltonian vector field is given. An equivalent condition for the momentum map to be constant on the orbits is given. We establishes a link between the symmetry of a point and the rank of the momentum map at the point, called bifurcation lemma. Also proved that the level sets of the momentum map is locally arcwise connected. [32], [2], [3], [35].

Proposition 1.5.1. Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold (M, ω). Assume that this action admits a momentum map $J : M \to \mathcal{G}^*$. Then the annihilator in $\mathcal{G}$ of $ImT_mJ \subset \mathcal{G}^*$ is $\mathcal{G}_m$.

Proof: The tangent map $T_mJ : T_mM \to \mathcal{G}$ is the transpose of $T_mJ^t : \mathcal{G} \to T_m^*M$; by $\xi \to (i_{\xi_M}\omega)_m$. To prove the proposition it is enough to prove that the annihilator in $\mathcal{G}^*$ of $\mathcal{G}_m$ is the subspace ImT_mJ.

We have ImT_mJ is the annihilator in $\mathcal{G}^*$ of $KerT_mJ^t$. But

$$\begin{aligned} KerT_mJ^t &= \{\xi \in \mathcal{G} | (i_{\xi_M}\omega)_m = 0\} \\ &= \{\xi \in \mathcal{G} | (\xi_M)_m = 0\} \\ &= \text{ Lie algebra } \mathcal{G}_m \text{ of } G_m. \end{aligned}$$

•

Proposition 1.5.2. Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold (M, ω). Assume that this action admits a momentum map $J : M \to \mathcal{G}^*$. The momentum map J is a submersion at the point m if and only if the stabilizer G_m is discrete.

Proof: From the above proposition the rank of T_mJ is the dimension of the annihilator in $\mathcal{G}^*$ of $\mathcal{G}_m$. That is, rank of $T_mJ = dim(\mathcal{G}/\mathcal{G}_m) = dim(G/G_m)$. Thus J is a submersion at the point m if and only if the stabilizer G_m is discrete. •

Corollary 1.5.3. Let G be a commutative Lie group acting effectively on a symplectic manifold (M, ω). Assume that this action admits a momentum map $J : M \to \mathcal{G}^*$. Then J is a submersion on the open dense subset of principal orbits in M.

Proof: If G is commutative and the action is effective, then the stabilizer of the principal orbits is discrete subgroup of G. Using the definition 1.1.28 and theorem 1.1.27 we have the result. •

If G is not commutative, it may happen that the momentum mapping J is nowhere submersive, even if the action is effective, as the next example shows.

Example 1.5.4. Let the group $SO(3)$ of rotations act on $S^2 \times M$ by the usual(effective) action on S^2 and by the trivial action on M. it preserves any "product" symplectic form $\omega_1 \oplus \omega_2$, and the image of the momentum mapping

$$\mu : S^2 \times M \to so(3)^* = \Re^3$$

is a sphere S^2 and, in particular ,contains no open dense subset of $so(3)^*$ so that J is nowhere submersive.

Proposition 1.5.5. Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold (M, ω). Assume that this action admits a momentum map $J : M \to \mathcal{G}^*$. Then the kernel T_mJ is the orthogonal (for ω_m) on the tangent space to the orbit through m.

Proof: We have $T_mJ(Y) = 0$ if and only if $< T_mJ(Y), \xi >= 0$ for all $\xi \in \mathcal{G}$, that is, if and only if $\omega_m((\xi_M)_m, Y) = 0$ for all $\xi \in \mathcal{G}$, that is, if and only if Y is orthogonal to the subspace generated by the infinitesimal generators . •

Theorem 1.5.6 (Noether's Theorem)**.** Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold $(M, \{.,.\})$. Assume that this action admits a momentum map $J : M \to \mathcal{G}^*$. Then the momentum map is a constant of the motion for the Hamiltonian vector field associated to any $\mathcal{G}$ - invariant function $h \in C^\infty(M)^{\mathcal{G}}$, that is, it satisfies Noether's condition.

Proof: It suffices to notice that for any $\xi \in \mathcal{G}$ we have

$$\begin{aligned} \{f, J^{\xi}\}(m) &= df(m).X_{J^{\xi}}(m) \\ &= df(m).\xi_M(m) = \xi_M[f](m) = 0. \end{aligned}$$

•

Remark 1.5.7. The above theorem means that the Hamiltonian vector field X_h is tangent to the level set $J^{-1}(\mu)$ of the momentum map.

Theorem 1.5.8. Let $\mathcal{G}$ be a Lie algebra acting canonically on the Poisson manifold $(M, \{.,.\})$ and $J : M \to \mathcal{G}^*$ an associated momentum map. Let $m \in M$ and let $\mathcal{L} \subset M$ be the symplectic leaf containing m. Then

$$T_m J(T_m \mathcal{L}) = (\mathcal{G}_m)^o,$$

where $(\mathcal{G}_m)^o$ denotes the annihilator in $\mathcal{G}^*$ of the isotropy subalgebra $\mathcal{G}_m$ of m. When M is a symplectic manifold then the above expression can be written as

$$range(T_m J) = (\mathcal{G}_m)^o.$$

This result is sometimes known as *bifurcation lemma* since it establishes a link between the symmetry of a point and the rank of the momentum map at the point.

Proof : Let $\upsilon_m \in T_m\mathcal{L}, \xi \in \mathcal{G}_m$, and let $f \in C^{\infty}(M)$ be such that $\upsilon_m = X_f(m)$. Then,

$$\begin{aligned} < T_m J.\upsilon_m, \xi > &= < T_m J.X_f(m), \xi > \\ &= dJ^{\xi}(m).X_f(m) = \{J^{\xi}, f\}(m) \\ &= -df(m).X_{J^{\xi}}(m) = -df(m).\xi_M(m) = 0. \end{aligned}$$

which proves that

$$T_m J(T_m \mathcal{L}) \subset (\mathcal{G}_m)^o.$$

We now show that

$$(\mathcal{G}_m)^o \subset T_m J(T_m \mathcal{L})$$

or, equivalently,

$$[T_m J(T_m \mathcal{L})]^o \subset \mathcal{G}_m.$$

Let $\xi \in [T_m J(T_m \mathcal{L})]^o$, that is $\xi \in \mathcal{G}$ is such that for any $f \in C^\infty(M)$ we have that

$$0 =< T_m J.X_f(m), \xi >= dJ^\xi(m).X_f(m) = \{J^\xi, f\}(m) = df(m).\xi_M(m).$$

Since the function f is arbitrary $\xi_M(m) = 0$ necessarily, and hence $\xi \in \mathcal{G}_m$.

If M is a symplectic manifold, then $T_m \mathcal{L} = T_m M$. •

Corollary 1.5.9. Let $\mathcal{G}$ be a Lie algebra acting canonically and locally free on the Poisson manifold $(M, \{.,.\})$ and $J : M \to \mathcal{G}^*$ an associated momentum map. Then J is a submersion onto some open subset of $\mathcal{G}^*$.

Proof: Since the action is locally free we have that $\mathcal{G}_m = \{0\}$ for any $m \in M$. Therefore the above proposition guarantees that J is a submersion and thereby an open map. In particular $J(M)$ is an open subset of $\mathcal{G}^*$. •

Note 1.5.10. For a regular value $\mu \in \mathcal{G}^*$ of J, call $V_\mu = J^{-1}(\mu)$. As J is equivariant, the subgroup $G_\mu \subset G$ keeps the set V_μ invariant : if $m \in V_\mu$ and $g \in G_\mu$, then $J(g.m) = g.J(m) = g.\mu = \mu$.

Look now at what happens to the symplectic form when we restrict by the inclusion map $i_\mu : V_\mu \subset M$

Lemma 1.5.11. Let $\mathcal{G}$ be a Lie algebra acting canonically on the symplectic manifold (M,ω). Assume that this action admits a momentum map $J : M \to \mathcal{G}^*$. The kernel of the pulled back 2-form $i_\mu^*\omega$ at a point m is the subspace $T_m(G_\mu.m)$. The rank of this 2-form is constant (namely, it does not depend on m) and

$$rank(i_\mu^*\omega) \;\; = \;\; 2dimV_\mu + dim(G.\mu) - dimM$$

Proof: We have , $Ker(i_\mu^*\omega)_m = T_mV_\mu \cap (T_mV_\mu)^o$.
Also $T_mV_\mu = KerT_mJ$ and from Proposition 1.5.5

$$KerT_mJ = (T_m(G.m))^o.$$

Thus

$$Ker(i_\mu^*\omega)_m \;\; = \;\; KerT_mJ \cap T_m(G.m).$$

The tangent space $T_m(G.m)$ to the orbit is generated by the infinitesimal generators ξ_M. From the above proof $\xi_M(m)$ is in $KerT_mJ$ if and only if the corresponding infinitesimal generator in $\mathcal{G}^*$ vanishes at $J(m)$, that is, if and only if ξ belongs to the Lie algebra $\mathcal{G}_{J(m)}$ of the stabilizer $G_{J(m)}$. Thus $Ker(i_\mu^*\omega)_m = T_m(G_\mu.m)$, its dimension is

$$dimG_\mu - dimG_m \;\; = \;\; dimG_\mu$$

(because μ is a regular value so that G_m is discrete) . Its rank is

$$\begin{aligned} rank(i_\mu^*\omega) &= dimV_\mu - (dimG_\mu - dimG_m) \\ &= dimV_\mu - dimG_\mu + dimG - dimM + dimV_\mu \\ &= 2dimV_\mu - dimM + dim(G.\mu) \end{aligned}$$

and thus is constant on V_μ. •

Proposition 1.5.12. Let G be a connected Lie group acting effectively on a symplectic manifold (M,ω) with momentum map J. The following three properties are equivalent.

(i) The orbits of G are isotropic .

(ii) The momentum map J is constant on orbits.

(iii) The group G is commutative.

In this case , $dimM \geq 2dimG$.

Proof : The orbit $G.m$ is isotropic if and only if

$$T_m(G.m) \subset (T_m(G.m))^o = KerT_mJ.$$

That is, if and only if $T_mJ(v_m) = 0, \forall v_m \in T_m(G.m)$. That is, if and only if the momentum map J is constant on $G.m$.

In this case, for any $\xi, \zeta \in \mathcal{G}$, the map $m \to -\omega_m(\xi_M(m), \zeta_M(m))$ is identically zero. But we have its differential is the 1-form $i_{[\xi_M,\zeta_M]}\omega$. Thus $i_{[\xi_M,\zeta_M]}\omega = 0$, for which it follows that $[\xi_M, \zeta_M] = 0$ and , due to the fact that the infinitesimal generator of the bracket is the bracket of infinitesimal generators , that is , $[\xi, \zeta] = 0$ for any $\xi, \zeta \in \mathcal{G}$, so that G is commutative.

Conversely, if G is commutative , $G_\mu = G$ for all $\mu \in \mathcal{G}^*$ and thus $Ker(i_\mu^*\omega)$ is the whole tangent space to the orbit which is thus isotropic. •

Proposition 1.5.13. Let (M, ω) be a symplectic manifold endowed with a Hamiltonian action of a compact connected Lie group G, with momentum map J. Let T be a maximal torus in G. If J is submersive in at least one point in W, then

$$dimG + dimT \quad \leq \quad dimM$$

Proof : It is sufficient to observe that the rank of $(i_\mu^*\omega)_m$ is a non negative integer, and to compute:

$$\begin{aligned} rank(i_\mu^*\omega) &= 2dimV_\mu + dim(G.\mu) - dimM \\ &= 2(dimM - dimG) + dim(G.\mu) - dimM \\ &= dimM - 2dimG + dimG - dimG_\mu. \end{aligned}$$

Hence $dimM \geq dimG + dimG_\mu \geq dimG + dimT$. •

Note 1.5.14. Next we show that the level set of the momentum map is locally arc wise connected. To do this we find a local normal form for a coadjoint equivariant momentum map of a proper Hamiltonian action near a fixed point in its zero level set.

Let $\phi : G \times M \to M$ be a proper Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) with coadjoint equivariant momentum map $J : M \to \mathcal{G}^*$. For $\mu \in \mathcal{G}^*$ let m be a point in the level set $J^{-1}(\mu)$. Let G_m be the isotropy group of m under the G-action ϕ and let G_μ be the isotropy group of μ under the coadjoint action of G on $\mathcal{G}^*$. Note that the linear $\omega(m)$-symplectic action

$$\widehat{\phi} : G_m \times T_mM \to T_mM : (h, v_m) \to T_m\phi_h.v_m$$

has a G_m-coadjoint equivariant momentum map $\widehat{J} : T_mM \to \mathcal{G}_m^*$, where

$$\widehat{J}(v_m)\xi = \frac{1}{2}\omega(m)(X^\xi(v_m), v_m),$$

for $\xi \in \mathcal{G}_m$ and $v_m \in T_mM$. Here the vector field X^ξ is the infinitesimal generator of the G_m-action $\widehat{\phi}$ in the direction ξ.

Proposition 1.5.15. We have the following decomposition for T_mM.

$$T_mM \quad = \quad \mathcal{H} \oplus (X \oplus T_m^*(G_\mu.m)),$$

where X is the complement to $W = (KerT_mJ) \cap (KerT_mJ)^{\omega(m)}$ in $KerT_mJ$ and $\mathcal{H}$ is a subspace of $\mathcal{G}$ which is isomorphic to $\frac{\mathcal{G}}{\mathcal{G}_m}$, that is , $\mathcal{H}$ is isomorphic to $T_m(G.m)$.

Proof : Using the Witt decomposition, starting with the subspace $KerT_mJ$ of the symplectic vector space $(T_mM, \omega(m))$, we obtain the decomposition

$$T_mM \quad = \quad X \oplus Y \oplus Z$$

where X, Y and Z are the $\omega(m)$-symplectic subspaces of T_mM defined by $KerT_mJ = X \oplus W$, $(KerT_mJ)^{\omega(m)} = Y \oplus W$ and $Z = (X \oplus Y)^{\omega(m)}$. From the Witt decomposition it follows that W is a Lagrangian subspace of Z. Therefore Z is isomorphic to $W \oplus W^*$. Thus we have

$$\begin{aligned} T_mM &= (Y \oplus W) \oplus (X \oplus W^*) \\ &= (KerT_mJ)^{\omega(m)} \oplus (X \oplus W^*). \end{aligned}$$

To prove the result it is enough to prove that

$$
\begin{aligned}
T_m(G.m) &= (KerT_mJ)^{\omega(m)} && (1.11)\\
T_m(G_\mu.m) &= (KerT_mJ) \cap (KerT_mJ)^{\omega(m)} = W. && (1.12)
\end{aligned}
$$

We have from the definition of the momentum map

$$
\begin{aligned}
\omega(m)(X^\xi(m), v) &= dJ^\xi(m)v\\
&= (T_mJ(v))\xi,
\end{aligned}
$$

for every $\xi \in \mathcal{G}$. Therefore if $v \in KerT_mJ$, then $\omega(m)(X^\xi(m), v) = 0$ for every $\xi \in \mathcal{G}$. In other words , $v \in (T_m(G.m))^{\omega(m)}$.

Conversely, if $v \in (T_m(G.m))^{\omega(m)}$, then $\omega(m)(X^\xi(m), v) = 0$ for every $\xi \in \mathcal{G}$. Hence we have $(T_mJ(v))\xi = 0$, for every $\xi \in \mathcal{G}$, that is ,$v \in KerT_mJ$. Thus $T_m(G.m) = (KerT_mJ)^{\omega(m)}$.

To prove 1.12 we begin by showing that

$$
T_m(G_\mu.m) = T_m(G.m) \cap (KerT_mJ) \qquad (1.13)
$$

Let $v_m \in T_m(G_\mu.m)$. Then for some $\xi \in \mathcal{G}_\mu$ $v_m = X^\xi(m)$. Differentiating the relation $J(\phi_{exps\xi}(m)) = Ad^t_{exp-s\xi}(J(m))$ with respect to s and then setting $s = 0$ gives

$$
\begin{aligned}
T_mJX^\xi(m) &= X^\xi_{\mathcal{G}^*}(J(m))\\
&= X^\xi_{\mathcal{G}^*}(\mu) = 0,
\end{aligned}
$$

where $X^\xi_{\mathcal{G}^*}(\mu) = \frac{d}{ds}/_{s=0}Ad^t_{exp-s\xi}(\mu)$ and the last equality follows because $\xi \in$

$\mathcal{G}_\mu$. Therefore $X^\xi(m) \in KerT_mJ$. That is , $v_m \in KerT_mJ$. Hence $T_m(G_\mu.m) \subseteq KerT_mJ$. Since $G_\mu \subseteq G$, $T_m(G_\mu.m) \subseteq T_m(G.m)$. Hence we have

$$T_m(G_\mu.m) \subseteq T_m(G.m) \cap KerT_mJ \tag{1.14}$$

To prove the reverse inclusion suppose that $v_m \in T_m(G.m) \cap KerT_mJ$. Then there is a $\xi \in \mathcal{G}$ such that $v_m = X^\xi(m)T_e\phi_m\xi$. Because $v_m \in KerT_mJ$ it follows from $T_mJX^\xi(m) = X^\xi_{\mathcal{G}^*}(\mu)$ that $X^\xi_{\mathcal{G}^*}(\mu) = 0$, that is, $\xi \in \mathcal{G}_\mu$. Consequently , $v_m \in T_m(G_\mu.m)$. Hence

$$T_m(G.m) \cap KerT_mJ \subseteq T_m(G_\mu.m) \tag{1.15}$$

Combining 1.14 and 1.15 we get 1.13. Substitute 1.13 in 1.11 we get 1.12. So we get the decomposition. •

Corollary 1.5.16. When $\mu = 0$ the above decomposition reads

$$T_mM = \mathcal{H} \oplus X \oplus W^* \tag{1.16}$$

where $\mathcal{H} = W = T_m(G.m)$.

Proof : When $\mu = 0$ the isotropy subgroup G_μ equals G. Therefore from (1) and (2) we obtain

$$\begin{aligned} T_m(G.m) &= (KerT_mJ)^{\omega(m)} \\ &= (KerT_mJ) \cap (KerT_mJ)^{\omega(m)} = W. \end{aligned}$$

Hence by definition $Y = \{0\}$. •

Corollary 1.5.17. The space X , Y , W and W^* in the Witt decomposition,

$$T_mM = X \oplus Y \oplus (W \oplus W^*) \tag{1.17}$$

can be chosen to be G_m invariant.

Proof : Since $J^{-1}(\mu)$ is invariant under the action $\phi/_{G_m \times M}$, $KerT_mJ) = T_mJ^{-1}(\mu)$ is invariant under the linear $\omega(m)$-symplectic action $\widehat{\phi}$. Consequently ,$(KerT_mJ)^{\omega(m)}$ and $W = (KerT_mJ)^{\omega(m)}$ are also invariant under $\widehat{\phi}$.

Let γ be an inner product on T_mM. Then averaging over G_m, which is compact because G-action is proper, we may assume that γ is $\widehat{\phi}$-invariant. Let X, Y be the orthogonal complement of W in $KerT_mJ$ and $(KerT_mJ)^{\omega(m)}$, respectively. Then X and Y are $\widehat{\phi}$-invariant $\omega(m)$-symplectic subspaces of $(T_mM, \omega(m))$. Therefore Z, which is orthogonal complement of $X \oplus Y$ in T_mM, is $\widehat{\phi}$-invariant $\omega(m)$-symplectic subspace. Hence we have obtained the $\widehat{\phi}$-invariant Witt decomposition

$$T_mM \quad = \quad X \oplus Y \oplus Z.$$

Since W is $\widehat{\phi}$-invariant Lagrangian subspace of Z, its orthogonal complement $W^{\perp}$ in Z is $\widehat{\phi}$-invariant. Since $W^{\perp}$ is isotropic and hence Lagrangian, it is isomorphic to W^*. Hence the corollary. •

Theorem 1.5.18. Let $m \in J^{-1}(0)$. Using the decomposition (6) we write the tangent space T_mM to M at m as the sum $\mathcal{H} \oplus X \oplus W^*$. Let (η, x, α) be coordinates on T_mM with respect to this decomposition. Then there is a local diffeomorphism $\vartheta : T_mM \to M$ with $\vartheta(0) = m$ and $T_o v = id_{T_mM}$ such that for every $v_m = (\eta, x, \alpha)$ sufficiently close to 0 ,

$$\vartheta^* J(\eta, x, \alpha) \quad = \quad Ad^t_{exp-\eta}(\widehat{J}(x) + \alpha).$$

Proof : We prove the theorem in 4 steps. In step 1 we symplectically identify a neighborhood of 0 in $(T_mM, \omega(m))$ with a neighborhood of m in (M, ω) in such a way that the G_m-action $\phi/_{G_m \times M}$ becomes the linear $\omega(m)$-symplectic G_m-action

$\widehat{J}$.

In step 2, after identifying the momentum map J with a locally defined G_m-coadjoint equivariant map $\mathcal{J} : T_mM \to \mathcal{G}^*$, we split $\mathcal{J}$ into a sum of two locally defined G_m-coadjoint equivariant maps $\mathcal{J}' : T_mM \to \mathcal{G}_m^*$, and $\mathcal{J}" : T_mM \to \mathcal{H}^*$ with $\mathcal{J}'(0) = 0, \mathcal{J}"(0) = 0$.

In step 3, we analyze the map $\mathcal{J}'$. Because $\mathcal{J}'(0) = 0$ we have $\mathcal{J}' = \widehat{J}$. We then use the decomposition

$$T_mM \quad = \quad X \oplus (W \oplus W^*), \tag{1.18}$$

where the summands are G_m-invariant and symplectically perpendicular symplectic subspaces of $(T_mM, \omega(m))$, to show that $W^* \subseteq Ker\widehat{J}$. Since $\mathcal{H} = W$, it follows that $\mathcal{H}^* = W^*$.

In step 4 we show that $\mathcal{J}"$ is a local submersion. Changing coordinates on T_mM by local diffeomorphism θ such that $\theta^*\mathcal{J}"$ is locally a projection on to $\mathcal{H}^*$ and $\theta/_{(X\oplus W)}$ is the identity map , we see that for $(x, 0, \alpha) \in T_mM = X\oplus(W\oplus W^*)$ near $(0, 0, 0)$ we have

$$\theta^*\mathcal{J}(x, 0, \alpha) \quad = \quad \widehat{J}(x) + \alpha. \tag{1.19}$$

Using the exponential map from $\mathcal{H}$ to G, the G-action ϕ, and the local diffeomorphism of step 1, we bring the momentum map J into the local normal form. •

Corollary 1.5.19. For every $\mu \in \mathcal{G}^*$ the level set $J^{-1}(\mu)$ is locally arc wise connected.

proof : Using the following device, called the shifting trick, we reduce the proof of the corollary to the case of the 0-level of a G-coadjoint equivariant momentum map $\mathcal{K}$ of a proper G-action. Consider the symplectic manifold $M \times \mathcal{O}_\mu$ with symplectic form $\Omega = \pi_1^*\omega + \pi_2^*\omega_{\mathcal{O}_\mu}$, where ω is a symplectic form on M and $\omega_{\mathcal{O}_\mu}$ is the symplectic form on the G-coadjoint orbit $\mathcal{O}_\mu$. Here π_i is the projection on the i^{th} factor of $M \times \mathcal{O}_\mu$. Define a G-action on $M \times \mathcal{O}_\mu$ by

$$((g,(m,v))) \to (\phi_g(m), Ad^t_{g^{-1}}(v)). \tag{1.20}$$

This action is proper and has coadjoint equivariant momentum map $\mathcal{K} : M\times\mathcal{O}_\mu \to \mathcal{G}^*$ given by $\mathcal{K}(m,v)\xi = J^\xi(m) - v(\xi)$ for every $\xi \in \mathcal{G}$.

First we can prove the level sets $J^{-1}(\mu) \times \mathcal{O}_\mu$ and $\mathcal{K}^{-1}(0)$ are locally diffeomorphic.

Let $\mathcal{U}$ be a neighborhood of $\mu \in \mathcal{O}_\mu$. Suppose that $\sigma : \mathcal{U} \to G$ is a local section of the bundle $G \to \frac{G}{G_\mu}$ such that $\sigma(\mu) = e$ and $Ad^t_{\sigma(v)^{-1}}\mu = v$. Observe that the map

$$\alpha : M \times \mathcal{U} \to M \times \mathcal{U} : (m,v) \to (\phi_{\sigma(v)}(m), v) \quad = \quad (\sigma(v).m, v)$$

is local diffeomorphism. For every $\xi \in \mathcal{G}$ we have

$$\begin{aligned} \mathcal{K}(\alpha(m,v))\xi &= \mathcal{K}(\sigma(v).m, v)\xi \\ &= (J(\sigma(v).m) - v)(\xi) \\ &= (Ad^t_{\sigma(v)^{-1}}J(m) - v)\xi. \end{aligned}$$

Thus $\mathcal{K}(\alpha(m,v)) = 0$ if and only if $J(m) = Ad^t_{\sigma(v)}v = \mu$ and $v \in \mathcal{U}$. Hence the level sets $J^{-1}(\mu) \times \mathcal{O}_\mu$ and $\mathcal{K}^{-1}(0)$ are locally diffeomorphic.

We have $\mathcal{O}_\mu$ is locally arc wise connected. therefore $\mathcal{K}^{-1}(0)$ is locally arc wise connected if and only if $J^{-1}(\mu)$ is. Thus it suffices to prove the corollary when $\mu = 0$. Applying the normal form to the value 0 of the coadjoint equivariant momentum map J, we see that

$$\begin{aligned} u = J(x,\alpha,\eta) \in J^{-1}(0) &\Leftrightarrow 0 = Ad^{t}_{exp-\eta}(\widehat{J}/_X(x) + \alpha) \\ &\Leftrightarrow 0 = \widehat{J}/_X(x) and \alpha = 0. \end{aligned}$$

But $\widehat{J}$ is the canonical quadratic momentum map of a linear symplectic G_m-action. Hence $(\widehat{J}/_X)^{-1}(0)$ is a cone in X, which is locally arc wise connected. Therefore $J^{-1}(0)$ is locally arc wise connected. •

Remark 1.5.20. A striking aspect of momentum map is the convexity properties of its image. The convexity results for torus actions on compact manifolds can be done using Morse theory. Torus actions are important because an integrable dynamical system is local torus action. The convexity theorem to actions of non-abelian compact groups on compact manifolds also done using Morse theory. We discuss it in Chapter 2. But, in general, Morse theory is not sufficient to study convexity properties of the image of the momentum map. In Chapter 3 we discuss the developments in this area in general cases.

Chapter 2

Torus Actions on Symplectic Manifolds

Among the group actions torus group action is of special interest. The momentum map of an effective Hamiltonian torus action on a symplectic manifold of the dimension double that of the torus is an integrable system. So Hamiltonian actions of tori of maximal dimension are a special case of integrable systems. More than that they are the local form of all integrable systems with compact level sets.

In this chapter we consider the action of a torus T^n on a symplectic manifold (M, ω). In the first section we define Hamiltonian torus action and give examples of it. Then proved that a Hamiltonian circle action on a compact symplectic manifold has fixed point.

One of the most striking aspects of momentum maps is the convexity properties of its image. In 1982 Atiyah and, independently, Guillemin and Sternberg proved that the image of the momentum map J associated to the action of a torus T on a compact symplectic manifold is a compact convex polytope, called the T-momentum polytope and it is equal to the convex hull of the image of the fixed point set of the T-action. Also the fibers of J are connected. In 1984 Guillemin and Sternberg proved that if a non-abelian compact Lie group G acts on a compact symplectic manifold M with associated equivariant momentum map $J : M \to \mathcal{G}^*$ and let T be a maximal torus of G. $J(M) \cap \mathcal{T}^*$ is a union of compact convex polytopes and Kirwan showed that this set is connected thereby concluding that $J(M) \cap \mathcal{T}^*_+$ is a compact convex polytope. We will refer to this as the G-momentum polytope. In section 2 we discuss these convexity properties of the momentum map.

2.1 Hamiltonian Torus Actions

In this section we define Hamiltonian torus action and give examples of it. Then proved that a Hamiltonian circle action on a compact symplectic manifold has fixed point. [31] , [3], [2]

Consider the action of a torus T^n on a symplectic manifold (M, ω). The coadjoint action is trivial on a torus. Hence if $G = T^n$ is an n-dimensional torus with Lie algebra and its dual both identified with Euclidean space, $\mathcal{G} \simeq \Re^n$ and $\mathcal{G}^* \simeq \Re^n$, we can define the momentum map as follows.

Definition 2.1.1. A momentum map for torus action is a map $J : M \longrightarrow \Re^n$ satisfying, for each basis vector ξ of $\Re^n$, the function J^ξ is a Hamiltonian function for ξ_M and is invariant under the action of the torus.

If J is the momentum map for the torus action, then clearly any of its translation $J + c$, $c \in \Re^n$ is also a momentum map for the action.

Example 2.1.2. On $(\mathbf{C}, \omega_o = \frac{i}{2} dz \wedge d\overline{z})$, consider the action of circle $S^1 = \{t \in \mathbf{C} :\mid t \mid= 1\}$ by rotation $\psi_t(z) = t^k z, t \in S^1$, where $k \in \mathbf{Z}$ fixed. The action $\phi : S^1 \longrightarrow Diff(\mathbf{C})$ is Hamiltonian with momentum map $J : \mathbf{C} \longrightarrow \Re$ by $J(z) = \frac{-1}{2} k|z|^2$.

Example 2.1.3. Let $T^n = \{(t_1, t_2, ..., t_n) \in \mathbf{C}^n : |t_i| = 1, \forall i\}$ be a torus acting diagonally on $\mathbf{C}^n$ by

$$(t_1, t_2, ..., t_n).(z_1, z_2, ..., z_n) = (t_1^{k_1} z_1, t_2^{k_2} z_2, ..., t_n^{k_n} z_n),$$

where $k_1, ..., k_n \in \mathbf{Z}$ are fixed. Then this action is Hamiltonian with momentum map $J : \mathbf{C}^n \longrightarrow \Re^n$ by

$$J(z_1, z_2, ..., z_n) = \frac{-1}{2}(k_1|z_1|^2, , ..., k_n|z_n|^2) + constant.$$

Example 2.1.4. Consider the circle S^1-action on any smooth manifold M. Since the Lie algebra of S^1 is one dimensional, one only needs to consider a single associated vector field. That is, if $G = S^1$ is acting smoothly on a closed smooth manifold, M, then the associated vector field, X, is defined by

$$X(x) = \frac{d}{dt}[exp(2\pi it)_x] \mid_{t=0},$$

for all $x \in M$.

If the action is Hamiltonian then $i_X\omega = dh$ for some $h \in C^\infty(M)$; and h is the momentum map.

Note 2.1.5. Let $G = S^1$ and suppose that G is acting smoothly on a closed symplectic manifold (M, ω). Let Ω^*_{inv} be the subcomplex of the de Rham complex of M consisting of invariant forms: that is, a differential form θ is in Ω^*_{inv} if $\mathcal{L}_X\theta = 0$, where X is the associated vector field of the circle action. Let t be an indeterminate of degree 2, and form the polynomial ring $\Omega^*_{inv}[t]$. Define a derivation D, on $\Omega^*_{inv}[t]$ by setting $D(t) = 0$ and $D(\theta) = d\theta + i_X(\theta)t$, for any $\theta \in \Omega^*_{inv}$, where d is the ordinary de Rham differential. Then $D^2 = 0$.

Proposition 2.1.6. Let (M, ω) be a closed symplectic manifold with a symplectic action of the circle group $G = S^1$. Let $\omega = [\omega] \in H^2(M, \Re)$. Then the action is Hamiltonian if and only if

$$\omega \in im[i^* : H^2_G(M, \Re) \longrightarrow H^2(M, \Re)].$$

Proof : If $\omega \in im[i^* : H^2_G(M, \Re) \longrightarrow H^2(M, \Re)]$, then there is an invariant 1-form θ and an invariant function $h \in C^\infty(M)$ such that $D(\omega + d\theta + ht) = 0$. So $i_X\omega = -dh - i_X d\theta = d(-h + i_X\theta$.

Conversely, if $i_X\omega = dh$, then $D(\omega - ht) = 0$; and so ω is in the image of i^*. •

Note 2.1.7. If $G = S^1 \times S^1 .... \times S^1 = T^r$, the r-torus then an action of G on a symplectic manifold is Hamiltonian if and only if the actions of each of the factor circles is Hamiltonian, and hence, if and only if the action of every subcircle is Hamiltonian. Indeed, it is not hard to show that this generalizes to any compact connected Lie group G : that is, the action of G is Hamiltonian if and only if the action of every subcircle is Hamiltonian; and it is sufficient that the action of at least one maximal torus is Hamiltonian.

Theorem 2.1.8. Let (M, ω) be a compact symplectic manifold endowed with an action of S^1. Assume action is Hamiltonian. Then it has fixed point.

Proof : A momentum map for an S^1-action is simply a function $H : M \longrightarrow \Re$. The manifold being compact, any function on it must have critical points. Let x be such a point, namely x is such that $(dH)_x = 0$, and hence such that the Hamiltonian vector field X_H vanishes at x. But the latter is the fundamental vector field associated with the action. Thus, x is a fixed point. •

Remark 2.1.9. Since M is compact, if the action is non-trivial, then H must have a maximum and minimum distinct from one another; and so M^{S^1} must be nonempty with at least two components. There are symplectic S^1-actions without fixed points. For instance, rotations on one factor S^1 on the torus $S^1 \times S^1$, namely the formula $t.(x, y) = (t.x, y)$ defines a symplectic action (which is not Hamiltonian).

2.2 Convexity Property of Torus Actions

In this section we discuss the convexity theorem for compact symplectic manifold M on which a torus acts in a Hamiltonian fashion and also convexity theorem to actions of non-abelian compact groups on compact symplectic manifolds. First we strengthen the Poincare lemma to deal with invariant forms. Then Darboux theorem for momentum maps is given using the G-relative Darboux theorem. Next shown that locally the image of the momentum map of a torus action near a fixed point is a convex set and proved for any point rather than fixed points. Now to improve it again, that is, for M compact $J(M)$ is a compact convex polytope, Morse theory is used. Then proved the convexity theorem to actions of non-abelian compact groups on compact manifolds. [24], [14], [15], [21], [34], [16], [3], [2].

For this we need to assume the group G is compact in addition to being connected, and we make this assumption in this section.

Theorem 2.2.1 (Equivariant Poincare Lemma)**.** If m_o is a fixed point of the action of G on M and β is any invariant closed p-form then there is an invariant neighborhood U of m_o and an invariant $(p-1)$-form α on U with $\beta = d\alpha$ on U.

Proof : Suppose G is compact and connected. The set of Riemannian metrics on a manifold M is a convex set, so the average of a metric is a metric, and in addition is invariant. Thus invariant metrics always exist, and can be used to give invariant geodesic neighborhoods. If m_o is a fixed point of the G-action then a sufficiently small geodesic ball around m_o will be a contractible invariant neighborhood U of m_o. Then by Poincare Lemma there exist an invariant $(p-1)$-form α on U with $\beta = d\alpha$ on U. •

Note 2.2.2. If β is any invariant closed p-form, then its G-average $\int_G \sigma_g^* \beta dg$ is also an invariant p-form. Differentiating the above we have $d\int_G \sigma_g^* \beta dg = \int_G \sigma_g^* d\beta dg$. Thus if an invariant form is exact then it is the differential of an invariant form by averaging the equation.

Theorem 2.2.3 (The G-relative Darboux Theorem)**.** Let M be a manifold and ω_o and ω_1 be two symplectic forms on it. Let G be a Lie group acting properly and symplectically with respect to both ω_o and ω_1. Let $m \in M$ and assume that

$$\omega_o(g.m)(v_{g.m}, w_{g.m}) = \omega_1(g.m)(v_{g.m}, w_{g.m}) \tag{2.1}$$

for all $g \in G$ and $v_{g.m}, w_{g.m} \in T_{g.m}M$. Then there exist two open G-invariant neighborhoods U_o and U_1 of $G.m$ and a G-equivariant diffeomorphism $\Psi : U_o \longrightarrow U_1$ such that $\Psi \mid_{G.m} = Id$ and $\Psi^* \omega_1 = \omega_o$.

Proof : Construct a smooth map $\phi : [0,1] \times U \longrightarrow U$ where U is a G-invariant neighborhood of $G.m$ satisfying the following properties:
(i) $\phi_t : \phi(t,.) : U \longrightarrow U$ is G-equivariant,
(ii) $\phi_t \mid_{G.m}$ is the identity map on $G.m$,
(iii) ϕ_0 is the identity map on U,
(iv) $\phi_1(U) = G.m$,
(v) ϕ_t is a diffeomorphism for $t \neq 1$.

We recall that in the proof of the Theorem 1.1.18, there exist a G_m-invariant Riemannian metric on some G_m-invariant neighborhood of m with associated exponential map Exp_m and such that the mapping

$$\begin{aligned} \tau := G \times_{G_m} V_m \longrightarrow M \text{ given by,} \\ \tau([g, \upsilon]) = g.Exp_m \upsilon, \end{aligned}$$

is a G-equivariant diffeomorphism onto some open G-invariant neighborhood U of the orbit $G.m$. We recall that V_m is some G_m-invariant neighborhood of the origin in the orthogonal complement to $\mathcal{G}.m$. Define for any $u = g.Exp_m \upsilon \in U$ the map $\phi_t(u) := g.Exp_m(1-t)\upsilon$. This map clearly satisfies properties (i) through (v).

We now use the diffeomorphisms ϕ_t, for $t \neq 1$, to construct a one-form α on a G-invariant neighborhood W of the orbit $G.m$, $W \subset U$ satisfying:
(a) $\omega_0 - \omega_1 = d\alpha$.
(b) $\alpha(g.m) = 0$, for any $g \in G$.
(c) α is G-invariant.

Let Y_t be the time-dependent vector field whose flow is ϕ_t for $t \neq 1$. Now, by the property (iv) of ϕ_t and the hypothesis 2.1 we have that

$$\begin{aligned} \omega_0 - \omega_1 &= \phi_1^*(\omega_1 - \omega_0) - (\omega_1 - \omega_0) \\ &= \int_0^1 \frac{d}{dt} \phi_t^*(\omega_1 - \omega_0) dt \\ &= \int_0^1 \phi_t^*(\pounds_{Y_t}(\omega_1 - \omega_0)) dt \\ &= \int_0^1 \phi_t^*(d i_{Y_t}(\omega_1 - \omega_0)) dt \\ &= d \int_0^1 \phi_t^*(i_{Y_t}(\omega_1 - \omega_0)) dt. \end{aligned}$$

Define $\alpha := \int_0^1 \phi_t^*(i_{Y_t}(\omega_1 - \omega_0))dt$. This form satisfies (a) by construction. Hypothesis 2.1 and property (ii) of ϕ_t guarantee that (b) is also satisfied. Property (c) is trivially verified.

We now define $\omega_t := \omega_o + t(\omega_1 - \omega_o)$. By equation 2.1 we have $\omega_t(g.m) = \omega_o(g.m)$ for any $g \in G$ and any $t \in [0,1]$, and consequently $\omega_t(g.m)$ is non-degenerate. The G-invariance of ω_t implies the existence of a G-invariant neighborhood $U_t \subset U$ of $G.m$ and a real number $\epsilon_t > 0$ such that $\omega_s(z)$ is non-degenerate for every $s \in I_t := (t - \epsilon_t, t + \epsilon_t)$ and $z \in U_t$. Cover the interval $[0,1]$ with a finite number of such intervals $\{I_{t_1}, ..., I_{t_n}\}$ and let $\{U_{t_1}, ..., U_{t_n}\}$ be the corresponding G-invariant neighborhoods of $G.m$. Then, the form ω_t is non-degenerate on $W := \cap_{i=1}^{n} U_{t_i}$ for every $t \in [0,1]$, guarantees the existence of a G-equivariant time-dependent vector field X_t on W satisfying

$$i_{X_t}\omega_t = \alpha. \tag{2.2}$$

Let Ψ_t be the flow of X_t. Therefore,

$$\begin{aligned}
\frac{d}{dt}\Psi_t^*\omega_t &= \Psi_t^*(\pounds_{X_t}\omega_t + \frac{d}{dt}\omega_t) \\
&= \Psi_t^*(i_{X_t}d\omega_t + di_{X_t}\omega_t + \omega_1 - \omega_0) \\
&= \Psi_t^*(di_{X_t}\omega_t + \omega_1 - \omega_0) \\
&= \Psi_t^*(d\alpha + \omega_1 - \omega_0) = 0,
\end{aligned}$$

where in the last two equalities we used equation 2.2 and the property (a) of the one-form α. Since $\Psi_0 = Id$ we get $\Psi_1^*\omega_1 = \omega_0$. If we take $U_0 = W, U_1 = \Psi_1(W)$ and $\Psi = \Psi_1 : U_0 \longrightarrow U_1$, the theorem is proved. •

Corollary 2.2.4 (Equivariant Darboux Theorem)**.** Let $(M, \omega_i), i = 0, 1$ be symplectic G-spaces, m a fixed point for the G-action such that ω_o and ω_1 agree at m, then there is an invariant neighborhood U of m and an equivariant diffeomorphism f of U into M with $f^*\omega_1 = \omega_o$ and $f(m) = m$.

Proof : Here $\omega_1 - \omega_o$ is invariant and vanishes at m, it is exact on some invariant neighborhood of m for an invariant form α also vanishing at m. If $\omega^t = \omega_o + t(\omega_1 - \omega_o)$ it is symplectic for $t \in [0,1]$ on some G-invariant neighborhood of m and so α gives an invariant vector field X_t which can be integrated to give a necessarily equivariant map f with $f(m) = m$. •

Theorem 2.2.5. Let (M, ω) be a symplectic G-manifold with m a fixed point for the G-action. Then there is an invariant neighborhood U of m and an equivariant diffeomorphism ϕ of U onto a neighborhood of 0 in T_mM with $\phi(m) = 0$, and such that if $\widehat{\omega_m}$ is the constant extension of ω_m to a symplectic structure on T_mM then $\phi^*\widehat{\omega_m} = \omega$ on U.

Proof : By averaging if necessary we can take an invariant Riemannian metric on M. Its exponential map $\epsilon = Exp_m : T_mM \to M$ is equivariant, sends 0 to m and is a diffeomorphism on some neighborhood of 0 with differential at 0 the identity map on T_mM. Then $\epsilon^*\omega$ is a G-invariant symplectic form on an invariant neighborhood of 0 agreeing with $\widehat{\omega_m}$ at 0. Thus there is an invariant neighborhood of 0 and an equivariant diffeomorphism f with $f(0) = 0$ and $f^*\epsilon^*\omega = \widehat{\omega_m}$. Taking $\phi = (\epsilon \circ f)^{-1}$ gives the required map on the appropriate neighborhood of m. •

Theorem 2.2.6 (Darboux Theorem for Momentum Maps)**.** If J is the momentum map for a Hamiltonian action of G on (M, ω) and m a fixed point then there is a connected invariant neighborhood U of m and an equivariant diffeomorphism ϕ of U onto an open neighborhood of 0 in T_mM with $\phi(m) = 0$, $\phi^*\widehat{\omega_m} = \omega$ and $J = J(m) + \rho^* \circ J_o \circ \phi$ on U where $J_o : T_mM \to \mathcal{SP}\ (T_mM, \omega_m)^*$ is the momentum map for the linear action of symplectic group $SP(T_mM, \omega_m)$ on (T_mM, ω_m) and ρ is the Lie group homomorphism between G and $SP(T_mM)$.

Proof : Here $J_o : T_mM \to \mathcal{SP}\ (T_mM, \omega_m)^*$ be the momentum map for the linear action of $SP(T_mM, \omega_m)$ on (T_mM, ω_m). Then $\rho^* \circ J_o$ is a momentum map for the linear action of G on $(T_mM, \widehat{\omega}_m)$ when m is a fixed point for G. Thus J and $\rho^* \circ J_o \circ \phi$ are both momentum maps for G on (U, ω) where U and ϕ are as in the above theorem. Taking U to be connected, these momentum maps must differ by a constant in $\mathcal{G}^*$. Therefore $J(y) - \rho^* \circ J_o \circ \phi(y) = J(m) - \rho^* \circ J_o \circ \phi(m) = J(m)$, since $\phi(m) = 0$ and J_o vanishes at the origin. Hence for every $y \in U$, $J(y) = J(m) + \rho^* \circ J_o \circ \phi(y)$. •

Definition 2.2.7. If J is the momentum map for a Hamiltonian action of G on (M, ω) and m a fixed point then the form $J = J(m) + \rho^* \circ J_o \circ \phi$ is called the *Normal form* for the momentum map J at a fixed point.

Note 2.2.8. Any action near a fixed-point is equivalent to a linear action, we examine such linear actions when $G = T$ is abelian and hence a torus. Thus let (V, Ω) be a symplectic vector space with a linear symplectic action of a torus group T. We can find an endomorphism $\mathcal{J}$ of V with $\mathcal{J}^2 = -1$, defines a complex inner product $(.,.)$ on V with Ω as its imaginary part. We say $\mathcal{J}$ is a positive complex structure compatible with Ω. Then there exist an orthonormal basis $f_1, f_2, ...f_n$ of V with respect to $(.,.)$ and elements $\alpha_1, \alpha_2, ..., \alpha_n$ of $\mathcal{T}^*$, where $\mathcal{T}^*$ is the dual of the Lie algebra $\mathcal{T}$ of T such that $\xi f_i = -\sqrt{-1}\alpha_i(\xi) f_i$, $\xi \in \mathcal{T}$. We call $\alpha_1, \alpha_2, ...\alpha_n$ the weights of the action of T on V. Since they are eigenvalues of endomorphisms, they are independent of the choice of the basis, but may depend on the choice of the complex structure $\mathcal{J}$.

It is possible for some of the α_i to be zero, there can be repetitions and the non-zero, distinct elements need not be linearly independent or spanning. Obviously, we can always choose a subset of them which is a basis for the subspace of $\mathcal{T}^*$ which they all span.

Consider these weights from a real point of view. If $f_1, f_2, ...f_n$ is a complex orthonormal basis as above then setting $e_i = \mathcal{J} f_i$, we get a real basis $(e_1, e_2, ...e_n, f_1, f_2, ...f_n)$ for V and

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \qquad \Omega(e_i, f_j) = \delta_{ij},$$

so we have a symplectic basis for V.

Theorem 2.2.9. If J is the momentum map for a Hamiltonian action of a torus T on the symplectic manifold (M, ω) and m a T-fixed point then there is an invariant neighborhood U of m in M and a neighborhood U' of $J(m)$ in $\mathcal{T}^*$ such that $J(U)$ is $U' \cap (J(m) + C(\alpha_1, \alpha_2, ..., \alpha_n))$ where $C(\alpha_1, \alpha_2, ..., \alpha_n)$ is the positive cone spanned by the weights $\alpha_1, \alpha_2, ..., \alpha_n$ of the action of T on M.

Proof : If $\rho : T \to SP(T_m M, \omega_m)$ is the map giving the action of T, then the momentum map

$$\rho^* \circ J_o : T_m M \to \mathcal{T}^* \text{given by}$$
$$< \rho^* \circ J_o(v), \xi >= \frac{1}{2}\omega_m(v, \rho_* \xi(v)).$$

If $v = \sum_i q_i e_i + p_i f_i$, where $f_1, f_2, ...f_n$ is a complex orthonormal basis and $e_i = \sqrt{-1} f_i$, gives symplectic co-ordinates $q_1, q_2, ...q_n, p_1, p_2, ...p_n$ then

$$\rho_* \xi f_i = -\sqrt{-1}\alpha_i(\xi) f_i = -\alpha_i(\xi) e_i.$$
$$\text{So,} < \rho^* \circ J_o(v), \xi >= \frac{1}{2}\sum_i (p_i^2 + q_i^2)\alpha_i(\xi).$$

Thus the momentum map is

$$\rho^* \circ J_o(v) = \frac{1}{2}\sum_i (p_i^2 + q_i^2)\alpha_i.$$

That is, the image of the momentum map of a linear symplectic torus action is the positive cone spanned by the weights denoted by $C(\alpha_1, \alpha_2, ..., \alpha_n)$.

From the quadratic nature of the momentum map $\rho^* \circ J_o$ we have the image of an open neighborhood U of the origin in $T_m M$ is the intersection of an open set $U' \subset \mathcal{T}^*$ with $C(\alpha_1, \alpha_2, ..., \alpha_n)$. Then using the Darboux Theorem for Momentum Maps we have the result. •

Theorem 2.2.10. Let T be a torus with Hamiltonian action on (M, ω) with momentum map J. Then each point $m \in M$ has a neighborhood U for which there is a neighborhood U' of $J(m)$ in $\mathcal{T}^*$ that $J(U) = U' \cap C(m)$ where

$$C(m) = J(m) + (i_m^*)^{-1} C(\alpha_1, \alpha_2, ..., \alpha_n),$$

and $\alpha_1, \alpha_2, ..., \alpha_n$ are the weights of the linear action of the stabilizer T_m on $T_m M$, and i_m be the inclusion map of T_m in T.

Proof : Since i_m be the inclusion map of T_m in T, then $i_m^* \circ J$ is the momentum map for T_m. Then from the above theorem we have a neighborhood U_1 of $m \in M$ and a neighborhood U'' of $i_m^*(J(m))$ in $\mathcal{T}_m^*$ such that $i_m^*(J(U))$ is $U'' \cap (i_m^*(J(m)) + C(\alpha_1, \alpha_2, ..., \alpha_n))$. If we let $U' = (i_m^*)^{-1}(U'')$ and $U = J^{-1}(U')$ then obviously

$$J(U) = U' \cap J(m) + (i_m^*)^{-1} C(\alpha_1, \alpha_2, ..., \alpha_n).$$

Since i_m^* is a linear map, $(i_m^*)^{-1} C(\alpha_1, \alpha_2, ..., \alpha_n)$ is convex, and is a subset of the fixed space $\mathcal{T}^*$ as m varies. Set

$$C(m) = J(m) + (i_m^*)^{-1} C(\alpha_1, \alpha_2, ..., \alpha_n),$$

a convex cone in $\mathcal{T}^*$ with vertex $J(m)$. Hence the theorem. •

Note 2.2.11. In Theorem 2.2.9 we have seen that locally the image of the momentum map of a torus action near a fixed point is a convex set and in Theorem 2.2.10 it is proved for any point rather than fixed points. Now to improve it again, that is, for M compact $J(M)$ is a compact convex polytope, Morse theory is used.

Definition 2.2.12. Let M be a manifold and $f \in C^\infty(M)$ then $c \in \Re$ is a *local maximum* of f if there is $m \in M$ with $c = f(m)$, and a neighborhood U of m such that $f(y) \leq c$ for all $y \in U$.

Definition 2.2.13. Let M be a manifold and $f \in C^{\infty}(M)$ then $c \in \Re$ is a *critical value* of m if there is $m \in M$ with $c = f(m)$, and $(df)_m = 0$. Such a point m is called a *critical point*. The set of all critical points is called the *critical set* of f and denoted by C_f.

Obviously, local maxima are critical values.

Definition 2.2.14. For $m \in C_f$, define *Hessian* of f at m to be the quadratic form $H_f\mid_m : T_mM \longrightarrow \Re$ that satisfies $H_f\mid_m(\upsilon) = \frac{1}{2}(f \circ \gamma)''(0)$ when $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ is a smooth curve satisfying $\gamma(0) = m$ and $\gamma'(0) = \upsilon$.

Definition 2.2.15. A critical point $m \in C_f$ is said to be nondegenerate if the quadratic form $H_f\mid_m : T_mM \longrightarrow \Re$ is nondegenerate.

Definition 2.2.16. Let M be a manifold and $f \in C^{\infty}(M)$. A submanifold N of M is called a *critical submanifold* of f if each component of N is a connected component of C_f. A *nondegenerate critical submanifold* of f is a critical submanifold of N such that the Hessian $H_f\mid_m$ is nondegenerate in normal directions at each point of N.

Definition 2.2.17. A function is said to be a *Morse function* if all of its critical points are nondegenerate.

Definition 2.2.18. A smooth function $f : M \longrightarrow \Re$ is said to be a *Morse-Bott function* on M if

(i) C_f is a smooth submanifold of M.

(ii) For each $m \in C_f$, the null space of the quadratic form $H_f\mid_m : T_mM \longrightarrow \Re$ is T_mC_f.

Definition 2.2.19. Let M be a manifold and $f \in C^{\infty}(M)$ then f is said to be *clean* if each connected component of C_f is a nondegenerate critical submanifold.

If f is clean and N is a connected component of C_f then the rank of the Hessian $H_f\mid_m$ is constant (equal to the co-dimension of N) and so the index is also constant. It is called the *index* of the critical submanifold N.

Note 2.2.20. Suppose M is equipped with a Riemannian metric. Then f defines a gradient vector field on M which is complete if M is compact. Let $\phi_t : M \longrightarrow M, -\infty < t < \infty$ be the flow generated by this gradient vector field. If we let C_i be the connected components of C_f then we let W_i be all points m such that $\phi_t(m) \to C_i$ as $t \to \infty$. That is,

$$W_i = \{m \in M \mid \phi_t(m) \to C_i \quad as \quad t \to \infty\}.$$

A basic result of Morse theory is that, if f is clean then each W_i is a cell bundle over C_i with fiber dimension equal to the index C_i and M is the disjoint union of the W_i. It follows that when C_i has index 0 then W_i is an open subset of M and in general W_i has codimension index C_i.

Proposition 2.2.21. Let $f \in C^\infty(M)$ be clean where M is compact and connected. If each component C_i of C_f has even dimension and index then f has a unique local maximum.

Proof : Let $C_1, C_2, ...C_k$ be the connected components of C_f of index 0 and $C_{k+1}, C_{k+2}....., C_N$ the remaining components. Then f is a constant on each C_i, say c_i. These $c_1, ...c_k$ are the local maxima of f and

$$M = W_1 \cup ..... \cup W_k \cup W_{k+1} ...... \cup W_N$$

with W_i open for $i \leq k$ and for $i > k$, each W_i has codimension at least 2. But if W_i is of codimension ≥ 2 it cannot disconnect M. It follows that

$$M - \cup_{i>k} W_i = \cup_{i=k} W_i$$

is connected and hence that $k = 1$ so that f has only one local maximum. •

Theorem 2.2.22. If M is compact and the interval $[a, b] \in \Re$ contains no critical value of f, then $U_b := \{m \in M \mid f(m) \leq b\}$ is diffeomorphic to U_a.

Proof : Choose a Riemannian metric on M, thus getting a gradient vector field grad f, which does not vanish on $f^{-1}[a, b]$. Modify it with the help of a differentiable function g on M which takes the value $1/\|gradf\|^2$ on $f^{-1}[a, b]$ and which vanishes outside a neighborhood as $X = g.gradf$. Let φ_t be the flow of X, then φ_{b-a} sends U_b to U_a. •

Note 2.2.23. If $c \in [a, b]$ is the unique critical value of f in this interval, the homotopy type of U_b is described by the addition to U_a of the negative normal bundle of the critical submanifold at level c.

Theorem 2.2.24. Let f be a nondegenerate function which has no critical submanifold of index 1 or $n-1$. Then it has a unique local minimum and a unique local maximum. Moreover, all of its nonempty levels are connected.

Proof : From the above note the homotopy type of U_a can change only by crossing a critical level, in which case it changes by adding to U_a the negative normal bundle of the critical submanifold. If the critical submanifold has index 0, that is if this is a local minimum, we add a connected component. To connect all pieces later on, as M connected, we must go through a new critical level, for which the sphere bundle of the negative bundle must be connected. But this is impossible except if the submanifold is 1. Thus it is seen that there can be only one local minimum, and applying the result to $-f$, only one local maximum.

Moreover, U_a is connected, and, for the same reasons,

$$\overline{M - U_a} = \{x \mid f(x) \geq a\}$$

is connected as well. Assume $V_c = f^{-1}(c)$ is nonconnected level. Any component of V_c thus defines a nontrivial element in $H_{n-1}(U_c)$. But this group is zero: Indeed, if c is strictly contained between the minimum and the maximum of f, the critical submanifolds of critical levels lower than c all have negative normal bundles of dimension $\leq n-2$ which cannot create nonzero elements of H_{n-1}. •

Theorem 2.2.25. Let J be the momentum map of a Hamiltonian action of a compact Lie group G on the compact symplectic manifold (M, ω). For each $\xi \in \mathcal{G}$ the function J^ξ has a unique local maximum on M.

Proof : We shall show that J^ξ is clean and has only critical manifolds which are even dimensional with even indices.

Let $f = J^\xi$, then f is the momentum map for the action $t.m = expt\xi.m$ of $\Re$ generated by ξ on M. In particular, any critical point m of f is a fixed point of this action. We can replace $\{expt\xi | t \in \Re\}$ by its closure in G (which will be a torus). Any critical point of f will be a fixed point for this torus and vice-versa. Then we apply Darboux Theorem for Momentum Maps to this torus and conclude that the action near a critical point m is symplectically equivalent to the linearized action near 0 in the tangent space at m by a diffeomorphism which takes the momentum map into a quadratic function. The fixed points near m get mapped to the critical points of this quadratic function, and these form a linear subspace, and hence a submanifold. This shows that f is clean.

We have $< \rho^* \circ J_o(v), \xi >= \frac{1}{2}\sum_i (p_i^2 + q_i^2)\alpha_i(\xi)$. Therefore

$$Rankd^2f = 2 \times \#\{\alpha_i | \alpha_i(\xi) \neq 0\} \quad \text{and} \quad Indexd^2f = 2 \times \#\{\alpha_i | \alpha_i(\xi) < 0.\}$$

Hence the dimension and index of C_i are even. Then using Proposition 2.2.21 J^ξ has a unique local maximum on M. •

Corollary 2.2.26. For any $m \in M$ we have $J(M) \subset C(m)$.

Proof : The cone $C(\alpha_1, \alpha_2, ..., \alpha_n) \subset \mathcal{T}_m^*$ is a closed convex set and as such it is an intersection of half-spaces. In other words, there is a set $\triangle$ of elements ξ of

$\mathcal{T}_m$ such that

$$\alpha \in C(\alpha_1, \alpha_2, ..., \alpha_n) \leftrightarrow \alpha(\xi) \geq 0, \forall \xi \in \triangle .$$

It follows that

$$C(x) = J(x) + \{\alpha \in \mathcal{T}^* | \alpha(\xi) \geq 0, \forall \xi \in \triangle\}.$$

For $\xi \in \triangle$ consider J^ξ. By Theorem 2.2.10, $J^\xi(m)$ is a local maximum and by Theorem 2.2.25 this must be a global maximum. Thus

$$< J(m) - J(y), \xi > \;\; \geq \;\; 0, \forall \xi \in \triangle$$

and hence $J(y) \in C(m)$. •

Theorem 2.2.27 (Atiyah - Guillemin -Sternberg)**.** Let (M, ω) be a compact connected symplectic manifold, and let T be a torus acts in a Hamiltonian fashion with associated invariant momentum map $J : M \to \mathcal{T}^*$. Then the image $J(M)$ of J is a compact convex polytope, called the T-momentum polytope. Moreover, it is equal to the convex hull of the image of the fixed point set of the T-action. The fibers of J are connected.

Proof: Each $C(m)$ is a convex polyhedron, so to prove the first part of the theorem it is enough to prove $J(M)$ is an intersection of finite number of these.

For each $m \in M$ we know that there are open sets $U_m \subset M$ and $U'_m \subset \mathcal{T}^*$ with $m \in U_m$ and $J(U_m) = U'_m \cap C(m)$. Clearly, $\{U_m\}_{m \in M}$ forms an open covering of M, so by compactness there are points $m_1, m_2, ...m_N$ in M such that $M = \cup_i U_{m_i}$.

From the above corollary we have $J(M) \subset \cap_i C(m_i)$. We claim that in fact $J(M) = \cap_i C(m_i)$. If not there is some $f \in \cap_i C(m_i)$ $J(M)$, and a nearest point $f_o \in J(M)$ to f in some Euclidean structure. Let $f_o = J(m)$ then $m \in U_{m_j}$ for some j. Then f_o and f are both in $C(m_j)$ along with the segment joining them. Since $m \in U_{m_j}$, $f_o \in U'_{m_j}$, so part of the segment from f_o to f is in U'_{m_j} and hence is in $U'_{m_j} \cap C(m_j) = J(U_{m_j})$, which contradicts the assumption that f_o is nearest to f. Thus f cannot exist and we have the desired equality. Thus $J(M)$ is a convex polyhedron.

In the proof of Theorem 2.2.25 we have seen that J^ξ has only critical manifolds which are even dimensional with even indices. Then using Theorem 2.2.24, the fibers of J are connected. •

Corollary 2.2.28. Let T^r be the torus of dimension r. Under the conditions of the Theorem 2.2.27, if the T^r -action is effective, then there must be at least $r+1$ fixed points.

Proof : At a point p of an r-dimensional orbit the momentum map is a submersion, that is, $(dJ_1)_p, ..., (dJ_r)_p$ are linearly independent. Hence $J(p)$ is an interior point of $J(M)$ and $J(M)$ is a non-degenerate convex polytope. Any nondegenerate convex polytope in $\Re^r$ must have at least $r+1$ vertices. The vertices of $J(M)$ are images of fixed points. •

Theorem 2.2.29. Let (M, ω, T^r, J) be a Hamiltonian T^r-space. If T^r- action is effective, then $dimM \geq 2r$.

Proof : Since the momentum map is constant on an orbit $\mathcal{O}$, for $p \in \mathcal{O}$ the exterior derivative $dJ_p : T_pM \longrightarrow \mathcal{G}^*$ maps $T_p\mathcal{O}$ to zero.

$$\text{Then, } T_p\mathcal{O} \subseteq KerdJ_p = (T_p\mathcal{O})^\omega,$$

where $(T_p\mathcal{O})^\omega$ is the symplectic orthogonal to $T_p\mathcal{O}$. This shows that orbits $\mathcal{O}$ of a Hamiltonian torus action are always isotropic submanifolds of M. In particular by symplectic linear algebra we have that $dim\mathcal{O} \leq 1/2dimM$. If we consider an r-dimensional orbit we get the theorem. •

Suppose a compact Lie group G acts on a compact symplectic manifold M with associated equivariant momentum map $J : M \to \mathcal{G}^*$ and let T be a maximal torus of G. In this case Guillemin and Sternberg proved that $J(M) \cap \mathcal{T}^*$ is a union of compact convex polytopes and Kirwan showed that this set is connected thereby concluding that $J(M) \cap \mathcal{T}^*_+$ is a compact convex polytope which will be called the G-momentum polytope.

Theorem 2.2.30 (Guillemin-Sternberg-Kirwan)**.** Let M be a compact connected symplectic manifold on which the compact connected Lie group G acts in a Hamiltonian fashion with associated equivariant momentum map $J : M \to \mathcal{G}^*$. Let T be a maximal torus of G, $\mathcal{T}$ its Lie algebra, $\mathcal{T}^*$ its dual, and $\mathcal{T}^*_+$ the positive Weyl chamber relative to a fixed ordering of the roots. Then $J(M) \cap \mathcal{T}^*_+$ is a compact convex polytope, called the G-momentum polytope. The fibers of J are connected.

Proof : First fix a G-invariant inner product on $\mathcal{G}$ and use it to identify $\mathcal{G}^*$ with $\mathcal{G}$. Let $\|\|$ be the associated norm on $\mathcal{G}$.

If M is given a Riemannian metric we can consider the function $\|J\|^2$ as a Morse function on M. Although $\|J\|^2$ is not nondegenerate, Kirwan showed that nevertheless $\|J\|^2$ induces a smooth stratification $\{S_\beta \mid \beta \in \mathcal{B}\}$ of M for some appropriate choice of G-invariant metric on M. The stratum to which a point of M belongs is determined by the limit set of its positive trajectory under the flow $-grad\|J\|^2$. The indexing set $\mathcal{B}$ is a finite subset of the positive Weyl chamber $\mathcal{T}_+$.

Also the strata S_β are all locally closed submanifolds of M of even dimension. Since it is impossible to disconnect a manifold by removing submanifolds of codimension at least two, there must be a unique open stratum. Hence the subset of points of M where $\|J\|^2$ takes its minimum value is connected.

Consider the product $M \times G/T$, or more generally at $M \times G/H$ where H is a centralizer of the torus in G. If α is any point in $\mathcal{G}$ there is a natural G-invariant symplectic structure on the coadjoint orbit of α in $\mathcal{G}$. This orbit has the form G/H where H is the centralizer in G of α or the torus generated by α. The inclusion of the orbit in $\mathcal{G}$ is a momentum map for the action of G.

Let ω_α denote the negative of the kähler form on G/H associated to α. This also gives G/H a symplectic structure, and thus the product $M \times G/H$ becomes a symplectic manifold. The map

$$J^\alpha : M \times G/H \longmapsto \mathcal{G}^* \text{ by}$$
$$J^\alpha(m, gH) = J(m) - Ad(g)\alpha$$

is a momentum map for the action of G on this manifold.

From now on identify $\mathcal{G}^*$ with $\mathcal{G}$ and $\mathcal{T}^*$ with $\mathcal{T}$ using the fixed inner product. Suppose that $J(M) \cap \mathcal{T}_+$ is not convex. Our aim is to obtain a contradiction. First we prove a Lemma

Lemma 2.2.31. For any sufficiently small $\varepsilon > 0$ there exist $\alpha \in \mathcal{T}_+$ such that the ball of radius ε and centre α meets $J(M) \cap \mathcal{T}_+$ in precisely two points α_1 and α_2 neither of which lies in the interior of the ball. We may assume that the centralizer of α in G fixes both α_1 and α_2.

Proof: Guillemin and Sternberg proved that $J(M) \cap \mathcal{T}_+$ is a finite union of compact convex polytopes, $P_1, P_2, \ldots\ldots, P_m$ say. Therefore the intersection of $J(M) \cap \mathcal{T}_+$ with any ball of small radius centered at a point $\xi \in J(M) \cap \mathcal{T}_+$ is a cone with vertex ξ. By assumption $J(M) \cap \mathcal{T}_+$ is not convex. It is compact and also connected since if it were the disjoint union of closed sets A and B then $J(M)$ would be the disjoint union of the closed sets $Ad(G)A$ and $Ad(G)B$. Therefore there must exist some ξ such that $J(M) \cap \mathcal{T}_+$ is not convex in any neighborhood of ξ. (Otherwise the shortest path in $J(M)\cap\mathcal{T}_+$ between any two points would be a straight line.) The set S of all such $\xi \in J(M)\cap\mathcal{T}_+$ is a union of convex compact polytopes.

Choose $\xi \in S$ such that the number of the polytopes $P_1, P_2, \ldots\ldots, P_m$ containing ξ is minimal. Then every point of S sufficiently close to ξ is contained in precisely the same polytopes P_j as ξ. We may assume that ξ belongs to the interior of S, so that there is a linear subspace V of $\mathcal{T}$ whose translation $V+\xi$ coincides with S in a neighborhood of ξ.

If P is any polytope containing $V+\xi$ near ξ then P decomposes as

$$P = ((V^{\perp}+\xi)\cap P)\oplus V$$

in a sufficiently small neighborhood of ξ. Hence the same is true of $J(M)\cap\mathcal{T}_+$. The choice of V implies that $(V^{\perp}+\xi)\cap J(M)\cap\mathcal{T}_+$ is locally a cone C with vertex ξ which is not convex but is such that every point other than ξ has a convex neighborhood in C. It follows from this local convexity property that for small $\delta > 0$ there exist distinct points $\eta, \zeta \in C-\{\xi\}$ with

$$\|\zeta-\xi\| = \|\eta-\xi\| = \delta$$

such that the line segment $L(\eta,\zeta)$ joining η to ζ meets C only at its endpoints and such that $\|\eta-\zeta\|$ is minimal among all η, ζ satisfying these conditions.

To see this let K be the infimum of all values of $\|\eta-\zeta\|$ for η and ζ satisfying these conditions. Then $K < 2\delta$. Choose η_n and ζ_n satisfying the conditions for $n \geq 1$ such that $\|\eta_n-\zeta_n\|$ converges to K as $n \longrightarrow \infty$. Without loss of generality there exist $\eta, \zeta \in C$ with $\|\zeta-\xi\| = \|\eta-\xi\| = \delta$ such that $\eta_n \longrightarrow \eta$ and $\zeta_n \longrightarrow \zeta$. It is enough to show that $L(\eta,\zeta)$ meets C only at its endpoints. Suppose not.

Then since $\|\eta - \zeta\| = K$ the whole line segment $L(\eta, \zeta)$ must be contained in C by the definition of K. Since η and ζ have convex neighborhoods in C, if n is large enough the line segments $L(\eta, \eta_n)$ and $L(\zeta, \zeta_n)$ are contained in C. Therefore if n is large there is a path from η_n to ζ_n in C of length less than 2δ, and hence the shortest path in C from η_n to ζ_n does not pass through ξ. Since every point of $C - \{\xi\}$ has a convex neighborhood in C this path must be a straight line, that is, $L(\eta_n, \zeta_n) \subseteq C$ which is a contradiction.

Let $\alpha = \frac{1}{2}(\eta + \zeta)$ and let ε be the distance from α to $L(\eta, \xi)$ or equivalently from $L(\zeta, \xi)$. Then the ball $B_\varepsilon(\alpha)$ of radius ε and centre α meets $J(M) \cap \mathcal{T}_+$ at a point α_1 on the line $L(\eta, \xi)$ and a point α_2 on the line $L(\zeta, \xi)$.

To prove the first statement it is enough to prove that no points of $B_\varepsilon(\alpha)$ other than α_1 and α_2 lie in the cone C.

Now, we know that if a Line L through ξ meets $B_\varepsilon(\alpha)$ at a point distinct from α_1 and α_2 then the angle which it makes with each of $L(\eta, \xi)$ and $L(\zeta, \xi)$ is strictly smaller than the angle between $L(\eta, \xi)$ and $L(\zeta, \xi)$. So if L lies in C by the minimality assumption every line segment joining a point of L to a point of either $L(\eta, \xi)$ or $L(\zeta, \xi)$ is contained in C. But this implies that $L(\eta, \zeta)$ also contained in C by the local convexity property. This is a contradiction.

It remains to check that the centralizer H of α in G fixes both α_1 and α_2. This is equivalent to requiring that if τ is the unique face of $\mathcal{T}_+$ containing α in its interior then the closure of τ contains α_1 and α_2. By restricting our attention to a sufficiently small neighborhood of ξ we may assume that if any face of $\mathcal{T}_+$ contains a point of $C - \{\xi\}$ in its interior then it contains the entire open line segment joining the point to ξ in its interior. In particular if the interior of a face contains α then it also contains $\frac{1}{2}(\alpha_1 + \alpha_2)$. Since α_1 and α_2 both lie in $\mathcal{T}_+$ and since $\mathcal{T}_+$ is convex, this can only happen if the closure of the face contains both

α_1 and α_2.

This completes the proof of the Lemma.

Now consider the function $\|J^\alpha\|^2$ on $M \times G/H$ where α satisfies the conditions of the above Lemma for some small $\varepsilon > 0$. To obtain a contradiction and thus to prove the theorem it is enough to prove that this function takes its minimum value precisely at those points (m, gH) such that $J(g^{-1}m) = \alpha_j$ for $j = 1, 2$. For the set of all such points is the disjoint union of the nonempty closed subsets

$$G(J^{-1}(\alpha_j) \times \{H\}), \qquad j = 1, 2$$

of $M \times G/H$ which is a contradiction the fact that subset of points of $M \times G/H$, where $\|J^\alpha\|^2$ takes its minimum value, is connected.

So it remains to prove

Lemma 2.2.32. The function $\|J^\alpha\|^2$ on $M \times G/H$ takes its minimum value precisely at those points (m, gH) such that $J(g^{-1}m) = \alpha_j$ for $j = 1, 2$.

Proof : Since $\|J\|^2$ is a G-invariant function we need consider points of the form (m, H). Then the point (m, H) is critical for $\|J\|^2$ if and only if the vector field on $M \times G/H$ induced by $J^\alpha(m, H) \in \mathcal{G}$ has a fixed point at (m, H). In particular

$$J^\alpha(m, H) = J(m) - \alpha$$

lies in the Lie algebra $\mathcal{H}$ of H. Since $\alpha \in \mathcal{T} \subseteq \mathcal{H}$ this happens if and only if $J(m) \in \mathcal{H}$. Then as T is a maximal torus of H there exists $h \in H$ such that $J(hm) \in \mathcal{T}$. Thus a point of $M \times G/H$ is critical if and only if it lies in the G-orbit of a critical point of the form (m, H) with $J(m) \in \mathcal{T}$. Therefore it is enough to show that all points $m \in M$ with $J(m) \in \mathcal{T}$ satisfy

$$\|J(m) - \alpha\| \geq \varepsilon$$

with equality if and only if $J(m) = \alpha_1$ or α_2.

Since $\alpha \in \mathcal{T}_+$ we have

$$\|w\xi - \alpha\| > \|\xi - \alpha\|$$

for any $\xi \in \mathcal{T}_+$ and w in the Weyl group such that $w\xi \neq \xi$. Therefore it is enough to consider the case when $J(m) \in \mathcal{T}_+$. But from the above lemma we have $\|J(m) - \alpha\| \geq \varepsilon$. This completes the proof of the Lemma, and hence also the proof of the Theorem. •

Remark 2.2.33. The case of compact symplectic manifolds is rich but quite particular. For noncompact manifolds the results in this chapter no longer hold. In general, Morse theory is not sufficient to study convexity properties of the image of the momentum map. The convexity results on compact group actions on non compact manifolds with proper momentum maps were proved by Condevaux, Dazord, and Molino [30]. Hilgert, Neeb and Plank [18] also proved it in many interesting situations. The underlying ideas in their proof is of topological in nature. Karshon and Marshall [20] generalized this technique (local to global convexity) and recovered the convexity theorems of Atiyah [24], Guillemin and Sternberg [14]. Petre Birtea, Juan-Pablo Ortega and Tudor S.Ratiu [4] also generalized the lokal-global prinzip and worked out numerous application in symplectic geometry. In fact in [4] they give generalization of several convexity results that are available in the literature. We discuss these things in the next chapter.

Chapter 3

Convexity - Topological Approach

In chapter 2 we have discussed the convexity property of torus actions using Morse theory. In general Morse theory is not sufficient to study convexity properties of the image of the momentum map. The case of compact symplectic manifolds is rich but quite particular. For noncompact manifolds the results in the previous chapter no longer hold. Convexity results to compact group actions on noncompact manifolds with proper momentum maps were given by Condevaux, Dazord, and Molino [30] and later by Hilgert, Neeb and Plank [18]. The Lokal-global-prinzip is the main tool in these works. Yael Karshon And Christina Marshall [20] gave a generalization of Lokal-global-prinzip for a proper map. But Petre Birtea, Juan-Pablo Ortega and Tudor S.Ratiu [4] gave a generalization of Lokal-global-prinzip for a closed map. Using this, many stronger results in convexity are obtained.

The essential attributes underlying the convexity theorems for momentum maps are the openness of the map onto its image and the local convexity data. The classical convexity theorems we have given in chapter 2 are also satisfy these conditions. In this chapter more general theorems on convexity are given using the topological ingredients.

To do convexity results using topological properties we need normal form for the momentum map which we have discussed in section 1. Most of the technical behavior of proper Lie group action is a direct consequence of the existence of slices and tubes; they provide a privileged system of semiglobal coordinates in which the group action takes on a particularly simple form. Proper symplectic Lie group actions turnout to behave similarly: the tubular chart can be constructed in such a way that the expression of the symplectic form is very natural and, moreover, if there is a momentum map associated to this canonical action, this construction provides a normal form for it. We start with the Witt-Artin decomposition of the tangent space. Then we construct a symplectic tube at a point of a symplectic manifold. The statement and proof of the symplectic slice theorem is given. Define tubewise Hamiltonian action and gives sufficient conditions for the action to be tubewise Hamiltonian. Then the expression of the momentum map in the slice coordinate, which is usually referred to as the Marle-Guillemin- Sternberg normal form is given.

In section 2 we discuss the convexity properties of the image of the momentum map using some topological vector space results. We give the statement of Lokal-global-prinzip and a generalization of it for a closed map using some topological vector space results. Using this we obtained that the convexity is rooted on the map being open onto its image and having local convexity data. Next we look at the convexity for momentum maps. Then a generalization of Atiyah-Guillemin-Sternberg Convexity theorem for non compact manifolds is given. Much more generalization of convexity results are obtained in two cases: when the momentum map has connected fibers and the case when the momentum map has only the locally fiber connectedness property. Then we give a generalization of Kirwan's convexity result.

Pull backs by J of smooth functions on $\mathcal{G}^*$ are called collective functions. A collective function is clearly constant on the level sets of the momentum map. The converse need not be true. A momentum map has the division property if any smooth function on M that is locally constant on the level set of J is a collective function. In section 3 we generalize a result on division property of momentum map by replacing the compactness of the Lie group with proper and effective action. Then proved that torus action has division property if J_T is closed and semi-proper. Also we prove that for a paracompact connected symplectic manifold with G a compact connected Lie group. If the associated momentum map J is closed and semi proper as a map into some open subset of $\mathcal{G}^*$, then J has the division property if the image $J(M)$ is contained the $\mathcal{G}^*_{reg}$, where denote $\mathcal{G}^*_{reg}$ the elements of $\mathcal{G}^*$ whose stabilizers under the coadjoint action of G are tori.

3.1 Normal Form

Most of the technical behavior of proper Lie group action is a direct consequence of the existence of slices and tubes; they provide a privileged system of semiglobal coordinates in which the group action takes on a particularly simple form. Proper symplectic Lie group actions turnout to behave similarly: the tubular chart can be constructed in such a way that the expression of the symplectic form is very natural and, moreover, if there is a momentum map associated to this canonical action, this construction provides a normal form for it.

To construct a normal form we use some reduction theory namely, the Marsden-Weinstein-Meyer theorem for symplectic reduction. We start with the Witt-Artin decomposition of the tangent space. Then we construct a symplectic tube at a point of a symplectic manifold. The statement and proof of the symplectic slice theorem is given. Define tubewise Hamiltonian action and gives sufficient conditions for the action to be tubewise Hamiltonian. Then the expression of the momentum map in the slice coordinate, which is usually referred to as the Marle-Guillemin- Sternberg normal form is given. [32], [13].

Theorem 3.1.1 (Symplectic point reduction). Let $\Phi : G \times M \longrightarrow M$ be a free proper canonical action of the Lie group G on the connected symplectic manifold (M, ω). Suppose that this action has an associated momentum map $J : M \longrightarrow \mathcal{G}^*$ with non-equivariance one-cocycle $\sigma : G \longrightarrow \mathcal{G}^*$. Let $\mu \in \mathcal{G}^*$ be a value of J and denote by G_μ the isotropy group of μ under the affine action of G on $\mathcal{G}^*$. Then:

(i) The space $M_\mu := J^{-1}(\mu)/G_\mu$ is regular quotient manifold and, moreover, it is a symplectic manifold with symplectic form ω_μ uniquely characterized by the relation

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega. \tag{3.1}$$

The maps $i_\mu : J^{-1}(\mu) \hookrightarrow M$ and $\pi_\mu : J^{-1}(\mu) \longrightarrow J^{-1}(\mu)/G_\mu$ denote the inclusion and the projection, respectively. The pair (M_μ, ω_μ) is called the *symplectic point reduced space.*

(ii) Let $h \in C^\infty(M)^G$ be a G-invariant Hamiltonian. The flow F_t of the Hamiltonian vector field X_h leaves the connected components of $J^{-1}(\mu)$ invariant and commutes with the G-action, so it induces a flow F_t^μ on M_μ defined by

$$\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu.$$

(iii) The vector field generated by the flow F_t^μ on (M_μ, ω_μ) is Hamiltonian with associated *reduced Hamiltonian function* $h_\mu \in C^\infty(M_\mu)$ defined by

$$h_\mu \circ \pi_\mu = h \circ i_\mu.$$

The vector fields X_h and X_{h_μ} are π_μ-related. The triple $(M_\mu, \omega_\mu, h_\mu)$ is called the *reduced Hamiltonian system.*

(iv) Let $k \in C^\infty(M)^G$ be another G-invariant function. Then $\{h, k\}$ is also G-invariant and $\{h, k\}_\mu = \{h_\mu, k_\mu\}_{M_\mu}$, where $\{.,.\}_{M_\mu}$ denotes the Poisson bracket associated to the symplectic form ω_μ on M_μ.

Proof : From Bifurcation lemma we know that for a symplectic manifold $range(T_m J) = (\mathcal{G}_m)^o$. Also here we have the freeness of the action, therefore J is a submersion onto some open subset of $\mathcal{G}^*$ and consequently, the level set $J^{-1}(\mu)$ is a G_μ-invariant(by the equivariance of J), closed and embedded submanifold of M. The free and proper G-action on M restricts to a free and proper G_μ-action on $J^{-1}(\mu)$ and from Theorem 1.1.12 the quotient $M_\mu := J^{-1}(\mu)/G_\mu$ is regular quotient manifold.

Then we show that for any $m \in J^{-1}(\mu)$, every vector $v_m \in T_m J^{-1}(\mu)$ can be written as $T_m i_\mu(v_m) = X_f(m)$, where $f \in C^\infty(M)^G$, and $i_\mu : J^{-1}(\mu) \hookrightarrow M$ is the canonical injection. This is indeed a consequence of the freeness of the action. Also we have

$$T_m J^{-1}(\mu) = ker T_m J = (\mathcal{G}.m)^\omega = \{X_f(m) \mid f \in C^\infty(M)^G\}.$$

We use this fact to define a two-form ω_μ on M_μ by

$$\omega_\mu([m]_\mu)(T_m\pi_\mu(v), T_m\pi_\mu(w)) = \{f, g\}(m),$$

where $[m]_\mu := \pi_\mu(m)$, $\{.,.\}$ is the Poisson bracket associated to the symplectic form ω on M , and $f, g \in C^\infty(M)^G$ are two G-invariant functions such that $T_m i_\mu(v) = X_f(m)$ and $T_m i_\mu(w) = X_g(m)$.

Now we check that the above expression is a good definition for the form ω_μ

on the quotient M_μ. Let $m, m' \in J^{-1}(\mu)$ be such that $\pi_\mu(m) = \pi_\mu(m')$ and let $\upsilon', w' \in T_{m'}J^{-1}(\mu)$ be such that

$$\begin{aligned} T_m\pi_\mu(\upsilon) &= T_{m'}\pi_\mu(\upsilon'), \\ T_m\pi_\mu(w) &= T_{m'}\pi_\mu(w'). \end{aligned}$$

Let $f', g' \in C^\infty(M)^G$ be such that

$$\upsilon' = X_{f'}(m') \quad and \quad m' = X_{g'}(m').$$

The condition $\pi_\mu(m) = \pi_\mu(m')$ implies there exist $k \in G_\mu$ such that $m' = \Phi_k(m)$. We also have that

$$T_m\pi_\mu = T_{m'}\pi_\mu \circ T_m\Phi_k.$$

Analogously, because of the equalities

$$\begin{aligned} T_m\pi_\mu(\upsilon) &= T_{m'}\pi_\mu(\upsilon'), \\ T_m\pi_\mu(w) &= T_{m'}\pi_\mu(w'), \end{aligned}$$

there exists two elements $\xi^1, \xi^2 \in \mathcal{G}_\mu$ such that

$$\begin{aligned} X_{f'}(m') - T_m\Phi_k.X_f(m) &= \xi^1_{J^{-1}(\mu)}(m') = X_{J\xi^1}(m') \\ \text{and, } X_{g'}(m') - T_m\Phi_k.X_g(m) &= \xi^2_{J^{-1}(\mu)}(m') = X_{J\xi^2}(m') \\ \text{or, analogously, } X_{f'}(m') = X_{J\xi^1 + f\circ\Phi_{k^{-1}}}(m') &= X_{J\xi^1+f}(m'), \\ \text{and, } X_{g'}(m') = X_{J\xi^2 + g\circ\Phi_{k^{-1}}}(m') &= X_{J\xi^2+g}(m'). \end{aligned}$$

$$\begin{aligned}
\text{Hence, } \omega_\mu(\pi_\mu(m'))(\upsilon', w') &= \{f', g'\}(m') = \{J^{\xi^1} + f, J^{\xi^2} + g\}(m') \\
&= \{J^{\xi^1} + f, J^{\xi^2} + g\}(\Phi_k(m)) \\
&= \{f \circ \Phi_k, g \circ \Phi_k\}(m) + \{f \circ \Phi_k, J^{\xi^2} \circ \Phi_k\}(m) \\
&\quad + \{J^{\xi^1} \circ \Phi_k, g \circ \Phi_k\}(m) + \{J^{\xi^1} \circ \Phi_k, J^{\xi^2} \circ \Phi_k\}(m). --(*)
\end{aligned}$$

Given that for any $\xi \in \mathcal{G}$ and any $k \in G$

$$J^\xi \circ \Phi_k = J^{Ad_{k^{-1}}\xi} + < \sigma(k), \xi >,$$

we can conclude that the functions $J^\xi \circ \Phi_k$ and $J^{Ad_{k^{-1}}\xi}$ differ by a constant. As the Poisson bracket depends only on the first derivative and the functions f and g are G-invariant, we get

$$\begin{aligned}
\omega_\mu(\pi_\mu(m'))(\upsilon', w') &= \{f, g\}(m) + \{f, J^{Ad_{k^{-1}}\xi^2}\}(m) \\
&\quad + \{J^{Ad_{k^{-1}}\xi^1}, g\}(m) + \{J^{Ad_{k^{-1}}\xi^1}, J^{Ad_{k^{-1}}\xi^2}\}(m).
\end{aligned}$$

The G-invariance of the functions f and g implies by Noether's Theorem that $\{f, J^{Ad_{k^{-1}}\xi^2}\} = \{J^{Ad_{k^{-1}}\xi^1}, g\} = 0$. Moreover, as the Lie algebra $\mathcal{G}_\mu$ is invariant by the adjoint action of the group G_μ, we have that $Ad_{k^{-1}}\xi^2 \in \mathcal{G}_\mu$ and hence that $(Ad_{k^{-1}}\xi^2)_M(m) \in T_m J^{-1}(\mu) = ker T_m J$. Consequently,

$$\begin{aligned}
\{J^{Ad_{k^{-1}}\xi^1}, J^{Ad_{k^{-1}}\xi^2}\}(m) &= dJ^{Ad_{k^{-1}}\xi^1}(m)(X_{J^{Ad_{k^{-1}}\xi^2}}(m)) \\
&= dJ^{Ad_{k^{-1}}\xi^1}(m)((Ad_{k^{-1}}\xi^2)_{J^{-1}(\mu)}(m)) \\
&= < T_m J((Ad_{k^{-1}}\xi^2)_{J^{-1}(\mu)}(m)), Ad_{k^{-1}}\xi^1 >= 0.
\end{aligned}$$

All these facts inserted in (*) imply that

$$\omega_\mu(\pi_\mu(m'))(\upsilon', w') = \{f, g\}(m) = \omega_\mu(\pi_\mu(m))(\upsilon, w)$$

and hence ω_μ defines indeed a two form on M_μ.

Next we want to prove $\pi_\mu^*\omega_\mu = i_\mu^*\omega$. Let $m \in J^{-1}(\mu)$ and let $v, w \in T_m J^{-1}(\mu)$. We can write $T_m i_\mu(v) = X_f(m)$ and $T_m i_\mu(w) = X_g(m)$, for some $f, g \in C^\infty(M)^G$. Then,

$$(\pi_\mu^*\omega_\mu)(m)(v,w) = \{f,g\}(m) = \omega(m)(X_f(m), X_g(m)) = (i_\mu^*\omega)(m)(v,w).$$

Conversely, the above chain of equalities shows that

$$\omega_\mu([m]_\mu)(T_m\pi_\mu(v), T_m\pi_\mu(w)) = \{f,g\}(m).$$

Since π_μ is a surjective submersion, this shows that $\pi_\mu^*\omega_\mu = i_\mu^*\omega$, which is equivalent to the above, which uniquely determines ω_μ and that ω_μ is a smooth two form on the quotient M_μ. The Jacobi identity for the bracket $\{.,.\}$ on M and its antisymmetry imply that ω_μ is closed and antisymmetric.

Finally we show that ω_μ is nondegenerate. Indeed, if

$$\omega_\mu([m]_\mu)(T_m\pi_\mu(v), T_m\pi_\mu(w)) = 0$$

for any $w \in T_m J^{-1}(\mu)$, then $\omega(m)(v,w) = 0$, for any for any $w \in T_m J^{-1}(\mu)$, which implies

$$w \in kerT_m J \cap (kerT_m J)^\omega = (\mathcal{G}.m)^\omega \cap ((\mathcal{G}.m)^\omega)^\omega = (\mathcal{G}.m)^\omega \cap (\mathcal{G}.m) = (\mathcal{G}_\mu.m).$$

This shows that $T_m\pi_\mu.v = 0$, as required. The pair (M_μ, ω_μ) is therefore a symplectic manifold.

(ii) By Noether's Theorem the flow F_t leaves invariant the connected component of $J^{-1}(\mu)$. The G-invariance of h and the canonical character of the action imply that F_t is G-equivariant from the Note 1.1.45. Let F_t^μ be the flow on M_μ that makes the following diagram commutative:

$$\begin{array}{ccc} J^{-1}(\mu) & \xrightarrow{F_t \circ i_\mu} & J^{-1}(\mu) \\ \pi_\mu \downarrow & & \downarrow \pi_\mu \\ M_\mu & \xrightarrow{F_t^\mu} & M_\mu . \end{array}$$

(iii) Due to the G-invariance of h, the function $h_\mu \in C^\infty(M_\mu)$ is uniquely determined by the identity $h_\mu \circ \pi_\mu = h \circ i_\mu$. Let $Y \in \mathcal{X}(M_\mu)$ be the vector field on M_μ whose flow is F_t^μ. By construction, Y is π_μ-related to X_h. Indeed differentiating the relation

$$\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu$$

with respect to the time t, we obtain

$$T\pi_\mu \circ X_h \circ i_\mu = Y \circ \pi_\mu .$$

We now verify that Y is a Hamiltonian vector field with Hamiltonian function h_μ, that is, $Y = X_{h_\mu}$. Let $m \in J^{-1}(\mu)$ and $v \in T_m J^{-1}(\mu)$ be arbitrary. Then,

$$\begin{aligned} \omega_\mu(\pi_\mu(m))(Y(\pi_\mu(m)), T_m\pi_\mu(v)) &= \omega_\mu(\pi_\mu(m))(T_m\pi_\mu(X_h(m)), T_m\pi_\mu(v)) \\ &= \omega(m)(X_h(m), v) = dh(m).v \\ &= d(h_\mu \circ \pi_\mu)(m).v \\ &= dh_\mu(\pi_\mu(m))(T_m\pi_\mu(v)) \\ &= \omega_\mu(\pi_\mu(m))(X_{h_\mu}(\pi_\mu(m)), T_m\pi_\mu(v)), \end{aligned}$$

which shows that $Y = X_{h_\mu}$.

(iv) The G-invariance of $\{h, k\}$ is a straightforward consequence of the canonical character of the action. Indeed, for any $g \in G$ we have

$$\{h, k\} \circ \Phi_g = \Phi_g^*\{h, k\} = \{\Phi_g^* h, \Phi_g^* k\} = \{h, k\}.$$

The function $\{h, k\}_\mu \in C^\infty(M_\mu)$ is uniquely characterized by the identity $\{h, k\}_\mu \circ \pi_\mu = \{h, k\} \circ i_\mu$. By the definition of the Poisson bracket on (M_μ, ω_μ), π_μ-relatedness of the relevant Hamiltonian vector fields, so we have for any $m \in J^{-1}(\mu)$

$$\begin{aligned}
\{h_\mu, k_\mu\}_{M_\mu}([m]_\mu) &= \omega_\mu([m]_\mu)(X_{h_\mu}([m]_\mu), X_{k_\mu}([m]_\mu)) \\
&= \omega_\mu([m]_\mu)(T_m i_\mu(X_h(m)), T_m i_\mu(X_k(m))) \\
&= \{h, k\}(m),
\end{aligned}$$

that is, the function $\{h_\mu, k_\mu\}_{M_\mu}$ also satisfies the relation $\{h_\mu, k_\mu\}_{M_\mu} \circ \pi_\mu = \{h, k\} \circ i_\mu$ which proves the desired equality $\{h_\mu, k_\mu\}_{M_\mu} = \{h, k\}_\mu$. •

Lemma 3.1.2. Let (E, ω) be a symplectic representation space of the compact Lie group H. Then any H-invariant Lagrangian subspace of (E, ω) admits an H-invariant Lagrangian complement.

Proof : Let $< ., . >$ be an H-invariant inner product on E always available by the compactness of H. Let $J : E \longrightarrow E$ be the H-equivariant map uniquely determined by the relation $< u, v >= \Omega(u, J(v))$ for any $u, v \in V$.
By construction $J^2 = -I_E$ and $\Omega(u, J(v)) = \Omega(J(u), v)$ for all $u.v \in E$. Let F be an H-invariant Lagrangian subspace of E. We will now show that $J(F)$ is an H-invariant Lagrangian complement of F in E.

First, the H-invariance of $J(F)$ is obvious by the H-equivariance of J and the

H-invariance of F. Second, we have that

$$dimJ(F)^{\Omega} = dimE - dimJ(F) = dimE - dimF = dimF,$$

where the last equality we used the Lagrangian character of F.
Third, $J(F) \subset J(F)^{\Omega}$. Indeed, for any $u, v \in F$ we have that

$$\Omega(J(u), J(v)) = -\Omega(u, J^2(v)) = \Omega(u, v) = 0.$$

So $J(F) = J(F)^{\Omega}$ which shows that $J(F)$ is Lagrangian.
Finally, since $J(F) \subset$.

$$\begin{aligned} J(F)^{\Omega} &= \{v \in E \mid \Omega(v), J(u)) = 0 forallu \in F\} \\ &= \{v \in E \mid < v, u >= 0 forallu \in F\} = F^{\perp} \end{aligned}$$

it follows that $J(F) = F^{\perp}$ is a complement to F.

Hence the proof of the Lemma. •

Note 3.1.3. Next we discuss the Witt-Artin decomposition of the tangent space T_mM at a point $m \in M$ of the symplectic manifold (M, ω) acted upon properly and canonically by a Lie group G. The first step in the construction of the Witt-Artin decomposition is the splitting of the Lie algebra $\mathcal{G}$ of G into three parts. The first summand is defined by

$$\mathcal{L} = \{\xi \in \mathcal{G} \mid \xi_M(m) \in (\mathcal{G}.m)^{\omega(m)}\};$$

$\mathcal{L}$ is clearly a vector subspace of $\mathcal{G}$ that contains the Lie algebra $\mathcal{G}_m$ of the isotropy subgroup G_m of the point $m \in M$. Hence we can fix an Ad_{G_m}-invariant inner product $< .,. >$ on $\mathcal{G}$ (always available by the compactness of G_m) and write

$$\mathcal{L} = \mathcal{G}_m \oplus \mathcal{M} \quad and \quad \mathcal{G} = \mathcal{G}_m \oplus \mathcal{M} \oplus \mathcal{Q}.$$

where $\mathcal{M}$ is the $< .,. >$- orthogonal complement of $\mathcal{G}_m$ in $\mathcal{L}$ and $\mathcal{Q}$ is the $< .,. >$-orthogonal complement of $\mathcal{L}$ in $\mathcal{G}$. The above splittings induce similar ones on the duals

$$\mathcal{L}^* = \mathcal{G}_m^* \oplus \mathcal{M}^* \quad and \quad \mathcal{G}^* = \mathcal{G}_m^* \oplus \mathcal{M}^* \oplus \mathcal{Q}^*.$$

Each of these spaces in this decomposition should be understood as the set of covectors in $\mathcal{G}^*$ that can be written as $< \xi, . >$, with ξ in the corresponding subspace. For example, $\mathcal{Q}^* = \{< \xi, . >| \ \xi \in \mathcal{Q}\}$.

Theorem 3.1.4 (Witt-Artin decomposition)**.** Let (M, ω) be a symplectic manifold and let G be a Lie group acting properly and canonically on it. Then for any $m \in M$

$$T_m M = \mathcal{L}.m \oplus \mathcal{Q}.m \oplus V \oplus W.$$

The definitions and properties of the spaces in this splitting are the following:

(i) $\mathcal{L} := \{\xi \in \mathcal{G} \mid \xi_M(m) \in (\mathcal{G}.m)^{\omega(m)}\}$ a Lie subalgebra of $\mathcal{G}$.

(ii) $\mathcal{Q}.m = \{\eta_M(m) \mid \eta \in \mathcal{Q}\}$ is a symplectic vector subspace of $(T_m M, \omega(m))$.

(iii) Let $<< .,. >>$ be a G_m invariant innerproduct in $T_m M$. Define V as the orthogonal complement to $\mathcal{G}.m \cap (\mathcal{G}.m)^{\omega(m)} = \mathcal{L}.m$ in $(\mathcal{G}.m)^{\omega(m)}$ with respect to $<< .,. >>$, that is,

$$(\mathcal{G}.m)^{\omega(m)} = (\mathcal{G}.m \cap (\mathcal{G}.m)^{\omega(m)}) \oplus V = \mathcal{L}.m \oplus V.$$

The subspace V is a symplectic G_m-invariant subspace of $(T_mM, \omega(m))$ such that $V \cap \mathcal{Q}.m = \{0\}$.

(iv) $\mathcal{L}.m := \{\xi_M(m) \mid \xi \in \mathcal{G}\}$ is a Lagrangian subspace of $(V \oplus \mathcal{Q}.m)^{\omega(m)}$.

(v) W is a G_m-invariant Lagrangian complement to $\mathcal{L}.m$ in $(V \oplus \mathcal{Q}.m)^{\omega(m)}$.

(vi) The map $f : W \longrightarrow \mathcal{M}^*$ defined by

$$< f(w), \eta >:= \omega(m)(\eta_M(m), w), \quad forall \quad \eta \in \mathcal{M}$$

is a G_m-equivariant isomorphism.

Proof : (i) $\mathcal{L}$ is clearly a vector subspace of $\mathcal{G}$. To show it is a Lie subalgebra, we shall prove first that for any $\xi, \eta, \zeta \in \mathcal{G}$ we have

$$d\omega(\xi_M, \eta_M, \zeta_M) = \omega([\xi, \eta]_M, \zeta_M) + \omega([\eta, \zeta]_M, \xi_M) + \omega([\zeta, \xi]_M, \eta_M).$$

To verify this identity we start by computing $d\omega$ for the infinitesimal generators ξ_M, η_M, ζ_M :

$$\begin{aligned} d\omega(\xi_M, \eta_M, \zeta_M) &= \xi_M[\omega(\eta_M, \zeta_M)] + \eta_M[\omega(\zeta_M, \xi_M)] + \zeta_M[\omega(\xi_M, \eta_M)] \\ &\quad -\omega([\xi_M, \eta_M], \zeta_M) - \omega([\eta_M, \zeta_M], \xi_M) + \omega([\zeta_M, \xi_M], \eta_M). \end{aligned}$$

However

$$\begin{aligned} \xi_M[\omega(\eta_M, \zeta_M)] &= (\mathcal{L}_{\xi_M}\omega)(\eta_M, \zeta_M) + \omega(\eta_M, [\xi_M, \zeta_M]) \\ &= \omega([\xi_M, \eta_M], \zeta_M) + \omega(\eta_M, [\xi_M, \zeta_M]) \end{aligned}$$

since $\mathcal{L}_{\xi_M}\omega = 0$, the G-action being canonical. Replacing this and the analogous two identities, we get the result.

In particular, since $d\omega = 0$, if $\xi, \eta \in \mathcal{L}$ and any $\zeta \in \mathcal{G}$,

$$\omega([\xi, \eta]_M, \zeta_M) = -\omega([\eta, \zeta]_M, \xi_M) - \omega([\zeta, \xi]_M, \eta_M) = 0,$$

which shows that $[\xi, \eta] \in \mathcal{L}$ and hence that $\mathcal{L}$ is a Lie subalgebra of $\mathcal{G}$.

(ii) We show that $\omega(m) \mid_{\mathcal{Q}.m}$ is nondegenerate. Let $\xi \in \mathcal{Q}$ be such that for any $\eta \in \mathcal{Q}$

$$\omega(m)(\xi_M(m), \eta_M(m)) = 0.$$

Since any element $\zeta \in \mathcal{G}$ can be written as $\zeta = \eta + \sigma$, with $\eta \in \mathcal{Q}, \sigma \in \mathcal{L}$ we have that

$$\omega(m)(\xi_M(m), \zeta_M(m)) = \omega(m)(\xi_M(m), \eta_M(m)) + \omega(m)(\xi_M(m), \sigma_M(m)) = 0.$$

Consequently, $\xi_M(m) \in (\mathcal{G}.m)^\omega$ and hence $\xi \in \mathcal{L} \cap \mathcal{Q} = \{0\}$.

(iii) We start by noting that both $\mathcal{G}.m$ and $(\mathcal{G}.m)^{\omega(m)}$ are G_m-invariant. Therefore, the G_m-invariance of $<< .,. >>$ guarantees that V is G_m-invariant. We now prove that V is symplectic.

Let $v \in V$ be such that $\omega(m)(v, v') = 0$, for any $v' \in V$, that is, $v \in V^{\omega(m)}$. Since $V \subset (\mathcal{G}.m)^{\omega(m)} \subset (\mathcal{L}.m)^{\omega(m)}$, we have

$$v \in V^{\omega(m)} \cap (\mathcal{L}.m)^{\omega(m)} = (V \oplus \mathcal{L}.m)^{\omega(m)} = ((\mathcal{G}.m)^{\omega(m)})^{\omega(m)} = \mathcal{G}.m.$$

Hence, $v \in V \cap \mathcal{G}.m \subset (\mathcal{G}.m)^{\omega(m)} \cap \mathcal{G}.m = \mathcal{L}.m$ and, therefore, $v \in \mathcal{L}.m \cap V = \{0\}$.

We now show that $V \cap \mathcal{Q}.m = \{0\}$. First notice that $\mathcal{L}.m \cap \mathcal{Q}.m = \{0\}$. Indeed, if $\xi \in \mathcal{L}$ ad $\eta \in \mathcal{Q}$ are such that $\xi_M(m) = \eta_M(m)$, then $\xi - \eta \in \mathcal{G}_m \subset \mathcal{L}$

and therefore, $\eta \in \mathcal{L} \cap \mathcal{Q} = \{0\}$.

$$\text{Therefore, } \mathcal{G}.m = \mathcal{L}.m \oplus \mathcal{Q}.m,$$
$$(\mathcal{G}.m)^{\omega(m)} = (\mathcal{L}.m)^{\omega(m)} \cap (\mathcal{Q}.m)^{\omega(m)}.$$

$$\begin{aligned}
\text{Now, } V \cap \mathcal{Q}.m &= V \cap \mathcal{Q}.m \oplus (\mathcal{L}.m \cap \mathcal{Q}.m) \\
&\subset (V \oplus \mathcal{L}.m) \cap \mathcal{Q}.m \\
&= (\mathcal{G}.m)^{\omega(m)} \cap \mathcal{Q}.m \\
&= (\mathcal{L}.m)^{\omega(m)} \cap (\mathcal{Q}.m)^{\omega(m)} \cap (\mathcal{Q}.m) \\
&= (\mathcal{L}.m)^{\omega(m)} \cap \{0\} = \{0\}.
\end{aligned}$$

(iv) Part (iii) guarantees that the sum $V + \mathcal{Q}.m$ is direct. Moreover, since V and $\mathcal{Q}.m$ are symplectic subspaces of T_mM, so is $V \oplus \mathcal{Q}.m$. This provides the first step in the Witt - Artin decomposition of T_mM into symplectic subspaces:

$$T_mM = V \oplus \mathcal{Q}.m \oplus (V \oplus \mathcal{Q}.m)^{\omega(m)}.$$

We now show that $\mathcal{L}.m \subset (V \oplus \mathcal{Q}.m)^{\omega(m)}$, which is equivalent to $V \subset (\mathcal{L}.m)^{\omega(m)}$ and $\mathcal{Q}.m \subset (\mathcal{L}.m)^{\omega(m)}$.

The first inclusion holds because $V \subset (\mathcal{G}.m)^{\omega(m)} \subset (\mathcal{L}.m)^{\omega(m)}$. As the second one, note that $\mathcal{Q}.m \subset \mathcal{G}.m \subset (\mathcal{L}.m)^{\omega(m)}$ since $\mathcal{G}.m \cap (\mathcal{G}.m)^{\omega(m)} = \mathcal{L}.m$.

Finally we show that $\mathcal{L}.m$ is a Lagrangian subspace of $(V \oplus \mathcal{Q}.m)^{\omega(m)}$, that is,

$$(\mathcal{L}.m)^{\omega(m)} \mid_{(V \oplus \mathcal{Q}.m)^{\omega(m)}} = \mathcal{L}.m,$$
$$\text{or, equivalently, } (\mathcal{L}.m)^{\omega(m)} \cap (V \oplus \mathcal{Q}.m)^{\omega(m)} = \mathcal{L}.m.$$

To prove this equality note that the definition of the subspace V implies that

$$\mathcal{G}.m = (\mathcal{L}.m)^{\omega(m)} \cap V^{\omega(m)}.$$

Additionally, recall that if A, B, and C are vector subspaces of a vector space E, such that $A \subset B$, and $A \cap C = \{0\}$, then

$$(B \cap C) \oplus A = B \cap (C \oplus A).$$

$$
\begin{aligned}
\text{Therefore, } (\mathcal{L}.m)^{\omega(m)} \cap (V \oplus \mathcal{Q}.m)^{\omega(m)} &= (\mathcal{L}.m + (V \oplus \mathcal{Q}.m))^{\omega(m)} \\
&= (\mathcal{G}.m + V)^{\omega(m)} \\
&= (\mathcal{G}.m)^{\omega(m)} \cap V^{\omega(m)} \\
&= ((\mathcal{L}.m)^{\omega(m)} \cap V^{\omega(m)})^{\omega(m)} \cap V^{\omega(m)} \\
&= [((\mathcal{L}.m)^{\omega(m)} \cap V^{\omega(m)}) \oplus V]^{\omega(m)} \\
&= [(\mathcal{L}.m)^{\omega(m)} \cap (V^{\omega(m)} \oplus V)]^{\omega(m)} \\
&= [(\mathcal{L}.m)^{\omega(m)} \cap T_m M]^{\omega(m)} \\
&= \mathcal{L}.m.
\end{aligned}
$$

(v) The existence of W is a consequence of the Lie algebraic lemma 3.1.2. Hence, we have that $(V \oplus \mathcal{Q}.m)^{\omega(m)} = \mathcal{L}.m \oplus W$.
So we have the decomposition

$$T_m M = \mathcal{L}.m \oplus \mathcal{Q}.m \oplus V \oplus W.$$

(vi) The map f is clearly linear and H-equivariant. In order to show that f is injective let $w \in W$ be such that $f(w) = 0$. This means that $\omega(m)(\eta_M(m), w) = 0$, for all $\eta \in \mathcal{M}$ or equivalently, $w \in (\mathcal{M}.m)^{\omega(m)} = ((\mathcal{G}_m \oplus \mathcal{M}).m)^{\omega(m)} = (\mathcal{L}.m)^{\omega(m)}$.

Now the definition of W in (v) gives $(V \oplus \mathcal{Q}.m)^{\omega(m)} = \mathcal{L}.m \oplus W$ and hence

$$w \in (\mathcal{L}.m)^{\omega(m)} \cap W \subset (\mathcal{L}.m)^{\omega(m)} \cap (V \oplus \mathcal{Q}.m)^{\omega(m)} = \mathcal{L}.m.$$

Therefore $w \in \mathcal{L}.m \cap W = \{0\}$. Hence f is surjective.

Since W and $\mathcal{L}.m$ are Lagrangian complements in $(V \oplus \mathcal{Q}.m)^{\omega(m)}$, it follows that

$$dim W = dim \mathcal{L}.m = dim \mathcal{L} - dim \mathcal{G}_m = dim \mathcal{M} = dim \mathcal{M}^*,$$

which shows that f is an isomorphism. •

Definition 3.1.5. Let (M, ω) be a symplectic manifold and G be a Lie group acting properly and canonically on it. Let $m \in M$, V be a symplectic G_m space constructed in part (iii) of Theorem 3.1.4. Any such space will be called a *symplectic normal space* at m. Since the G_m action on $(V, \omega(m) \mid_V)$ is linear and canonical it has a standard associated momentum map to be denoted by $J_V : V \longrightarrow \mathcal{G}_m^*$.

Proposition 3.1.6. Let (M, ω) be a symplectic manifold and G be a Lie group acting properly and canonically on it. Let $m \in M$, V be a symplectic normal space at m, and $\mathcal{M} \subset \mathcal{G}$ the subspace introduced in the splitting $\mathcal{L} = \mathcal{G}_m \oplus \mathcal{M}$. Then there exist G_m-invariant neighborhoods $\mathcal{M}_r^*$ and V_r of the origin in $\mathcal{M}^*$ and V respectively such that the twisted product

$$Y_r := G \times_{G_m} (\mathcal{M}_r^* \times V_r)$$

is a symplectic manifold with the two form ω_{Y_r} defined by

$$\begin{aligned}\omega_{Y_r}\quad ([g,\rho,v])\quad & (T_{(g,\rho,v)}\pi(T_eL_g(\xi_1),\alpha_1,u_1),T_{(g,\rho,v)}\pi(T_eL_g(\xi_2),\alpha_2,u_2))\\ := \qquad & <\alpha_2+T_vJ_V(u_2),\xi_1> - <\alpha_1+T_vJ_V(u_1),\xi_2>\\ & + <\rho+J_V(v),[\xi_1,\xi_2]> +\Psi(m)(\xi_1,\xi_2)+\omega(m)(u_1,u_2), \quad (3.2)\end{aligned}$$

where $\Psi : M \longrightarrow Z^2(\mathcal{G})$ is the Chu map in the Definition **??** associated to the G-action on (M,ω), $\pi : G\times(\mathcal{M}_r^*\times V_r) \longrightarrow G\times_{G_m}(\mathcal{M}_r^*\times V_r)$ is the projection, $[g,\rho,v]\in Y_r, \xi_1,\xi_2\in\mathcal{G},\alpha_1,\alpha_2\in\mathcal{M}^*$, and $u_1,u_2\in V$.

The Lie group G acts canonically on (Y_r,ω_{Y_r}) by $g.[h,\eta,\nu] := [gh,\eta,\nu]$, for any $g\in G$ and any $[h,\eta,\nu]\in Y_r$.

The symplectic manifold (Y_r,ω_{Y_r}) is called a *symplectic tube* of (M,ω) at the point m.

Proof : First we construct a G-invariant symplectic form on a neighborhood of the zero section of the trivial bundle $G\times\mathcal{L}^*$.

The splitting

$$\mathcal{L}^* = \mathcal{G}_m^*\oplus\mathcal{M}^* \quad \textit{and} \quad \mathcal{G}^* = \mathcal{G}_m^*\oplus\mathcal{M}^*\oplus\mathcal{Q}^*.$$

provides an injection $i : G\times\mathcal{L}^* \hookrightarrow G\times\mathcal{G}^*$ that will be used to pull back the canonical symplectic form of $G\times\mathcal{G}^*\cong T^*G$ to $G\times\mathcal{L}^*$ in order to obtain a closed two form ω_1 on $G\times\mathcal{L}^*$.

Define on $G\times\mathcal{L}^*$ the skew symmetric two form ω_2 by

$$\omega_2(g,\nu)((T_eL_g(\xi),\rho),(T_eL_g(\eta),\sigma)) = \omega(m)(\xi_M(m),\eta_M(m)) = \Psi(m)(\xi,\eta),$$

for any $(g,\nu) \in G \times \mathcal{L}^*$, $\xi, \eta \in \mathcal{G}$, $\rho, \sigma \in \mathcal{L}^*$.

We now prove that there exists a G_m-invariant neighborhood $\mathcal{L}^*_r$ of the origin in $\mathcal{L}^*$ such that the restriction of the form

$$\Omega = \omega_1 + \omega_2 \tag{3.3}$$

to $T_r = G \times \mathcal{L}^*_r$ is a symplectic form.

Ω is closed: The two form ω_1 is clearly closed. In order to show that ω_2 is closed we define for any $\xi \in \mathcal{G}$, and $\rho \in \mathcal{L}^*$, the vector field $(\xi,\rho) \in \mathcal{X}(G \times \mathcal{L}^*)$ given by $(\xi,\rho)(g,\nu) = (T_eL_g(\xi),\nu)$, whose flow is
$F_t(g,\nu) = (g.expt\xi, \rho + t\nu)$. It is easy to see that the Lie bracket of two such vector fields is given by $[(\xi,\rho),(\eta,\sigma)] = ([\xi,\eta],0)$.

We have

$$\begin{aligned} d\omega_2((\xi,\rho),(\eta,\sigma),(\lambda,\tau)) &= (\xi,\rho)[\omega_2((\eta,\sigma),(\lambda,\tau))] - (\eta,\sigma)[\omega_2((\xi,\rho),(\lambda,\tau))] \\ &+ (\lambda,\tau)[\omega_2((\xi,\rho),(\eta,\sigma))] - \omega_2([(\xi,\rho),(\eta,\sigma)],(\lambda,\tau)) \\ &+ \omega_2([(\xi,\rho),(\lambda,\tau)],(\eta,\sigma)) - \omega_2([(\eta,\sigma),(\lambda,\tau)],(\xi,\rho)) \end{aligned}$$

for any $\xi, \eta, \lambda \in \mathcal{G}$ and any $\rho, \sigma, \tau \in \mathcal{L}^*$. Now note that for any $(g,\nu) \in G \times \mathcal{L}^*$ we have, for instance, that

$$\begin{aligned} &((\xi,\rho)[\omega_2((\eta,\sigma),(\lambda,\tau))])(g,\nu) \\ &\quad = \frac{d}{dt}\,|_{t=0}\, \omega_2(gexpt\xi, \nu + t\rho) \quad (\quad (\eta,\sigma)(gexpt\xi, \nu + t\rho), (\lambda,\tau)(gexpt\xi, \nu + t\rho)) \\ &= \frac{d}{dt}\,|_{t=0}\, \omega(m)(\eta_M(m), \lambda_M(m)) \quad = \quad 0, \end{aligned}$$

and also,

$$\omega_2([(\xi,\rho),(\eta,\sigma)],(\lambda,\tau))(g,\nu)=\omega(m)([\xi,\eta]_M(m),\lambda_M(m)).$$

The last three equalities imply that

$$\begin{aligned}
&d\omega_2(g,\nu)((T_eL_g(\xi),\rho),(T_eL_g(\eta),\sigma),(T_eL_g(\lambda),\tau))\\
&=-\omega(m)([\xi,\eta]_M(m),\lambda_M(m))+\omega(m)([\xi,\lambda]_M(m),\eta_M(m))\\
&\quad-\omega(m)([\eta,\lambda]_M(m),\xi_M(m))\\
&=d\omega(m)(\xi_M(m),\eta_M(m),\lambda_M(m))=0.
\end{aligned}$$

which guarantees the closedness of ω_2.

Ω is nondegenerate on $G\times\{0\}$: Let $\xi=\xi_1+\xi_2$ and $\eta=\eta_1+\eta_2$ be arbitrary elements in $\mathcal{G}$, with $\xi_1,\eta_1\in\mathcal{L}$ and $\xi_2,\eta_2\in\mathcal{Q}$. Let also $(g,\nu)\in G\times\mathcal{L}^*$, $\rho,\sigma\in\mathcal{L}^*$ and suppose that for all $\eta\in\mathcal{G}$, and $\sigma\in\mathcal{L}^*$ we have that

$$\Omega(g,\nu)((T_eL_g(\xi),\rho),(T_eL_g(\eta),\sigma))=0,$$

or equivalently

$$<g,\xi_1>-<\rho,\eta_1>+<\nu,[\xi,\eta]>+\omega(m)((\xi_2)_M(m),(\eta_2)_M(m))=0.$$

We show that when $\nu=0$ this implies that $\xi=0$ and $\rho=0$. Indeed suppose that $\nu=0$. Setting $\eta=0$ in the last equality and letting σ vary, we obtain $\xi_1=0$. Also, setting $\eta_2=0$ and letting η_1 vary, we obtain $\rho=0$. Finally, since $\omega(m)((\xi_2)_M(m),(\eta_2)_M(m))=0$ for all $\eta\in\mathcal{G}$, this implies that $(\xi_2)_M(m)\in\mathcal{G}.m\cap(\mathcal{G}.m)^\omega$ and hence $\xi_2\in\mathcal{L}\cap\mathcal{Q}=\{0\}$ and, consequently, $\xi=0$ and $\rho=0$.

Since nondegeneracy is an open condition, we can choose an $Ad(G_m)$-invariant neighborhood $\mathcal{L}_r^*$ of 0 in $\mathcal{L}^*$ where the last expression is nondegenerate. Also, as the expression does not depends on G (the form Ω is by construction G-invariant), it follows that Ω is nondegenerate on $T_r = G \times \mathcal{L}_r^*$ and hence (T_r, Ω) is a symplectic manifold.

The symplectic form ω_{Y_r} on Y_r is obtained by a symplectic reduction of the symplectic forms of T_r and V. First consider the left action $\mathcal{R}$ of G_m on T_r defined by $\mathcal{R}_h(g, \nu) = (gh^{-1}, Ad^*_{h^{-1}}\nu)$, $h \in G_m$ and $(g, \nu) \in T_r$.

Using the definition of Ω it is straightforward to verify that this action is globally Hamiltonian on T_r with equivariant momentum map $J_{\mathcal{R}} : T_r \longrightarrow \mathcal{G}_m^*$, given by

$$J_{\mathcal{R}}((g, (\eta, \rho))) = -\eta, \text{ for any } (\eta, \rho) \in \mathcal{G}_m^* \oplus \mathcal{M}_r^* = \mathcal{L}_r^*.$$

As we already pointed out, the G_m-action on V is globally Hamiltonian with momentum map $J_V : V \longrightarrow \mathcal{G}_m^*$. Putting together these two actions, we construct a product action of G_m on the symplectic manifold $T_r \times V$, which is Hamiltonian, with G_m-equivariant momentum map $K : T_r \times V \cong G \times \mathcal{M}_r^* \times (\mathcal{G}_m^*)_r \times V \longrightarrow \mathcal{G}_m^*$, given by the sum $J_{\mathcal{R}} + J_V$ from [10], that is,

$$\begin{array}{rcl} K : G \times \mathcal{M}_r^* \times (\mathcal{G}_m^*)_r \times V & \longrightarrow & \mathcal{G}_m^*, \\ (g, \rho, \eta, \upsilon) & \longmapsto & J_V(\upsilon) - \eta. \end{array}$$

The G_m-action on $T_r \times V$ is free and proper and $0 \in \mathcal{G}_m^*$ is clearly a regular value of K. Therefore from Theorem 3.1.1 $K^{-1}(0)/G_m$ is a well defined symplectic point reduced space that can be identified with $Y_r = G \times_{G_m} (\mathcal{M}_r^* \times V_r)$ by means of the quotient diffeomorphism L induced by the G_m-equivariant diffeomorphism l

$$
\begin{array}{rcl}
l : G \times \mathcal{M}_r^* \times V_r & \longrightarrow & K^{-1}(0) \subset G \times \mathcal{M}_r^* \times (\mathcal{G}_m^*)_r \times V_r, \\
(g, \rho, \upsilon) & \longmapsto & (g, \rho, J_{V_m}(\upsilon), \upsilon),
\end{array}
$$

where the G_m-invariant neighborhood of the origin V_r has been chosen so that $J_{V_m}(V_r) \subset (\mathcal{G}_m^*)_r$. We define the symplectic form ω_{Y_r} on Y_r as the pull back by L of the reduced symplectic form Ω_0 on $K^{-1}(0)/G_m$. Thus we have the following commutative diagram with the lower arrow a symplectic diffeomorphism

$$
\begin{array}{rcl}
G \times \mathcal{M}_r^* \times V_r & \stackrel{l}{\longrightarrow} & K^{-1}(0) \subset G \times \mathcal{M}_r^* \times (\mathcal{G}_m^*)_r \times V_r, \\
\pi \downarrow & & \downarrow \pi_0 \\
(G \times_{G_m} (\mathcal{M}_r^* \times V_r), \omega_{Y_r}) & \longrightarrow & (K^{-1}(0)/G_m, \Omega_0).
\end{array}
$$

We now show that the symplectic form ω_{Y_r} that we just defined coincides with the one given in the statement, namely, with expression 3.2. First notice that by definition $\omega_{Y_r} = L^*\Omega_0$. As the projection π is a submersion, this is equivalent to $\pi^*\omega_{Y_r} = \pi^*(L^*\Omega_0)$. Using the maps in the above diagram we can express this equality as $\pi^*\omega_{Y_r} = l^*(\pi_0^*\Omega_0) = l^* i_0^*(\Omega + \omega_V)$, where $\omega_V := \omega(m) \mid_V$ is the symplectic form on the symplectic normal space V. We now check that this coincides with the expression 3.2. Indeed, for any $[g, \rho, v] \in Y_r, \xi_1, \xi_2 \in \mathcal{G}, \alpha_1, \alpha_2 \in \mathcal{M}^*$, and

$u_1, u_2 \in V$, we have

$$\begin{aligned}
\omega_{Y_r} \quad & ([\quad g,\rho,v])(T_{(g,\rho,v)}\pi(T_eL_g(\xi_1),\alpha_1,u_1),T_{(g,\rho,v)}\pi(T_eL_g(\xi_2),\alpha_2,u_2)) \\
= \quad & (\pi^*\omega_{Y_r})(g,\rho,v)((T_eL_g(\xi_1),\alpha_1,u_1),(T_eL_g(\xi_2),\alpha_2,u_2)) \\
= \quad & ((i_0\circ l)^*(\Omega+\omega_V))(g,\rho,v)((T_eL_g(\xi_1),\alpha_1,u_1),(T_eL_g(\xi_2),\alpha_2,u_2)) \\
= \quad & (\Omega+\omega_V)(g,\rho,J_V(v),v)((T_eL_g(\xi_1),\alpha_1,T_vJ_V(u_1),u_1), \\
& (T_eL_g(\xi_2),\alpha_2,T_vJ_V(u_2),u_2)) \\
= \quad & \Omega(g,\rho,J_V(v))((T_eL_g(\xi_1),\alpha_1,T_vJ_V(u_1)),(T_eL_g(\xi_2),\alpha_2,T_vJ_V(u_2))) \\
& +\omega_V(v)(u_1,u_2) \\
= \quad & <\alpha_2+T_vJ_V(u_2),\xi_1> - <\alpha_1+T_vJ_V(u_1),\xi_2> \\
& + <\rho+J_V(v),[\xi_1,\xi_2]> +\Psi(m)(\xi_1,\xi_2)+\omega(m)(u_1,u_2).
\end{aligned}$$

•

Theorem 3.1.7 (The Symplectic slice Theorem). Let (M,ω) be a symplectic manifold and G be a Lie group acting properly and canonically on it. Let $m \in M$, and let (Y_r, ω_{Y_r}) be the G-symplectic tube at that point constructed in Proposition 3.1.6. Then there is a G- invariant neighborhood U of m in M and a G-equivariant symplectomorphism $\phi : U \longrightarrow Y_r$ satisfying $\phi(m) = [e,0,0]$.

Proof : Consider the tube

$$Y_r := G \times_{G_m} (\mathcal{M}_r^* \times V_r)$$

and $f : W \longrightarrow \mathcal{M}^*$ the G_m-equivariant linear isomorphism introduced in part(vi) of Theorem 3.1.4.

Let $f_r^{-1} : \mathcal{M}_r^* \longrightarrow W_r$ where $W_r = f^{-1}(\mathcal{M}_r^*) \subset W$ be the restriction of the inverse f^{-1} of f to $\mathcal{M}_r^*$.

In the proof of the Theorem 1.1.18, we established the existence of a G_m-invariant Riemannian metric on some G_m-invariant neighborhood of m with associated exponential map Exp_m such that, for r small enough, the mapping

$$\begin{aligned} \tau := G \times_{G_m} (W_r \times V_r) \longrightarrow M \text{ given by,} \\ \tau([g, w, \upsilon]) = g.Exp_m(w + \upsilon), \end{aligned}$$

is a G-equivariant diffeomorphism onto some open G-invariant neighborhood of the orbit $G.m$. Therefore the composed map

$$\begin{aligned} \Psi : Y_r = G \times_{G_m} (\mathcal{M}_r^* \times V_r) &\longrightarrow U \text{ given by,} \\ \Psi([g, \rho, \upsilon]) &= g.Exp_m(f_r^{-1}(\rho) + \upsilon) \end{aligned}$$

has the same properties. Consequently, the open G-invariant neighborhood U can be endowed with two symplectic forms $\omega \mid_U$ and $\Psi_* \omega_{Y_r}$.

We now prove that these two symplectic forms coincide on $G.m$. Since both forms are G-invariant it suffices to show that $\omega(m) = \Psi_* \omega_{Y_r}(m)$. Let $u_1, u_2 \in T_m M$ arbitrary. The Witt-Artin decomposition guarantees the existence of $\xi_1, \xi_2 \in \mathcal{G}$, $w_1, w_2 \in W$, and $v_1, v_2 \in V$ such that $u_i = (\xi_i)_M(m) + w_i + v_i, i = 1, 2$.

Hence Proposition 3.1.6 and the definition of f_r imply

$$
\begin{aligned}
\Psi_* \omega_{Y_r}(m)(u_1,u_2) &= \omega_{Y_r}([e,0,0,])(T_m\Psi^{-1}((\xi_1)_M(m)+w_1+v_1),\\
&\quad T_m\Psi^{-1}((\xi_2)_M(m)+w_2+v_2)))\\
&= \omega_{Y_r}([e,0,0,])(T_{(e,0,0)}\pi(\xi_1, f_r(w_1), v_1), T_{(e,0,0)}\pi(\xi_2, f_r(w_2), v_2))\\
&= < f_r(w_2), \xi_1 > - < f_r(w_1), \xi_2 > +\\
&\quad \omega(m)((\xi_1)_M(m),(\xi_2)_M(m)) + \omega(m)(v_1,v_2)\\
&= \omega(m)((\xi_1)_M(m), w_2) - \omega(m)((\xi_2)_M(m), w_1) +\\
&\quad \omega(m)((\xi_1)_M(m),(\xi_2)_M(m)) + \omega(m)(v_1,v_2)\\
&= \omega(m)(u_1,u_2).
\end{aligned}
$$

The last equality follows from the fact that $\omega(m)((\xi_1)_M(m), v_2) = \omega(m)((\xi_2)_M(m), v_1) = 0$ since $V \subset (\mathcal{G}.m)^\omega$. Moreover, $\omega(m)(w_1,w_2) = 0$ since W is Lagrangian in $(V \oplus \mathcal{Q}.m)^{\omega(m)}$, and $\omega(m)(w_1,v_2) = \omega(m)(w_2,v_1) = 0$ because $W \subset V^{\omega(m)}$.

In these circumstances, Theorem 2.2.3 guarantees the existence of two open G-invariant neighborhoods U_o and U_1 of $G.m$ in U and a G-equivariant symplectomorphism $\Delta : (U_o, \Psi_*\omega_{Y_r} \mid_{U_o}) \longrightarrow (U_1, \omega \mid_{U_1})$ which is the identity on $G.m$. Take, without loss of generality, $U_1 = U$. Then the composed map $\Delta \circ \Psi : (Y_r, \omega_{Y_r}) \longrightarrow (U, \omega \mid_U)$ gives us, for $r > 0$ small enough, the inverse of the map needed in the statement of the theorem. •

Definition 3.1.8. Let (M,ω) be a symplectic manifold and G be a Lie group acting canonically on it. For any point $m \in M$ we say that the G-action on M is *tubewise Hamiltonian* at m if there exists a G-invariant open neighborhood U of the orbit $G.m$ such that the restriction of the action to the symplectic manifold $(U, \omega \mid_U)$ has an associated momentum map.

Proposition 3.1.9. Let (M,ω) be a symplectic manifold and let G be a Lie group with Lie algebra $\mathcal{G}$ acting properly and canonically on M. For $m \in M$ let $Y_r := G \times_{G_m} (\mathcal{M}_r^* \times V_r)$ be the slice model around the orbit $G.m$. If the G-equivariant $\mathcal{G}^*$-valued one form $\gamma \in \Omega^1(G; \mathcal{G}^*)$ defined by

$$< \gamma(g)(T_e L_g(\eta)), \xi >:= -\omega(m)((Ad_{g^{-1}}\xi)_M(m), \eta_M(m)) \tag{3.4}$$

for any $g \in G$ and $\xi, \eta \in \mathcal{G}$ is exact, then the G-action on Y_r given by $g.[h, \eta, v] := [gh, \eta, v]$, for any $g \in G$ and any $[h, \eta, v] \in Y_r$, has an associated momentum map and thus the G-action on (M, ω) is tubewise Hamiltonian at m.

Proof: The construction of the symplectic form ω_{Y_r} on Y_r in Proposition 3.1.6 reveals that the existence of a standard momentum map for the G-action on Y_r is guaranteed by the existence of a momentum map for the G-action on the symplectic manifold $(G \times \mathcal{L}_r^*, \Omega)$ introduced in the proof of Proposition 3.1.6. This action is given by $g.(h, \eta) := (gh, \eta)$, for any $g, h \in G, \eta \in \mathcal{L}_r^*$. The existence of this momentum map is in turn equivalent to the vanishing of the map in Proposition 1.4.8

$$[\xi] \in \mathcal{G}/[\mathcal{G}, \mathcal{G}] \longmapsto [i_{\xi_{G\times\mathcal{L}^*}}\Omega] \in H^1(G \times \mathcal{L}_r^*), \text{for any } \xi \in \mathcal{G}. \tag{3.5}$$

By the definition of Ω we have that for any $\xi, \eta \in \mathcal{G}, g \in G$, and $\nu, \sigma \in \mathcal{G}^*$

$$\begin{aligned} i_{\xi_{G\times\mathcal{L}^*}}\Omega(g, \nu)(T_e L_g(\eta), \sigma) =&< \sigma, Ad_{g^{-1}}\xi > + < \nu, [Ad_{g^{-1}}\xi, \eta] > \\ &+\omega(m)((Ad_{g^{-1}}\xi)_M(m), \eta_M(m)). \end{aligned}$$

The first two terms on the right-hand side of the previous expression are the differential of the real function $f \in C^\infty(G \times \mathcal{L}_r^*)$ given by

$$f(g, \nu) :=< \nu, Ad_{g^{-1}}\xi >;$$

hence the vanishing of 3.5 is equivalent to the exactness of the $\mathcal{G}^*$-valued one form γ in the statement. •

The following proposition provides a characterization of the exactness of 3.4 and therefore gives another sufficient condition for the tubewise Hamiltonian character of the action.

Proposition 3.1.10. Let (M, ω) be a symplectic manifold and let G be a Lie group with Lie algebra $\mathcal{G}$ acting properly and canonically on M. For $m \in M$, let $Y_r := G \times_{G_m} (\mathcal{M}_r^* \times V_r)$ be the slice model around the orbit $G.m$. Let Σ_C : $\mathcal{G} \times \mathcal{G} \longrightarrow \Re$ be the two cycle induced by the Chu map, that is,

$$\Sigma_C(\xi, \eta) := \omega(m)(\xi_M(m), \eta_M(m)), \qquad \xi, \eta \in \mathcal{G}, \tag{3.6}$$

and let $\Sigma_C^b : \mathcal{G} \longrightarrow \mathcal{G}^*$ be defined by $\Sigma_C^b(\xi) = \Sigma_C(\xi, .), \xi \in \mathcal{G}$. Then the form 3.4 is exact if and only if there exists a $\mathcal{G}^*$-valued group one-cocycle $\theta : G \longrightarrow \mathcal{G}^*$ such that

$$T_e\theta = \Sigma_C^b \tag{3.7}$$

In such a case the action is tubewise Hamiltonian at the point m. Also, in the presence of this cocycle, the map $J_\theta : G \times \mathcal{L}^* \longrightarrow \mathcal{G}^*$ given by

$$J_\theta(g, \nu) := Ad_{g^{-1}}^* \nu - \theta(g) \tag{3.8}$$

is a momentum map for the G-action on the presymplectic manifold $G \times \mathcal{L}^*$ with non-equivariance cocycle equal to $-\theta$.

Proof: Suppose first that the form γ in equation 3.4 is exact. In such a case, there exists a function $\theta : G \longrightarrow \mathcal{G}^*$ such that $\gamma(g) = d\theta(g)$, that is for any

$\xi, \eta \in \mathcal{G}$ and $g \in G$ we have

$$\begin{aligned} < T_g\theta(T_eL_g(\eta)), \xi > &= < \gamma(g)(T_eL_g(\eta)), \xi > \\ &= -\omega(m)((Ad_{g^{-1}}\xi)_M(m), \eta_M(m)). \end{aligned} \tag{3.9}$$

This expression determines uniquely the derivative of θ and hence choosing $\theta(e) = 0$ fixes the map $\theta : G \longrightarrow \mathcal{G}^*$. We now show that θ is a cocycle by checking that it satisfies the cocycle identity. Indeed, for any $g, h \in G$ and any $\xi, \eta \in \mathcal{G}$, we have

$$\begin{aligned} < T_g(\theta \circ L_h)(T_eL_g(\eta), \xi > &= < (T_{hg}\theta \circ T_eL_{hg})(\eta)), \xi > \\ &= < \gamma(hg)(T_eL_{hg}(\eta)), \xi > \\ &= < Ad^*_{h^{-1}}(\gamma(g)(T_eL_g(\eta))), \xi > \\ &= < T_g(Ad^*_{h^{-1}} \circ \theta)(T_eL_g(\eta)), \xi > . \end{aligned}$$

Therefore, for any $g, h \in G$ we have $T_g(\theta \circ L_h) = T_g(Ad^*_{h^{-1}} \circ \theta)$ and consequently,

$$\theta \circ L_h = Ad^*_{h^{-1}} \circ \theta + c(h, n),$$

where $c(h, n) \in \mathcal{G}^*$, for any $h \in G$, and any $n \in [1, Card(G/G^o)], n \in \aleph$. Equivalently, for any $g, h \in G$ we can write

$$\theta(hg) = Ad^*_{h^{-1}}\theta(g) + c(h, n). \tag{3.10}$$

Set in this equality $g = e$ and use $\theta(e) = 0$ to get $\theta(h) = c(h, n)$, for all $h \in G$, and $n \in [1, Card(G/G^o)], n \in \aleph$. Hence 3.10 becomes

$$\theta(hg) = Ad^*_{h^{-1}}\theta(g) + \theta(h), \tag{3.11}$$

which is the group one-cocycle identity.

Finally, note that from equation 3.9 it is easy to see that $T_e\theta = \gamma(e) = \Sigma_C^b$ and therefore θ is the one-cocycle in the statement of the Proposition. The converse is straightforward.

The fact that the expression 3.8 produces a momentum map for the G-action on $G \times \mathcal{L}^*$ follows from the equality

$$\begin{aligned} i_{\xi_{G\times\mathcal{L}^*}}\Omega(g,\nu)(T_eL_g(\eta),\sigma) &=< \sigma, Ad_{g^{-1}}\xi > + < \nu, [Ad_{g^{-1}}\xi, \eta] > \\ &\quad +\omega(m)((Ad_{g^{-1}}\xi)_M(m), \eta_M(m)) \\ &=< \sigma, Ad_{g^{-1}}\xi > + < \nu, [Ad_{g^{-1}}\xi, \eta] > \\ &\quad - < T_g\theta(T_eL_g(\eta)), \xi >, \text{ from 3.9} \\ = d < \nu, Ad_{g^{-1}}\xi > (T_eL_g(\eta),\sigma) &- d < \theta(g), \xi > (g,\nu)(T_eL_g(\eta),\sigma) \\ &= d < J_\theta, \xi > (g,\nu)(T_eL_g(\eta),\sigma). \end{aligned}$$

•

Theorem 3.1.11. Let (M,ω) be a symplectic manifold and G be a Lie group with Lie algebra $\mathcal{G}$ acting properly and canonically on it. If either
(i) $H^1(G) = 0$, or
(ii) the orbit $G.m$ is isotropic,
then the G-action on (M,ω) is tubewise Hamiltonian at m.
Proof : (i) If $H^1(G) = 0$, the map 3.5 vanishes. Hence the G-action is tubewise Hamiltonian at m.
(ii) If the orbit $G.m$ is isotropic then by Proposition 1.5.12 the group G is commutative. Therefore $\mathcal{G} = [\mathcal{G},\mathcal{G}]$. Hence the map 3.5 vanishes. Hence the G-action is tubewise Hamiltonian at m. •

Theorem 3.1.12 (The Marle-Guillemin- Sternberg normal form)**.** Let (M,ω) be a connected symplectic manifold and G be a Lie group acting properly and canonically on it. Suppose that this action has an associated momentum map J :

$M \longrightarrow \mathcal{G}^*$ with non equivariance cocycle $\sigma : G \longrightarrow \mathcal{G}^*$. Let $m \in M$, and (Y_r, ω_{Y_r}) be the symplectic tube at m that models a G-invariant open neighborhood U of the orbit $G.m$ via the G-equivariant symplectomorphism $\phi : (U, \omega_U) \longrightarrow (Y_r, \omega_{Y_r})$. Then the canonical left G-action on (Y_r, ω_{Y_r}) admits a momentum map $J_{Y_r} : Y_r \longrightarrow \mathcal{G}^*$ given by the expression

$$\begin{gathered} J_{Y_r} : Y_r = G \times_{G_m} (\mathcal{M}_r)^* \times V_r \longrightarrow \mathcal{G}^* \text{given by} \\ J_{Y_r}([g, \rho, \upsilon]) = Ad^*_{g^{-1}}(J(m) + \rho + J_V(\upsilon)) + \sigma(g). \end{gathered}$$

The map $J_{Y_r} \circ \phi$ is a momentum map for the canonical G-action on (U, ω_U). Moreover, if the group G is connected, this momentum map satisfies $J\mid_U = J_{Y_r} \circ \phi$.

Proof : Define $\theta : G \longrightarrow \mathcal{G}^*$ by $\theta(g) := -Ad^*_{g^{-1}} J(m) - \sigma(g), g \in G$. We can prove that θ satisfies the condition $T_e\theta = \Sigma^b_C$. Indeed, for any $\xi, \eta \in \mathcal{G}$ we have that

$$\begin{aligned} < T_e\theta(\xi), \eta > &= -\frac{d}{dt}\mid_{t=0} < Ad^*_{exp(-t\xi)} J(m), \eta > - < T_e\sigma(\xi), \eta > \\ &= < ad^*_\xi J(m), \eta > -\Sigma(\xi, \eta) \text{by } 1.10 \\ &= - < T_m J.\xi_M(m), \eta > \text{ by } 1.9 \\ &= -\omega(m)(\eta_M(m), \xi_M(m)) = < \Sigma^b_C(\xi), \eta > . \end{aligned}$$

Then by using the characterization in Proposition 3.1.10 the action is tubewise Hamiltonian. Therefore the canonical left G-action on (Y_r, ω_{Y_r}) have an associated standard momentum map $J_{Y_r} : Y_r \longrightarrow \mathcal{G}^*$ exists.

We now check that J_{Y_r} is well defined. For any $[g, \rho, \upsilon] \in Y_r$ and $h \in G_m$ we have

$$\begin{aligned} J_{Y_r}([gh, h^{-1}.\rho, h^{-1}.\upsilon]) &= Ad^*_{(gh)^{-1}}(J(m) + Ad^*_h\rho + J_V(h^{-1}.\upsilon)) + \sigma(gh) \\ &= Ad^*_{g^{-1}}(Ad^*_h(J(m) + Ad^*_h\rho + Ad^*_h J_V(\upsilon))) + \sigma(g) + Ad^*_{g^{-1}}\sigma(h) \\ &= Ad^*_{g^{-1}}(J(m) + \rho + J_V(\upsilon)) + \sigma(g) = J_{Y_r}([g, \rho, \upsilon]), \end{aligned}$$

where we used the fact that as $h \in G_m$, we have that $\sigma(h) = J(h.m) - Ad^*_{h^{-1}} J(m) = J(m) - Ad^*_{h^{-1}} J(m)$.

It only remains to show that J_{Y_r} is a momentum map for the left canonical G-action on (Y_r, ω_{Y_r}). Let $[g, \rho, \upsilon] \in Y_r$, $\xi, \zeta \in \mathcal{G}$, $\alpha \in \mathcal{M}^*$, and $u \in V$ arbitrary. Then on the one hand we have that

$$\begin{aligned}
< \; & dJ_{Y_r}[g, \rho, \upsilon](T_e L_g(\xi), \alpha, u), \zeta > \\
= \; & \frac{d}{dt}\,|_{t=0}\, [< Ad^*_{(gexpt\xi)^{-1}}(J(m) + \rho + t\alpha + J_V(\upsilon + tu)), \zeta > \\
& + < \sigma(gexpt\xi), \zeta >] \\
= \; & \frac{d}{dt}\,|_{t=0}\, [< Ad^*_{g^{-1}} Ad^*_{(expt\xi)^{-1}}(J(m) + \rho + t\alpha + J_V(\upsilon + tu)), \zeta > \\
& + < \sigma(g), \zeta > + < Ad^*_{g^{-1}}\sigma(expt\xi), \zeta >] \\
= \; & < J(m) + \rho + J_V(\upsilon), [Ad_{g^{-1}}\zeta, \xi] > + < \alpha + T_\upsilon J_V(u), Ad_{g^{-1}}\zeta > \\
& + \Sigma(\xi, Ad_{g^{-1}}\zeta).
\end{aligned} \tag{3.12}$$

On the other hand, notice that the infinitesimal generators associated to the G-action on Y_r take the form

$$\zeta_{Y_r}[g, \rho, \upsilon] = \frac{d}{dt}\,|_{t=0}\, [expt\zeta, \rho, \upsilon] = T_{(g,\rho,\upsilon)}\pi(T_e L_g(Ad_{g^{-1}}\zeta), 0, 0),$$

where $\pi : G \times (\mathcal{M}_r)^* \times V_r \longrightarrow G \times_{G_m} (\mathcal{M}_r)^* \times V_r$ is the projection. By the expression for the symplectic form ω_{Y_r} on Y_r, we have

$$\begin{aligned}
& \omega_{Y_r}([g, \rho, \upsilon])(\zeta_{Y_r}[g, \rho, \upsilon], (T_e L_g(\xi), \alpha, u)) \\
= & < \alpha + T_\upsilon J_V(u), Ad_{g^{-1}}\zeta > + < \rho + J_V(\upsilon), [Ad_{g^{-1}}\zeta, \xi] > \\
& + \omega(m)((Ad_{g^{-1}}\zeta)_M(m), \xi_M(m)).
\end{aligned} \tag{3.13}$$

But we have

$$\omega(m)((Ad_{g^{-1}}\zeta)_M(m), \xi_M(m)) = dJ^{Ad_{g^{-1}}\zeta}(m)(\xi_M(m))$$
$$=< -ad^*_\xi J(m) + \Sigma(\xi, .), Ad_{g^{-1}}\zeta >=< J(m), [Ad_{g^{-1}}\zeta, \xi] > +\Sigma(\xi, Ad_{g^{-1}}\zeta),$$

which substituted in the above expression 3.13 and compare to 3.12 gives

$$i_{\zeta_{Y_r}}\omega_{Y_r} = dJ^{\zeta}_{Y_r},$$

for any $\zeta \in \mathcal{G}$, that is J_{Y_r} is a momentum map for the G-action on (Y_r, ω_{Y_r}).

Finally the map $J_{Y_r} \circ \phi$ is a momentum map for the canonical G-action on $(U, \omega\mid_U)$. The map J_U also has this property and coincides with $J_{Y_r} \circ \phi$ at the point m. We now recall that two different momentum maps for the same canonical action on a connected manifold differ by a constant. Consequently, if the group G is connected, so are Y_r and U, which implies that $J_{Y_r} \circ \phi = J_U$. •

3.2 Convexity using Topology

In chapter 2 we discussed the convexity property of torus actions using Morse theory. In general Morse theory is not sufficient to study convexity properties of the image of the momentum map. The case of compact symplectic manifolds is rich but quite particular. For noncompact manifolds the results in the previous chapter no longer hold. The topological approach gives many stronger results in convexity.

In this section first the definitions of local convexity data and locally fiber connected condition are given. We give the statement of Lokal-global-prinzip and a generalization of it for a closed map using some topological vector space results.

Using this it is obtained that the convexity is rooted on the map being open onto its image and having local convexity data. Next we look at the convexity for momentum maps. Then a generalization of Atiyah-Guillemin-Sternberg Convexity theorem for non compact manifolds is given. Much more generalization of convexity results are obtained in two cases: when the momentum map has connected fibers and the case when the momentum map has only the locally fiber connectedness property. Then we give a generalization of Kirwan's convexity result. We also give non-abelian analogues of the above two cases. [4] , [19], [18], [12], [8], [19] , [36] , [9].

Definition 3.2.1. Let V be a topological vector space. A set $C \subset V$ is called a *cone with vertex* v_0 if for each $\lambda \geq 0$ and for each $v \in C, v \neq v_o$, we have $(1-\lambda)v_o + \lambda v \in C$. If the set C is, in addition, convex then C is called a *convex cone*. Not that, by definition, the vertex $v_o \in C$.

Definition 3.2.2. The continuous map $f : X \to V$ defined on a connected, locally connected Hausdorff topological space X with values in a locally convex topological vector space V is said to have *local convexity data* if for each $x \in X$ and every sufficiently small neighborhood U_x of x there exists a convex cone $C_{x,f(x),U_x}$ in V with vertex at $f(x)$ such that

$(1) f(U_x) \subset C_{x,f(x),U_x}$ is a neighborhood of the vertex $f(x)$ in the cone $C_{x,f(x),U_x}$ and

(2) $f/_{U_x} : U_x \to C_{x,f(x),U_x}$ is an open map and for any neighborhood $U'_x \subset U_x$ of x, the set $f(U'_x)$ is a neighborhood of the vertex $f(x)$ in the cone $C_{x,f(x),U_x}$, where the cone $C_{x,f(x),U_x}$ is endowed with the subspace topology induced from V. If the associated cones $C_{x,f(x),U_x}$ are such that $C_{x,f(x),U_x} \cap f(X)$ is closed in $f(X)$, then we say that f has *local convexity data with closed cones.*

Definition 3.2.3. Let X and Y be two topological space and $f : X \to Y$ a continuous map. The subset $A \subset X$ satisfies the *locally fiber connected condition*

(LFC) if A does not intersect two different connected components of the fiber $f^{-1}(f(x))$, for any $x \in A$.

Let X be a connected, locally connected, Hausdorff topological space and V a locally convex topological vector space. The continuous map $f : X \to V$ is said to be *locally fiber connected* if for each $x \in X$, any open neighborhood of x contains a neighborhood U_x of x such that U_x satisfies the (LFC) condition.

Let $f : X \to V$ be a continuous map defined on a connected Hausdorff topological space X with values in a locally convex vector space V. On the topological space X define the following equivalence relation: two points $x, y \in X$ to be equivalent if and only if $f(x) = f(y) = v$ and they belong to the same connected component of $f^{-1}(v)$. The topological quotient space will be denoted by $X_f := X/R$, the projection map by $\pi_f : X \to X_f$, and the induced map on X_f by $\widetilde{f} : X_f \to V$. Thus $\widetilde{f} \circ \pi_f = f$ uniquely characterizes $\widetilde{f}$. The map $\widetilde{f}$ is continuous and if the fibers of f are connected then it is also injective. Note that a subset $A \subset X$ satisfies *locally fiber connected condition* if and only if $\widetilde{f}\,|_{\pi_f(A)}$ is injective. Similarly f is locally fiber connected if and only if for any $x \in X$, any open neighborhood of x contains an open neighborhood U_x of x such that the restriction of $\widetilde{f}$ to $\pi_f(U_x)$ is injective.

Theorem 3.2.4. Let $f : X \to V$ be a closed map onto its image that has local convexity data. If f has connected fibers then it is open onto $f(X)$.

Proof: The hypothesis implies that the induced map $\widetilde{f} : X_f \to f(X)$ uniquely determined by the equality $\widetilde{f} \circ \pi_f = f$, is a homeomorphism. Indeed, closedness of $\widetilde{f}$ follows from the identity $\widetilde{f}(A) = f(\pi_f^{-1}(A))$ for any subset A of X_f. Since $\widetilde{f}$ is open onto $f(X)$ it follows that $f = \widetilde{f} \circ \pi_f$ is also open onto its image. •

Theorem 3.2.5 (Lokal-global-prinzip)**.** Let $f : X \to V$ be a locally fiber connected map from a connected locally connected Hausdorff topological space X

to a finite dimensional vector space V, with local convexity data $(C_x)_{x\in X}$ such that all convex cones C_x are closed in V. Suppose that f is a proper map. Then $f(X)$ is a closed locally polyhedral convex subset of V, the fibers $f^{-1}(v)$ are all connected, and $f : X \to f(X)$ is an open mapping.

In spite of its generality the theorem 3.2.5 cannot be applied to situations where the fibers $f^{-1}(v)$ are either not compact or the map f is not closed because both conditions are necessary for f to be a proper map.

Note 3.2.6. To generalize the Lokal-global-prinzip for a map that is closed and having normal topological space as domain, instead of being proper the following results are obtained first.

Lemma 3.2.7. Let $f : X \to Y$ be a continuous map between two topological spaces. Assume that f has connected fibers and is open or closed. Then for every connected subset C of Y the inverse image $f^{-1}(C)$ is connected.

Proof : Suppose that f is an open map, $C \subset Y$ is connected, and $f^{-1}(C)$ is not connected. Then there exist two open sets U_1, U_2 in X such that $f^{-1}(C) = (U_1 \cap f^{-1}(C)) \cup (U_2 \cap f^{-1}(C)), U_1 \cap f^{-1}(C) \neq \phi, U_2 \cap f^{-1}(C) \neq \phi$, and $U_1 \cap U_2 \cap f^{-1}(C) = \phi$. Note that $C = f(f^{-1}(C)) = f((U_1 \cap f^{-1}(C)) \cup (U_2 \cap f^{-1}(C))) = f(U_1 \cap f^{-1}(C)) \cup f(U_2 \cap f^{-1}(C)) \subset (f(U_1) \cap C) \cup (f(U_2) \cap C)$.

Conversely, since $f(U_1) \cap C \subset C$ and $f(U_2) \cap C \subset C$ it follows that $(f(U_1) \cap C) \cup (f(U_2) \cap C) \subset C$ which proves that $(f(U_1) \cap C) \cup (f(U_2) \cap C) = C$. Also, $f(U_1) \cap C \supset f(U_1 \cap f^{-1}(C)) \neq \phi$ and $f(U_2) \cap C \supset f(U_2 \cap f^{-1}(C)) \neq \phi$. By openness of f, the sets $f(U_1)$ and $f(U_2)$ are open in Y so that connectedness of C implies that $f(U_1) \cap f(U_2) \cap C \neq \phi$.

If $c \in f(U_1) \cap f(U_2) \cap C$ then $f^{-1}(c) = (U_1 \cap f^{-1}(c)) \cup (U_2 \cap f^{-1}(c))$. The inclusion $\supset$ is obvious. To prove the reverse inclusion $\subset$, let $x \in f^{-1}(c) \subset f^{-1}(C)$. Thus $x \in U_1 \cap f^{-1}(C)$ or $x \in U_2 \cap f^{-1}(C)$. Since $x \in f^{-1}(c)$ by hypothesis, this implies that $x \in U_1 \cap f^{-1}(C)$ or $x \in U_2 \cap f^{-1}(C)$ which proves the inclusion $\subset$. Note also that $U_1 \cap f^{-1}(C) \neq \phi$ since $c \in f(U_1)$. Similarly, $U_2 \cap f^{-1}(C) \neq \phi$. Finally, $U_1 \cap U_2 \cap f^{-1}(c) \subset U_1 \cap U_2 \cap f^{-1}(C) = \phi$. Thus the fiber $f^{-1}(c)$ can be written as the disjoint union of the two nonempty sets $U_1 \cap f^{-1}(c)$ and $U_2 \cap f^{-1}(c)$, which contradicts the connectedness hypothesis of the fibers of f.

The proof for f a closed map is identical to the one above by repeating the same argument for U_1 and U_2 closed subsets of X. •

Theorem 3.2.8. Let X be a connected Hausdorff topological space, V a locally convex topological vector space, and $f : X \to V$ a continuous map that has local convexity data. If f is open on to its image then $f(X)$ is a locally convex subset of V. Moreover, if $f(X)$ is closed in a convex subset of V then it is convex.

Proof : Let $v \in f(X)$ be arbitrary and take $x \in f^{-1}(v)$. By the condition (1) of the definition of local convexity data, there exists a neighborhood $U_x \subset X$ of x such that $f(U_x) \subset C_{x,f(x)}$ is open in $f(X)$; $C_{x,f(x)}$ is the convex cone with vertex at $v = f(x)$ given in the definition of local convexity data 3.2.2. Then shrinking U_x if necessary, using condition (2) of 3.2.2, and the local convexity of the topological vector space V, we can find a convex neighborhood V_v of v in V such that $f(U_x) = V_v \cap C_{x,f(x)}$. Since f is open onto its image, the neighborhood V_v can be shrunk further to a convex neighborhood of v, also denoted by V_v, such that $f(U_x) = V_v \cap f(X)$. Taking this as the neighborhood of v and shrinking U_x if necessary, we get $V_v \cap C_{x,f(x)} = f(U_x) = V_v \cap f(X)$. Since the intersection of two convex sets is convex, it follows that $V_v \cap C_{x,f(x)}$ is also convex. Thus the point $v \in f(X)$ has a neighborhood V_v in V such that $V_v \cap f(X)$ is convex, that is, $f(X)$ is locally convex.

If, in addition, $f(X)$ is closed, then it is also connected and locally convex. Since each closed connected and locally convex subset of a topological vector space is convex, $f(X)$ is convex. •

Corollary 3.2.9. Let $f : X \to V$ be a closed map onto its image that has local convexity data. If f has connected fibers then it is locally convex. Moreover if $f(X)$ is closed then it is also convex.

Proof : By Theorem 3.2.4 f is open onto its image. Then using the above theorem we have the result. •

Note 3.2.10. We have seen that a necessary condition for the map f that has connected fibers to be open onto its image is that the inverse image of any connected set in $f(X)$ is connected in X. The next proposition states that if f has local convexity data with closed cones, this condition is also sufficient.

Proposition 3.2.11. Let $f : X \to V$ be a continuous map that has local convexity data with closed cones. If the fibers of f are connected and for every point $v \in f(X)$ and for all small neighborhoods V_v of v the set $f^{-1}(V_v)$ is connected, then f is open onto its image.

Proof : Suppose that f is not open onto its image, has local convexity data with closed cones, and connected fibers. So there exists a point $x \in X$ and an open neighborhood U_x in the definition of local convexity data such that $V_{f(x)} \cap C_{x,f(x)} = f(U_x) \subseteq V_{f(x)} \cap f(X)$ for some open neighborhood $V_{f(x)}$ of $f(x)$ in V. Consequently, $(V_{f(x)} \cap f(X))\ f(U_x) \neq \phi$ is open in $f(X)$ since $f(U_x) = V_{f(x)} \cap C_{x,f(x)} \cap f(X)$ is closed in the topology of $V_{f(x)} \cap f(X)$ due to the fact that $C_{x,f(x)} \cap f(X)$ is closed in $f(X)$. We can also choose $V_{f(x)}$ small enough so that the hypothesis holds for it, that is, $f^{-1}(V_{f(x)} \cap f(X))$ is connected.

Note that connectedness of the fibers, and thus bijectivity of $\widetilde{f}$, implies that $\widetilde{f}^{-1}(f(A)) = \pi_f(A)$ for any subset A of X.
The sets that enter in the equality

$$\widetilde{f}^{-1}(V_{f(x)} \cap f(X)) = \widetilde{f}^{-1}((V_{f(x)} \cap f(X)) \backslash f(U_x)) \cup \widetilde{f}^{-1}(f(U_x))$$

or equivalently

$$\pi_f(f^{-1}(V_{f(x)} \cap f(X))) = \widetilde{f}^{-1}((V_{f(x)} \cap f(X)) \backslash f(U_x)) \cup \pi_f(U_x)$$

are all open because π_f is open and we also have that $\widetilde{f}^{-1}((V_{f(x)} \cap f(X)) \backslash f(U_x)) \cap \widetilde{f}^{-1}(f(U_x)) = \phi$. But this contradicts the connectivity of $\pi_f(f^{-1}(V_{f(x)} \cap f(X)))$. •

Remark 3.2.12. Let $f : X \to V$ be a continuous map such that f has local convexity data and is locally fiber connected. Then for every point $[x] \in X_f$ there is a neighborhood $\widetilde{U_{[x]}}$ of $[x]$ such that $\widetilde{f} : \widetilde{U_{[x]}} \longrightarrow C_{x,f(x)}$ is a homeomorphism onto its open image. Eventually, after shrinking $\widetilde{U_{[x]}}$, we can suppose that its image is convex. Note that X_f is connected since X is connected. Also X_f is locally path connected. Therefore X_f is path connected.

Definition 3.2.13. Let V be a Banach space. Define a distance d on X_f in the following way: for $[x], [y] \in X_f$ let $d([x], [y])$ be the infimum of all the lengths $l(f \circ \gamma)$, where γ is a curve in X_f that connects $[x]$ and $[y]$. The length is calculated with respect to the distance d_V defined by the norm on V. From the definition it follows that $d_V(\widetilde{f}([x]), \widetilde{f}([y])) \leq d([x], [y])$ and from the above note equality holds for $[x]$ and $[y]$ sufficiently close.

Theorem 3.2.14. [Generalization of the lokal-global-prinzip] Let $f : X \to V$ be a closed map with values in a finite dimensional Euclidean vector space V and X a connected, locally connected, first countable, and normal topological space. Assume that f has local convexity data and is locally fiber connected. Then
(i) All the fibers of f are connected.
(ii) f is open on to its image.
(iii) The image $f(X)$ is a closed convex set.

Proof : Let $[x]_0, [x]_1 \in X_f$ be two arbitrary points and $c := d([x]_0, [x]_1)$. By the definition of d, we have that for every $n \in \aleph$ there exist a curve γ_n defined on the interval $[a,b]$ connecting $[x]_0$ and $[x]_1$, and satisfying $l(\widetilde{f} \circ \gamma_n) \leq c + \frac{1}{n}$. Also, for every $n \in \aleph$, let $v_n = (\widetilde{f} \circ \gamma_n)(t_0)$ be the point on the curve $\widetilde{f} \circ \gamma_n$ such that $l(\widetilde{f} \circ \gamma_n \mid_{[a,t_0]}) = \frac{1}{2} l(\widetilde{f} \circ \gamma_n)$. Then there exists a finite set of points $\{[x]_1^n, ..., [x]_{k_n}^n\}$in X_f such that $\widetilde{f}^{-1}(v_n) \cap range(\gamma_n) = \{[x]_1^n, ..., [x]_{k_n}^n\} \subset B_{c+1}([x]_0)$, where $B_{c+1}([x]_0) := \{[x] \in X_f \mid d([x]_0, [x]) \leq c+1\}$ is compact. Relabelling the elements of the set $\cup_{n \in \aleph}\{[x]_1^n, ..., [x]_{k_n}^n\}$ we obtain a sequence included in the compact set $B_{c+1}([x]_0)$ and consequently, it will have an accumulation point denoted by $[x]_{\frac{1}{2}}$.

The definition of d implies that $d([x]_0, [x]_{\frac{1}{2}}) = d([x]_{\frac{1}{2}}, [x]_1) = \frac{c}{2}$. Repeating this process for the pair of points $([x]_0, [x]_{\frac{1}{2}})$ and $([x]_{\frac{1}{2}}, [x]_1)$ we obtain the points $[x]_{\frac{1}{4}}$ and $[x]_{\frac{3}{4}}$ satisfying $d([x]_0, [x]_{\frac{1}{4}}) = d([x]_{\frac{1}{4}}, [x]_{\frac{1}{2}}) = d([x]_{\frac{1}{2}}, [x]_{\frac{3}{4}}) = d([x]_{\frac{3}{4}}, [x]_1) = \frac{c}{4}$. Inductively, we obtain points $[x]_{\frac{n}{2^m}}, [x]_{\frac{n'}{2^{m'}}}$ for $0 \leq n \leq 2^m, 0 \leq n' \leq 2^{m'}$, such that

$$d([x]_{\frac{n}{2^m}}, [x]_{\frac{n'}{2^{m'}}}) = c \mid \frac{n}{2^m} - \frac{n'}{2^{m'}} \mid . \tag{3.14}$$

We can extend the map $\frac{n}{2^m} \longrightarrow [x]_{\frac{n}{2^m}}$ to a continuous map $\gamma : [0,1] \longrightarrow X_f$ such that

$$d(\gamma(t), \gamma(t')) = c \mid t - t' \mid . \tag{3.15}$$

To see this, note that every $t \in [0,1]$ can be approximated by a sequence of the type $\frac{n_k}{2^{m_k}}$. The corresponding points are contained in the compact set $B_{c+1}([x]_0)$ and hence they have an accumulation point $[x]_t$. It is now easy to see, using 3.14, that $[x]_t$ does not depend on the sequence $\frac{n_k}{2^{m_k}}$ and that the curve γ constructed in this way is continuous.

Remark 3.2.12 and equation 3.15 imply that locally $d_V((\widetilde{f} \circ \gamma)(t), (\widetilde{f} \circ \gamma)(t')) = c \mid t - t' \mid$ which shows that $\widetilde{f} \circ \gamma$ is locally a straight line. Due to equation 3.15, $\widetilde{f} \circ \gamma$ is necessarily a straight line that goes through $\widetilde{f}([x]_0)$ and $\widetilde{f}([x]_1)$. This proves the convexity of $f(X)$. Since f is a closed map the set $f(X)$ is closed in V which proves (iii).

In order to prove the connectedness of the fibers of f let $[x]_0, [x]_1 \in X_f$ be two arbitrary points such that $v := \widetilde{f}([x]_0) = \widetilde{f}([x]_1)$ and $c := d([x]_0, [x]_1)$. Any curve that connects these two points is mapped by $\widetilde{f}$ into a loop based at v. We shall prove that $c = 0$, which implies that $[x]_0 = [x]_1$and hence that the fibers of f are connected. Let γ be the curve constructed above. Then the range of $\widetilde{f} \circ \gamma$ is a segment that contains v. We will prove by contradiction that this segment consists of just one point which is v itself.

Suppose that this is not true. Since $\widetilde{f} \circ \gamma$ is a loop based at V and at the same time a straight line, there exists a turning point $v_0 := (\widetilde{f} \circ \gamma)(t_0)$ on the segment $\widetilde{f} \circ \gamma$ such that for $t \leq t_0$ we approach v_0 and for $t' \geq t_0$ we move away from v_0 staying on the same segment which is the range of $\widetilde{f} \circ \gamma$. Otherwise stated, $range(\widetilde{f} \circ \gamma \mid_{[t,t_0]}) = range(\widetilde{f} \circ \gamma \mid_{[t_0,t']})$ and hence in a neighborhood of $\gamma(t_0)$ the map $\widetilde{f}$ is not injective. However, since f is locally fiber connected the map $\widetilde{f}$ is locally injective, which is a contradiction. This proves (i). Note that from $c = d([x]_0, [x]_1)$ and $d_V(\widetilde{f}([x]_0), \widetilde{f}([x]_1)) = d_V(v, v) = 0$ we cannot conclude that $c = 0$ since the equality between the two metrics holds only locally.

The openness of f is implied by Corollary 3.2.9. •

Remark 3.2.15. The Theorem 3.2.14 remains true if we replace the vector space V by a convex subset C of V. This is a fact used in the generalization of several classical convexity theorems. We have seen that the convexity is rooted on the map being open onto its image and having local convexity data. Now we look at the convexity for momentum maps. First a generalization of Atiyah-Guillemin-Sternberg Convexity theorem for non compact manifolds is given.

For the remaining part in this section we assumed that all manifolds considered to be are Hausdorff.

Definition 3.2.16. Let V be a finite dimensional vector space.

(i) A subset $K \subset V$ is called *polyhedral* if it is the intersection of a finite family of closed half spaces of V. Consequently, a polyhedral subset of V is closed and convex.

(ii) A subset $K \subset V$ is called *locally polyhedral* if for every $x \in K$ there exists a polytope P_x in V such that $x \in int(P_x)$ and $K \cap P_x$ is a polytope.

Note 3.2.17. The original version of the Marle-Guillemin-Sternberg Normal form Theorem 3.1.12 provides the twisted product $(T_0 \times T_1) \times_{T_0} (\mathcal{T}_1^* \times V)$ as a local T-invariant local model for M. This is equivariantly diffeomorphic to $T_1 \times \mathcal{T}_1^* \times V)$ via the map

$$\begin{array}{rcl} (T_0 \times T_1) \times_{T_0} (\mathcal{T}_1^* \times V) & \longrightarrow & T_1 \times \mathcal{T}_1^* \times V \\ ((t_1, t_0), \eta, v) & \longmapsto & (t_1, \eta, t_0.v). \end{array}$$

Therefore Marle-Guillemin-Sternberg Normal form Theorem 3.1.12 to the case of torus actions can be stated as follows.

Theorem 3.2.18. Let (M, ω) be a symplectic manifold and let T be a torus acting properly on M in a globally Hamiltonian fashion with invariant momentum map $J_T : M \to \mathcal{T}^*$. Let $m \in M$ and $T_0 = (T_m)^0$ be the connected component of the stabilizer T_m. Let $T_1 \subset T$ be a subtorus such that $T = T_0 \times T_1$. Then:

(i) There exists a symplectic vector space (V, ω_V), a T-invariant open neighborhood $U \subset M$ of the orbit $T.m$, and a symplectic covering of a T-invariant open subset U' of $(T_1 \times \mathcal{T}_1^*) \times V$ onto U under which the T-action on M is modelled by

$$(T_0 \times T_1) \times (T_1 \times \mathcal{T}_1^*) \times V \to (T_1 \times \mathcal{T}_1^*) \times V$$
$$((t_0, t_1), (t_1', \beta, v)) \mapsto (t_1 t_1', \beta, \pi(t_0)v)$$

where $\pi : T_0 \to Sp(V)$ is a symplectic representation.

(ii) There exist a complex structure I on V such that $< v, \omega >:= \omega_V(I_v, \omega)$ defines a positive scalar product on V. Then $V = \oplus_{\alpha \in \mathcal{P}_V} V_\alpha$, where $V_\alpha := \{v \in V \mid Y.v = \alpha(Y)Iv$, for all $Y \in t_0\}$ and
$\mathcal{P}_V := \{\alpha \in \mathcal{T}_0^* \mid V_\alpha \neq \{0\}\}$. The corresponding T-momentum map $\Phi : (T_1 \times \mathcal{T}_1^*) \times V \to \mathcal{T}_1^* \times \mathcal{T}_0^* \simeq \mathcal{T}^*$ is given by

$$\Phi((t_1, \beta), \sum_\alpha v_\alpha) = \Phi(1, 0, 0) + (\beta, \frac{1}{2} \sum_{\alpha \in \mathcal{P}_V} \|v_\alpha\|^2 \alpha).$$

Theorem 3.2.19. Let (M, ω) be a symplectic manifold and let T be a torus acting on M in a globally Hamiltonian fashion with invariant momentum map $J_T : M \to \mathcal{T}^*$. Then there exists an arbitrarily small neighborhood U of m and a convex polyhedral cone $C_{J(m)} \subset \mathcal{T}^*$ with vertex $J_T(m)$ such that
(i) $J_T(U) \subset C_{J_T(m)}$ is an open neighborhood of $J_T(m)$ in $C_{J_T(m)}$.
(ii) $J_T : U \to C_{J_T(m)}$ is an open map.
(iii) If $\mathcal{T}_o$ is the Lie algebra of the stabilizer T_m of m, then $C_{J_T(m)} = J_T(m) + \mathcal{T}_o^\perp + cone(\mathcal{P}_V)$; where $P_V := \{\alpha \in \mathcal{T}_o^* | V_\alpha \neq \{0\}\}$.
(iv) $J_T^{-1}(J_T(m)) \cap U$ is connected for all $m \in U$.

Proof : Here is a sketch of the proof. For details see [18]. Recall that cone($\mathcal{P}_V$) $:= \{\sum_j a_j \alpha_j \mid a_j \geq 0\}$. By the above theorem it suffices to work with the momentum map J. For small neighborhoods $B_{\mathcal{T}_1^*}$ and B_V of the origin in $\mathcal{T}_1^*$ and V respectively, the restriction of J to $U := T_1 \times B_{\mathcal{T}_1^*} \times B_V$ takes values in the polyhedral closed convex cone $C_{J_T(m)} = J_T(m) + \mathcal{T}_1^* +$ cone($\mathcal{P}_V$), where *cone*$(\mathcal{P}_V)$ denotes the cone generated by the finite set $P_V := \{\alpha_1, \alpha_2, ...\alpha_n\}$ of T_o-weights for the action on V; cone$(\mathcal{P}_V)$ is clearly closed. In order to prove that J_T satisfies the conditions of the theorem, we decompose it in two maps $\varphi_1 : (t_1, \beta, \sum_\alpha v_\alpha) \mapsto (\beta, \|v_\alpha\|^2)$ and $\varphi_2 : (\beta, a_1, ...a_n) \mapsto (\beta, \frac{1}{2}\sum_j a_j\alpha_j\}$. We have $J_T = \varphi_2 \circ \varphi_1 + J_T(m)$. One proves that φ_1, φ_2 are open onto their images and have connected fibers so J_T have the same properties. •

Note 3.2.20. The above result shows that the momentum maps of globally Hamiltonian torus actions always have local convexity data with closed cones and are locally fiber connected. In fact the associated cones are closed in $\mathcal{T}^*$. Hence the following generalization of the Atiyah-Guillemin-Sternberg Convexity theorem for non compact manifolds.

Theorem 3.2.21. Let M be a paracompact connected symplectic manifold on which a torus T acts in a Hamiltonian fashion. Let $J_T : M \to \mathcal{T}^*$ be an associated momentum map which we suppose is closed. Then the image $J_T(m)$ is a closed convex locally polyhedral subset in $\mathcal{T}^*$. The fibers of J_T are connected and J_T is open on to its image.

Proof : This is a consequence of the above theorem and the generalization of the Lokal-global-prinzip. The image is locally polyhedral since J_T is open onto its image and the associated cones are polyhedral. •

This approach to convexity also generalizes a result due to Prato [9].

Theorem 3.2.22. Let M be a paracompact connected symplectic manifold on which a torus T acts in a Hamiltonian fashion. Let $J_T : M \to \mathcal{T}^*$ be an associated momentum map . If there exists $\xi \in \mathcal{T}$ such that the map $J_T^\xi \in C^\infty(M)$ defined by $J_T^\xi :=< J_T, \xi >$ is proper, then the image $J_T(m)$ is a closed convex locally polyhedral subset in $\mathcal{T}^*$. Moreover, the fibers of J_T are connected and J_T is open on to its image.

Proof : The proof is done using the generalization of the Lokal-global-prinzip 3.2.14. Then it is enough to prove J_T is a closed map.

Note that the map $J_T^\xi \in C^\infty(M)$ can be written as $J_T^\xi = b \circ \pi \circ J_T$, where $\pi : \mathcal{T}^* \to span\{\xi\}^*$ is the dual of the inclusion $span\{\xi\} \hookrightarrow \mathcal{T}$ and $b : span\{\xi\}^* \to \Re$ is the linear isomorphism obtained as the map that assigns to each element in $span\{\xi\}^*$ its coordinate in the dual basis of $\{\xi\}$ as a basis of $span\{\xi\}$. Let $\mu \in \overline{J_T(A)}$ be arbitrary and $\{\mu_n\}_{n\in N} \subset J_T(A)$ a sequence such that $\mu_n \to \mu$. Let $\{x_n\}_{n\in N} \subset A$ a sequence such that $J_T(x_n) = \mu_n$. By continuity we have $J_T^\xi(x_n) = b \circ \pi \circ J_T(x_n) \to b \circ \pi(\mu)$. Since by hypothesis J_T^ξ is a proper map there exists a convergent subsequence $x_{n_k} \to x \in \overline{A}$ and hence $J_T(x_{n_k}) \to J_T(x) = \mu$, which shows that $\mu \in J_T(\overline{A})$. Thus $\overline{J_T(A)} \subset J_T(\overline{A})$. •

Note 3.2.23. The openness of a map that has local convexity data is considered in two cases: when the map has connected fibers and the case when map has only the locally fiber connectedness property. First we give the necessary and sufficient topological conditions for J_T to be open on to its image when J_T has connected fibers.

Definition 3.2.24. Let M be a manifold and G a Lie group acting properly on it. The orbit $G.m$ is called *regular* if the dimension of nearby orbits coincides with the dimension of $G.m$. Let M^{reg} denote the union of all regular orbits. For every connected component M^o of M the subset $M^{reg} \cap M^o$ is connected, open and dense in M^o.

Proposition 3.2.25. Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action on the connected symplectic manifold (M, ω). Suppose that J_T is open on to its image. Then the complement $\mathbf{C}J_T(M^{reg}) := J_T(M) \backslash J_T(M^{reg})$ (where M^{reg} denotes the union of all regular orbits) does not disconnect any region in $J_T(M)$.

Proof : Suppose there exists a region $V \subset J_T(M)$ (relative to induced topology from $\mathcal{T}^*$) such that $V \backslash \mathbf{C}J_T(M^{reg})$ is disconnected. So

$V \backslash \mathbf{C}J_T(M^{reg}) = A \cup B$, where A and B are open in V and $A \cap B = \emptyset$. We have $J_T(M^{reg})$ is dense in $J_T(M)$, therefore

$$J_T(M^{reg}) \cup V = V \backslash \mathbf{C}J_T(M^{reg}) = A \cup B$$

is dense in $J_T(M) \cup V = V$. Hence

$$\overline{A \cup B} \cap V = V + (\overline{A} \cap V) \cup (\overline{B} \cap V).$$

Claim : There exists an element $v \in \mathbf{C}J_T(M^{reg}) \cap V$ such that any neighborhood $V_v \subset V$ of V in v is disconnected by $\mathbf{C}J_T(M^{reg}) \cap V$.

Suppose that this claim is false. Then for every $v \in \mathbf{C}J_T(M^{reg}) \cap V$ there would exist a neighborhood $V_v \subset V$ of V in v such that $V_v \backslash \mathbf{C}J_T(M^{reg})$ is connected.

Therefore we have either

$$V_v \backslash \mathbf{C}J_T(M^{reg}) \subset A \quad or \quad V_v \backslash \mathbf{C}J_T(M^{reg}) \subset B.$$

Thus, either

$$v \in \overline{A} \cap V \quad or \quad v \in \overline{B} \cap V.$$

But $v \in \mathbf{C}J_T(M^{reg}) \cap V$ is arbitrary, therefore,

$$(\overline{A} \cap V) \cap (\overline{B} \cap V) = \emptyset.$$

This contradicts the connectivity of V and hence there exists an element $v \in \mathbf{C}J_T(M^{reg}) \cap V$ such that any neighborhood $V_v \subset V$ of V in v is disconnected by $\mathbf{C}J_T(M^{reg})$.

Take an arbitrary element $x \in J_T^{-1}(v)$ and U_x a small neighborhood of x such that $J_T(U_x) \subset V$ is an open neighborhood of v in $J_T(M)$; this holds because J_T is open onto its image by hypothesis. Then by assumption $J_T(M) \cap J_T(M^{reg})$ is disconnected. Taking the T-saturation of U_x we get a T-invariant neighborhood whose image is in V since J_T is T-invariant. Thus we can assume that U_x is T-invariant and then the set of regular points for the induced T-action on U_x equals the set $U_x \cap M^{reg}$ which in turn is open, dense, and connected in U_x.

Let $E := \{z \in U_x | J_T(z) \in \mathbf{C} J_T(M^{reg})\}$.
Since we can write $U_x = E \cup D$ with $D := U_x \backslash E$, by the construction of E we have

$$J_T(E) = J_T(U_x) \cap \mathbf{C} J_T(M^{reg}) \quad and \quad J_T(D) = J_T(U_x) \cap J_T(M^{reg}).$$

Now since $E \subset U_x \backslash M^{reg}$, the inclusion $U_x \cap M^{reg} \subset D$, also holds. Because $U_x \cap M^{reg}$ is dense and connected in U_x so is D in U_x. But this is a contradiction with the fact that $J_T(D) = J_T(U_x) \cap J_T(M^{reg})$ is disconnected. This proves the result. •

Lemma 3.2.26. Let (M, ω) be a connected symplectic manifold and $J_T : M \to \mathcal{T}^*$ be the invariant momentum map associated to the canonical torus action on M. Then $J_T|_{M^{reg}} : M^{reg} \to J_T(M)$ is an open map. In particular $J_T(M^{reg})$ is an open dense subset of $J_T(M)$.

Proof : We shall prove that for each point in M^{reg} there is an open neighborhood such that the restriction of J_T to this neighborhood is an open map onto its image. Let $x_0 \in M^{reg}$ be an arbitrary point. By the openness and the T-invariance of M^{reg} we can find an open connected T-invariant neighborhood U_{x_0} of x_0 included in M^{reg}. Therefore, for $x \in U_{x_0}$, we have $dimT.x = dimT.x_0 = dimT/T_0 = dimT_1$, where $T_0 := (T_{x_0})^0$ and $T = T_0 \times T_1$. Eventually shrinking

U_{x_0}, using theorem 3.2.18, we can work with the normal form. Recall that the original action is symplectically and T-equivariantly transformed to the action

$$\begin{aligned}(T_0 \times T_1) \times (T_1 \times \mathcal{T}_1^*) \times V &\to (T_1 \times \mathcal{T}_1^*) \times V \\ ((t_0, t_1), (t_1', \beta, v)) &\mapsto (t_1 t_1', \beta, \pi(t_0)v)\end{aligned}$$

where $\pi : T_0 \to Sp(V)$ is a symplectic representation. Since the isotropy subgroup of this action at the point (t_1', β, v) equals $\{t_0 \in T_0 \mid \pi(t_0)v = v\} \times \{e\} \subset T_0 \times T_1$, the condition that it be equal to $T \times \{e\}$ implies that the representation π is trivial. Therefore all its weights are zero. By theorem 3.2.19 we conclude that $C_{J_T(x_0)} = J_T(x_0) + \mathcal{T}_1^* + cone(\mathcal{P}_V) = J_T(x_0) + \mathcal{T}_1^*$ and that $J_T : U_{x_0} \to C_{J_T(x_0)} = J_T(x_0) + \mathcal{T}_1^*$ is an open map.

Note that $J_T(M^{reg}) \subset J_T(x_0) + \mathcal{T}_1^*$ for some (and hence any) $x_0 \in M^{reg}$ and T_1 is the torus whose Lie algebra is $\mathcal{T}_0^{\perp}$, where $\mathcal{T}_0$ is the isotropy algebra of a regular point in M and the perpendicular is taken relative to an a priori chosen T-invariant inner product on $\mathcal{T}$. Indeed, using the well-chained property of M^{reg} any two points in M^{reg} can be linked by a finite chain formed by the open neighborhoods constructed above. The image of each such neighborhood lies in a translate of $\mathcal{T}_1^*$ and since the neighborhoods intersect pairwise, all these affine spaces coincide. Thus, $J_T(M^{reg})$ lies in just one translate of $\mathcal{T}_1^*$. By the density of M^{reg} in M and the closedness of the affine space in $\mathcal{T}^*$ it follows that $J_T(M)$ lies in the same affine space.

Hence, we have shown that for any $x_0 \in M^{reg}$ there exists an open neighborhood $U_{x_0} \subset M^{reg}$ such that $J_T(U_{x_0})$ is open in a given translate of $\mathcal{T}_1^*$. Therefore, $J_T(U_{x_0})$ is open in $J_T(M)$. •

Proposition 3.2.27. Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action. Assume that J_T has connected fibers and that $\mathbf{C}J_T(M^{reg}) := J_T(M)\backslash J_T(M^{reg})$

does not disconnect any region in $J_T(M)$. Then J_T is open on to its image.

Proof : If M has more than one connected component then the J_T-images of any two components do not intersect, for otherwise this would contradict the connectedness of the fibers. Since connected components of M are necessarily T-invariant, we can suppose without loss of generality that M is connected. The proof is done by using Proposition 3.2.11. In order to apply this result it is enough to prove for any $v \in J_T(M)$ and any neighborhood V_v, the pre image $J_T^{-1}(V_v)$ is connected in M.

From the above Lemma we have $J_T|_{M^{reg}} : M^{reg} \to J_T(M)$ is an open map. Let $\mathcal{T}_1^*$ be the dual of the subtorus whose translate contains $J_T(M)$.

Since M is path connected, $J_T(M)$ is also path connected and thus it is also locally connected. Let $J_T(x) \in J_T(M)$ be arbitrary. Choose a small neighborhood $V_{J_T(x)}$ of $J_T(x)$ in $\mathcal{T}_1^*$ such that $V := V_{J_T(x)} \cap J_T(M)$ is a region in $J_T(M)$. Then $V_o := V_{J_T(x)} \cap J_T(M^{reg})$ is connected due to the hypothesis that the region $V := V_{J_T(x)} \cap J_T(M)$ cannot be disconnected by removing $\mathbf{C}J_T(M^{reg})$. Note that $J_T|_{M^{reg}} : M^{reg} \to J_T(M)$ is an open map and just showed that $V_o \subset J_T(M^{reg})$ is connected. Any fiber of J_T is connected by hypothesis. Since such a fiber is T-invariant, the set of its regular points for the T-induced action is open dense and connected in it. If $v \in J_T(M^{reg})$, then $J_T^{-1}(v) \cap M^{reg}$ is connected. Thus $J_T|_{M^{reg}}$ is open and has connected fibers , therefore $J_T^{-1}(V_o) \cap M^{reg}$ is connected. Since $J_T^{-1}(V_o) \cap M^{reg}$ is dense in $J_T^{-1}(V_o)$ it follows that $U_o := J_T^{-1}(V_o)$ is connected.

Next show that U_o is dense in $U := J_T^{-1}(V)$. Indeed, if this is not true, then there exist an element $x_o \in U \backslash U_o$ and a neighborhood U_{x_o} that does not intersect U_o. For the open set $U' = U \cap U_{x_o} \cap M^{reg} \neq \emptyset$ we have that $J_T(U') \subset V_o$ is open in V_o. So there exist an element $v_o \in J_T(U')$ such that $J_T^{-1}(v_o) \cap U_{x_o} \neq \emptyset$ and $J_T^{-1}(v_o) \cap U_o \neq \emptyset$. But $J_T^{-1}(v_o) \subset U_o$ which contradicts the assumption $U_{x_o} \cap U_o = \emptyset$.

By the connectedness of U_o and the fact that it is dense in U we obtain that U is connected and hence the result follows from Proposition 3.2.11. •

Thus from Propositions 3.2.27 and 3.2.25 we have the characterization for J_T to be open onto its image in the case when J_T has connected fibers.

Theorem 3.2.28. Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action which has connected fibers. Then J_T is open on to its image if and only if $\mathbf{C}J_T(M^{reg}) := J_T(M)\backslash J_T(M^{reg})$ (where M^{reg} denotes the union of all regular orbits) does not disconnect any region in $J_T(M)$. Moreover, the image of the momentum map is locally convex and locally polyhedral.

Note 3.2.29. Next we discuss the above characterization when the map J_T has only the locally fiber connectedness property.

Definition 3.2.30. A metric space is called a *generalized continuum* if it is locally compact and connected. In a topological space a *quasi-component* of a point is the intersection of all closed and open sets that contain that point. A topological space is called *totally disconnected* if the quasi component of any point consists of the point itself. A continuous map $f : X \to Y$ is called *light* if all fibers $f^{-1}(y)$ are totally disconnected. We say that a subset of a topological space is *non-dense* if and only if it contains no open subsets.

Theorem 3.2.31 (Whyburn)**.** Let X and Y be locally connected generalized continua and let $f : X \to Y$ be an onto light mapping which is open on $X\backslash f^{-1}(F)$, where F is a closed non-dense set in Y which separates no region in Y and is such that $f^{-1}(F)$ is non-dense. Then f is open on X.

Definition 3.2.32. Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action of a connected symplectic manifold (M, ω). We say that J_T satisfies *the connected component fiber condition* if $J_T(x) = J_T(y)$ and $E_x \cap M^{reg} \neq \emptyset$, then $E_y \cap M^{reg} \neq \emptyset$, where E_x and E_y are the connected components of the fiber $J_T^{-1}(J_T(x))$ that contain x and y respectively.

Proposition 3.2.33. Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action of a connected symplectic manifold (M, ω). Suppose that M_{J_T} is a Hausdorff space, $J_T(M)$ is locally compact, $\mathbf{C}J_T(M^{reg})$ does not disconnect any region in $J_T(M)$, and J_T satisfies the connected component fiber condition. Then J_T is open on to its image.

Proof : Take $f := \widetilde{J_T} : M_{J_T} \to J_T(M) \subset \mathcal{T}^*$, that is the quotient map uniquely defined by $J_T = \pi_{J_T} \circ \widetilde{J_T}$ and $F := \mathbf{C}J_T(M^{reg})$, which is closed and non dense in $J_T(M)$ by Lemma 3.2.26.

By hypothesis M_{J_T} is a Hausdorff space. Using the fact that M is locally compact, and π_{J_T} is open we obtained that M_{J_T} is locally compact. Since M is connected, the quotient M_{J_T} is connected. Then M_{J_T} is a metric space since it is Hausdorff. Therefore, M_{J_T} is a generalized continuum. The same is true for $J_T(M)$. Both are locally connected since M is path connected.

Now we prove that $\widetilde{J_T} : M_{J_T} \to J_T(M)$ is a light map. For this take $v \in J_T(M)$. We want to show that $\widetilde{J_T}^{-1}(v)$ is totally disconnected. Let $[x] \in \widetilde{J_T}^{-1}(v)$ be arbitrary and choose $x \in M$ a representative of this class. Since J_T has local convexity data and locally fiber connected, we can find a small neighborhood U_x of x in M such that $\pi_{J_T}(U_x)$ is open in M_{J_T} and is such that $\widetilde{J_T}^{-1}(v) \cap \pi_{J_T}(U_x) = [x]$. Thus $\widetilde{J_T} : M_{J_T} \to J_T(M)$ is a light map.

Now we prove that $\widetilde{J_T}^{-1}(F)$ is non-dense in M_{J_T}. By contradiction, suppose that this is not true. Then there exists an open set $U \subset \widetilde{J_T}^{-1}(F)$. Because $\pi_{J_T}(M^{reg})$ is dense in M_{J_T} we have that $U \cap \pi_{J_T}(M^{reg}) \neq \emptyset$. Thus there exists an element $[x] \in \widetilde{J_T}^{-1}(F) \cap \pi_{J_T}(M^{reg})$ and hence there is an $x \in M^{reg}$ such that $J_T(x) = \widetilde{J_T}([x]) \in F$. This contradicts the definition of F.

Then we prove $\widetilde{J_T}$ restricted t $M_{J_T}\backslash\widetilde{J_T}^{-1}(F)$ is an open map. Note that

$$\widetilde{J_T}^{-1}(F) = \widetilde{J_T}^{-1}(\mathbf{C}J_T(M^{reg}))$$
$$= \mathbf{C}\widetilde{J_T}^{-1}(J_T(M^{reg})) \subset \mathbf{C}\pi_{J_T}(M^{reg})$$

shows that $M_{J_T}\backslash\widetilde{J_T}^{-1}(F) \supset \pi_{J_T}(M^{reg})$. Now we shall prove the reverse inclusion. Let $[x] \in M_{J_T}\backslash\widetilde{J_T}^{-1}(F)$. If the connected component E_x of the fiber $J_T^{-1}(J_T(x))$ intersects M^{reg}, then $[x] \in \pi_{J_T}(M^{reg})$. If not, then we have $J_T(x) = \widetilde{J_T}([x])$ is not element of F. that is, $J_T(x) \in \mathbf{C}F = J_T(M^{reg})$. Therefore, there is some $y \in M^{reg}$ such that $J_T(x) = J_T(y)$. By CCF condition $E_x \cap M^{reg} \neq \emptyset$, and hence $[x] \in \pi_{J_T}(M^{reg})$ which proves the equality$M_{J_T}\backslash\widetilde{J_T}^{-1}(F) = \pi_{J_T}(M^{reg})$. Thus, since $\widetilde{J_T}$ is open in $\pi_{J_T}(M^{reg})$, we have the requirements of the Whyburns Theorem. •

Theorem 3.2.34. Let $J_T : M \to \mathcal{T}^*$ be the momentum map of a torus action of a connected symplectic manifold (M, ω). Suppose that M_{J_T} is a Hausdorff space. Then J_T is open on to its image if and only if $J_T(M)$ is locally compact, $\mathbf{C}J_T(M^{reg})$ does not disconnect any region in $J_T(M)$, and J_T satisfies the connected component fiber condition . Moreover under these hypothesis the image of the momentum map is locally convex and locally polyhedral.

Proof : It is enough to prove that if $J_T : M \to J_T(M)$ is open onto its image then CCF condition holds. Suppose that the condition does not hold. So there exist a fiber with at least two connected components E_x and E_y such that $E_x \cap M^{reg} \neq \emptyset$ and $E_y \cap M^{reg} = \emptyset$.

Consequently we can suppose that $x \in M^{reg}$ and y is contained in a stratum of the T-action. Then we have the strict inclusion $C_{J_T(y)} = v + \mathcal{T}_0^{\perp} + cone(\mathcal{P}_V) \subset C_{J_T(x)} = v + \mathcal{T}_1^{\perp}$, where $v = J_T(x) = J_T(y)$. By condition (1) in the definition of the local convexity data there exists open neighborhoods U_x and U_y of x and y respectively, such that $J_T(U_x)$ is an open ball in $\mathcal{T}_1^*$ centered at v and $J_T(U_y)$ is the intersection of an open ball in $\mathcal{T}_1^*$ centered at v with a closed proper cone in $\mathcal{T}_1^*$ with vertex v. This contradicts the openness onto its image of J_T. •

Note 3.2.35. In the case of a compact, connected, and non-abelian group, the momentum map $J_G : M \to \mathcal{G}^*$ is, in general, not open onto its image even if it is a proper map. Nevertheless, it can be shown that results obtained in the abelian case hold for the quotient map $j_G : M \to \mathcal{G}^*/G \simeq \mathcal{T}_+^*$, where $j_G = \pi_G \circ J_G$ and $\pi_G : \mathcal{G}^* \to \mathcal{T}_+^*$ is the projection map which is always proper if G is compact. The quotient map j_G has local convexity data due to the following result of Sjamaar [36].

Theorem 3.2.36. Let M be a connected Hamiltonian G - manifold. Then for every $x \in M$ there exist a unique, closed, polyhedral convex cone C_x in $\mathcal{T}_+^*$ with vertex at $j_G(x)$ such that for every sufficiently small G-invariant neighborhood U of x the set $j_G(U)$ is an open neighborhood of $j_G(x)$ in C_x.

Using Lerman's symplectic cut technique [8], Knop [12] proved that the map j_G is locally fiber connected. So we have the following theorem.

Theorem 3.2.37. The map j_G is locally fiber connected and has local convexity data.

Using the above theorem and the generalization of lokal-global-prinzip we obtain the following generalization of Kirwan's convexity result.

Definition 3.2.38. The map $\widetilde{j_G} : M/G \to \mathcal{T}_+^*$ defined by the identity $j_G = \pi \circ \widetilde{j_G}$, where $\pi : M \to M/G$ is the projection. Also a G-equivariant momentum map $J_G : M \to \mathcal{G}^*$ is said to be *G-open* on to its image whenever $\widetilde{j_G}$ is open on to its image.

Theorem 3.2.39. Let M be a paracompact connected Hamiltonian G-manifold with G a compact connected Lie group. If the momentum map J_G is closed then

$J_G(M) \cap \mathcal{T}_+^*$ is a closed convex locally polyhedral set. Moreover, J_G is G-open on to its image and all its fibers are connected.

Proof : We have j_G is open on to its image and consequently that J_G is G-open on to its image. Additionally, the set $j_G(M) = J_G(M) \cap \mathcal{T}_+^*$ is a closed convex locally polyhedral set and j_G has connected fibers.

It remains to be proved that J_G has connected fibers. To see this, note that since j_G has connected fibers, the pre-images $J_G^{-1}(\mathcal{O}_\mu)$ are connected as topological subspaces of M for every coadjoint orbit $\mathcal{O}_\mu \subset J_G(M)$. Note now that $J_G^{-1}(\mathcal{O}_\mu)$ can also be endowed with the initial topology induced by the map

$$J_G^{\mathcal{O}_\mu} : J_G^{-1}(\mathcal{O}_\mu) \longrightarrow \mathcal{O}_\mu \quad by, \quad J_G^{\mathcal{O}_\mu}(z) = J_G(z),$$

where the orbit $\mathcal{O}_\mu$ comes with its orbit smooth structure induced by the homogeneous manifold G/G_μ. Since G is compact, the orbit $\mathcal{O}_\mu$ is an embedded submanifold of $\mathcal{G}^*$ and hence the initial topology for $J_G(\mathcal{O}_\mu)$ is weaker than the subspace topology. Indeed, the sets of the form

$$(J_G^{\mathcal{O}_\mu})^{-1}(U \cap \mathcal{O}_\mu) = J_G^{-1}(U) \cap J_G^{-1}(\mathcal{O}_\mu),$$

with U open in $\mathcal{G}^*$, form a sub basis of the initial topology of $J_G^{\mathcal{O}_\mu}$ and since they are open in M by the continuity of J_G, the claim follows. Therefore, $J_G^{-1}(\mathcal{O}_\mu$ is also connected for the initial topology. Also we have if $J_G^{-1}(\mathcal{O}_\mu$ is endowed with its initial topology then the map

$$f : G \times_{G_\mu} J_G^{-1}(\mu) \longrightarrow J_G^{-1}(\mathcal{O}_\mu) \quad by, \quad f([g, z]) = g.z,$$

is a homeomorphism, where $G \times_{G_\mu} J_G^{-1}(\mu)$ denotes the orbit space of the free and continuous action $h.(g, z) := (gh, h^{-1}.z), h \in G_\mu, g \in G, z \in J_G^{-1}(\mu)$, of the compact connected group G_μ on the product $G \times J_G^{-1}(\mu)$. The set $J_G^{-1}(\mu)$ is considered with its subspace topology. Let $\pi_\mu : G \times J_G^{-1}(\mu) \longrightarrow G \times_{G_\mu} J_G^{-1}(\mu)$ be the continuous and open projection. Since the fibers of π_μ are connected and π_μ is open it follows that the pre-image of any connected set is connected. Therefore, $G \times J_G^{-1}(\mu)$ is connected and hence so is $J_G^{-1}(\mu)$ since G is connected. •

Note 3.2.40. next we give non-abelian analogues of the Theorems 3.2.28 and 3.2.34.

Theorem 3.2.41. Let G be a compact connected Lie group and M be a connected Hamiltonian G-Manifold with equivariant momentum map $J_G : M \to \mathcal{G}^*$. Suppose that J_G has connected fibers. Then J_G is G-open on to its image if and only if $\mathbf{C}((\pi_G \circ J_G)(M^{reg}))$ does not disconnect any region in $J_G(M) \cap \mathcal{T}_+^*$. Moreover, in this context, the image $J_G(M) \cap \mathcal{T}_+^*$ is a locally convex and locally polyhedral set.

Proof : Suppose that J_G is G-open on to its image. That is, the map $\widetilde{j_G} : M/G \to \mathcal{T}_+^*$ defined by the identity $j_G = \widetilde{j_G} \circ \pi$ is open on to its image. Therefore $j_G : M \to \mathcal{T}_+^*$ is open on to its image. Then proceed as in the proof of Proposition 3.2.25 by replacing j_G instead of $J_T : M \to \mathcal{T}^*$ we get $\mathbf{C}(j_G(M^{reg}))$ does not disconnect any region in $j_G(M)$. But we have $j_G = \pi_G \circ J_G$. Hence $\mathbf{C}((\pi_G \circ J_G)(M^{reg}))$ does not disconnect any region in $j_G(M) = J_G(M) \cap \mathcal{T}_+^*$.

Conversely, assume that J_G has connected fibers and $\mathbf{C}((\pi_G \circ J_G)(M^{reg}))$ does not disconnect any region in $J_G(M) \cap \mathcal{T}_+^*$. That is, $\mathbf{C}(j_G(M^{reg}))$ does not disconnect any region in $j_G(M)$. Therefore by Proposition 3.2.27 j_G is open on to its image. Hence $\widetilde{j_G}$ is open on to its image. Then by definition J_G is G-open on to its image.

From Theorem 3.2.39 the image $J_G(M) \cap \mathcal{T}_+^*$ is a locally convex and locally polyhedral set. •

Theorem 3.2.42. Let G be a compact connected Lie group and M be a connected Hamiltonian G-Manifold with the momentum map $J_G : M \to \mathcal{G}^*$. Suppose that $(M/G)_{\widetilde{j_G}}$ is a Hausdorff space. Then J_G is G-open on to its image if and only if $J_G(M)$ is locally compact, $\mathbf{C}((\pi_G \circ J_G)(M^{reg}))$ does not disconnect any region in $J_G(M) \cap \mathcal{T}_+^*$ and satisfies the connect component fiber condition. Moreover, in this context, the image $J_G(M) \cap \mathcal{T}_+^*$ is a locally convex and locally polyhedral set.

Proof : Suppose that $(M/G)_{\widetilde{j_G}}$ is a Hausdorff space and $J_G(M)$ is locally compact, $\mathbf{C}((\pi_G \circ J_G)(M^{reg}))$ does not disconnect any region in $J_G(M) \cap \mathcal{T}_+^*$ and satisfies the connect component fiber condition. Then proceed as in the proof of Proposition 3.3.17 by replacing j_G instead of J_T we get j_G is open on to its image. Hence J_G is G-open on to its image.

Conversely, suppose that J_G is G-open on to its image. So $\widetilde{j_G}$, and hence $j_G : M \to \mathcal{T}_+^*$ is open on to its image. Then applying Theorem 3.2.34 to j_G we have the result. •

3.3 Division Property

In this section we look at the division property of the momentum maps. Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a Hamiltonian action of a compact Lie group G on a connected symplectic manifold (M, ω). If this map J is proper, Y.Karshon and E.Lerman proved that every formal pull back with respect to J is a collective function. We can generalize this theorem by replacing the compactness condition on the Lie group by proper and effective action. We prove that Torus action has division property if J_T is closed and semi-proper. Also we proved that for a paracompact connected symplectic manifold with G a compact connected Lie group. If the associated momentum map J is closed and semi proper as a map into some open subset of $\mathcal{G}^*$, then J has the division property if the image $J(M)$ is contained the $\mathcal{G}^*_{reg}$, where denote $\mathcal{G}^*_{reg}$ the elements of $\mathcal{G}^*$ whose stabilizers under the coadjoint action of G are tori. [11] , [41] , [7], [6] , [13].

Definition 3.3.1. Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a Hamiltonian action of a connected Lie group G on a symplectic manifold (M, ω). Pull backs by J of smooth functions on $\mathcal{G}^*$ are called *collective functions*. They form Poisson subalgebra of the algebra of smooth functions on M.

Note 3.3.2. A collective function is clearly constant on the level sets of the momentum map. The converse need not be true. For example, the standard linear action of the group $G = SU(2)$ on $\mathbf{C}^2$ has a momentum map $J(u, v) = (\overline{u}v, \frac{1}{2}(\mid u \mid^2 - \mid v \mid^2))$ when we identify the vector space $\mathcal{G}^*$ with $\Re \times \mathbf{C}$. The function $f(u, v) = \mid u \mid^2 + \mid v \mid^2$ is constant on the level sets of J because it is equal to $(\mid \overline{u}v \mid^2 + (\frac{1}{2}(\mid u \mid^2 - \mid v \mid^2))^2)^{\frac{1}{2}} = \frac{1}{2} \parallel J \parallel$. It is not collective because the function $\parallel x \parallel$ is not smooth on $\Re \times \mathbf{C}$.

As a corollary of Marle-Guillemin- Sternberg local normal form Theorem 3.1.12 we have

Theorem 3.3.3 (Local normal form near an isotropic orbit)**.** Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a proper Hamiltonian action of a Lie group G on a symplectic manifold (M, ω). Suppose that the orbit $G.m$ is isotropic in M. Let G_m denote the stabilizer of m in G, let $\mathcal{G}^0_m$ denote the annihilator of its Lie algebra in $\mathcal{G}^*$, and let $G_m \longrightarrow Sp(V)$ denote the symplectic slice representation.

Given a G_m-equivariant embedding, $i : \mathcal{G}^*_m \longrightarrow \mathcal{G}^*$, there exist a G-invariant closed two form, ω_Y, on the manifold $Y = G \times_{G_m} (\mathcal{G}^0_m \times V)$, such that
1. the form ω_Y is nondegenerate near the zero section of the bundle $Y \longrightarrow G/G_m$,
2. a neighborhood U_m of the orbit of m in M is equivariantly symplectomorphic

to a neighborhood of the zero section in Y, and

3. the action of G on (Y, ω_Y) is Hamiltonian with a momentum map $J_Y : Y \longrightarrow \mathcal{G}^*$ given by $J_Y([g, \eta, v]) = Ad^*(g)(\eta + i(J_V(v)))$ where Ad^* is the coadjoint action, and $J_V : V \longrightarrow \mathcal{G}_m^*$ is the momentum map for the slice representation of G_m.

Consequently the equivariant embedding $i : U_m \longrightarrow Y$ intertwines the two momentum maps, up to translation : $J\mid_{U_m} = J_Y \circ i + J(m)$.

Remark 3.3.4. Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a proper Hamiltonian action of a Lie group, G on a symplectic manifold (M, ω). Then any two points in a connected component of a level set of J can be joined by a piece-wise smooth curve that lies in the level set.

Note 3.3.5. A non-trivial consequence of the local normal form theorem is the image under the momentum map of a small invariant neighborhood of an orbit $G.m$ does not change as m varies along a connected component of the level set $J^{-1}(J(m))$.

Theorem 3.3.6. Let (M, ω) be a symplectic manifold and G be a Lie group acting properly and canonically on it. Suppose that this action is Hamiltonian with an associated momentum map $J : M \longrightarrow \mathcal{G}^*$. Also suppose that J be equivariant with respect to the given action of G on M and the coadjoint action on $\mathcal{G}^*$. Then the centralizer of the algebra of G-invariant functions in the Poisson algebra on M is the set of smooth functions that are locally constant on the level sets of the momentum map.

Proof: The Hamiltonian flow of an invariant function preserves the level sets of the momentum map, the Poisson bracket of an invariant function and a function that is locally constant on the level sets of the momentum map is zero. This shows the centralizer of the invariant functions contains the functions that are locally constant on the level sets of the momentum map.

Let h be a function in the centralizer of the invariant functions. Let $\gamma(t)$ be any smooth curve contained in the level set of the momentum map J. Since any two points in a connected component of a level set of J can be connected by a piece wise smooth curve, we are done if we can prove that the derivative of $h(\gamma(t))$ is zero for all t. This derivative is equal to $\omega(\dot{\gamma}, X_h)$ where X_h is the Hamiltonian vector field of h.

For any vector ξ in the Lie algebra $\mathcal{G}$ we have $0 =< \dot{\gamma}, dJ^{\xi} >= \omega(\dot{\gamma}, \xi_M)$. Hence if $\gamma(t)$ is a smooth curve contained in a level set of the momentum map, the tangent vectors $\dot{\gamma}$ lie in the symplectic perpendiculars to the G-orbits.

Now it suffices to show that the Hamiltonian vector field, X_h, of a function h in the centralizer of the invariant functions is tangent to the G-orbits. Let $\sigma(t)$ be an integral curve of the vector field X_h. Then for any G-invariant function, f, we have $\frac{d}{dt}(f(\sigma(t))) = (X_h f)(\sigma(t)) = 0$ that is, f is constant along $\sigma(t)$. Here the G-invariant functions separate orbits, the integral curve $\sigma(t)$ is contained in a single G-orbit. Hence the vector field X_h is tangent to G-orbits. •

Corollary 3.3.7. Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a proper Hamiltonian action of a connected Lie group G on a symplectic manifold (M, ω). The algebra of collective functions and the algebra of invariant functions are mutual centralizers in the Poisson algebra $C^\infty(M)$ if and only if every smooth function on M that is locally constant on the level sets of the momentum map is collective.

Definition 3.3.8. A smooth map $\psi : M \longrightarrow N$ between two smooth manifolds has the *division property* if any smooth function on M that is locally constant on the level sets of ψ is the pull back via ψ of a smooth function on N.

The Corollary 3.3.7 can be restated as follows:

Corollary 3.3.9. Let $J : M \longrightarrow \mathcal{G}^*$ be a momentum map associated to a proper Hamiltonian action of a connected Lie group G on a symplectic manifold (M, ω). The algebra of collective functions and the algebra of invariant functions are mutual centralizers in the Poisson algebra $C^\infty(M)$ if and only if the momentum map J has division property.

Definition 3.3.10. Let $\psi : M \longrightarrow N$ be a smooth map between two smooth manifolds. A smooth function f on M is a *formal pullback* with respect to ψ if for every point y in the image $\psi(M)$ there exists a function, φ on N such that $f - \psi^*\varphi$ is flat at all the points of $\psi^{-1}(y)$.

Since the pull back of functions induces a well defined pull back of Taylor's series, being a formal pull back with respect to a smooth function $\psi : M \longrightarrow N$ if and only if for every $y \in N$ there exists a power series φ on N, centered at y, such that for all x in the level set $\psi^{-1}(y)$, the power series of f at x is the pull back of the power series φ.

Remark 3.3.11. Every formal pull back with respect to ψ is constant on the level sets of ψ; if $f - \psi^*\varphi$ is flat, $f(x) = \varphi(y)$ for all $x \in \psi^{-1}(y)$.

Definition 3.3.12. Let $\psi : A \longrightarrow B$ is *semi-proper* if for every compact set $L \subset B$ there is a compact set $K \subset A$ such that $\psi(K) = L \cap \psi(A)$.

Theorem 3.3.13. Let G be a connected abelian Lie group acting properly and effectively on a connected symplectic manifold (M, ω). Let $J : M \longrightarrow \mathcal{G}^*$ be a proper momentum map associated to this action. Then J has the division property if and only if every smooth function on M that is locally constant on the level sets of J is a formal pull back with respect to J.

Proof. Let m be a point in M, and let G_α be the stabilizer of its image , $\alpha = J(m) \in \mathcal{G}^*$ under the coadjoint action. Since the action is effective and G is a connected abelian Lie group, the G-orbits are isotropic by Proposition 1.5.12. So α is fixed under the coadjoint action of G, for every $\alpha \in \mathcal{G}^*$. Since α is fixed, the translation $J - \alpha$ of the momentum map by $-\alpha$ is still a momentum map. So, without loss of generality we can assume that $\alpha = 0$.

For the proper action, by Theorem 3.3.3 we have a neighborhood of an isotropic orbit $G.m$,

$$Y = G \times_{G_m} (\mathcal{G}_m^o \oplus V),$$

where G_m is the stabilizer of m , $\mathcal{G}_m$ is its Lie algebra, $\mathcal{G}_m^0$ is the annihilator of $\mathcal{G}_m$ in $\mathcal{G}^*$, and V is the symplectic slice at m. The action of G on Y is Hamiltonian with a momentum map $J_Y : Y \longrightarrow \mathcal{G}^*$ given by

$$J_Y([g, \eta, \upsilon]) = Ad^*(g)(\eta + i(J_V(\upsilon))),$$

where Ad^* is the coadjoint action, and $J_V : V \longrightarrow \mathcal{G}_m^*$ is the quadratic momentum map for the slice representation of G_m and i is a G_m-equivariant embedding of $\mathcal{G}_m^*$ in $\mathcal{G}^*$. Moreover there exists a neighborhood U_m of the orbit $G.m$ in M and an equivariant embedding $i : U_m \longrightarrow Y$, of U_m onto a neighborhood of the zero section of the bundle $Y \longrightarrow G/G_m$, such that $J = J_Y \circ i$.

As a consequence of the normal form the image under the momentum map of a small neighborhood of an orbit $G.m$ does not change as m varies along a connected component of the level set $J^{-1}(J(m))$. Also this image is the intersection of the cone $J_Y(Y)$ with a neighborhood of the origin in $\mathcal{G}^*$.

Note that the hypothesis that the momentum map is proper can be replaced by the hypothesis that it is semi-proper as a map into some open subset of $\mathcal{G}^*$ and that its level sets are connected. So we can choose a neighborhood W_m of the origin in $\mathcal{G}^*$ and shrink the neighborhood U_m of $G.m$ so that $J(U_m) = J(M) \cap W_m = J_Y(Y) \cap W_m$.

The map J_Y is analytic with respect to the natural real analytic structures of the model Y and of the vector space $\mathcal{G}^*$. If we endow U_m with the real analytic structure induced by its embedding , i, into Y, then the restriction $J|_{U_m}: U_m \longrightarrow W_m$ is a real analytic map.

Consider the action of $\Re_+$ on Y given by $\lambda.[g, \eta, \upsilon] = [g, \lambda\eta, \sqrt{\lambda}\upsilon]$. The map $J_Y : Y \longrightarrow \mathcal{G}^*$ is homogeneous of degree one with respect to the action of $\Re_+$. After possibly shrinking U_m and W_m further, we can assume that the open set $i(U_m) \subseteq Y$ is preserved under multiplication by any $\lambda < 1$; for such λ we define $\lambda : U_m \longrightarrow U_m$ by $i(\lambda.m) = \lambda.i(m)$. Let K be a compact subset of the open set W_m. Then there exist a positive number $\lambda < 1$ such that K is contained in λW_m. By homogeneity $K \cap J(U_m) \subset J(\lambda.U_m)$. Then $L := closure(\lambda.U_m) \cap J^{-1}K$ is a compact subset of U_m whose image is $K \cap J(U_m)$. Thus the restriction $J|_{U_m}: U_m \longrightarrow W_m$ is semi-proper.

Since the map J_V is algebraic, its image $J_V(V)$ is a semialgebraic subset of $\mathcal{G}^*_m$. Furthermore, since $Ad^*(G) \subseteq GL(\mathcal{G}^*)$ is algebraic, the set $J_Y(Y) = Ad^*(G)(\mathcal{G}^0_m \times J_V(V))$ a semialgebraic subset of $\mathcal{G}^*$. Restricting to the open subset W_m, we see that $J(U_m) = J_Y(Y) \cap W_m$ is a semi-analytic subset of W_m.

Thus there exist a neighborhood U_m of the orbit $G.m$ in M and a neighborhood W_m of the point $J(m)$ in $\mathcal{G}^*$ with the following properties:
(1). $J(U_m) = J(M) \cap W_m$.
(2). The restriction $J|_{U_m}: U_m \longrightarrow W_m$ is semi-proper.
(3). There exist real analytic structures on U_m and on W_m, compatible with their smooth structures, such that the restriction $J|_{U_m}: U_m \longrightarrow W_m$ is a real analytic map and the image $J(U_m)$ is a semi analytic subset of W_m.

Moreover the neighborhoods U_m and W_m can be chosen to be arbitrarily small, that is, can be chosen to be contained in any given neighborhoods U' of $G.m$ and W' of $J(m)$.

Let N be an open subset of $\mathcal{G}^*$ containing the moment image $J(M)$ with the property that the momentum map $J : M \longrightarrow N$ is semi-proper. Also the image of any semi-proper map is closed. Therefore $J(M)$ is closed subset of N.

Then we can prove that the set of pull backs by the map J coincides with the set of formal pull backs with respect to J.

Clearly every pull back is a formal pull back. Conversely, let $f \in C^{\infty}(M)$ be a formal pull back with respect to J. Let m be a point in M, and let U_m and W_m be as obtained above. Since f is a formal pull back with respect to J, its restriction $f\mid_{U_m}$ is a formal pull back with respect to the map $J\mid_{U_m}: U_m \longrightarrow W_m$.

If M and N be real analytic manifolds. Let $\psi : M \longrightarrow N$ be a real analytic map that is semi-proper and whose image $\psi(M)$ is semi-analytic. Then a function f is a formal pull back with respect to ψ if and only if it is the pull back via ψ of a smooth function on N [5]. So we can apply this to the map $J\mid_{U_m}$ because of conditions (2) and (3). Hence there exists a smooth function φ_m on W_m such that $f = \varphi_m \circ J$ on U_m. This equality holds on all $J^{-1}(J(U_m))$ because f, being formal pull back with respect to J, is constant on the level sets of J.

Condition (1) implies that $J^{-1}(J(U_m)) = J^{-1}(W_m)$ so, $f = \varphi_m \circ J$ on all of $J^{-1}(W_m)$. The open sets W_m together with the complement of the image $J(M)$ form an open cover of N. Using a partition of unity subordinate to this cover we piece together the functions φ_m to form a function φ on N such that $f = \varphi \circ J$.

Then taking $N = \mathcal{G}^*$, we have the theorem. •

Note 3.3.14. Next we consider the torus actions. Let $J_T : M \longrightarrow \mathcal{T}^*$ be a momentum map associated to a Hamiltonian action of a torus T on a connected symplectic manifold (M, ω).

Theorem 3.3.15. Let M be a compact connected symplectic manifold on which a torus T acts in a Hamiltonian fashion. If the associated momentum map J_T :

$M \rightarrow \mathcal{T}^*$ is proper, then it has the division property.

Proof : Given that the momentum map $J_T : M \rightarrow \mathcal{T}^*$ is proper, therefore it is semi-proper as a map into some open subset of $\mathcal{T}^*$ and that its level sets are connected. If the level sets of J_T are connected, then centralizer of invariant functions is locally collective in a neighborhood of an orbit [6].

We know that the image of $Y = G \times_{G_m} (\mathcal{M}^* \times V)$ under the momentum map J_Y is a linear cone. So under the momentum map $J : M \longrightarrow \mathcal{G}^*$ the image of a sufficiently small neighborhood of the orbit $G.m$ is of the form $(\alpha + C) \cap W$ where W is an open set in $\mathcal{G}^*$ about $\alpha = J(m)$ Also the image of a small neighborhood of a point in $J^{-1}(\alpha)$ is an open subset of some cone $C^{'}$ translated to α.

Claim : There exist a neighborhood U of the level set $J^{-1}(\alpha)$ such that under J it is of the form $(\alpha + C) \cap W'$where W' is an open set in $\mathcal{G}^*$ about α. That is, the cone does not vary along the level set.

This follows from two observations, first of all $J^{-1}(\alpha)$ is connected, so it is enough to show that the cone does not vary locally. But the local behavior is modelled by (Y, J_Y), and along the zero level set of J_Y the cone does not vary. Therefore for any q in $J^{-1}(\alpha)$ there exists open sets U_q in M containing q and W_q in $\mathcal{G}^*$ containing α so that $J(U_q) = (\alpha + C) \cap W_q$. Since $J^{-1}(\alpha)$ is compact, there exists $q_1, ...q_L$ in $J^{-1}(\alpha)$ such that the corresponding sets $U_1, ...U_L$ cover $J^{-1}(\alpha)$. The claim follows with $W' = W_1 \cap W_2 \cap .... \cap W_L$.

Thus $J\,|_U$ is an open map into the translated cone $\alpha + C$. So for any open set $U_0 \subset U$ there exists an open set $W_0' \in \mathcal{G}^*$ such that $J(U_0) = (\alpha + C) \cap W_0'$.

Next we show that the centralizer of invariants is collective not just in a neighborhood of an orbit but in a neighborhood of the whole level set.

Take any $f \in (C^\infty(M)^G)^c$, that is, f commutes with all G-invariant functions on M. Choose an open set U in M about $J^{-1}(\alpha)$ as in the claim. Then $J(U) = (\alpha + C) \cap W'$ for some open set $W' \subset \mathcal{G}^*$. Since centralizer of invariants is locally collective, for any q in $J^{-1}(\alpha)$ there exist an open set $U(q)$ and a function $\phi_q \in C^\infty(\mathcal{G}^*)$ so that $f \mid_{U(q)} = \phi_q \circ J \mid_{U(q)}$. We may assume that $U(q) \subset U$ for all q. Since $J^{-1}(\alpha)$ is compact, we can cover $J^{-1}(\alpha)$ by finitely many $U(q)$, say $U_1 = U(q_1), ...U_L = U(q_L)$. Let $\phi_1, ..., \phi_L$ be the corresponding functions in $C^\infty(\mathcal{G}^*)$. Since $J^{-1}(\alpha)$ is connected, we may assume that $U_i \cap U_{i+1}$ is non empty for $i = 1, ...L$.

We proceed on induction on L. Suppose for simplicity that $L = 2$, so $J^{-1}(\alpha) \subset U_1 \cap U_2$. Now $f \mid_{U_i} = \phi_i \circ J \mid_{U_i}$ for $i = 1, 2$ implies that $\phi_1 \mid_{(U_1 \cap U_2)} = \phi_2 \mid_{(U_1 \cap U_2)}$. By 3.3.15 there exists an open set W'' in $\mathcal{G}^*$ so that $J(U_1 \cap U_2) = (\alpha + C) \cap W''$. Let $\phi = \phi_1$, and $U_\alpha = J^{-1}(W'') \cap (U_1 \cap U_2)$. Then $f \mid_{U_\alpha} = \phi \circ J \mid_{U_\alpha}$.

Thus we have for any point $\alpha \in J^{-1}(\alpha)$ there exists an open set U_α in M, $J^{-1}(\alpha) \subset U_\alpha$ an open set $W(\alpha) \in \mathcal{G}^*$, a cone C_α and a function $\phi_\alpha \in C^\infty(\mathcal{G}^*)$ so that

$$\begin{aligned} &(i) \quad J(U_\alpha) = (\alpha + C_\alpha) \cap W(\alpha) \text{ and} \\ &(ii) \quad f \mid_{U_\alpha} = \phi_\alpha \circ J \mid_{U_\alpha}. \end{aligned}$$

$J(M)$ is compact, hence there exist $\alpha_1, ...\alpha_s$ in $\mathcal{G}^*$ such that $W_1 = W(\alpha_1), ...W_s = W(\alpha_s)$ cover $J(M)$. Let $\phi_1, ..., \phi_s$ be the corresponding functions in $\mathcal{G}^*$. Let W_0 be the complement of $J(M)$ in $\mathcal{G}^*$, that is, let $W_0 = \mathcal{G}^* \backslash J(M)$. Choose a partition of unity $\{\rho_0, ...\rho_s\}$ on $\mathcal{G}^*$ subordinate to $\{W_0, ...W_s\}$, then $\{J^*\rho_1, ...J^*\rho_s\}$ is a partition of unity on M subordinate to $\{U_1, ...U_s\}$. Let $\phi = \Sigma\rho_i\phi_i$. Then $J^*\phi = \Sigma J^*\rho_i J^*\phi_i = \Sigma(J^*\rho_i)f = f$. Hence the Theorem. •

Remark 3.3.16. We give a generalization of theorem 3.3.15 for paracompact manifolds.

Theorem 3.3.17. Let M be a paracompact connected symplectic manifold on which a torus $\mathcal{T}$ acts in a Hamiltonian fashion. If the associated momentum map $J_{\mathcal{T}}$ is closed and semi proper as a map into some open subset of $\mathcal{T}^*$, then J has the division property.

Proof : Given that M be a paracompact connected symplectic manifold on which a torus $\mathcal{T}$ acts in a Hamiltonian fashion with the associated momentum map $J_{\mathcal{T}}$ is closed. Then by the Theorem 3.2.21 the level sets of J_T are connected. Also J_T is semi-proper as a map into some open subset of $\mathcal{T}^*$. Thus J_T is a proper momentum map. Then using the above theorem J_T has the division property. •

Note 3.3.18. Let G be a compact connected Lie group acting on a compact connected symplectic manifold M in a Hamiltonian fashion with a momentum map $J : M \longrightarrow \mathcal{G}^*$. Put a G-invariant metric on $\mathcal{G}^*$, and use it to identify $\mathcal{G}^*$ with $\mathcal{G}$. Let $\mathcal{G}_{reg}$ be the elements of $\mathcal{G}$ whose stabilizers under the coadjoint action of G are tori, that is, if,

$$\mathcal{G}_{reg} = \{\xi \in \mathcal{G} : \text{stabilizer of } \xi \text{ is a torus } \}.$$

Theorem 3.3.19. Let $J : M \longrightarrow \mathcal{G}^*$ be a proper momentum map associated to a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω). Suppose the image $J(M)$ is contained the $\mathcal{G}_{reg}$. Then J has the division property.

Proof : Given that the momentum map $J : M \to \mathcal{G}^*$ is proper, therefore it is semi-proper as a map into some open subset of $\mathcal{G}^*$ and that its level sets are connected.

By assumption $J(M) \subset \mathcal{G}_{reg}$. Fix a maximal torus T in G, let $\mathcal{T}$ be its Lie algebra and let R be a connected component of $\mathcal{T} \cap \mathcal{G}_{reg}$. G is a principal T-bundle over G/T. The map

$$G \times R \longrightarrow \mathcal{G}_{reg},$$
$$(g, \xi) \longrightarrow Ad_g(\xi)$$

is a surjection. It induces a G-equivariant bijection $G \times_T R \longrightarrow \mathcal{G}_{reg}$. Here $G \times_T R$ denotes the associated fiber bundle over G/T with fiber R. The momentum map J is transversal to R, so $F = J^{-1}(R)$ is a submanifold of M. Moreover F is symplectic. Since the inverse image of R under j equals the inverse image of its closure, F is closed. The fact that $M = J^{-1}(\mathcal{G}_{reg})$ and equivariance of J imply that M is diffeomorphic to $G \times_T F$ as a G-space. More explicitly let

$$p_1 : G \times F \longrightarrow G \times_T F$$

be the projection. Then the map $G \times_T F \longrightarrow M$ is given by $p_1(g, f) = g.f$. We know that G/T is simply connected. Since M is a connected fiber bundle with a simply connected base its fiber F is connected.

Let $j = J\mid_F$. Then F is a Hamiltonian T-space, and j is a corresponding T-momentum map. The map $id \times j : G \times F \longrightarrow G \times R$ induces a G-equivariant map of fiber bundles $G \times_T F \longrightarrow G \times_T R$. Since J is also G-equivariant the induced map equals J.

Let $p_2 : G \times R \longrightarrow G \times_T R$ be the projection. Then the map $G \times_T R \longrightarrow \mathcal{G}_{reg}$ is given by $p_2(g, r) = g.r$. Let $\mu : U \longrightarrow G$ be a local section of $G \longrightarrow G/T$. The map μ induces trivializations of $\pi_1 : G \times_T F \longrightarrow G/T$ and $\pi_2 : G \times_T R \longrightarrow G/T$:

$$\begin{aligned}
&\phi_1 : U \times F \longrightarrow G \times_T F, \ \ by \ \ (u,q) \longrightarrow p_1(\mu(u),q)\\
&\phi_2 : U \times R \longrightarrow G \times_T R, \ \ by \ \ (u,r) \longrightarrow p_2(\mu(u),r).
\end{aligned}$$

$$\begin{aligned}
\text{Now,} J(\phi_1((u,q))) &= J(p_1(\mu(u),q)) = J(\mu(u).q)\\
&= \mu(u).J(q) = \mu(u).j(q)\\
&= \phi_2(\mu(u), j(q)).
\end{aligned}$$

Thus with respect to the identifications, $J\mid_{\pi_1^{-1}(U)}: \pi_1^{-1}(U) \longrightarrow \pi_2^{-1}(U)$ is given by $J(u,q) = (u, J(q))$.

Since $M = G \times_T F$ there exists a 1-1 correspondence between G-invariant functions on M and T-invariant functions on F. In one direction the correspondence is simply restriction to the fiber. In the other direction, a T-invariant function on F pulls up to a G and T-invariant function on $G \times F$ and so descends to a G-invariant function $G \times_T F$.

This carries over to the correspondence between Hamiltonian vector fields. (Recall that F is a symplectic manifold.) That is, given a G-invariant function f, restricting it to F and taking the Hamiltonian vector field of the restriction is the same as taking the Hamiltonian vector field Ξ_f of f and restricting it to F. To prove this it is enough to show that Ξ_f is tangent to F. So let p be a point in $F \longrightarrow G \times_T F$. Since f is constant along the orbit $G.p$, $\Xi_f(p)$ lies in the symplectic perpendicular $T_p(G.p)^\perp$. We know that $T_p(G.p)^\perp = kerdJ_p$. But $F = J^{-1}(R)$ and J intersect R transversely. Hence T_pF contains $kerdJ_p$ and therefore $\Xi_f(p)$ lies in T_pF.

Consider h in $(C^\infty(M)^G)^c$. Assume for a moment that the support of h is contained in $\pi_1^{-1}(U)$ and $G \longrightarrow G/T$ is trivial over U. Then $\pi_1^{-1}(U) = U \times F$, and it follows from the discussions above that h is killed by the Hamiltonian vector fields of the T-invariant functions on F. Using Theorem 4.3 we can find a function h' in $C^\infty(U \times \mathcal{T})$ so that $h(u, f) = h'(u, j(f))$. Thus $H = J^*\phi$ for some ϕ in $C^\infty(\mathcal{G}) = C^\infty(\mathcal{G}^*))$.

In general let $\{U_i\}$ be a cover of G/T such that the $G \mid U_i$ are trivial. Choose a partition of unity $\{\sigma_i\}$ subordinate to the cover. Then $\{\pi_i^*\sigma_i\}$ is a partition of unity on $G \times_T F$ and each $\pi_i^*\sigma_i$ is supported in $\pi_1^{-1}(U_i)$. Moreover, since $\pi_1 = \pi_2 \circ J$, $\pi_i^*\sigma_i$ is collective. Therefore if h is in $(C^\infty(M)^G)^c$, then $(\pi_i^*\sigma_i).h$ are also in $(C^\infty(M)^G)^c$. But by the discussion above $(\pi_i^*\sigma_i).h$ are collective, and so $h = \Sigma(\pi_i^*\sigma_i).h$ is also collective. Hence the theorem. •

Note 3.3.20. We give a generalization of Theorem 3.3.19 by replacing the properness of the momentum map by closedness of it.

Theorem 3.3.21. Let M be a paracompact connected symplectic Hamiltonian G-manifold with G a compact connected Lie group. If the associated momentum map J is closed and semi proper as a map into some open subset of $\mathcal{G}^*$, then J has the division property if the image $J(M)$ is contained the $\mathcal{G}^*_{reg}$.

Proof : Given that M be a paracompact connected symplectic Hamiltonian G-manifold with G a compact connected Lie group with the associated momentum map J_G is closed. Then using the Theorem 3.2.39, the level sets of J_G are connected. Also J is semi-proper as a map into some open subset of $\mathcal{G}^*$. Thus J is a proper momentum map.. Then from the above theorem J has the division property if the image $J(M)$ is contained the $\mathcal{G}_{reg}$. •

Chapter 4

Generalizations of the Standard Momentum Maps

In this chapter we discuss certain generalizations of standard momentum map. The first section is on cylinder valued momentum maps, which has the important property of being always defined, unlike the standard momentum map. Cylinder valued momentum maps are genuine generalizations of the standard ones in the sense that whenever a Lie algebra action admits a standard momentum map, there is a cylinder valued momentum map that coincides with it. In section 2 we discuss Lie group valued momentum maps. For Abelian symmetries, cylinder valued momentum maps are closely related to the so- called Lie group valued momentum maps. This relation ship is discussed in detail. In the third section we give a generalization of momentum map in which the group action not involved. After giving a sufficient condition for the existence of momentum map, we have recaptured a generalization of standard momentum map by family of symplectomorphisms and the momentum map associated to Hamiltonian group action.

4.1 Cylinder valued Momentum Maps

To introduce cylinder valued momentum maps, we need connections on a principal fiber bundle. Then we define holonomy bundle and some properties are discussed. The definition of cylinder valued momentum map is given as a generalization of the standard momentum map. We look at certain properties of Cylinder valued momentum maps. Cylinder valued momentum maps are genuine generalizations of the standard ones in the sense that whenever a Lie algebra action admits a standard momentum map, there is a cylinder valued momentum map that coincides with it [32] , [33].

Definition 4.1.1. Let (P, M, π, G) be a principal fiber bundle. Denote by $R : P \times G \to P$ the right action whose orbit space is $\frac{P}{G} = M$. A *connection* on (P, M, π, G) is a $\mathcal{G}$ valued one form $A \in \Omega^1(P, \mathcal{G})$ such that for any $\xi \in \mathcal{G}, g \in G, z \in P$, and $v_z \in T_zP$, we have that
(i) $A(z) : T_zP \to \mathcal{G}$ is linear.
(ii) $A(z).\xi_M(z) = \xi$.
(iii) $A(R_gz).(T_zR_g.v_z) = Ad_{g^{-1}}(A(z).v_z)$

Note 4.1.2. The connection A provides a splitting of the tangent bundle $TP = V \oplus H$, where V is the bundle of vertical vectors defined by $V = KerT\pi$ and H, that of the horizontal vectors given by $H = KerA$, that is, $H(z) = \{v_z \in T_zP | A(z).v_z = 0\}$.

Definition 4.1.3. A curve $C : I \subset \Re \to P$ is horizontal if $C'(t) \in H(C(t))$ for any $t \in I$. Given a curve $d : [0, 1] \to M$ on M and a point $z \in P$, there exist a unique horizontal curve $C : [0, 1] \to P$ such that $C(0) = z$ and $(\pi \circ C)(t) = d(t), \forall t \in [0, 1]$. This curve C is called the *horizontal lift* of the curve d through z.

Definition 4.1.4. Each point $z \in P$ and each loop $d : [0, 1] \to M$ at the point

$\pi(z)$ determine an element in G. Indeed, let $C : [0,1] \to M$ be the horizontal lift of d through z. Since $d(0) = d(1) = \pi(z)$. We have that $z = C(0), C(1) \in \pi^{-1}(\pi(z))$ and hence there exists a unique element $g \in G$ such that $C(1) = R_g(C(0))$. The elements in G determined by all the loops at $\pi(z)$ form a closed subgroup $\hbar(z)$ of G called the *holonomy group* of connection A with reference point $z \in P$. If two points $z_1, z_2 \in P$ can be joined by a horizontal curve, then $\hbar(z_1) = \hbar(z_2)$. If two points $z_1, z_2 \in P$ are in the same fiber of π, then there exists $g \in G$ such that $z_2 = R_g z_1$ and hence $\hbar(z_2) = g^{-1}\hbar(z_1)g$.

Note 4.1.5. Let (P, M, π, G) be a principal fiber bundle. Let $i : Q \to P$ be an injectively immersed submanifold of P and H a Lie sub group of G (not necessarily embedded) that leaves Q invariant. If (Q, M, π', H) is a principal fiber bundle, where $\pi' : Q \to \frac{Q}{H} = M$ is the projection. We say that (Q, M, π', H) is the reduction of (P, M, π, G). Given a reduction (Q, M, π', H) of principal bundle (P, M, π, G), a connection $A \in \Omega^1(P, \mathcal{G})$ is said to be reducible to the connection $A' \in \Omega^1(Q, \mathcal{H})$ where $\mathcal{H}$ is the Lie algebra of H, if

$$A'(q)(u_q) \quad = \quad A(i(q))(T_q i(u_q)),$$

for every $q \in Q$ and all $u_q \in T_q Q$.

Definition 4.1.6. Let $A \in \Omega^1(P, \mathcal{G})$ be a connection on (P, M, π, G) where M is connected and paracompact. Let $P(z)$ be a set of points in P which can be joined to z by a horizontal curve. $P(z)$ is called *holonomy bundle* through z. The reduction theorem states that $(P(z), M, \pi', \hbar(z))$ is a reduction of (P, M, π, G) and that the connection A reducible to a connection in $(P(z), M, \pi', \hbar(z))$.

Note 4.1.7. We discuss certain properties of the holonomy bundle.

(i) Holonomy bundles are initial submanifolds of P : The tangent space to the holonomy bundle $P(z)$ at any point $y \in P(z)$ can be written as the direct sum

$$T_y P(z) \quad = \quad H(y) \oplus Lie(\hbar(z)).y.$$

where $H(y)$ is the horizontal space at $y \in P(z) \subset P$ of the connection $A \in \Omega^1(P, \mathcal{G})$.

The collection of the tangent spaces to the holonomy bundles form a smooth and involutive distribution on P whose maximal integral manifolds are the holonomy bundles themselves. This implies that the holonomy bundles are not only injectively immersed submanifolds but initial submanifolds of P.

(ii) All the holonomy bundles are isomorphic as principal bundles via the group action: that is, given any two points $z_1, z_2 \in P$ there exist an element $g \in G$ such that $R_g : P(z_1) \to P(z_2)$ is a principal bundle isomorphism whose associated structure group isomorphism is conjugation by g^{-1}. Then three possibilities are there.

(a) $z_1, z_2 \in P$ are two points that can be joined by a horizontal curve, then $P(z_1) = P(z_2)$ by definition.

(b) $z_1, z_2 \in P$ are two points in the same fiber, that is , $\exists g \in G \ni z_2 = R_g(z_1)$. Since the group action maps horizontal curves to horizontal curves we have $R_g(P(z_1)) = P(z_2)$. In addition $R_g : P(z_1) \to P(z_2)$ is principal bundle isomorphism relative to the group isomorphism $\hbar(z_1) \to \hbar(z_2)$ implemented by conjugation using the element $g^{-1} \in G$.

(c) If none of the above possibility holds, any two points $z_1, z_2 \in P$ are such that $\pi(z_1)$ and $\pi(z_2)$ can be joined by a smooth curve (connectedness of the base). Horizontally lift this curve through z_1. Its end point z_3 lies in the fiber of z_2. Therefore $P(z_1) = P(z_3)$ by point (a) and there exists a $g \in G$ such that $z_2 = R_g(z_3)$. Therefore, by point (b) $R_g : P(z_3) \to P(z_2)$ is principal bundle isomorphism between $P(z_1) = P(z_3)$ and $P(z_2)$.

Definition 4.1.8. Let (P, M, π, G) be a principal fiber bundle where M is a connected and paracompact manifold. Let now $A \in \Omega^1(P, \mathcal{G})$ be a connection on (P, M, π, G). Given any vector $v_z \in T_zP$ we will denote $v_z^H \in \hbar(z)$ its horizontal part. A curvature form $\Omega \in \Omega^2(P, \mathcal{G})$ of the connection form A is defined as

$$\Omega(z)(v_z, w_z) \;\; = \;\; dA(z)(v_z^H, w_z^H).$$

A connection A is said to be flat when its curvature form is identically zero.

Note 4.1.9. From the holonomy theorem given any point $z \in P$, the Lie algebra of the holonomy group $\hbar(z)$ of A with reference point z equals the subspace of $\mathcal{G}$ spanned by all the elements of the form $\Omega(p)(v_p, w_p), p \in P(z), v_p, w_p \in H(p)$.

The holonomy theorem implies the connection form is flat if and only if its holonomy groups are discrete. This is equivalent to the horizontal subbundle being an involutive distribution that has the holonomy bundles as maximal integral manifolds.

Proposition 4.1.10. Let A be a flat connection on the principal bundle (P, M, π, G) with connected and paracompact base M and let $(P(z), M, \pi^{'}, \hbar(z))$ be the holonomy reduced bundle at a point $z \in P$. Then $\pi^{'} : P(z) \to M$ is a covering map.

Proof: Since the connection is flat, the Lie algebra $Lie(\hbar(z))$ of the holonomy group is trivial by the holonomy theorem and hence $\hbar(z)$ is a discrete group. As $(P(z), M, \pi^{'}, \hbar(z))$ is locally trivial bundle, any point $m \in M$ has an open neighborhood U such that $(\pi^{'})^{-1}(U)$ is diffeomorphic to $U \times \hbar(z)$. Since $\hbar(z)$ is discrete, each subset $U \times \{g\}, g \in \hbar(z)$, is an open subset diffeomorphic to U. Hence $\pi^{'}$ is a covering map. •

Note 4.1.11. Let (M, ω) be a connected and paracompact symplectic manifold and $\mathcal{G}$ a Lie algebra acting canonically on it.

Let $\pi : M \times \mathcal{G}^* \to M$ be the projection onto M. Consider π as the bundle map of the trivial principal bundle $(M \times \mathcal{G}^*, M, \pi, \mathcal{G}^*)$ that has $(\mathcal{G}^*, +)$ as an abelian structure group. The group $(\mathcal{G}^*, +)$ acts on $M \times \mathcal{G}^*$ by $v.(m, \mu) := (m, \mu - v)$, with $m \in M$ and $\mu, v \in \mathcal{G}^*$. Let $\alpha \in \Omega^1(M \times \mathcal{G}^*, \mathcal{G}^*)$ be the connection one form defined by

$$< \alpha(m, \mu).(v_m, v), \xi > \;\; := \;\; (i_{\xi_M}\omega)(m).v_m - < v, \xi >,$$

where $(m, \mu) \in M \times \mathcal{G}^*$, $(v_m, v) \in T_mM \times \mathcal{G}^*$,$\xi \in \mathcal{G}$ and $< .,. >$ denotes the natural pairing between $\mathcal{G}^*$ and $\mathcal{G}$. Then α is a well-defined connection one form on $M \times \mathcal{G}^*$.

The vertical subbundle $V \subset T(M \times \mathcal{G}^*)$ of $\pi : M \times \mathcal{G}^* \to M$ is given for any $(m, \mu) \in M \times \mathcal{G}^*$ by

$$V(m, \mu) \;\; := \;\; \{(0, \rho) \in T_{(m,\mu)}(M \times \mathcal{G}^*) | \rho \in \mathcal{G}^*\}.$$

Also the horizontal subspace determined by α at the point $(m, \mu) \in M \times \mathcal{G}^*$ is given by

$$\begin{aligned} H(m, \mu) &= \{(v_m, v) \in T_{(m,\mu)}(M \times \mathcal{G}^*) | < \alpha(m, \mu).(v_m, v), \xi >= 0, \forall \xi \in \mathcal{G}\} \\ &= \{(v_m, v) \in T_{(m,\mu)}(M \times \mathcal{G}^*) | (i_{\xi_M}\omega)(m).v_m - < v, \xi >= 0, \forall \xi \in \mathcal{G}\} \qquad (4.1) \end{aligned}$$

consequently, given any vector $(v_m, v) \in T_{(m,\mu)}(M \times \mathcal{G}^*)$, its horizontal $(v_m, v)^H$ and the vertical$(v_m, v)^V$ parts are such that

$$(v_m, v)^H = (v_m, \rho) \quad and \quad (v_m, v)^V = (0, \rho'),$$

where $\rho, \rho^{'} \in \mathcal{G}^*$ are uniquely determined by the relations

$$< \rho, \xi >= (i_{\xi_M}\omega)(m).v_m \quad and \quad \rho^{'} = v - \rho, \forall \xi \in \mathcal{G}$$

Also α is a flat connection.

Definition 4.1.12. For $(z,\mu) \in M \times \mathcal{G}^*$, let $M \times \mathcal{G}^*(z,\mu)$ be the holonomy bundle through (z,μ) and let $\hbar(z,\mu)$ be the holonomy group of α with reference point (z,μ). The reduction theorem guarantees that $(M \times \mathcal{G}^*(z,\mu), M, \pi/_{M\times\mathcal{G}^*(z,\mu)}, \hbar(z,\mu))$ is a reduction of $(M \times \mathcal{G}^*, M, \pi, \mathcal{G}^*)$. For simplicity we use $(\widetilde{M}, M, \widetilde{P}, \hbar)$ instead of $(M \times \mathcal{G}^*(z,\mu), M, \pi/_{M\times\mathcal{G}^*(z,\mu)}, \hbar(z,\mu))$. Let $\widetilde{K} : \widetilde{M} \subset M \times \mathcal{G}^* \to \mathcal{G}^*$ be the projection into the $\mathcal{G}^*$-factor.

Consider now the closure $\overline{\hbar}$ of $\hbar$ in $\mathcal{G}^*$. Since $\overline{\hbar}$ is a closed subgroup of $(\mathcal{G}^*, +)$, the quotient $D := \frac{\mathcal{G}^*}{\overline{\hbar}}$ is a cylinder, that is, it is isomorphic to the abelian Lie group $\Re^a \times T^b$ for some $a, b \in \aleph$. Let $\pi_D : \mathcal{G}^* \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ be the projection. Define $K : M \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{K}} & \mathcal{G}^* \\ \widetilde{P}\downarrow & & \downarrow \pi_D \\ M & \xrightarrow{K} & \frac{\mathcal{G}^*}{\overline{\hbar}}. \end{array}$$

In other words, K is defined by $K(m) = \pi_D(v)$, where $v \in \mathcal{G}^*$ is any element such that$(m,v) \in C$. This is well defined because if we have two points $(m,v), (m,v^{'}) \in \widetilde{M}$, then $(m,v), (m,v^{'}) \in \widetilde{P}^{-1}(m)$, that is, there exists $\rho \in \hbar$ such that $v' = v + \rho$. So $\pi_D(v) = \pi_D(v')$.

Then the map $K : M \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ is referred as a *cylinder valued momentum map* associated to the canonical $\mathcal{G}$ action on (M,ω). The definition of K depends on the choice of the holonomy bundle, that is, if we let $\widetilde{M_1}$ and $\widetilde{M_2}$ are two holonomy bundles of $(M \times \mathcal{G}^*, M, \pi, \mathcal{G}^*)$. Then

$$K_{\widetilde{M_1}} = K_{\widetilde{M_2}} + \pi_D(\tau)$$

where $\tau \in \mathcal{G}^*$.

Example 4.1.13. Here we compute the cylinder valued momentum map for the canonical circle action on a torus that does not have a standard momentum map. Consider a torus $T^2 = \{(e^{i\theta_1}, e^{i\theta_2})\}$ as a symplectic manifold with the area form $\omega := d\theta_1 \wedge d\theta_2$ and a circle $S^1 = \{e^{i\phi}\}$ acting canonically on it by

$$e^{i\phi}.(e^{i\theta_1}, e^{i\theta_2}) = (e^{i(\theta_1+\phi)}, e^{i\theta_2}).$$

Consider the trivial principal bundle $T^2 \times \Re \to T^2$ with $(\Re, +)$ as structure group.

Now the connection form is

$$< \alpha(m,a).(v_m, b), \xi > := (i_{\xi_M}\omega)(m).v_m - < b, \xi > .$$

Therefore the Horizontal vectors in $T(T^2 \times \Re)$ with respect to the connection α are of the form $((a,b),b)$, with $a, b \in \Re$. Therefore the holonomy groups at any point is $(\mathcal{Z}, +)$. Thus we can define a cylinder valued momentum map $K : T^2 \to \frac{\Re}{\mathcal{Z}} \simeq S^1$ by using the diagram

$$\begin{array}{ccc} \widetilde{T^2} & \xrightarrow{\widetilde{K}} & \Re \\ \widetilde{P} \downarrow & & \downarrow \pi_D \\ T^2 & \xrightarrow{K} & \dfrac{\Re}{\mathcal{Z}} \simeq S^1. \end{array}$$

More specifically, we have that by $K(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_2}$, for any $(e^{i\theta_1}, e^{i\theta_2}) \in T^2$.

We look at certain properties of cylinder valued momentum maps

Theorem 4.1.14. Let (M, ω) be a connected and paracompact symplectic manifold and $\mathcal{G}$ a Lie algebra acting canonically on it. Then any cylinder valued momentum map $K : M \to D$ associated to this action has the following properties.

(i) K is a smooth Noether momentum map.

(ii) For any $v_m \in T_m M, m \in M$ we have that

$$T_m K(v_m) \quad = \quad T_\mu \pi_D(v)$$

where $\mu \in \mathcal{G}^*$ is any element such that $K(m) = \pi_D(\mu)$, and $v \in \mathcal{G}^*$ is uniquely determined by

$$< v, \xi > \quad = \quad (i_{\xi_M}\omega)(m).v_m$$

for any $\xi \in \mathcal{G}$.

(iii) $KerT_m K = ((Lie(\overline{\hbar}))^o.m)^\omega$.

(iv) Bifurcation Lemma :

$$range(T_m K) \quad = \quad T_\mu \pi_D((\mathcal{G}_m)^o),$$

where $\mu \in \mathcal{G}^*$ is any element such that $K(m) = \pi_D(\mu)$.

Proof : As $\frac{\mathcal{G}^*}{\overline{\hbar}}$ is a homogeneous manifold, we have that $\pi_D : \mathcal{G}^* \to \frac{\mathcal{G}^*}{\overline{\hbar}}$ is a surjective submersion. Moreover, $K \circ \widetilde{P} = \pi_D \circ \widetilde{K}$ is a smooth map and $\widetilde{P}$ is a surjective submersion, the map K is necessarily smooth.

(ii) Let $m \in M$ and $(m, \mu) \in \widetilde{P}^{-1}(m)$. If $v_m = T_{(m,\mu)}\widetilde{P}(v_m, v)$, then

$$\begin{aligned} T_m K(v_m) &= T_m K(T_{(m,\mu)}\widetilde{P}(v_m, v)) \\ &= T_\mu \pi_D(T_{(m,\mu)}\widetilde{K}(v_m, v)) \\ &= T_\mu \pi_D(v). \end{aligned}$$

(i) We now check that K satisfies Noether's condition. Let $h \in C^{\infty}(M)^{\mathcal{G}}$ and let F_t be the flow of the associated Hamiltonian vector field X_h. Using the expression for the derivative T_mK in (ii) we have $T_mK(X_h(m)) = T_\mu \pi_D(v)$ where $\mu \in \mathcal{G}^*$ is any element such that $K(m) = \pi_D(\mu)$, and $v \in \mathcal{G}^*$ is uniquely determined by

$$\begin{aligned} < v, \xi > &= (i_{\xi_M}\omega)(m).(X_h(m)) \\ &= -dh(m)(\xi_M(m)) \\ &= \xi_M[h](m) = 0, \end{aligned}$$

for any $\xi \in \mathcal{G}$, which proves that $v = 0$ and hence
$T_mK(X_h(m)) = 0, \forall m \in M$.

$$\text{Also, } \frac{d}{dt}(K \circ F_t)(m) = T_{F_t(m)}K(X_h(F_t(m))) = 0,$$

we have $K \circ F_t = K/_{Dom(F_t)}$.
Hence K satisfies the Noether's condition.
(iii) Due to the expression in (ii), a vector $v_m \in KerT_mK$ if and only if there exist unique element $v \in \mathcal{G}^*$ determined by

$$< v, \xi > \; = \; (i_{\xi_M}\omega)(m).v_m, \forall \xi \in \mathcal{G}.$$

Also $T_\mu \pi_D(v) = 0$,

$$\begin{aligned} \text{that is, } v \in Lie(\overline{\hbar}) &\Leftrightarrow \; < v, \xi >= 0, \forall \xi \in (Lie(\overline{\hbar}))^o \subset (\mathcal{G}^*)^* = \mathcal{G} \\ &\Leftrightarrow \; (i_{\xi_M}\omega)(m).v_m = 0, \forall \xi \in (Lie(\overline{\hbar}))^o \\ &\Leftrightarrow \; v_m \in ((Lie(\overline{\hbar}))^o.m)^\omega. \end{aligned}$$

Hence $KerT_mK = ((Lie(\overline{\hbar}))^o.m)^\omega$.

(iv) Let $T_mK(v_m) \in range(T_mK)$. Let $v \in \mathcal{G}^*$ determined by

$$\begin{aligned} < v, \xi > &= (i_{\xi_M}\omega)(m).v_m, \forall \xi \in \mathcal{G}. \\ \text{Thus, } T_mK(v_m) &= T_\mu \pi_D(v). \end{aligned}$$

Therefore for any $\xi \in \mathcal{G}_m$,

$$\begin{aligned} < v, \xi > &= (i_{\xi_M}\omega)(m).v_m \\ &= \omega(m)(\xi_M(m), v_m) = 0, \end{aligned}$$

which implies $v \in (\mathcal{G}_m)^o$.

$$\begin{aligned} \text{Thus, } range(T_mK) &\subset T_\mu \pi_D((\mathcal{G}_m)^o). \\ \text{Also, } rank(T_mK) &= dimM - dim(Ker(T_mK)) \\ &= dimM - dimM + dim((Lie(\overline{\hbar}))^o.m) \\ &= dim((Lie(\overline{\hbar}))^o) - dim(\mathcal{G}_m \cap (Lie(\overline{\hbar}))^o) \\ &= -dim((Lie(\overline{\hbar})) + dim(\mathcal{G}) \\ &\quad -dim([(Lie(\overline{\hbar})) + (\mathcal{G}_m)^o]^o) \\ &= dim(\mathcal{G}) - dim(\mathcal{G}_m) - dim(Lie(\overline{\hbar})) \\ &\quad -dim(\mathcal{G}_m)^o + dim(Lie(\overline{\hbar}) + (\mathcal{G}_m)^o) \\ &= dim(\mathcal{G}) - dim(\mathcal{G}_m) - dim(Lie(\overline{\hbar}) \cap (\mathcal{G}_m)^o) \\ &= dim(\mathcal{G}) - dim(\mathcal{G}_m) - dim(KerT_\mu \pi_D \cap (\mathcal{G}_m)^o) \\ &= dim(\mathcal{G}_m)^o - dim(KerT_\mu \pi_D/_{(\mathcal{G}_m)^o}) \\ &= dim(T_\mu \pi_D((\mathcal{G}_m)^o)). \\ \text{This implies, } range(T_mK) &= T_\mu \pi_D((\mathcal{G}_m)^o). \end{aligned}$$

•

Note 4.1.15. Cylinder valued momentum maps are generalizations of standard momentum maps. It can be proved that given a canonical Lie algebra action on a connected symplectic manifold, there exists a standard momentum map if and only if the holonomy $\hbar$ is trivial. Moreover in such situations the cylinder valued momentum maps are the standard momentum maps.

Proposition 4.1.16. Let (M, ω) be a connected paracompact symplectic manifold and $\mathcal{G}$ a Lie algebra acting canonically on it. Let $K : M \to \frac{\mathcal{G}^*}{\hbar}$ be a cylinder valued momentum map. Then there exists a standard momentum map if and only if $\hbar = \{0\}$. In this case K is a standard momentum map.

Proof : Suppose that the cylinder valued momentum map $K : M \to \frac{\mathcal{G}^*}{\hbar}$ has been constructed using the reduced bundle
$(M \times \mathcal{G}^*(z, \mu), M, \pi/_{M \times \mathcal{G}^*(z,\mu)}, \hbar(z, \mu)), (z, \mu) \in M \times \mathcal{G}^*$. We now show that if there exists a standard momentum map $J : M \to \mathcal{G}^*$ associated to this action, then $\hbar = \{0\}$. Indeed, if $\tau \in \hbar$, then there exist a loop $C : [0, 1] \to M$ at m, that is, $C(0) = C(1) = m$ such that one of its horizontal lifts $\widetilde{C} : [0, 1] \to \widetilde{M}$ given by the function $\widetilde{C}(t) = (C(t), \rho(t))$ is such that $\rho(0) = \rho \in \mathcal{G}^*$ and $\rho(1) = \rho + \tau, \rho \in \mathcal{G}^*$.

Now since $\widetilde{C}$ is horizontal we have that

$$
\begin{aligned}
< \alpha(C(t), \rho(t))(C'(t), \rho'(t)), \xi >= 0, \forall \xi \in \mathcal{G} \\
\Leftrightarrow (i_{\xi_M}\omega)(C(t))(C'(t)) =< \rho'(t), \xi > \\
\Leftrightarrow dJ^{\xi}(C(t))(C'(t)) =< \rho'(t), \xi > \\
\Leftrightarrow \frac{d}{dt} J^{\xi}(C(t)) = \frac{d}{dt} < \rho(t), \xi >
\end{aligned}
$$

Integrating we obtain

$$\begin{aligned} J^{\xi}(C(t)) - J^{\xi}(m) &= \int_0^t \frac{d}{ds} J^{\xi}(C(s)) ds \\ &= \int_0^t \frac{d}{ds} < \rho(s), \xi > ds \\ &= < \rho(t), \xi > - < \rho, \xi > \end{aligned}$$

If we take $t = 1$, then

$$\begin{aligned} < \tau, \xi > &= < \rho(1) - \rho(0), \xi > \\ &= J^{\xi}(C(1)) - J^{\xi}(C(0)) \\ &= J^{\xi}(m) - J^{\xi}(m) = 0. \end{aligned}$$

Since $\xi \in \mathcal{G}$ is arbitrary, we have $\tau = 0$ and consequently $\hbar = 0$.

Conversely, suppose that $\hbar = 0$. Let $C : [0, 1] \to M$ be a loop at an arbitrary point $z \in M$, that is, $C(0) = C(1) = z$. Let $v \in \widetilde{P}^{-1}(z)$ and let $\widetilde{C}(t) = (C(t), v(t))$ be the horizontal lift of C starting at the point $(z, v) \in \widetilde{M}$. Since (z, v) belongs to the same holonomy bundle as (m, v) we have that the holonomy group with reference at that point is zero. This implies that

$$\begin{aligned} 0 =< v(1) - v(0), \xi > &= \int_0^1 \frac{d}{ds} < v(s), \xi > ds \\ &= \int_0^1 (i_{\xi_M}\omega)(C(s))(C'(s)) ds \\ &= \int_C i_{\xi_M}\omega \end{aligned}$$

Since the equality $\int_C i_{\xi_M}\omega = 0$ holds for any loop C at any point M, the deRham theorem implies the cohomology class $[i_{\xi_M}\omega]$of the form $i_{\xi_M}\omega$ is trivial, that is, for any $\xi \in \mathcal{G}$, the existence of a standard momentum map is guaranteed by choosing $J : M \to \mathcal{G}^*$ such that $< J(m), \xi >= J^{\xi}(m)$, for any $\xi \in \mathcal{G}$ and $m \in M$.

Also the graph $Graph(J) := \{(m, J(m)) \in M \times \mathcal{G}^* | m \in M\}$ integrates the horizontal distribution associated to α. Indeed, choose J such that $J(m) = v$. Then by equation 4.1, we have for any $(z, \rho) \in \widetilde{M}$

$$\begin{aligned} H(z,\rho) &= \{(v_z, \tau) \in T_{(z,\rho)}(M \times \mathcal{G}^*) | < \tau, \xi >= (i_{\xi_M}\omega)(z)(v_z), \forall \xi \in \mathcal{G}\} \\ &= \{(v_z, \tau) \in T_{(z,\rho)}(M \times \mathcal{G}^*) | < \tau, \xi >= dJ^{\xi}(z)(v_z), \forall \xi \in \mathcal{G}\} \\ &= \{(v_z, \tau) \in T_{(z,\rho)}(M \times \mathcal{G}^*) | < \tau, \xi >=< T_z J(v_z), \xi >, \forall \xi \in \mathcal{G}\} \\ &= \{(v_z, T_z J(v_z)) | v_z \in T_z M\} = T_{(z,J(z))} Graph(J). \end{aligned}$$

Since J is defined up to a constant in $\mathcal{G}^*$, it can be chosen so that $Graph(J) = \widetilde{M}$ and hence the momentum map J can be chosen to coincide with K which make K a standard momentum map. •

4.2 Lie group valued Momentum Maps

In this section we discuss Lie group valued momentum maps. We define Lie group valued momentum maps and then show that it is a Noether Momentum Map. For abelian symmetries, cylinder valued momentum maps are closely related to the so- called Lie group valued momentum maps. This relation ship is discussed in detail. [1] , [32].

Definition 4.2.1. Let G be an abelian Lie Group whose Lie algebra $\mathcal{G}$ acts canonically on a symplectic manifold (M, ω). Let $(.,.)$ be some bilinear symmetric nondegenerate form on the lie algebra $\mathcal{G}$. The map $J : M \to G$ is called a G-valued momentum map for the $\mathcal{G}$ action on M whenever

$$i_{\xi_M}\omega(m).v_m = (T_m(L_{J(m)^{-1}} \circ J)(v_m), \xi),$$

for any $\xi \in \mathcal{G}$, $m \in M$, and $v_m \in T_mM$, where $L_{J(m)^{-1}} : G \longrightarrow G$.

Proposition 4.2.2. Let G be an abelian Lie Group whose Lie algebra $\mathcal{G}$ acts canonically on a symplectic manifold (M, ω). Let $J : M \to G$ be a G-valued momentum map for this $\mathcal{G}$ action on M. Then
(i) $J : M \to G$ is a Noether Momentum Map .
(ii) $KerT_mJ = (\mathcal{G}.m)^\omega$ for any $m \in M$.

Proof : (i) Let F_t be the flow of the Hamiltonian vector field X_h associated to a $\mathcal{G}$-invariant function $h \in C^\infty(M)^{\mathcal{G}}$. By the definition of Lie group valued momentum map we have for any $m \in M$ and any $\xi \in \mathcal{G}$

$$\begin{aligned}
((T_{J(F_t(m))}(L_{J(F_t(m))^{-1}}) \circ T_{F_t(m)}J)(X_h(F_t(m))), \xi) = \\
(T_{F_t(m)}(L_{J(F_t(m))^{-1}} \circ J))(X_h(F_t(m))), \xi) \\
= i_{\xi_M}\omega(F_t(m))(X_h(F_t(m))) \\
= -dh(F_t(m))(\xi_M(F_t(m))) = 0.
\end{aligned}$$

Consequently ,

$$\begin{aligned}
(T_{J(F_t(m))}(L_{J(F_t(m))^{-1}} \circ T_{F_t(m)}J)(X_h(F_t(m))) = 0 \\
\Leftrightarrow T_{F_t(m)}J(X_h(F_t(m))) = 0 \\
\Leftrightarrow \frac{d}{dt}(J \circ F_t)(m) = 0 \\
\Leftrightarrow J \circ F_t = J \mid_{Dom(F_t)},
\end{aligned}$$

since t and m are arbitrary elements.

(ii) A vector $v_m \in KerT_mJ$ if and only if $T_mJ(v_m) = 0$. This is equivalent to $((T_{J(m)}L_{J(m)^{-1}} \circ T_mJ)(v_m), \xi) = 0$, for any $\xi \in \mathcal{G}$ and by the definition of Lie group valued momentum maps

$$\begin{aligned} i_{\xi_M}\omega(m).v_m = 0, \forall \xi \in \mathcal{G} \\ \Leftrightarrow v_m \in (\mathcal{G}.m)^\omega. \end{aligned}$$

•

Proposition 4.2.3. Let (M, ω) be a connected paracompact symplectic manifold and $\mathcal{G}$ an abelian Lie algebra acting canonically on it. Let $\hbar \subset \mathcal{G}^*$ be the holonomy group associated to the connection α and $(.,.) : \mathcal{G} \times \mathcal{G} \to \Re$ some bilinear symmetric nondegenerate form on $\mathcal{G}$. Let $f : \mathcal{G} \to \mathcal{G}^*$ be the isomorphism given by $f(\xi) = (\xi, .), \xi \in \mathcal{G}$ and $\mathcal{T} := f^{-1}(\hbar)$. The map f induces an abelian group isomorphism $\widetilde{f} : \frac{\mathcal{G}}{\mathcal{T}} \to \frac{\mathcal{G}^*}{\hbar}$ by $f(\xi + \mathcal{T}) := (\xi, .) + \hbar$. Suppose that $\hbar$ is closed in $\mathcal{G}^*$ and define $J := \widetilde{f}^{-1} \circ K : M \to \frac{\mathcal{G}}{\mathcal{T}}$, where K is a cylinder valued momentum map for the $\mathcal{G}$-action on (M, ω). Then

$$i_{\xi_M}\omega(m).v_m \quad = \quad (T_m(L_{J(m)^{-1}} \circ J)(v_m), \xi), \tag{4.2}$$

for any $\xi \in \mathcal{G}$, $m \in M$, and $v_m \in T_mM$. Consequently, the map $J : M \to \frac{\mathcal{G}}{\mathcal{T}}$ constitutes a $\frac{\mathcal{G}}{\mathcal{T}}$ -valued momentum map for the canonical action of the Lie algebra $\mathcal{G}$ of $(\frac{\mathcal{G}}{\mathcal{T}}, +)$ of (M, ω).

Proof : We start by noticing that the right hand side of 4.2 makes sense due to the closedness hypothesis on $\hbar$. Indeed, this condition and the fact that $\hbar$ is discrete due to the flatness of α implies that $\frac{\mathcal{G}^*}{\hbar}$, and therefore $\frac{\mathcal{G}}{\mathcal{T}}$, are abelian Lie groups whose Lie algebras can be naturally identified with $\mathcal{G}^*$ and $\mathcal{G}$ respectively. This identification is used in 4.2, where $T_m(L_{J(m)^{-1}} \circ J)(v_m) \in Lie(\frac{\mathcal{G}}{\mathcal{T}})$ as an element of $\mathcal{G}$.

Given $\mu \in \mathcal{G}^*$ arbitrary, we denote $\xi_\mu \in \mathcal{G}$ by the unique element such that

$\mu = (\xi_\mu, .)$. Let $\mu + \hbar := K(m)$ and hence $J(m) = \xi_\mu + \mathcal{T}$. Then we have

$$\begin{aligned} T_m J(v_m) &= T_m(\widetilde{f}^{-1} \circ K)(v_m) \\ &= T_{\mu+\hbar}\widetilde{f}^{-1}(T_m K(v_m)) \\ &= T_{\mu+\hbar}\widetilde{f}^{-1}(T_\mu \pi_D(\nu)) \text{ where,} \\ < \nu, \eta > &= i_{\eta_M}\omega(m).v_m, \forall \eta \in \mathcal{G}. \end{aligned}$$

Since $(\widetilde{f}^{-1} \circ \pi_D)(\rho) = \xi_\rho + \mathcal{T}$ for any $\rho \in \mathcal{G}^*$, we can write

$$\begin{aligned} T_{\mu+\hbar}\widetilde{f}^{-1}(T_\mu \pi_D(\nu)) &= T_\mu(\widetilde{f}^{-1} \circ \pi_D)(\nu) \\ &= \frac{d}{dt}/_{t=0}(\widetilde{f}^{-1} \circ \pi_D)(\mu + t\nu) \\ &= \frac{d}{dt}/_{t=0}(\xi_\mu + t\xi_\nu + \mathcal{T}). \end{aligned}$$

Hence ,

$$T_m J(v_m) = \frac{d}{dt}/_{t=0}(\xi_\mu + t\xi_\nu + \mathcal{T}) \in T_{\xi_\mu+\mathcal{T}}(\frac{\mathcal{G}}{\mathcal{T}}).$$

Now,

$$\begin{aligned} (T_m(L_{J(m)^{-1}} \circ J)(v_m), \xi) &= (T_{J(m)}L_{J(m)^{-1}}(T_m J(v_m)), \xi) \\ &= (\frac{d}{dt}/_{t=0}(\xi_\mu + \mathcal{T}) + (\xi_\mu + t\xi_\nu + \mathcal{T}), \xi) \\ &= (\xi_\nu, \xi) =< \nu, \xi >= i_{\xi_M}\omega(m).v_m. \end{aligned}$$

•

Next we shall isolate hypothesis that guarantees that a Lie group valued momentum map naturally induces a cylinder valued momentum map.

Theorem 4.2.4. Let (M, ω) be a connected paracompact symplectic manifold and $\mathcal{G}$ abelian Lie algebra acting canonically on it. Let $\hbar \subset \mathcal{G}^*$ be the holonomy group associated to the connection α associated to the $\mathcal{G}$-action and let $(.,.) : \mathcal{G} \times \mathcal{G} \to \Re$ be a bilinear symmetric non degenerate form on $\mathcal{G}$. Let $f : \mathcal{G} \to \mathcal{G}^*$, $\widetilde{f} : \frac{\mathcal{G}}{\mathcal{T}} \to \frac{\mathcal{G}^*}{\hbar}$ where $\mathcal{T} := f^{-1}(\hbar)$. Let G be a connected abelian Lie group whose Lie algebra is $\mathcal{G}$ and suppose that there exists a G-valued momentum map $\mathbf{J} : M \to G$ associated to the $\mathcal{G}$-action whose definition uses the form $(.,.)$

(i) If $exp : \mathcal{G} \to G$ is the exponential map , then $\hbar \subset f(Ker\ exp)$.

(ii) $\hbar$ is closed in $\mathcal{G}^*$.

Let $J := \widetilde{f}^{-1} \circ K : M \to \frac{\mathcal{G}}{\mathcal{T}}$, where K is a cylinder valued momentum map for the $\mathcal{G}$-action on (M, ω). If $f(Ker\ exp) \subset \hbar$, then $J : M \to \frac{\mathcal{G}}{\mathcal{T}} = \frac{\mathcal{G}}{Ker\ exp} \simeq G$ is a G-valued momentum map that differs from $\mathbf{J}$ by a constant in G.

Conversely, if $\hbar = f(Ker\ exp)$, then $J : M \to \frac{\mathcal{G}}{Ker\ exp} \simeq G$ is a G-valued momentum map.

Proof : Assume that the $\mathcal{G}$-action on (M, ω) has an associated G-valued momentum map $\mathbf{J} : M \to G$. Then we can prove $\hbar \subset f(Ker\ exp)$.

Let $\mu \in \hbar$. Then there exists a piecewise smooth loop $m : [0, 1] \to M$ at the point m, that is, $m(0) = m(1) = m \in M$, whose horizontal lift $\widetilde{m}(t) = (m(t), \mu(t))$ starting at the point $(m, 0)$ satisfies $\mu = \mu(1)$. From the horizontality of $\widetilde{m}(t)$, we have

$$< \dot{\mu}(t), \xi >= i_{\xi_M}\omega(m(t))(\dot{m}(t))$$
$$= (T_{m(t)}(L_{\mathbf{J}(m(t))^{-1}} \circ \mathbf{J})(\dot{m}(t)), \xi), \forall \xi \in \mathcal{G}$$
$$\Leftrightarrow \dot{\mu}(t) = f(\frac{d}{ds}/_{s=0}\mathbf{J}(m(t))^{-1}\mathbf{J}(m(s))).$$

Fix $t_o \in [0, 1]$. Since the exponential map $exp : \mathcal{G} \to G$ is a local diffeomorphism, there exists a smooth curve $\xi : I_{t_o} := (t_o - \epsilon, t_o + \epsilon) \to \mathcal{G}$, for $\epsilon > 0$ sufficiently small such that for any $s \in (-\epsilon, \epsilon)$

$$\mathbf{J}(m(t_0+s)) = exp\xi(t_0+s)\mathbf{J}(m(t_0)).$$

Then we have $\dot{\mu}(t) = f(\dot{\xi}(t))$.
We now cover the interval $[0,1]$ with a finite number of intervals $I_1, I_2, ...I_n$ such that in each of them we define a function $\xi_i : I_i \to \mathcal{G}$ that satisfies the above two expressions. We now write $I_i = [a_i, a_{i+1}]$, with $i \in \{1, 2, ...n\}, a_1 = 0$, and $a_{n+1} = 1$. Using these intervals, since $\mu(0) = 0$, we have

$$\mu = f(\xi_1(a_2) - \xi_1(a_1) + ... + \xi_n(a_{n+1}) - \xi_n(a_n)).$$

But from the construction of the intervals I_i we have

$$\begin{aligned} \mathbf{J}(m(a_i)) &= exp\xi_i(a_i)\mathbf{J}(m(a_i)) \\ \Leftrightarrow exp\xi_i(a_i) = e & \\ \Leftrightarrow \xi_i(a_i) \in Ker\ exp,\ \forall i \in \{1, 2, ...n\}. & \end{aligned}$$

Also we have

$$\mathbf{J}(m(1)) = exp(\xi_1(a_2) + \xi_2(a_3) + ... + \xi_n(a_{n+1}))\mathbf{J}(m(0)).$$

Since $m(0) = m(1) = m$ we have $\mathbf{J}(m(0)) = \mathbf{J}(m(1))$ and therefore

$$\begin{aligned} exp(\xi_1(a_2) + \xi_2(a_3) + ... + \xi_n(a_{n+1})) &= e \\ \Leftrightarrow \xi_1(a_2) + \xi_2(a_3) + ... + \xi_n(a_{n+1}) \in Ker\ exp. & \end{aligned}$$

Thus we get $\hbar \subset f(Ker\ exp)$.

(ii) To show that $\hbar$ is closed in $\mathcal{G}^*$. The closedness of Ker exp in $\mathcal{G}$, the fact that

f is an isomorphism and (i) imply that $\bar{\hbar} \subset \overline{f(Ker\ exp)} = f(Ker\ exp)$. Because G is abelian, Ker exp is discrete subgroup of $(\mathcal{G}, +)$ and hence $\bar{\hbar}$ is discrete subgroup of $\mathcal{G}^*$. This implies that $\bar{\hbar} \subset \hbar$. Hence $\hbar$ is closed in $\mathcal{G}^*$.

Assume that $f(Ker\ exp) \subset \hbar$. Therefore from (i) $f(Ker\ exp) = \hbar$ and that $\hbar$ is closed in $\mathcal{G}^*$. Hence $J : M \to \frac{\mathcal{G}}{Ker\ exp} \simeq G$ is a G-valued momentum map for the $\mathcal{G}$-action on (M, ω). We now show that $\mathbf{J}$ and J are differ by a constant in G. We have for any $\xi \in \mathcal{G}$ and $v_m \in T_m M$,

$$\begin{aligned}
(T_m(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J})(v_m), \xi) &= i_{\xi_M}\omega(m)(v_m) \\
&= (T_m(L_{J(m)^{-1}} \circ J)(v_m), \xi)
\end{aligned}$$

which implies that $TJ = T\mathbf{J}$. Since the manifold M is connected, we have that $\mathbf{J}$ and J coincide up to a constant element in G. •

4.3 Another Generalization of the Standard Momentum Map

Here we discuss a generalization of the standard momentum map not involving the group action. The classical notion of momentum map from Weinstein's point of view is given first. Then we look at the standard momentum map in a more general set up as a map $\widetilde{J} : M \times G \to \mathcal{G}^*$. In this case we have shown that $\widetilde{J}$ is a momentum map. Then introduce the notion of generalization of Hamiltonian actions using Hamiltonian symplectomorphisms. We discuss the generalization of the momentum map, where the group action is replaced by a family of symplectomorphisms. Then we give a more general set up which does not contains the group action. After giving a sufficient condition for the existence of momentum map, we have recaptured a generalization of standard momentum map by family of symplectomorphisms and the momentum map associated to Hamiltonian group action. [38] , [10], [17].

To give the classical notion of momentum map from Weinstein's point of view we recall some ideas related to the symplectic category.

Definition 4.3.1. If V is a symplectic vector space, let V^- denote the same vector space but with the form ω of V replaced by $-\omega$. If V_1 and V_2 are symplectic vector spaces, let $V_1^- \times V_2$ denote the symplectic vector space with the direct sum symplectic structure. A Lagrangian subspace Γ of $V_1^- \times V_2$ is called a *linear canonical relation* from V_1 to V_2. Then define the category, *LinSymp* whose objects are symplectic vector spaces, whose morphisms are linear canonical relations and whose composition law is given by composition of relations. More explicitly, if V_3 is a third symplectic vector space and Γ_1 is a Lagrangian subspace of $V_1^- \oplus V_2$ and Γ_2 is a Lagrangian subspace of $V_2^- \oplus V_3$, then as a set the composition

$$\Gamma_2 \circ \Gamma_1 \subset V_1 \times V_3$$

is defined by

$$(x, z) \in \Gamma_2 \circ \Gamma_1 \Leftrightarrow \exists y \in V_2 \quad \text{such that} \quad (x, y) \in \Gamma_1 and (y, z) \in \Gamma_2.$$

Then the composition defined above is a Lagrangian subspace of $V_1^- \times V_3$. Also the diagonal subspace of $V^- \times V$ acts as the identity morphism and that the associative law holds. Thus the composite of two linear canonical relations is a linear canonical relation. So *Linsymp* is a category.

Definition 4.3.2. Let $(M_i, \omega_i), i = 1, 2$ be symplectic manifolds. A Lagrangian submanifold Γ of $M_1^- \times M_2$ is called a *canonical relation.* For example, if f :

$M_1 \to M_2$ is a symplectomorphism, then $\Gamma_f = graph f$ is a canonical relation.

If $\Gamma_1 \subset M_1^- \times M_2$ and $\Gamma_2 \subset M_2^- \times M_3$, then take the composition as in the above definition.

Definition 4.3.3. Let $\pi : \Gamma_1 \longrightarrow M_2$ denote the restriction to Γ_1 of the projection of $M_1 \times M_2$ onto the second factor, and let $\rho : \Gamma_2 \longrightarrow M_2$ denote the restriction to Γ_2 of the projection of $M_2 \times M_3$ onto the first factor. Let F as the subset $M_1 \times M_2 \times M_3$ consisting of all points (m_1, m_2, m_3) with $(m_1, m_2) \in \Gamma_1$ and $(m_2, m_3) \in \Gamma_2$. The *clean intersection hypothesis* says that F is a manifold and $T_m F$ consists of all $(v_1, v_2, v_3) \in T_{m_1} M_1 \oplus T_{m_2} M_2 \oplus T_{m_3} M_3$ such that $(v_1, v_2) \in T_{(m_1,m_2)}\Gamma_1$ and $(v_2, v_3) \in T_{(m_2,m_3)}\Gamma_2$.

Note 4.3.4. In general $\Gamma_2 \circ \Gamma_1$ need not be a Lagrangian submanifold of $M_1^- \times M_3$. But if the canonical relations $\Gamma_1 \subset M_1^- \times M_2$ and $\Gamma_2 \subset M_2^- \times M_3$ intersect cleanly, then their composition $\Gamma_2 \circ \Gamma_1$ is an immersed Lagrangian submanifold of $M_1^- \times M_3$.

In the symplectic category, choose point object to be the unique connected zero dimensional symplectic manifold and call it "*pt.*". Then a canonical relation between *pt.* and a symplectic manifold M is a Lagrangian submanifold of $pt. \times M$ which may be identified with a Lagrangian submanifold of M. These are the points in our symplectic category.

Suppose that Λ is a Lagrangian submanifold of M_1 and $\Gamma \in Morph(M_1, M_2)$ is a canonical relation. Consider Λ as an element of $Morph(pt., M_1)$, then if Γ and Λ are composible, form $\Gamma \circ \Lambda \in Morph(pt., M_2)$ which may be identified with a Lagrangian submanifold of M_2.

Note 4.3.5. We can modify the Definition 1.4.1 as follows:

Let (M, ω) be a symplectic manifold. G a connected Lie group and ϕ an action of G on M preserving the symplectic form. From Definition 1.1.4, corresponding

to ϕ there is an infinitesimal action $\dot{\phi} : \mathcal{G} \to \mathcal{X}(\mathrm{M})$ by $\dot{\phi}(\xi) = \xi_M$.
In particular, for $p \in M$, there exists a linear map

$$d\phi_p : \mathcal{G} \to T_pM, \qquad \xi \to \xi_M(p);$$

and from ω_p, a linear isomorphism,

$$T_p \to T_p^*, \qquad v \to i_v\omega_p;$$

thus there exists a linear map

$$\widetilde{d\phi_p} : \mathcal{G} \to T_p^*M, \qquad \xi \to i_{\xi_M}\omega_p.$$

Therefore we can redefine Definition 1.4.1 as :

Definition 4.3.6. A G-equivariant map $J : M \to \mathcal{G}^*$ is a *momentum map*, if for every $p \in M$,

$$dJ_p : T_pM \to \mathcal{G}^*$$

is the transpose of the map $\widetilde{d\phi_p}$.

Definition 4.3.7. A symplectomorphism $f : M \to M$ is *Hamiltonian* if there exists a family of symplectomorphisms, $f_t : M \to M, 0 \leq t \leq 1$, depending smoothly on t with $f_o = id_M$ and $f_1 = f$, such that the vector field

$$v_t = f_t^{-1}\frac{df_t}{dt}$$

is Hamiltonian for all t.

Proposition 4.3.8. ξ_M is Hamiltonian for all $\xi \in \mathcal{G}$ if and only if the symplectomorphism, ϕ_g, is Hamiltonian for all $g \in G$.

Proof ξ_M is Hamiltonian for all $\xi \in \mathcal{G}$ if and only if $\xi_M = X_f$, for some $f \in C^\infty(M)$. Let $\rho_t : M \longrightarrow M, t \in \Re$ be the one parameter family of diffeomorphisms generated by X_f :

$$\rho_0 = id_M$$
$$X_f(\rho_t(m)) = \frac{d\rho_t}{dt}(m).$$

Here each diffeomorphism ρ_t preserves ω, that is, ρ_t is a symplectomorphism. Also $X_f = \xi_M = \frac{d}{dt} |_{t=0} \phi_{expt\xi}$. This implies $(\phi_g)_t = \rho_t$. So the family of symplectomorphisms $\rho_t : M \to M, 0 \leq t \leq 1$, makes ϕ_g Hamiltonian for all $g \in G$. •

Remark 4.3.9. From the left action of G on T^*G, one gets a trivialization,

$$T^*G \;\; = \;\; G \times \mathcal{G}^*$$

and using this trivialization one gets a Lagrangian submanifold

$$\Gamma_\phi \;\; = \;\; \{(m, \phi_g(m), g, J(m)); m \in M, g \in G\},$$

of $M \times M^- \times T^*G$, which is called the *moment Lagrangian.* This can be viewed as a canonical relation between $M \times M^-$ and T^*G, or as a map

$$\Gamma_\phi : M^- \times M \to T^*G.$$

From the modulo clean intersection hypotheses, such a map maps Lagrangian submanifolds of $M^- \times M$ onto Lagrangian submanifolds of T^*G and vice versa. Also the diagonal in $M^- \times M$ gets mapped by Γ_ϕ into a disjoint union of Lagrangian submanifolds of T^*G.

Note 4.3.10. Next we look at momentum maps in a more general set up.

Definition 4.3.11. Let G be a Lie group acting canonically on the symplectic manifold (M, ω) via the action ϕ. Suppose there exist a momentum map $J : M \to \mathcal{G}^*$ for the associated Lie algebra action. Then J can be viewed as $J : M \times \{e\} \to \mathcal{G}^*$. Using right translation we can extend J to the whole of $M \times G$. We know that if the action is symplectic $r_g^* = (Ad_{g^{-1}})^*$. Therefore we can define $\widetilde{J} : M \times G \to \mathcal{G}^*$ as

$$< \widetilde{J}(mg), \xi >:=< J(m), Ad_{g^{-1}}\xi >, \forall \xi \in \mathcal{G}.$$

Theorem 4.3.12. The map $\widetilde{J} : M \times G \to \mathcal{G}^*$ satisfies

$$d < \widetilde{J}(mg), \xi >= i_{(\phi_g)_* \xi_M} \omega, \forall \xi \in \mathcal{G}.$$

Proof: For any $m \in M, g \in G$ and $\xi \in \mathcal{G}$,

$$\begin{aligned} d < \widetilde{J}, \xi > (mg) = d < J, Ad_{g^{-1}}\xi > (m) \\ = i_{(Ad_{g^{-1}}\xi)_M} \omega \quad & \text{(since J is a momentum map)} \\ = i_{(\phi_g)_* \xi_M} \omega. \end{aligned}$$

Thus $\widetilde{J}$ is a momentum map. •

Note 4.3.13. We discuss the generalization of the momentum map, where the group action is replaced by a family of symplectomorphisms.

Let (M, ω) be a symplectic manifold, S an arbitrary manifold and $f_s, s \in S$, a family of symplectomorphisms of M depending smoothly on s. For $p \in M$ and $s_o \in S$, let $g_{s_o,p} : S \to M$ be the map, $g_{s_o,p}(s) = f_s \circ f_{s_o}^{-1}(p)$. Then the derivative at s_o is given by

$$(dg_{s_o,p})_{s_o} : T_{s_o}S \to T_pM.$$

From this we get the linear map

$$\widetilde{(dg_{s_o,p})}_{s_o} : T_{s_o}S \to T_p^*M.$$

Now, let J be the map of $M \times S$ into T^*S which is compatible with the projection, $M \times S \to S$ in the sense

$$\begin{array}{ccc} M \times S & \xrightarrow{J} & T^*S \\ & \searrow & \downarrow \\ & & S \end{array}$$

commutes; and for $s_o \in S$ let

$$J_{s_o} : M \to T^*_{s_o}S$$

be the restriction of J to $M \times \{s_o\}$.

Definition 4.3.14. Let (M, ω) be a symplectic manifold, S an arbitrary manifold and $f_s, s \in S$, a family of symplectomorphisms of M depending smoothly on s. The map of $J : M \times S \longrightarrow T^*S$ is a *momentum map* if, for all s_o and p,

$$(dJ_{s_o})_p : T_pM \to T^*_{s_o}S$$

is the transpose of the map $\widetilde{(dg_{s_o,p})}_{s_o}$.

Note 4.3.15. Now we analyze in a more general set up which does not involve group action and prove a sufficient condition for the existence of momentum map.

We will generalize by assuming that the f_s's are canonical relations rather than canonical transformations. Then replace $M \times M^-$ by M itself and canonical relations by Lagrangian submanifolds of M.

Let (M, ω) be a symplectic manifold . Let Z, X and S be manifolds and suppose that

$$\pi : Z \to S$$

is a fibration with fibers diffeomorphic to X. Let

$$G : Z \to M$$

be a smooth map and let

$$g_s : Z_s \to M, Z_s := \pi^{-1}s$$

denote the restriction of G to Z_s. We assume that g_s is a Lagrangian embedding and let

$$\Lambda_s := g_s(Z_s)$$

denote the image of g_s. Thus, for each $s \in S$, G imbeds the fiber, $Z_s = \pi^{-1}s$, into M as the Lagrangian submanifold, Λ_s. For $z \in Z_s$ and $w \in T_z Z_s$ tangent to the fiber Z_s,

$$dG_z w = (dg_s)_z w \in T_{G(z)}\Lambda_s.$$

So, dG_z induces a map, denoted again by dG_z

$$dG_z : \frac{T_z Z}{T_z Z_s} \to \frac{T_m M}{T_m \Lambda_s}, \qquad m = G(z).$$

But $d\pi_z$ induces an identification

$$\frac{T_z Z}{T_z Z_s} \quad = \quad T_s S.$$

From the linear isomorphism

$$T_m M \longrightarrow T_m^* M \quad by \quad u \longrightarrow \omega_m(u, .) \tag{4.3}$$

we have an identification

$$\frac{T_m M}{T_m \Lambda_s} \quad = \quad T_m^* \Lambda_s.$$

Using the identifications, we have

$$dG_z : T_s S \longrightarrow T_z^* Z_s. \tag{4.4}$$

Now, let $J : Z \to T^*S$ be a lifting of $\pi : Z \to S$, so that

$$\begin{array}{ccc} Z & \xrightarrow{J} & T^*S \\ \pi & \searrow & \downarrow \\ & & S \end{array}$$

commutes, and for $s \in S$, let

$$J_s : Z_s \to T_s^* S$$

be the restriction of J to Z_s.

Definition 4.3.16. J is a momentum map if, for all s and all $z \in Z_s$,

$$(dJ_s)_z : T_z Z_s \to T_s^* S$$

is the transpose of dG_z.

Note that this condition determines J_s up to an additive constant $\nu_s \in T_s^* S$ and hence, determines J up to a section $s \to \nu_s$, of $T^* S$.

We have an embedding

$$(G, J) : Z \to M \times T^* S.$$

from the momentum map $J : Z \to T^* S$.

Theorem 4.3.17. Let (M, ω) be a symplectic manifold . Let Z, X and S be manifolds and suppose that $\pi : Z \to S$ is a fibration with fibers diffeomorphic to X. Let $G : Z \to M$ be a smooth map and J is a momentum map. The pull back by (G, J) of the symplectic form on $M \times T^* S$ is the pull back by π of a closed two form ρ on S. If $[\rho] = 0$, there exists a momentum map, J, for which the imbedding (G, J) is Lagrangian.

Proof: Consider the map $dG_z : T_s S \longrightarrow T_z^* Z_s$.

Claim I: If s fixed, but let z vary over Z_s, then for each $\xi \in T_s S$ gives rise to a one form τ^ξ on Z_s with $d\tau^\xi = 0$.

Proof of Claim I: Let us choose a trivialization of the bundle around Z_s to give an identification $H : Z_s \times U \longrightarrow \pi^{-1}(U)$ where U is a neighborhood of s in S. If $t \longrightarrow s(t)$ is any curve on S with $s(0) = s, s'(0) = \xi$, we get a curve of maps $h_{s(t)}$ of $Z_s \longrightarrow M$ where $h_{s(t)} = g_{s(t)} \circ H$. We thus get a vector field v^ξ along the map

h_s

$$v^\xi : Z_s \longrightarrow TM, \qquad v^\xi(z) = \frac{d}{dt} h_{s(t)}(z) \mid_{t=0} .$$
$$\text{Then define, } \tau^\xi \;=\; h_s^*(i_{v^\xi}\omega).$$

Now, the general form of the Weil formula and the fact that $d\omega = 0$ gives

$$(\frac{d}{dt} h_{s(t)}^* \omega) \mid_{t=0} = dh_s^* i(v^\xi)\omega$$

and the fact that Λ_s is Lagrangian for all s implies that the left hand side is zero. Therefore $d\tau^\xi = 0$. Hence claim I.

Assume that for all s and ξ the one form τ^ξ is exact. Then $\tau^\xi = dJ^\xi$ for some C^∞ function J^ξ on Z_s. The function J^ξ is uniquely determined up to an additive constant (if Z is connected) which can fix so that it depends smoothly on s and linearly on ξ.

Then for a fixed $z \in Z_s$, the number $J^\xi(z)$ depends linearly on ξ. Hence we get a map

$$J_o : Z \to T^*S, \quad with$$
$$J_o(z) = \lambda \Leftrightarrow \lambda(\xi) = J^\xi(z).$$

If Z is connected, the choice determines J^ξ up to an additive constant $\mu(s, \xi)$ which we can assume to be smooth in s and linear in ξ. Replacing J^ξ by $J^\xi + \mu(s, \xi)$ has the effect of making the replacement

$$J_o \longmapsto J = J_o + \mu \circ \pi$$

where $\mu : S \to T^*S$ is the one form $< \mu(s), \xi >= \mu(s, \xi)$. Thus we get a map

$J : Z \to T^*S$ defined by $J := J_o + \mu \circ \pi$.

Claim II: J is a momentum map.

Proof of Claim II: For, the restriction of J to the fiber Z_s maps $Z_s \to T^*_s S$. Hence, for $z \in Z_s$,

$$dJ_z : T_z Z_s \to T^*_s S$$

is a linear map. Also we have the map

$$dG_z : T_s S \longrightarrow T^*_z Z_s.$$

Now, for each $\xi \in T_s S$ gives rise to a one form τ^ξ on Z_s, the value of this one form at $z \in Z_s$ is exactly $dG_z(\xi)$.
Indeed, for any $w \in T_z Z_s$

$$\begin{aligned} \tau^\xi(w) &= h^*_s(i_{v^\xi}\omega)(w) \\ &= \frac{d}{dt} h_{s(t)}(z) \mid_{t=0} \text{by the identification 4.3} \\ &= (dg_s)_z(\xi)(w) = dG_z(\xi)(w). \end{aligned}$$

The function J^ξ was defined on Z_s so as to satisfy $dJ^\xi = \tau^\xi$. In other words, for $v \in T_z Z$

$$< dG_z(\xi), v >=< dJ_z(v), \xi > .$$

Thus the maps dJ_z and dG_z defined above are transposes of one another.

Also the Kernel of dG_z is the annihilator of the image of the map dJ_z. In particular , z is a regular point of the map $J : Z_s \to T^*_s S$ if the map dG_z is injective. Also the Kernel of the map dJ_z is the annihilator of the image dG_z.

Hence claim II.

Claim III: Let ω_S denote the canonical two form on T^*S. Then there exists a closed two form ρ on S such that

$$G^*\omega + J^*\omega_S = \pi^*\rho. \tag{4.5}$$

If $[\rho] = 0$, then there is a one form ν on S such that if we set $J = J_o + \nu \circ \pi$, then $G^*\omega + J^*\omega_S = 0$. As a consequence, the map $\widetilde{G} : Z \to M \times T^*S$, given by $\widetilde{G}(z) = (G(z), J(z))$ is a Lagrangian embedding.

Proof of Claim III : We first prove a local version of the statement. Locally we may assume that $Z = X \times S$. This means that we have an identification of Z_s with X for all s. We may assume that $M = T^*X$ and that for a fixed $s_0 \in S$ the Lagrangian submanifold Λ_{s_0} is the zero section of T^*X and that the map

$$G : X \times S \longrightarrow T^*X \text{ is given by } G(x,s) = d_X\psi(x,s)$$

where $\psi \in C^\infty(X \times S)$. So, in terms of these choices, the maps $h_{s(t)}$ used in the proof of claim I are given by $h_{s(t)}(x) = d_X\psi(x, s(t))$ and hence the one form τ^ξ is given by

$$d_S d_X \psi(x, \xi) = d_X < d_S\psi, \xi > .$$

So, we may choose $J(x,s) = d_S\psi(x,s)$.
Thus, $G^*\alpha_X = d_X\psi$ and $J^*\alpha_S = d_S\psi$. Hence, $G^*\omega_X + J^*\omega_S = -dd\psi = 0$, which proves the local version of the statement.

We now pass from the local to global: By uniqueness, our global J_0 must agree with our local J up to the replacement $J \longrightarrow J + \mu \circ \pi$. Therefore, we know that

$$G^*\omega + J_0^*\omega_S = (\mu \circ \pi)^*\omega_S = \pi^*\mu^*\omega_S.$$

Here μ is a one form on S regarded as a map $S \longrightarrow T^*S$. But

$$d\pi^*\mu^*\omega_S = \pi^*\mu^*d\omega_S = 0.$$

So, we know that $G^*\omega + J_0^*\omega_S$ is a closed two form which is locally and hence globally of the form $\pi^*\rho$ where $d\rho = 0$.

Now, suppose that $[\rho] = 0$ and hence $\rho = d\nu$ for some one form ν on S. Replacing J_0 by $J_0 + \nu$ replaces ρ by $\rho + \nu^*\omega_S$, but

$$\nu^*\omega_S = -\nu^*d\alpha_S = -d\nu = -\rho.$$

Hence claim III.

From the equation 4.5 the pull back by (G, J) of the symplectic form on $M \times T^*S$ is the pull back by π of a closed two form ρ on S. If $[\rho] = 0$, from claim II J is a momentum map and from claim III the imbedding (G, J) is Lagrangian. •

Theorem 4.3.18. Let J be a map of Z into T^*S lifting the map, π, of Z into S. Then, if the imbedding (G, J) is Lagrangian, J is a momentum map.

Proof It suffices to prove for $Z = X \times S, M = T^*X$ and $G(x, s) = d_X\psi(x, s)$ where $\psi \in C^\infty(X \times S)$. If $J : X \times S \to T^*S$ is a lifting of the projection $X \times S \to X$, then (G, J) can be viewed as a section of $T^*(X \times S)$, that is as a one form β on $X \times S$. If (G, J) is a Lagrangian embedding, then β is closed. Moreover the $(1, 0)$ component of β is $d_X\psi$ so $\beta - d\psi$ is a closed form and hence is of the form $\mu \circ \pi$

for some closed form on S. This shows that

$$J = d_S\psi + \pi^*\mu$$

and hence, as above, J is a momentum map. •

Note 4.3.19. The Definition 4.3.14 can be treated as a special case of Definition 4.3.16.

Let (M, ω) be a symplectic manifold, S a manifold and $F : M \times S \to M$ a smooth map such that $f_s : M \to M$ is a symplectomorphism for each s, where $f_s(m) = F(m, s)$. Let $\Lambda_s \subset M \times M^-$ is the graph of f_s and G is the map

$$G : M \times S \to M \times M^-, \quad by \quad G(m, s) = (m, F(m, s)).$$

Apply the results in the above section there exists a map

$$J : M \times S \to T^*S.$$

Assume that J exists, then consider the analogue for J of Weinstein's moment Lagrangian, $\Gamma_J = \{(m, f_s(m), J(m, s)); m \in M, s \in S\}$, and consider the imbedding of $M \times S$ into $M \times M^- \times T^*S$ given by the map J of Weinstein's moment Lagrangian,

$$G : M \times S \to M \times M^- \times T^*S \quad where \quad G(m, s) = (m, f_s(m), J(m, s)).$$

From Theorem 4.3.17 we get the following theorems.

Theorem 4.3.20. The pull back by G of the symplectic form on $M \times M^- \times T^*S$ is the pull back by the projection, $M \times S \to S$ of a closed two-form, μ, on S.

If μ is exact, that is, if, $\mu = d\nu$, we can modify J by setting

$$J_{new}(m,s) \;\;=\;\; J_{old}(m,s) - \nu_s,$$

and for this modified J, the pull back by G on the symplectic form on $M \times M^- \times T^*S$ will be zero; so, the theorem:

Theorem 4.3.21. If μ is exact, there exist a momentum map, $J : M \times S \to T^*S$, for which Γ_J is Lagrangian.

The following converse result is also true.

Theorem 4.3.22. Let J be a map of $M \times S$ into T^*S which is compatible with the projection of $M \times S$ onto S. Then, if Γ_J is Lagrangian, J is a momentum map.

Note 4.3.23. The definition of the momentum map for Hamiltonian group actions can be treated as a special case of Definition 4.3.16.

Suppose that a compact Lie group K acts as fiber bundle automorphisms of $\pi : Z \to S$ and act as symplectomorphisms of M. Suppose further that the fibers of Z are compact and equipped with a density along the fiber which is invariant under the group action. Also the map G is equivariant for the group actions of K on Z and on M. Then, the map J is equivariant for the actions of K on Z and the induced action of K on $M \times T^*S$.

Assume that S is a Lie group K and that $F : M \times K \to M$ is a Hamiltonian group action. This gives a map

$$G : M \times K \to M \times M^-, \qquad G(m,a) = (m, am).$$

Let K act on $Z = M \times K$ via its left action on K. Thus $a \in K$ acts on Z as $a(m,b) = (m, ab)$. To say that the action, F, is Hamiltonian with momentum map

$J : M \to \mathcal{K}^*$ is to say that $i(\xi_M)\omega = -d < J, \xi >$ where $\mathcal{K}$ is the Lie algebra of K.

Thus under the left invariant identification of T^*K with $K \times \mathcal{K}^*$, J determines a momentum map

$$\widetilde{J} : M \times K \to T^*K, \qquad \widetilde{J}(m, a) = (a, J(m)).$$

So $\widetilde{J}$ is indeed a generalization of the momentum map for Hamiltonian group actions.

Remark 4.3.24. Mikami and Weinstein [27] shown that some of the momentum maps above introduced can be interpreted as the momentum maps associated to some groupoid action naturally defined on the symplectic manifold.

Thus a Hamiltonian action of a Lie group should be seen as a special case of more flexible notion of symmetry-the action of a symplectic groupoid. Such an action always comes equipped with a momentum map. It is natural to ask whether properties of momentum maps of Hamiltonian group actions extend to the groupoid case. In this direction Weinstein [39] has given an extension of the Theorem 2.2.30.

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