Ph.D THESIS **MATHEMATICS**

SOME LATTICE THEORETIC PROBLEMS RELATED TO GENERAL AND FUZZY TOPOLOGY

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CERTIFICATE

I hereby certify that the work presented in the thesis entitled " ***Some lattice theoretic problems related to general and fuzzy topology*** " is a bona fide work carried out by Sri. Baby Chacko, under my guidance for the award of the Degree of Ph.D in Mathematics of the University of Calicut, and that this work has not been included in any other thesis submitted previously for the award of any degree.

Dr. Ramachandran P.T.
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DECLARATION

I declare that the work presented in the thesis entitled " ***Some lattice theoretic problems related to general and fuzzy topology*** " is based on the original work done by me under the guidance of Dr. Ramachandran P.T., Reader, Department of Mathematics, University of Calicut and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

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Some Lattice Theoretic Problems Related to General and Fuzzy Topology

CONTENTS

CHAPTER

0

Introduction

As the title indicates, this thesis discusses some lattice theoretic problems related to general topology and fuzzy topology. The content of the thesis comprises of four chapters. The general topological part (Chap.1), concentrates mainly on the automorphisms of the lattice of Čech closure operators on a fixed set. The fuzzy topological part (Chap.2, 3 & 4), deals with problems on the automorphisms of the lattice of fuzzy topologies on a set and that of the lattice of fuzzy Čech closure operators and some other topics related to fuzzy topologies and fuzzy Čech closure operators.

The introduction of the idea of metric spaces by Frechet (1906) marked the beginning of a new discipline called set topology. The works of people like Hausdorff, Kuratowski [21], Tychonoff, Stone A. H. and Dieudonne were pioneering contributions to this area. A topological space is defined to be an ordered pair

(X, τ) such that X is a set and τ is a collection of subsets of X which is closed for the operations of finite intersection and arbitrary union and containing the empty set ϕ and the set X. The elements of τ are called open sets.

A Čech closure operator V on a set X is a function $V : \wp(X) \longrightarrow \wp(X)$ such that,

(i) $V(\phi) = \phi$

(ii) $A \subseteq V(A)$ for all $A \in \wp(X)$

(iii) $V(A \cup B) = V(A) \cup V(B)$ for all $A, B \in \wp(X)$, where $\wp(X)$ denotes the power set of X.

The pair (X, V) is called a Čech closure space. A subset A in a Čech closure space (X, V) is said to be open if $V(X - A) = X - A$. The set of all open sets in (X, V) form a topology on X, called the topology associated with V.

On the other hand, to every topology τ on X, we can associate a closure operator 'cl' on X called the Kuratowski closure operator on $\wp(X)$ by defining '$cl(A)$' as the smallest closed set in (X, τ) containing A. This closure operator 'cl' satisfies all the three conditions of a Čech closure operator and an additional condition $cl(cl(A)) = cl(A)$ for every $A \in \wp(X)$.

A Čech closure operator on a set is not uniquely determined by its associated topology. That is, two different Čech closure operators on a set can have the same topology as the associated topology. If a Čech closure operator V on a set X satisfies the additional condition $V(V(A)) = V(A)$ for every $A \in \wp(X)$, then the Kuratowski closure operator of the associated topology of V is V itself and in this case, V is also said to be topological. Thus Čech closure spaces can be considered as a generalization of topological spaces.

It is natural to ask why we search for more general spaces other than topological spaces, namely Čech closure spaces, for topological investigations. The main answer is that there are some important Čech closure spaces which are not topological. Also a great deal of basic definitions and theorems for topological spaces carries over to Čech closure spaces and hence studies of such a general structure will help one to do further investigations in topological spaces. Finally, Čech closure spaces can be adopted effectively as a general background for the study of various continuous structures, such as uniform spaces, proximity spaces etc. The consideration of such general spaces is also useful frequently in the study of topological spaces.

A subset A of the universal set X can be equivalently represented by its characteristic function - a mapping χ_A from X to the 2-valued set $\{0, 1\}$ defined by $\chi_A(x) = 1$ if x belongs to A and 0, otherwise. Hence there are only two grades of membership for the elements of X namely, 1 and 0 representing either belonging to A or not. Since the nineteenth century, mathematicians have commonly used the concepts of sets and functions to represent physical problems. In many circumstances these representations seem to be meaningless or imprecise. For example, the class of " animals " includes dogs, cows, horses etc. as its members and clearly excludes such objects as rocks, fluids, plants etc. However such objects as starfish, bacteria etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 40 in relation to the class of " all real numbers which are much greater than 1 ". Clearly, the class of " all real numbers which are much greater than 1 " or the class of " all beautiful women ", or the class of " all tall men " do not constitute classes or sets in usual mathematical sense of these terms. Yet the fact remains that such imprecisely defined " classes " play an important role in human thinking, particularly in the domains of pattern recognition and communication of information.

In 1965, Zadeh, L. A. [40], an American electrical engineer introduced a concept which could be used in dealing with the " classes " of the type mentioned above. The concept is that of a fuzzy set, that is, a " class " with a continuum of grades of membership. The concept of fuzzy sets parallels in many respects the ordinary sets, but is more general and has a wider scope of applicability. Almost all concepts in mathematics have been redefined using fuzzy sets.

In, " fuzzy " case the " belonging to " relation $\chi_A(x) = 1$ between x and A is no longer " 0 or otherwise 1 ", it has a degree of " belonging to ", i.e. a membership degree such as 0.6 or 0.7. Therefore, the range has to be extended from the set $\{0, 1\}$ to the closed interval $[0, 1]$; or more generally, to a lattice L, because all the membership degrees, in mathematical view, form an ordered structure, a lattice.

A fuzzy set on a universal set X is simply a mapping from X to a lattice L. Thus, the fuzzy set has extended the basic mathematical concept of a set. Many mathematicians, while developing fuzzy set theory, have used different lattice structures for the membership sets like, complete and completely distributive lattice with order reversing involution by Hutton, B. & I. Reilly [18], complete and completely distributive non-atomic Boolean algebra by Mira Sarkar [31], complete and distributive lattice by Rodabaugh, S. E. [36] and Lou, S. P. [28], complete Boolean algebra by Hohle, Ulrich [17], complete chain by Conard, F. [12] and Badard, Robert [7].

In view of the fact that the set theory is the cornerstone of mathematics and fuzzy set theory being a generalization of set theory, a new and more general frame work of mathematics was established. Fuzzy mathematics is just a kind of mathematics developed in this frame work, and fuzzy topology is just a kind of topology defined using fuzzy sets. As the theoretical development is concerned, fuzzy topology seems to be one of the most fruitful branches of fuzzy mathematics. Denote the family of all the fuzzy sets on the universe X, which takes lattice L as the range, by L^X. From the partial order in L, it is easy to equip L^X with a partial order pointwise, then L^X is also a lattice. Substituting inclusion relation by the order relation in L^X, we introduce a topological structure naturally into L^X. Thus fuzzy topology is a common carrier of two important structures in mathematics, namely ordered structure and topological structure, or more specifically, it is a kind of topology on lattice. Therefore, even if we consider only its pure mathematical significance but not its practical background, fuzzy topology has an important value for research.

In 1968, Chang, C. L. [11] introduced, a new branch in mathematics, namely, the fuzzy topology. He used the unit closed interval $I = [0, 1]$ as the membership set. Since then an extensive study of fuzzy topological spaces has been carried out by many mathematicians. The works of Lowen, R. [29], Goguen, J. A. [14], Hutton, B. & I. Reilly [18], Ying Ming Liu [26], Ulrich Hole [17], Lou S. P.[28] and Mira Sarkar [31] were remarkable in this area. In this thesis, we take the definition of fuzzy topology given by Chang, with a complete and distributive lattice with an order reversing involution (F-lattice) as the membership set.

The first chapter deals with Čech closure operators on a fixed set. The set of all Čech closure operators on a set X, denoted by $LC(X)$, is a complete lattice under the partial order '$\leq$' defined by $V_1 \leq V_2$ if and only if $V_2(A) \subseteq V_1(A)$ for every subset A of X. We show that when X is finite, the group of automorphisms of the lattice $LC(X)$ is in one to one correspondence with the group of bijections on the set $(X \times X) - \Delta$ where $\Delta = \{(x, x) : x \in X\}$. We also discuss some automorphisms of $LC(X)$ when X is infinite. The automorphisms of the lattice $LQ(X)$ of all quasi-discrete Čech closure operators and that of the sublattice $[I, C_0]$ of $LC(X)$ are also determined.

The second chapter is concerned with fuzzy topologies. In 1989, Babusundar S. [2] defined t-irreducible subsets in lattices and established that ultra fuzzy topologies exist if and only if minimal t-irreducible subsets exist in the corresponding membership lattice. In the beginning of this chapter, we travel through this direction and prove some results connecting fuzzy topologies and t-irreducible subsets. Using the notion of ultra fuzzy filters, we have defined principal ultra fuzzy topologies and some of its properties are studied. We have also discussed some relations between fuzzy topologies on a set X with membership lattice L and pre-order relations on $\mathrm{Pt}(L^X)$, the set of all fuzzy points in L^X.

In the third chapter, we mainly discuss the automorphisms of the lattice of all fuzzy topologies on a fixed set. The set of all fuzzy topologies on a set X with membership lattice L form a complete lattice under the partial order of set inclusion. This lattice has been denoted by $LFT(X, L)$ and some classes of automorphisms of this lattice have been determined. Also we determine the group of automorphisms of the lattice $LFT(X, L)$ in the case of the particular membership lattice namely, $L = \{0, 1/2, 1\}$ with usual order.

Let L^X denote the set of all fuzzy subsets of X with an F-lattice as the membership set. A fuzzy Čech closure operator ψ on X is a function $\psi : L^X \longrightarrow L^X$ such that,

(i) $\psi(\underline{0}) = \underline{0}$ where $\underline{0}$ is the smallest element of L^X

(ii) $A \leq \psi(A)$ for all $A \in L^X$

(iii) $\psi(A \vee B) = \psi(A) \vee \psi(B)$ for all $A, B \in L^X$.

The pair (X, ψ) or (L^X, ψ) is called a fuzzy Čech closure space. A subset A in a fuzzy Čech closure space (X, ψ) is said to be open if $\psi(A') = A'$ where A' denotes the pseudo-complement of A in L^X. The set of all open sets in (X, ψ) is a fuzzy topology on X, called the fuzzy topology associated with ψ. The fuzzy Čech closure operators are generalizations of fuzzy topological spaces.

The final chapter is a study of fuzzy Čech closure operators. In 1985, Mashhour A. H. and Ghanim M. H. [30] defined fuzzy Čech closure spaces and extended the concept of Čech proximity to fuzzy topology. In 1990, Johnson T. P. [19] made an attempt to study the lattice structure of the set of all fuzzy Čech closure spaces on a fixed set. The set of all fuzzy Čech closure operators on a set X is a complete lattice under the partial order '$\leq$' defined by $\psi_1 \leq \psi_2$ if and only if $\psi_2(A) \leq \psi_1(A)$ for every fuzzy subset A of X. We have denoted this lattice by $LFC(X, L)$ where L is the membership lattice and have determined some classes of automorphisms of this lattice. Also, we have defined quasi-discrete fuzzy Čech closure operators on a set as an extension of Čech's [10] definition of quasi-discrete Čech closure operators and have shown that the study of quasi-discrete fuzzy Čech closure operators on a set X is equivalent to the study of reflexive relations on the set of all fuzzy points in L^X. Further, we have defined the fuzzy topological modification $\tau\psi$ of a fuzzy Čech closure operator ψ on a set X and proved some results related to it. Finally, we have defined a fuzzy pseudo metric and have shown that to every fuzzy pseudo metric, a fuzzy Čech closure operator is associated and proved that the fuzzy Čech closure operator induced by a fuzzy pseudo metric is fuzzy topological.

For definitions and details we refer the following texts : For topology, Čech, E. [10] and Willard, Stephen [39] ; for fuzzy topology, Liu, Ying-Ming & Mao-Kang, Luo [27] ; for fuzzy set theory, Bojadziev, G. & Bojadziev, M. [9] ; for lattice theory, Birkhoff, Garrett [8] and Gratzer, George [15].

CHAPTER

1

Some Problems Related to Čech Closure Operators

1.1 Introduction

The set of all Čech closure operators on a set X is a complete lattice which is denoted by $LC(X)$. In this chapter, we discuss some basic properties of Čech closure operators and characterize the group of automorphisms of the lattice $LC(X)$ when X is finite. We also determine the group automorphisms of some other lattices related to $LC(X)$ namely, the lattice of quasi-discrete closure operators $LQ(X)$ and the interval $[I, C_0]$ of $LC(X)$. The definitions and theorems quoted in this section, are mainly taken from [10] and [33] and will be used in the subsequent sections.

Notation 1.1.1. $\wp(X)$ denotes the power set of a set X.

Definition 1.1.2. A Čech closure operator on a set X is a function $V : \wp(X) \longrightarrow \wp(X)$ such that,

(i) $V(\phi) = \phi$

(ii) $A \subseteq V(A)$ for all $A \in \wp(X)$

(iii) $V(A \cup B) = V(A) \cup V(B)$ for all $A, B \in \wp(X)$.

For brevity, we call V a closure operator on X and the pair (X, V) a closure space.

Definition 1.1.3. Let (X, V) be a closure space. A subset A of X is said to be closed, if $V(A) = A$ and open, if $V(X - A) = X - A$. The closed sets of (X, V) are exactly the fixed points of V. By definition itself, $A \subseteq V(A)$ for all $A \subseteq X$. Thus a subset A of X is closed, if $A \supseteq V(A)$.

Remark 1.1.4. In a closure space, the closure of a set need not be closed. For example, let $X = \{x, y, z\}$ and V be the closure operator defined by $V(\{x\}) = \{x, y\}$, V({y})={y,z}, $V(\{z\}) = \{z, x\}$. (Note that a closure operator on a finite set is determined by the closures of singleton sets). Then the closure of $\{x\}$ is not closed, since $V(V(\{x\})) = V(\{x, y\}) = \{x, y, z\} \neq V(\{x\})$.

Remark 1.1.5. The set of all open sets in (X, V) is a topology on X, called the topology associated with V. On the other hand, to every topology τ on X, we can associate a closure operator V on X (the Kuratowski closure operator) defined by $V(A) = cl(A)$ where $cl(A)$ denotes the closure of A in (X, τ). We say that V is the closure operator associated with τ. Note that a closure operator need not be the closure operator associated with the topology associated with it.

Definition 1.1.6. A closure operator V on a set X is said to be topological, if $V(V(A)) = V(A)$ for every $A \in \wp(X)$.

Example 1.1.7. Let $V : \wp(X) \longrightarrow \wp(X)$ be defined by $V(A) = A$ for all $A \in \wp(X)$. Then V is a closure operator on X, called the discrete closure operator. This closure operator is topological and defines the discrete topology on X. The discrete closure operator is usually denoted by D.

Example 1.1.8. Let $V : \wp(X) \longrightarrow \wp(X)$ be defined by

$$V(A) = \begin{cases} A & \text{if } A = \phi, \\ X & \text{otherwise.} \end{cases}$$

Then V is a closure operator on X, called the indiscrete closure operator. This closure operator is topological and defines the indiscrete topology on X. The indiscrete closure operator is usually denoted by I.

Example 1.1.9. Let X be an infinite set. Define $V : \wp(X) \longrightarrow \wp(X)$ by

$$V(A) = \begin{cases} A & \text{if } A \text{ is finite,} \\ X & \text{otherwise.} \end{cases}$$

Then V is a closure operator on X, called the co-finite closure operator. This closure operator is topological and defines the co-finite topology on X. The co-finite closure operator is usually denoted by C_0.

Example 1.1.10. The closure operator mentioned in Remark 1.1.4 is not topological.

Example 1.1.11. Let X be the set of all functions on the unit interval $I = [0, 1]$ of real numbers and let $V(A)$ where $A \subseteq X$, be the set of all functions f such that some sequence $\{f_n\}$ in A converges pointwisely to f. Then V is a closure operator on X which is not topological.

Remark 1.1.12. Note that the closure operator associated with a topology is a topological closure operator.

Example 1.1.13. Let V_1 be the closure operator mentioned in Remark 1.1.4 and V_2 be the indiscrete closure operator I defined in Example 1.1.8. Then $V_1 \neq V_2$, but both have the indiscrete topology as the associated topology.

Definition 1.1.14. Let (X, V) be a closure space. A subset B of X is said to be dense in a subset A of X, if $B \subseteq A$ and $A \subseteq V(B)$. A subset A of (X, V) is said to be dense in (X, V), if $V(A) = X$.

Remark 1.1.15. The relation $\{(A, B) : A, B$ subsets of (X, V) such that A is dense in $B\}$, completely determines the closure structure of the closure space (X, V). Indeed, $V(A)$ is the union of all sets in which A is dense.

Theorem 1.1.16. *A closure space (X, V) is topological, if and only if the relation $\{(A, B) : A, B$ subsets of (X, V) such that A is dense in $B\}$ is transitive (see [10]).*

Remark 1.1.17. Let V be a closure operator on X and let τ be the associated topology of V. If K is the closure operator associated with τ (the Kuratowski closure operator), then $K \leq V$, because for $A \in \wp(X)$,

$$\begin{aligned} K(A) &= \bigcap\{F \in \wp(X) : V(F) = F \text{ and } F \supseteq A\} \\ &\supseteq \bigcap\{F \in \wp(X) : V(F) = F \text{ and } V(F) \supseteq V(A)\} \\ &= \bigcap\{F \in \wp(X) : F \supseteq V(A)\} \\ &= V(A). \end{aligned}$$

Remark 1.1.18. If a closure operator V is topological, then the closure operator corresponding to the associated topology of V is V itself, for, if K is the closure operator corresponding to the associated topology of V, then $K \leq V$ (Remark 1.1.17). On the other hand, for $A \in \wp(X)$, $K(A)$ is the smallest closed set containing A. But if V is topological, $V(V(A)) = V(A)$ so that $V(A)$ is a closed set containing A. Therefore $K(A) \subseteq V(A)$ for all $A \in \wp(X)$. That is, $V \leq K$ and hence $K = V$.

Since the topological closure operators are the closure operators associated with topology associated with it, we can consider a closure operator as a generalization of a topological space.

Remark 1.1.19. By induction, we obtain at once from the condition (iii) of the definition of a closure operator that

$\bigcup V(A_j) = V(\bigcup(A_j))$ ....(*)

for every finite family of subsets of a closure space (X, V). Formula (*) need not be true if the family is infinite. For example, let X be an infinite set and fix a point x of X. Define a closure operator V on X by setting

$$V(A) \quad = \quad \begin{cases} A & \text{if } A \text{ is finite,} \\ A \cup \{x\} & \text{otherwise.} \end{cases}$$

Then the family $\{A_b : b \in B\}$ where $A_b = \{b\}$ and $B \subseteq X - \{x\}$ fulfils the condition (*) if and only if B is finite.

Definition 1.1.20. Let V_1 and V_2 be two closure operators on a set X. Then V_1 is said to " coarser than " V_2 (or V_2 is said to finer than V_1) if $V_1(A) \supseteq V_2(A)$ for all $A \in \wp(X)$. In this case we write $V_1 \leq V_2$.

Remark 1.1.21. The relation " coarser than " is a partial order on the set of all closure operators on X. We denote the set of all closure operators on a set X by $LC(X)$. Then $LC(X)$ is a lattice under the relation " coarser than ". The least element of $LC(X)$ is I, the indiscrete closure operator and the greatest element is D, the discrete closure operator.

Theorem 1.1.22. *$LC(X)$ is a complete lattice.*

Remark 1.1.23. Let V_1, V_2 be two closure operators on a set X. If we define $V : \wp(X) \longrightarrow \wp(X)$ by $V(A) = V_1(A) \cup V_2(A)$ for $A \in \wp(X)$, then V is a closure operator on X. But if $V : \wp(X) \longrightarrow \wp(X)$ is defined by,

$V(A) = V_1(A) \cap V_2(A)$ for $A \in \wp(X)$, ...(*)

then V need not be a closure operator on X, as V need not satisfy the third condition for a closure operator. For example, let $X = \{1, 2, 3\}$ and $V_i, i = 1, 2$ be the closure operators for which $V_1(\{1\}) = \{1, 3\}, V_2(\{2\}) = \{2, 3\}$ and $V_i(\{j\}) =$

$\{j\}$ in the remaining cases. Let V be defined by (*). Then $V(\{j\}) = \{j\}$ for each j = 1,2,3. But $V(\{1,2\}) = \{1,2,3\}$ and hence $V(\{1\}) \cup V(\{2\}) \neq V(\{1,2\})$ so that V is not a closure operator.

Remark 1.1.24. If A, B are subsets of a closure space (X, V) such that $A \subseteq B$, then $V(A) \subseteq V(B)$.

Remark 1.1.25. There cannot exist two different closure operators on a set X with the discrete topology as the the same associated topology, because if V_1, V_2 are two closure operators on a set X having the discrete topology as the associated topology and, if K is the associated closure operator (the Kuratowski closure operator) of the discrete topology, then K is the discrete closure operator and therefore $K \leq V_1$ and $K \leq V_2$, which implies $V_1 = V_2 = K$.

On the other hand, there can exist two different closure operators on a set X with the indiscrete topology as the same associated topology. See Example 1.1.13.

Definition 1.1.26. A closure operator V on X is said to be T_1 if $V(\{x\}) = \{x\}$ for all $x \in X$.

Remark 1.1.27. The discrete closure operator is the only T_1 closure operator on a finite set.

Definition 1.1.28. With any closure operator V on a set X, there is associated the interior operator int_V, which is a mapping from $\wp(X) \longrightarrow \wp(X)$ such that for each $A \subseteq X$, $int_V(A) = X - V(X - A)$. The set $int_V(A)$ is called the interior of A in (X, V) or the V-interior of A in X.

Remark 1.1.29. From the definitions of a closure operator and an interior operator, we immediately obtain the following assertion : In any space the following three conditions are fulfilled,

(i) $int(X) = X$

(ii) $int(A) \subseteq A$ for all $A \subseteq X$

(iii) $int(A \cap B) = int(A) \cap int(B)$ for all $A, B \subseteq X$.

Theorem 1.1.30. *If int is a mapping from $\wp(X) \longrightarrow \wp(X)$ satisfying the above three conditions and if we define, $V(A) = X - int(X - A)$ for each $A \subseteq X$, then V is a closure operator on X and $int_V = int$.*

Remark 1.1.31. If V is a closure operator on X, and int_V is the corresponding interior operator, then $int_V(A) = X - V(X - A)$ and $V(A) = X - int_V(X - A)$. Thus the closure operator on a set is uniquely determined by the interior operator and the interior operator on a set is uniquely determined by the closure operator. The notion of a neighbourhood of a subset of X is closely related to the interior operator on X.

Definition 1.1.32. A neighbourhood of a subset A in the space X is any subset U of X such that $A \subseteq int(U)$. A neighbourhood of a point $x \in X$ is any subset U of X such that $x \in int(U)$.

Definition 1.1.33. The neighbourhood system of a subset A (or a point x) in the space X is the collection of all neighbourhoods of the set A (or the point x).

Theorem 1.1.34. *Let (X, V) be a closure space. A subset A of X is open if and only if it is a neighbourhood of all of its points, or equivalently it is a neighbourhood of itself.*

Remark 1.1.35. By above theorem, a subset A of X is open if and only if $A \subseteq int(A)$. But $int(A) \subseteq A$ holds for all $A \subseteq X$. Thus A is open if and only if $int(A) = A$.

Theorem 1.1.36. *Let (X, V) be a closure space. Then the neighbourhood system of a nonempty subset A of X (or a point x of X) is a filter on X.*

Remark 1.1.37. The following theorem shows that the closure of a set is completely determined by neighbourhoods of points of the space.

Theorem 1.1.38. *Let (X, V) be a closure space. Then for each subset A of X, $x \in V(A)$ if and only if each neighbourhood of x in (X, V) intersects A.*

Theorem 1.1.39. *Let U and V be two closure operators on a set X. In order that U should be coarser than V, it is necessary and sufficient that, for each $x \in X$, every U-neighbourhood of x be a V-neighbourhood of x.*

Definition 1.1.40. A closure operator on X other than I is called an infra closure operator, if the only closure operator on X strictly smaller than it is I. Note that the infra closure operators on X are precisely the atoms of the lattice $LC(X)$.

Definition 1.1.41. A closure operator on X other than D is called an ultra closure operator, if the only closure operator on X strictly larger than it is D. Note that the ultra closure operators on X are precisely the dual atoms of the lattice $LC(X)$.

Remark 1.1.42. The set of all topologies $LT(X)$ on a set X is a complete lattice under set inclusion. The indiscrete topology is the smallest element and the discrete topology is the largest element of this lattice. The atoms of this lattice are called infra topologies and the dual atoms are called ultra topologies (see [38]).

Remark 1.1.43. *Otto Frölich* [13] , characterized the ultra topologies on a set X as $\wp(X - \{x\}) \cup \mathcal{F}$ where $x \in X$ and $\mathcal{F}$ is an ultra filter on X not containing $\{x\}$.

Definition 1.1.44. The closure operator V associated with an ultra topology $\wp(X - \{a\}) \cup \mathcal{F}$ is given by

$$V(A) = \begin{cases} A & \text{if } A = \phi,\ a \in A \text{ or } X - A \in \mathcal{F}, \\ A \cup \{a\} & \text{otherwise.} \end{cases}$$

Theorem 1.1.45. *A closure operator on X is an ultra closure operator if and only if, it is the closure operator associated with some ultra topology on X.*

Definition 1.1.46. An ultra topology $\wp(X - \{a\}) \cup \mathcal{F}$ is called a principal ultra topology or non-principal ultra topology, according as the ultra filter $\mathcal{F}$ is principal or not (see [24]). The closure operator associated with an ultra topology is called principal ultra closure operator or non-principal ultra closure operator, according as the ultra topology is principal or non-principal.

Definition 1.1.47. For $a, b \in X, a \neq b$, define $V_{a,b}$ by,

$$V_{a,b}(A) = \begin{cases} A & \text{if } A = \phi, \\ X - \{b\} & \text{if } A = \{a\}, \\ X & \text{otherwise.} \end{cases}$$

Then $V_{a,b}$ is an infra closure operator. For our convenience, we use the notation $V_{(a,b)}$ in place of $V_{a,b}$ that is used in [33].

Theorem 1.1.48. *A closure operator on X is an infra closure operator if and only if it is of the form $V_{(a,b)}$ for some $a, b \in X$ such that $a \neq b$.*

Notation 1.1.49. We use the notation Ω to denote the atoms of the lattice $LC(X)$. Then by the Theorem 1.1.48, the members of Ω are of the form $V_{(a,b)}$ where $a, b \in X$; $a \neq b$.

1.2 Automorphisms of the lattice $LC(X)$

In this section, we determine the group of automorphisms of the lattice $LC(X)$ when X is finite.

Definition 1.2.1. [*15*] A homomorphism from a lattice L into another lattice M is a function $f : L \longrightarrow M$ such that $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in L$.

Definition 1.2.2. [*15*] An isomorphism between two lattices L and M is a bijection $f : L \longrightarrow M$ which satisfies $x \leq y \Longleftrightarrow f(x) \leq f(y)$ for all $x, y \in L$. An isomorphism from a lattice L onto itself is called an automorphism of L. A dual isomorphism between two lattices L and M is a bijection $f : L \longrightarrow M$ which satisfies $x \leq y \Longleftrightarrow f(x) \geq f(y)$ for all $x, y \in L$.

Remark 1.2.3. An isomorphism between two lattices is a one to one and onto homomorphism.

Definition 1.2.4. [*15*] A lattice L with 0 is said to be atomistic, if for every element $a \in L, a \neq 0$ is the supremum of atoms smaller than or equal to a. A lattice L with 1 is said to be dually atomistic, if for every element $b \in L, b \neq 1$ is the infimum of dual atoms greater than or equal to b.

Notation 1.2.5. $S(X)$ denotes the group of all bijections on X. The set of all atoms of the lattice $\wp(X)$ is $\{\{x\} : x \in X\}$. We shall use the notation $\mathcal{D}$ to represent the collection $\{\{x\} : x \in X\}$.

Lemma 1.2.6. *For $p \in S(X)$, define the mapping $A_p : \wp(X) \longrightarrow \wp(X)$ by $A_p(S) = p(S)$ where $p(S) = \{p(x) : x \in S\}$. Then the group of automorphisms of the lattice $\wp(X)$ is precisely given by the collection $\{A_p : p \in S(X)\}$.*

Proof : For $S_1, S_2 \in \wp(X)$,

$$\begin{aligned} A_p(S_1) = A_p(S_2) &\Longrightarrow p(S_1) = p(S_2). \\ \text{Then} \quad x \in S_1 &\Longleftrightarrow p(x) \in p(S_1) \\ &\Longleftrightarrow p(x) \in p(S_2) \\ &\Longleftrightarrow x \in S_2. \end{aligned}$$

Therefore $S_1 = S_2$ and hence A_p is one to one.

For $S \in \wp(X)$, let $T = p^{-1}(S)$. Then, $A_p(T) = S$ so that A_p is onto. Also for

$S, T \in \wp(X)$,

$$\begin{aligned} S \subseteq T &\iff p(S) \subseteq p(T) \\ &\iff A_p(S) \subseteq A_p(T). \end{aligned}$$

Hence A_p is an automorphism of the lattice $\wp(X)$.

Now let A be any automorphism of $\wp(X)$. Then A maps $\mathcal{D}$ onto $\mathcal{D}$. For $\{x\} \in \mathcal{D}$, let $A(\{x\}) = \{y\}$ for some $y \in X$. Then y is unique. Define $p(x) = y$. Then $p \in S(X)$.

$$\begin{aligned} \text{Now, } A(\{x\}) &= \{y\} \\ &= \{p(x)\} \\ &= p(\{x\}) \\ &= A_p(\{x\}). \end{aligned}$$

Thus $A = A_p$ on $\mathcal{D}$. Since $\wp(X)$ is atomistic (see Definition. 1.2.4), it follows that $A = A_p$ on $\wp(X)$. □

Remark 1.2.7. In the following discussions of this section, the set X will be finite.

Lemma 1.2.8. [33] *Let V be a closure operator on X. Define a relation ρV on X by $(a, b) \in \rho V$ if and only if $b \in V(\{a\})$. Then ρV is a reflexive relation on X.*

Lemma 1.2.9. [33] *Let R be a reflexive relation on X. Define a mapping $\nu R : \wp(X) \longrightarrow \wp(X)$ by $\nu R(A) = \{y : (x, y) \in R \text{ for some } x \in A\}, A \in \wp(X)$. Then νR is a closure operator on X.*

Notation 1.2.10. Denote the lattice of all reflexive relations on X under set inclusion by $LR(X)$.

Lemma 1.2.11. [33] *Let X be a finite set and $V \in LC(X)$. Then the mapping ρ defined by $\rho(V) = \rho V$ is a dual isomorphism of $LC(X)$ onto $LR(X)$.*

Lemma 1.2.12. [33] *Let R be a reflexive relation on X. Then the mapping ν defined by $\nu(R) = \nu R$ is a dual isomorphism of $LR(X)$ onto $LC(X)$.*

Lemma 1.2.13. *Let $R \in LR(X)$ and $\Delta = \{(x, x) : x \in X\}$. Then $R - \Delta \in \wp((X \times X) - \Delta)$ and the correspondence $R \longrightarrow R - \Delta$ is an isomorphism of $LR(X)$ onto $\wp((X \times X) - \Delta)$.*

Proof : Proof is obvious. □

Lemma 1.2.14. *Let $G \subseteq \wp((X \times X) - \Delta)$. Then $G \cup \Delta$ is a reflexive relation on X and the correspondence $G \longrightarrow G \cup \Delta$ is an isomorphism of $\wp((X \times X) - \Delta)$ onto $LR(X)$.*

Proof : Proof is obvious. □

Theorem 1.2.15. *Let X be a set and p be a bijection on $(X \times X) - \Delta$. For $V \in LC(X)$, let $R_{p,V} = \{p(a, b) : b \in V(\{a\}), a \neq b\} \cup \Delta$. Then $R_{p,V}$ is a reflexive relation on X. Further, let $T_pV = \nu R_{p,V}$. Then $T_pV \in LC(X)$ and the mapping T_p defined by $T_p(V) = T_pV$ for $V \in LC(X)$ is an automorphism of $LC(X)$.*

Proof : For $V \in LC(X)$, $\rho V - \Delta = \{(a, b) : b \in V(\{a\}), a \neq b\} \in \wp((X \times X) - \Delta)$, by Lemma 1.2.13. For $p \in S((X \times X) - \Delta)$, $p(\rho V - \Delta) = \{p(a, b) : b \in V(\{a\}), a \neq b\} \in \wp((X \times X) - \Delta)$ by Lemma 1.2.6. Then $R_{p,V} = p(\rho V - \Delta) \cup \Delta \in LR(X)$, by Lemma 1.2.14 and $T_pV = \nu R_{p,V} \in LC(X)$, by Lemma 1.2.9. Also for $V_1, V_2 \in LC(X)$,

$$\begin{aligned} V_1 \leq V_2 &\iff \rho V_2 \subseteq \rho V_1 \text{ by Lemma 1.2.11} \\ &\iff R_{p,V_2} \subseteq R_{p,V_1} \text{ by Lemma 1.2.14} \\ &\iff T_p V_1 \leq T_p V_2 \text{ by lemma 1.2.12} \\ &\iff T_p(V_1) \leq T_p(V_2). \end{aligned}$$

Further, since the correspondences $V \longrightarrow \rho V$, $\rho V \longrightarrow \rho V - \Delta$, $\rho V - \Delta \longrightarrow p(\rho V - \Delta)$, $p(\rho V - \Delta) \longrightarrow p(\rho V - \Delta) \cup \Delta$ $(= R_{p,V})$ and $R_{p,V} \longrightarrow T_p V$ are bijections, it follows that, $T_p : V \longrightarrow T_p V$ is a bijection. Hence T_p is an automorphism of the lattice $LC(X)$. □

Theorem 1.2.16. [35] *The lattice $LC(X)$ is atomistic if and only if X is finite.*

Theorem 1.2.17. *When X is a finite set, the set of all automorphisms of the lattice $LC(X)$ is precisely the set $\{T_p : p \in S((X \times X) - \Delta)\}$.*

Proof : Let A be any automorphism of the lattice $LC(X)$. We want to show that $A = T_p$ for some $p \in S((X \times X) - \Delta)$. For $V_{(a,b)} \in \Omega$ (see Notation 1.1.49), let $A(V_{(a,b)}) = V_{(a,b)'}$ for some $(a,b)' \in (X \times X) - \Delta$. Then $(a,b)'$ is unique. Define $p((a,b)) = (a,b)'$. Then $p \in S((X \times X) - \Delta)$.
For $(a,b) \neq (c,d) \in (X \times X) - \Delta$, let $A(V_{(a,b)}) = V_{(a,b)'}$ and $A(V_{(c,d)}) = V_{(c,d)'}$. (This is because automorphisms map atoms onto atoms). Then,

$$\begin{aligned} (a,b) \neq (c,d) &\implies V_{(a,b)} \neq V_{(c,d)} \\ &\implies A(V_{(a,b)}) \neq A(V_{(c,d)}) \\ &\implies V_{(a,b)'} \neq V_{(c,d)'} \\ &\implies (a,b)' \neq (c,d)' \\ &\implies p((a,b)) \neq p((c,d)). \end{aligned}$$

Therefore p is one to one. For $(c,d) \in (X \times X) - \Delta$, consider $V_{(c,d)}$. Since A is onto, there exists $(a,b) \in (X \times X) - \Delta$ such that $A(V_{(a,b)}) = V_{(c,d)}$. Then $p((a,b)) = (c,d)$ and therefore p is onto.

Now for $V_{(a,b)} \in \Omega$,

$$\begin{aligned} A(V_{(a,b)}) &= V_{p(a,b)} \\ &= \nu[(\rho V_{p(a,b)} - \Delta) \cup \Delta] \\ &= \nu[\, p(\rho V_{(a,b)} - \Delta) \cup \Delta] \\ &= T_p V_{(a,b)} \\ &= T_p(V_{(a,b)}). \end{aligned}$$

Hence $A = T_p$ on Ω. Since $LC(X)$ is atomistic, it follows that $A = T_p$ on $LC(X)$. $\square$

1.3 Automorphisms of some other lattices related to the lattice $LC(X)$

In this section, we determine the automorphisms of the sublattice $[I, C_0]$ of the lattice $LC(X)$ and the automorphisms of the lattice $LQ(X)$ of all quasi-discrete closure operators (see Definition. 1.3.2) on a set X.

Definition 1.3.1. [*10*] The quasi-discrete modification of a closure operator U on a set X is a function $V : \wp(X) \longrightarrow \wp(X)$ defined by

$$\begin{aligned} V(A) &= \bigcup \{y \in X : y \in U(\{x\}) \text{ for some } x \in A\},\ A \in \wp(X) \\ &= \bigcup \{U(\{x\}) : x \in A\} \text{ for all } A \in \wp(X). \end{aligned}$$

Definition 1.3.2. [*10*] A closure operator V on a set X is called a quasi-discrete closure operator, if V coincides with its quasi-discrete modification. That is, if $V(A) = \bigcup \{V(\{x\}) : x \in A\}$ for all $A \in \wp(X)$.

Remark 1.3.3. A closure operator associated with a topology is quasi-discrete if and only if the topology is principal. The concept of quasi-discrete closure operator is thus a natural generalization of a principal topology (see [3]). It can be verified that a closure operator V on a set X is quasi-discrete, if $V(\bigcup A_\alpha) = \bigcup V(A_\alpha)$ for every collection $\{A_\alpha\}$ of subsets of X. When X is finite, every closure operator on X is quasi-discrete.

Theorem 1.3.4. [*10*] *The set of all quasi-discrete closure operators on a set X under the partial order " $\leq$ " defined by $V_1 \leq V_2 \Longleftrightarrow V_2(A) \subseteq V_1(A)$ for every $A \in \wp(X)$ is a complete lattice.*

Remark 1.3.5. We denote the set of all quasi-discrete closure operators on a set X by $LQ(X)$.

Remark 1.3.6. $LQ(X)$ is not a complete sublattice of the lattice $LC(X)$, because the supremum in $LC(X)$ of all the atoms $\{V_{(a,b)} : a, b \in X, a \neq b\}$ of $LQ(X)$ is the co-finite closure operator C_0 which is not quasi-discrete.

Lemma 1.3.7. [*33*] *For $V \in LQ(X)$, define a relation ρV on X by $\rho V = \{(x, y) : y \in V(\{x\})\}$. Then ρV is a reflexive relation on X.*

Lemma 1.3.8. [*33*] *For $R \in LR(X)$, define $\nu R : \wp(X) \longrightarrow \wp(X)$ by $\nu R(A) = \{y \in X : xRy$ for some $x \in A\}, A \in \wp(X)$. Then νR is a quasi-discrete closure operator on X.*

Remark 1.3.9. It can be easily verified that, the mapping $\nu : LR(X) \longrightarrow LQ(X)$ defined by $\nu(R) = \nu R$ and the inverse mapping $\rho : LQ(X) \longrightarrow LR(X)$ defined by $\rho(V) = \rho V$ are dual isomorphisms.

Theorem 1.3.10. *Let X be a non-empty set. For $V \in LQ(X)$ and $p \in S((X \times X) - \Delta)$, let $R_{p,V} = p(\rho V - \Delta) \cup \Delta$. Then $R_{p,V} \in LR(X)$. Further, let $T_pV = \nu R_{p,V}$. Then $T_pV \in LQ(X)$ and the mapping T_p defined by $T_p(V) = T_pV$ for $V \in LQ(X)$ is an automorphism of $LQ(X)$.*

Proof : From the Lemmas 1.3.7, 1.3.8, and Remark 1.3.9, it follows that for $V_1, V_2 \in LQ(X)$,

$$\begin{aligned} V_1 \leq V_2 &\iff \rho V_2 \subseteq \rho V_1 \\ &\iff R_{p,V_2} \subseteq R_{p,V_1} \\ &\iff T_p V_1 \leq T_p V_2 \\ &\iff T_p(V_1) \leq T_p(V_2). \end{aligned}$$

Further, since the correspondences $V \longrightarrow \rho V$, $\rho V \longrightarrow \rho V - \Delta$, $\rho V - \Delta \longrightarrow p(\rho V - \Delta)$, $p(\rho V - \Delta) \longrightarrow p(\rho V - \Delta) \cup \Delta$ $(= R_{p,V})$ and $R_{p,V} \longrightarrow T_p V$ are bijections, it follows that $T_p : V \longrightarrow T_p V$ is a bijection. Hence T_p is an automorphism of the lattice $LQ(X)$. $\square$

Remark 1.3.11. Since the lattice $LR(X)$ is atomistic, it follows that the lattice $LQ(X)$ is also atomistic. Obviously the set of atoms of $LQ(X)$ are precisely the set $\Omega = \{V_{(a,b)} : a, b \in X \,\&\, a \neq b\}$.

Theorem 1.3.12. *Let X be a non-empty set. Then the group of automorphisms of the lattice $LQ(X)$ is precisely the set* $\{T_p : p \in S((X \times X) - \Delta)\}$.

Proof : Let A be any automorphism of the lattice $LQ(X)$. We want to show that $A = T_p$ for some $p \in S((X \times X) - \Delta)$. For $V_{(a,b)} \in \Omega$, let $A(V_{(a,b)}) = V_{(a,b)'}$ for some $(a,b)' \in (X \times X) - \Delta$. Then $(a,b)'$ is unique. Define $p(a,b) = (a,b)'$. Then $p \in S((X \times X) - \Delta)$. Now for $V_{(a,b)} \in \Omega$,

$$\begin{aligned} A(V_{(a,b)}) &= V_{p(a,b)} \\ &= \nu[(\rho V_{p(a,b)} - \Delta) \cup \Delta] \\ &= \nu[\, p(\rho V_{(a,b)} - \Delta) \cup \Delta] \\ &= T_p V_{(a,b)} \\ &= T_p(V_{(a,b)}). \end{aligned}$$

Hence $A = T_p$ on Ω. Since the lattice $LQ(X)$ is atomistic, it follows that $A = T_p$ on $LQ(X)$. $\square$

Remark 1.3.13. The interval $[I, C_0]$ where I is the indiscrete closure operator and C_0 is the co-finite closure operator is a complete sublattice of the lattice $LC(X)$.

Definition 1.3.14. *[33]* For $R \in LR(X)$, define $\mu R : \wp(X) \longrightarrow \wp(X)$ by

$$\mu R(A) = \begin{cases} \{y : xRy & \text{for some } x \in A\} \text{ if } A \text{ is finite,} \\ X & \text{otherwise.} \end{cases}$$

Then μR is a closure operator on X and $\mu R \leq C_0$. That is, $\mu R \in [I, C_0]$.

Remark 1.3.15. For a closure operator V on X such that $V \leq C_0$, define the mapping ρ by $\rho V = \{(x, y) : y \in V(\{x\})\}$. Then ρV is a reflexive relation on X.

Remark 1.3.16. It can be easily verified that the mapping $\mu : LR(X) \longrightarrow [I, C_0]$ defined by $\mu(R) = \mu R$ and the mapping $\rho : [I, C_0] \longrightarrow LR(X)$ defined by $\rho(V) = \rho V$ are dual isomorphisms.

Theorem 1.3.17. *Let X be a non-empty set and let ρV be the reflexive relation on X corresponding to the closure operator $V \in [I, C_0]$. For $p \in S((X \times X) - \Delta)$, let $R_{p,V} = p(\rho V - \Delta) \cup \Delta$. Then $R_{p,V} \in LR(X)$. Further, let $T_p V = \mu R_{p,V}$. Then $T_p V \in [I, C_0]$ and the mapping T_p defined by $T_p(V) = T_p V$ for $V \in [I, C_0]$ is an automorphism of the lattice $[I, C_0]$.*

Proof : From the Remark 1.3.16, it follows that, for $V_1, V_2 \in [I, C_0]$,

$$\begin{aligned} V_1 \leq V_2 &\iff \rho V_2 \subseteq \rho V_1 \\ &\iff R_{p,V_2} \subseteq R_{p,V_1} \\ &\iff T_p V_1 \leq T_p V_2 \\ &\iff T_p(V_1) \leq T_p(V_2). \end{aligned}$$

Further, since the correspondences $V \longrightarrow \rho V$, $\rho V \longrightarrow \rho V - \Delta$, $\rho V - \Delta \longrightarrow p(\rho V - \Delta)$, $p(\rho V - \Delta) \longrightarrow p(\rho V - \Delta) \cup \Delta$ $(= R_{p,V})$ and $R_{p,V} \longrightarrow T_p V$ are bijections, it follows that, $T_p : V \longrightarrow T_p V$ is a bijection. Hence T_p is an automorphism of the lattice $[I, C_0]$. $\square$

Remark 1.3.18. Since the lattice $LR(X)$ is atomistic, it follows that the lattice $[I, C_0]$ is also atomistic. Obviously the set of atoms of $LQ(X)$ is the set $\Omega = \{V_{(a,b)} : a, b \in X \,\&\, a \neq b\}$.

Theorem 1.3.19. *The group of automorphisms of the lattice* $[I, C_0]$ *is precisely the set* $\{T_p : p \in S((X \times X) - \Delta)\}$.

Proof : Let A be any automorphism of the lattice $[I, C_0]$. We want to show that $A = T_p$ for some $p \in S((X \times X) - \Delta)$. For $V_{(a,b)} \in \Omega$, let $A(V_{(a,b)}) = V_{(a,b)'}$ for some $(a, b)' \in (X \times X) - \Delta$. Then $(a, b)'$ is unique. Define $p(a, b) = (a, b)'$. Then $p \in S((X \times X) - \Delta)$. Now for $V_{(a,b)} \in \Omega$,

$$\begin{aligned} A(V_{(a,b)}) &= V_{p(a,b)} \\ &= \mu[(\rho V_{p(a,b)} - \Delta) \cup \Delta] \\ &= \mu[\,p(\rho V_{(a,b)} - \Delta) \cup \Delta] \\ &= T_p V_{(a,b)} \\ &= T_p(V_{(a,b)}). \end{aligned}$$

Hence $A = T_p$ on Ω. Since the lattice $[I, C_0]$ is atomistic, it follows that $A = T_p$ on $[I, C_0]$. $\square$

Definition 1.3.20. For $p \in S(X)$ and $A \in \wp(X)$, define $p(A)$ by $p(A) = \{p(x) : x \in A\}$. Then $p(A) \in \wp(X)$.

Theorem 1.3.21. [33] *Let X be a non-empty set. For a closure operator V on X, define $T'_p V$ by $T'_p V(A) = p^{-1}(V(p(A)))$, $A \in \wp(X)$. Then $T'_p V \in LC(X)$ and* *the mapping $T'_p : LC(X) \longrightarrow LC(X)$ defined by $T'_p(V) = T'_pV$ is an automorphism of $LC(X)$.*

Remark 1.3.22. For the sake of our convenience, we use the notation T'_p in place of T_p that is used in [33].

Definition 1.3.23. [*34*] For an infinite subset A of X and $x \notin A$, define $V_{A,x} : \wp(X) \longrightarrow \wp(X)$ by

$$V_{A,x}(S) = \begin{cases} S & \text{if } S \text{ is finite,} \\ X - \{x\} & \text{if } S \text{ is infinite}, S - A \text{ finite and } x \notin S, \\ X & \text{otherwise.} \end{cases}$$

Then $V_{A,x}$ is a closure operator on X.

Notation 1.3.24. Denote the set of all closure operators on X of the form $V_{A,x}$ by £.

Remark 1.3.25. (a) Let A be an infinite subset of X such that $x \notin A$ and B an infinite subset of A. Then $V_{B,x} \leq V_{A,x}$.

(b) £ has no minimum and maximum elements.

(c) £ has no minimal elements.

(d) Maximal elements of £ are of the form $V_{X-\{x\},x}$ where $x \in X$.

(e) Let A, B be infinite subsets of X such that $x \notin A, B$. Then $V_{A,x} = V_{B,x}$ if and only if $(A - B) \cup (B - A))$ is finite.

(f) The closure operators $V_{X-\{x\},x}$ are topological. The topology corresponding to $V_{X-\{x\},x}$ is $C_0 \cup \{x\}$ where C_0 is the co-finite topology of X.

Theorem 1.3.26. *The automorphism T'_p, $p \in S(X)$ of the lattice $LC(X)$ defined as in the Theorem 1.3.21, maps the closure operator $V_{A,x}$ into $V_{p^{-1}(A),p^{-1}(x)}$.*

Proof : Let $A \subseteq X$ be infinite and $x \notin A$. Then for $S \in \wp(X)$, $T_p'V_{A,x}(S) = p^{-1}(V_{A,x}(p(S)))$. Now,

$$V_{A,x}(p(S)) = \begin{cases} p(S) & \text{if } p(S) \text{ is finite,} \\ X - \{x\} & \text{if } p(S) \text{ is infinite}, p(S) - A \text{ finite and } x \notin p(S), \\ X & \text{otherwise.} \end{cases}$$

$$= \begin{cases} p(S) & \text{if } S \text{ is finite,} \\ X - \{x\} & \text{if } S \text{ is infinite}, S - p^{-1}(A) \text{ finite and } p^{-1}(x) \notin S, \\ X & \text{otherwise.} \end{cases}$$

Therefore,

$$p^{-1}(V_{A,x}(p(S))) = \begin{cases} S & \text{if } S \text{ is finite,} \\ X - \{p^{-1}(x)\} & \text{if } S \text{ is infinite, } S - p^{-1}(A) \text{ finite} \\ & \text{and } p^{-1}(x) \notin S, \\ X & \text{otherwise.} \end{cases}$$

$$= V_{(p^{-1}(A), p^{-1}(x))}(S).$$

Since A is infinite, $p^{-1}(A)$ is also infinite. Further, $x \notin A$ implies $p^{-1}(x) \notin p^{-1}(A)$. That is, $T_p'V_{A,x}(S) = V_{B,y}(S)$ for all $S \in \wp(X)$ where $B = p^{-1}(A)$ and $y = p^{-1}(x)$. Therefore the automorphism T_p' ; $p \in S(X)$ of the lattice $LC(X)$ maps the closure operator $V_{A,x}$ into $V_{(p^{-1}(A),\, p^{-1}(x))}$. □

Theorem 1.3.27. *The automorphism T_p', $p \in S(X)$ of the lattice $LC(X)$ defined as in the Theorem 1.3.21, maps the closure operator $V_{(a,b)}$ into $V_{(p^{-1}(a),\, p^{-1}(b))}$.*

Proof : For $a, b \in X; a \neq b$ and $S \in \wp(X)$, $T_p'V_{(a,b)}(S) = p^{-1}(V_{(a,b)}(p(S)))$. Now,

$$V_{(a,b)}(p(S)) = \begin{cases} p(S) & \text{if } p(S) = \phi, \\ X - \{b\} & \text{if } p(S) = \{a\}, \\ X & \text{otherwise.} \end{cases}$$

$$= \begin{cases} p(S) & \text{if } S = \phi, \\ X - \{b\} & \text{if } S = p^{-1}(\{a\}), \\ X & \text{otherwise.} \end{cases}$$

Therefore,

$$p^{-1}(V_{(a,b)}(p(S))) = \begin{cases} S & \text{if } S = \phi, \\ X - \{p^{-1}(b)\} & \text{if } S = p^{-1}(\{a\}), \\ X & \text{otherwise.} \end{cases}$$

$$= V_{(p^{-1}(a), p^{-1}(b))}(S)$$

That is, $T_p' V_{(a,b)}(S) = V_{(x,y)}(S)$ for all $S \in \wp(X)$ where $x = p^{-1}(a)$ and $y = p^{-1}(b)$. Therefore the automorphism $T_p', p \in S(X)$ of the lattice $LC(X)$ maps the closure operator $V_{(a,b)}$ into $V_{(p^{-1}(a),\, p^{-1}(b))}$. □

Remark 1.3.28. The image of $V_{A,x}$ under any automorphism of $LC(X)$ is a T_1 closure operator. For,

Let α be an automorphism of $LC(X)$. Since the closure operators I, D, and C_0 are left fixed by any automorphism (see [33]), $\alpha([C_0, D]) = [C_0, D]$. But as the interval $[C_0, D]$ of $LC(X)$is precisely the set of all the T_1 closure operators of $LC(X)$ (see [33]), it follows that α maps a T_1 closure operator onto a T_1 closure operator. Now since $V_{A,x}$ is T_1, it follows that $\alpha(V_{A,x})$ is also T_1.

Remark 1.3.29. We have determined the group of automorphisms of the lattice $LC(X)$ of all closure operators on a fixed set X, when X is finite. We have also determined the automorphisms of the lattice $LQ(X)$ of all quasi-discrete closure operators and that of the sublattice $[I, C_0]$ of $LC(X)$ on any fixed set X. But when X is infinite, the problem of determining the group of automorphisms of the lattice $LC(X)$ is only partially answered.

CHAPTER

2

Some Problems Related to Fuzzy Topology

2.1 Introduction

In this section, the basic concepts in fuzzy set theory and fuzzy topology are discussed.

Definition 2.1.1. [27] Let X be a nonempty set, L a complete lattice. An L-fuzzy subset A of X is a mapping $A : X \longrightarrow L$. The family of all L-fuzzy subsets of X is denoted by L^X. For brevity, we call an L-fuzzy subset of X as a fuzzy subset of X.

Definition 2.1.2. [27] Define the partial order " $\leq$ " in L^X by : For all $A, B \in L^X$, $A \leq B \iff A(x) \leq B(x)$ for all $x \in X$. With this partial order, L^X is a complete lattice. The smallest and the greatest elements of L^X are the constant functions taking the values 0 and 1 respectively and are denoted by $\underline{0}$ and $\underline{1}$.

Remark 2.1.3. The lattice L^X is distributive if and only if L is distributive.

Definition 2.1.4. [*2*] For $A, B \in L^X$, the lattice operations $\wedge$ and $\vee$ are defined by $(A \wedge B)(x) = \min\,\{A(x), B(x)\}$ for all $x \in X$ and $(A \vee B)(x) = \max\,\{A(x), B(x)\}$ for all $x \in X$. For $\{A_\alpha : \alpha \in J\} \subseteq L^X$, the operations $\bigwedge_{\alpha\in J} A_\alpha$ and $\bigvee_{\alpha\in J} A_\alpha$ are defined by $(\bigwedge_{\alpha\in J} A_\alpha)(x) = \bigwedge_{\alpha\in J} A_\alpha(x)$ for all $x \in X$ and $(\bigvee_{\alpha\in J} A_\alpha)(x) = \bigvee_{\alpha\in J} A_\alpha(x)$ for all $x \in X$.

Definition 2.1.5. [*27*] A fuzzy point on X is a fuzzy subset x_l in L^X defined as, for all $y \in X$,

$$x_l(y) = \begin{cases} l & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

where $x \in X$ and l is a non-zero element of L. The set of all the fuzzy points of X is denoted by $\mathrm{Pt}(L^X)$.

Example 2.1.6. Let $X = R$ and $L = [0, 1]$ and consider the fuzzy set on X defined as follows :

$$A(x) \quad = \quad \begin{cases} 1/2 & \text{if } x = 50, \\ 0 & \text{otherwise.} \end{cases}$$

Then this fuzzy set A is a fuzzy point and is denoted by $50_{1/2}$.

Remark 2.1.7. For a fuzzy point $x_l \in Pt(L^X)$ and a fuzzy subset $A \in L^X$ such that $x_l \leq A$, we also say x_l is in A and denote it as $x_l \in A$ sometimes. Every fuzzy set $A \in L^X$ can be expressed as the join of all fuzzy points which belong to A. It is interesting to see that in ordinary set theory, point (singleton set) is the smallest element of a set whereas in fuzzy set theory, a fuzzy point may contain other smaller fuzzy points. For example, when $L = [0, 1]$, the fuzzy point $x_{1/4} < x_{1/2}$.

Lemma 2.1.8. [*37*] *Let A and B be fuzzy subsets of X. Then the followings are equivalent :*
(i) $A \leq B$
(ii) $x_l \in A \Longrightarrow x_l \in B$ *for all* $x_l \in Pt(L^X)$.

Definition 2.1.9. [*27*] Let L be a lattice. A mapping $' : L \longrightarrow L$ is called order-reversing if, for all $a, b \in L$, $a \leq b \Longrightarrow a^{'} \geq b^{'}$; called an involution on L if, $a^{''} = a$ for all $a \in L$. It is obvious that an involution is always a bijection.

Definition 2.1.10. [*27*] A complete and distributive lattice L is called an F-lattice, if L has an order reversing involution $' : L \longrightarrow L$. Let X be a non-empty ordinary set, L an F- lattice, $'$ the order reversing involution on L. For all $A \in L^X$, use the order reversing involution $'$ on L^X by $A'(x) = (A(x))'$ for all $x \in X$. Call $' : L^X \longrightarrow L^X$ the pseudo-complementary operation on L^X, A' the pseudo-complementary set of A (or the pseudo-complement of A) in L^X.

Lemma 2.1.11. [*27*] *Let X be a nonempty ordinary set, L an F-lattice, then the pseudo-complementary operation $' : L^X \longrightarrow L^X$ is an order reversing involution.*

Definition 2.1.12. [*27*] Let X be a nonempty ordinary set, L an F- lattice, $\delta \subseteq L^X$. Then δ is called a fuzzy topology on X and (X, δ) or (L^X, δ) is called a fuzzy topological space, if δ satisfies the following three conditions :

(i) $\underline{0}, \underline{1} \in \delta$

(ii) For all $\mathcal{A} \subseteq \delta$, $\bigvee \mathcal{A} \in \delta$.

(iii) For all $A, B \in \delta$, $A \wedge B \in \delta$.

Definition 2.1.13. [*27*] An element of δ is called an open set in L^X, a pseudo-complement of an open set is called a closed set in L^X.

Definition 2.1.14. [*32*] Let (X, δ) be a fuzzy topological space. A subfamily $\mathcal{B}$ of δ is called a base for δ if and only if, for each $A \in \delta$, there exists $(A_i)_{i \in I} \subseteq \mathcal{B}$ such that $A = \bigvee_{i \in I} A_i$.

Definition 2.1.15. [*27*] Let X be a nonempty set and δ_1, δ_2 be two fuzzy topologies on X. We say δ_1 is " coarser than " δ_2 (or δ_2 is finer than δ_1) if $\delta_1 \subseteq \delta_2$.

Remark 2.1.16. The relation " coarser than ", denoted by $\subseteq$ is a partial order relation. The set of all fuzzy topologies on X denoted by $LFT(X, L)$ is a complete lattice under the relation $\subseteq$ defined above. The smallest element of $LFT(X, L)$ is the indiscrete fuzzy topology $\delta = \{\underline{0}, \underline{1}\}$ and the greatest element is the discrete fuzzy topology $\delta = L^X$.

Example 2.1.17. Suppose T is an ordinary topology on X. Then $\delta = \{\chi_A : A \in T\} \subseteq L^X$ is a fuzzy topology on X.

Remark 2.1.18. Let L be a complete and distributive lattice. Since, $2 = \{0, 1\}$ is isomorphic to a sublattice of L, 2^X is isomorphic to a sublattice of L^X. Therefore all topological spaces may be considered as fuzzy topological spaces. In other words, a fuzzy topological space may be considered as a generalization of a topological space.

Definition 2.1.19. [*27*] Let (X, δ) be a fuzzy topological space, $A \in L^X$. Define the interior of A as the join of all the open subsets contained in A and is denoted by int(A) or A°.

Definition 2.1.20. [*27*] Define the closure of a fuzzy set $A \in L^X$ as the meet of all the closed sets containing A and is denoted by $cl(A)$ or $\bar{A}$.

Remark 2.1.21. The arbitrary meet of closed sets is closed. The interior of a fuzzy set $A \in L^X$ is just the largest open subset contained in A, and the closure of $A \in L^X$ is the smallest closed set containing A.

Definition 2.1.22. [*2*] The atoms of $LFT(X, L)$ are called infra fuzzy topologies and the dual atoms are called ultra fuzzy topologies.

Definition 2.1.23. [*32*] A fuzzy set A in a fuzzy topological space (X, δ) is said to be a neighbourhood of a fuzzy point x_l if and only if there exists some $B \in \delta$ such that $x_l \in B \leq A$. The family consisting of all neighbourhoods of x_l is called the system of neighbourhoods of x_l. This is a generalization of that defined in general topological spaces.

Remark 2.1.24. In general topology, the neighbourhood structure of points can be decomposed as " Structure of open sets + Membership relation between points and sets."

In fuzzy topology, if we regard the ordinary ' belonging to ' as the membership relation, we obtain the neighbourhood structure called " neighbourhood system " mentioned above. This structure appeared very early in the study of fuzzy topology and encountered many difficulties. For example, the notions of accumulation point, derived set, net and compactness based on this membership relation will cause some absurd results, such as " every closed subset must be crisp, " and so on. For some years, this problem was an obstacle in the study of fuzzy topology. It was solved by Ying Ming Liu [27] in 1977, by introducing the concept of quasi-coincident neighbourhood system which overcomes these problems well and provides a stable fundamental neighbourhood structure for fuzzy topology.

In fact the cause of these shortages is the failure of the ' Multiple Choice Principle ' on this membership function which states that " if a point has relation R with the union of family of sets then it has also R with one of these sets. " This principle is a fundamental fact in set theory. But in fuzzy set theory, it does not hold for the relation ' belonging to.' This can be illustrated through the following example. Let for all $x \in X$, for every $n = 2, 3, 4....$, $x_{1-1/n} \in L^X$ where $L = [0, 1]$. Then consider $\bigvee\{x_{1-1/n} : n \geq 2\}$. Clearly $x_1 \in \bigvee\{x_{1-1/n} : n \geq 2\}$ as this union is the fuzzy point x_1 itself. But $x_1 \notin x_{1-1/n}$; $n \geq 2$. Hence $x \in \bigcup A_i$ need not implies that $x \in A_i$ for atleast one A_i. A reasonable membership relation between fuzzy sets and fuzzy points should satisfy the ' Membership Relation Determination Principles ' which includes the followings :

• Extension Principle

Restricted to the ordinary set theory, membership relation R should become the ordinary belonging relation.

• Range Determination Principle

The fact that $x_\lambda \mathrm{R}\, A$ or not is completely determined by a system of formula about λ and $A(x)$ expressed in terms of the order relation and involution in L.

• Maximum and Minimum Principle

For every fuzzy point p, $p\mathrm{R}\underline{0}$ and $p\mathrm{R}\underline{1}$ where $\underline{0}$ and $\underline{1}$ are the smallest fuzzy subset and the largest one on X respectively.

• Multiple Choice Principle

For every family $\{A_t : t \in T\}$ of fuzzy subsets of X, the following implication holds : $x_\lambda \mathrm{R} \bigvee\{A_t : t \in T\} \Longrightarrow$ there exists $t \in T$, such that $x_\lambda \mathrm{R} A_t$.

Ying Ming Liu [26] proved that the quasi-coincidence relation is the unique membership relation satisfying all these four principles. Thus the new concept of quasi-coincidence and quasi neighbourhood structure was introduced.

Definition 2.1.25. *[27]* Let (X, δ) be a fuzzy topological space. We say that the fuzzy point x_a is quasi-coincident with the fuzzy set A, denoted by $x_a\, q\, A$, if $x_a \not\leq A'$; say A is quasi-coincident with B at x if $A(x) \not\leq B'(x)$; say A is quasi-coincident with B, if A is quasi-coincident with B at some point $x \in X$.

Definition 2.1.26. *[27]* Let (X, δ) be a fuzzy topological space. $U \in \delta$ is called a quasi-coincident neighbourhood of a fuzzy point x_a, shortened as Q-neighbourhood, if x_a is quasi-coincident with U. The family of all the Q-neighbourhoods of x_a is called the Q-neighbourhood system of x_a, denoted by $Q(x_a)$.

Remark 2.1.27. The following example shows that a Q-neighbourhood of a fuzzy point is not necessary to contain the fuzzy point itself.

Example 2.1.28. Let $X = L = [0, 1]$. Let the pseudo-complement of $A \in L^X$ be defined by $A'(x) = 1 - A(x)$ for all $x \in X$. Consider the fuzzy topology $\{\underline{0}, B, \underline{1}\}$ where $B \in L^X$ is given by $B(x) = 1/2$ for all $x \in X$. Then B is a Q-neighbourhood for the fuzzy point $e = x_{3/4}$. But $e \notin B$, since $3/4 > 1/2$.

Theorem 2.1.29. [27] *Let (X, δ) be a fuzzy topological space, $A, B, C \in L^X$, $x \in X$. Then*

(i) AqB at $x \Longleftrightarrow BqA$ at x

(ii) $AqB \Longleftrightarrow BqA$.

2.2 Fuzzy topologies and t-irreducible subsets

In 1980, Babusundar S. [2] characterized the ultra fuzzy topologies on a set X using minimal t-irreducible subsets of the lattice L and ultra fuzzy filters on X. In this section, we study and add a few results in this direction.

Definition 2.2.1. [2] A nonempty subset $\mathcal{F}$ of L^X is said to be a fuzzy filter on X if,

(i) $\underline{0} \notin \mathcal{F}$

(ii) $A, B \in \mathcal{F} \Longrightarrow A \wedge B \in \mathcal{F}$

(iii) $A \in \mathcal{F}$, $B \in L^X$ and $B \geq A \Longrightarrow B \in \mathcal{F}$.

Definition 2.2.2. [2] A fuzzy filter on X is said to be an ultra fuzzy filter, if it is not properly contained in any other fuzzy filter on X.

Theorem 2.2.3. [2] *Every fuzzy filter is contained in an ultra fuzzy filter.*

Definition 2.2.4. [2] A non-empty subset $\mathcal{A}$ of L^X is said to be a fuzzy filter base, if $\underline{0} \notin \mathcal{A}$ and $A, B \in \mathcal{A}$ implies that there exists $C \in \mathcal{A}$ such that $C \leq A \wedge B$.

A subset $\mathcal{A}$ of L^X is said to be a base for a fuzzy filter $\mathcal{F}$, if $\mathcal{F} = \{A \in L^X : A \geq B$, for some B in $\mathcal{A}\}$.
Note that any fuzzy filter base $\mathcal{A}$ is a base for the fuzzy filter $\mathcal{F}$ as defined above. $\mathcal{F}$ is called the filter generated by the fuzzy filter base $\mathcal{A}$.

Theorem 2.2.5. [2] *Let $\mathcal{F}$ be an ultra fuzzy filter on X and A be a complemented fuzzy subset in L^X. Then either $A \in \mathcal{F}$ or $A' \in \mathcal{F}$.*

Definition 2.2.6. [2] A nonempty subset R, not containing 0 of a lattice L is said to be t-irreducible if no element of R can be written as the finite meet or arbitrary join of members of $L - R$. If further, no proper subset of R is t-irreducible, then R is said to be minimal t-irreducible.

Definition 2.2.7. [2] A nonzero element l in a lattice L is said to be a t-irreducible element, if $\{l\}$ is a t-irreducible subset of L.

Definition 2.2.8. [2] A subset E of a lattice L is said to be t-complete, if E is closed for the operations of finite meet and arbitrary join.

Notation 2.2.9. Let $R \subseteq L$. We denote $L - R$ by R^c.

Remark 2.2.10. $R \subseteq L$ is t-irreducible $\Longrightarrow R^c$ is t-complete. R is t-complete and $R \neq L \Longrightarrow R^c$ is t-irreducible.
In [11], Babusundar S. characterized the minimal t-irreducible subsets of the Boolean lattice $\wp(X)$. In the following two theorems, we shall give a simpler proof of that result using t-complete subsets.

Theorem 2.2.11. [2] *Let X be any set with atleast two elements and $x, y \in X$ such that $x \neq y$. Then $R_{x,y} = \{A \subseteq X : x \in A$ and $y \notin A\}$ is a minimal t-irreducible subset of the Boolean lattice $\wp(X)$.*

Proof : Let $R_{x,y} = \{A \subseteq X : x \in A \text{ and } y \notin A\}$.

$$\begin{aligned} \text{Then} \quad R^c_{x,y} &= \{A \subseteq X : x \notin A \text{ or } y \in A\} \\ &= \wp(X - \{x\}) \cup \mathcal{F}(y) \text{ where } \mathcal{F}(y) \text{ is the principal ultra filter at } y. \end{aligned}$$

Therefore $R^c_{x,y}$ is an ultra topology on X (see[13]). Therefore $R^c_{x,y}$ is t-complete. Also $R^c_{x,y} \neq \wp(X)$ as $\{x\} \notin R^c_{x,y}$. Therefore $R_{x,y}$ is t-irreducible.

Now we show that $R_{x,y}$ is minimal t-irreducible. Suppose that G is a t-irreducible subset of $R_{x,y}$ such that $G \neq R_{x,y}$. G is t-irreducible implies G^c is t-complete. Also G^c contains X, the largest element of $\wp(X)$ as $X \notin R_{x,y}$. Further, $G^c \neq \wp(X)$. Therefore G^c is a topology on X other than $\wp(X)$. Also $G^c \supset R^c_{x,y}$. This is a contradiction to the fact that $R^c_{x,y}$ is an ultra topology. Therefore $R_{x,y}$ is minimal t-irreducible. □

Theorem 2.2.12. [2] *If X contains atleast two elements, then a subset R of $\wp(X)$ is minimal t-irreducible if and only if R^c is an ultra topology on X.*

Proof : R is minimal t-irreducible implies R^c is t-complete. Also R^c contains X. (If X, the largest element of $\wp(X)$ belongs to R, then by Theorem 2.1.5 of [2], $R = X$. But X is not t-irreducible subset of $\wp(X)$, since X contains atleast two elements). Therefore R^c is a topology on X. Suppose there exists a topology G on X other than $\wp(X)$ such that $G \supset R^c$. Now since G is a topology, G is t-complete and since $G \neq \wp(X)$, G^c is t-irreducible. Further, $G^c \subset R$. This leads to a contradiction to the fact that R is minimal t-irreducible. Therefore R^c is an ultra topology on X.

Conversely, suppose R^c is an ultra topology on X. Then as in the proof of the Theorem 2.2.11, R is minimal t-irreducible. □

Remark 2.2.13. We shall now establish certain relations connecting fuzzy topologies and t- irreducible subsets of the corresponding membership lattice.

In the Theorem 4.3.2 of [2], it is proved that, " Corresponding to each ultra fuzzy topology T on X, there exists unique $x \in X$ such that the set $R = \{l \in L : x_l \notin T\}$ is minimal t-irreducible "

From the proof of this result, one can easily deduce that : if T is a fuzzy topology on X other than the discrete fuzzy topology, then there exists atleast one $x \in X$ such that the set $R_x = \{l \in L : x_l \notin T\}$, is t-irreducible.

Remark 2.2.14. In the Theorem 4.3.4 of [2], it is proved that, " if X is a set with at least two elements and if R is a minimal t-irreducible subset of L, then $U(x, R) = \{A \in L^X : A(x) \notin R\}$ is an ultra fuzzy topology on X, if $1 \notin R$, and $\mathcal{U} = U(x, 1) \cup \mathcal{F}$ where $\mathcal{F}$ is an ultra fuzzy filter on X not containing the fuzzy point x_1, is an ultra fuzzy topology on X, if $1 \in R$. "

From the proof of this result, one can deduce that : if X is a set with atleast two elements and R is a t-irreducible subset of L such that $1 \notin R$, then $U(x, R) = \{A \in L^X : A(x) \notin R\}$ is a fuzzy topology on X. Also if $1 \in R$ it is easy to prove that $\mathcal{U} = U(x, 1) \cup \{\underline{1}\}$ where $U(x, 1) = \{A \in L^X : A(x) \neq 1\}$ is a fuzzy topology on X.

Remark 2.2.15. In Theorem 2.1.4. of [2], it is proved that," If 1 belongs to a minimal t-irreducible subset R of L, then $R = \{1\}$. "

From the proof this result, immediately it follows that : if R is a t-irreducible subset of L containing 1 and $R \neq \{1\}$, then $R - \{1\}$ is t-irreducible.

Theorem 2.2.16. *Every fuzzy topology on X (other than the discrete fuzzy topology) is contained in an ultra fuzzy topology on X, if and only if every t-irreducible subset of L contains a minimal t-irreducible subset.*

Proof : First assume that every t-irreducible subset of L contains a minimal t-irreducible subset. We prove that every fuzzy topology on X (other than the discrete fuzzy topology) is contained in an ultra fuzzy topology on X. Let T be a fuzzy topology on X such that $T \neq L^X$. Then there exists a fuzzy point $x_l \in L^X$ such that $x_l \notin T$. Let $R = \{m \in L : x_m \notin T\}$. Then R is a t-irreducible subset of L and by assumption R contains a minimal t-irreducible subset R^* of L.

Case 1 : $1 \notin R$.

Then $1 \notin R^*$ and consider $\mathcal{U}(x, R^*) = \{A \in L^X : A(x) \notin R^*\}$. Then $U(x, R^*)$ is an ultra fuzzy topology on X by Remark 2.2.14.

Claim : $T \subseteq U(x, R^*)$

Suppose $A \in T$ and $A \notin U(x, R^*)$. Then $A(x) \in R^*$ and hence $A(x) \in R$. Clearly $A(x) \neq 0$. Since $1 \notin R$, $x_1 \in T$. Therefore $A \wedge x_1 = x_{A(x)} \in T$ so that $A(x) \notin R$, a contradiction. Hence $T \subseteq U(x, R^*)$.

Case (2): $1 \in R$

Then $x_1 \notin T$.

$$\begin{aligned} \text{Let } \mathcal{G} &= \{A \in T : A(x) = 1\} \\ &= \{A \in T : x_1 \leq A\}. \end{aligned}$$

Then $\mathcal{G}$ is a fuzzy filter on X, namely, the neighbourhood system at x_1.

Clearly $x_1 \notin \mathcal{G}$, for, if $x_1 \in \mathcal{G}$, then $x_1 \in T \Longrightarrow 1 \notin R$, a contradiction. Let $D \in L^X$ be such that $D(y) = 1$ if $y \neq x, D(y) = 0$ if $y = x$. Then D is the complement of x_1 in L^X.

Then the collection $\{C \wedge D : C \in \mathcal{G}\}$ is a fuzzy filter base on X, [Clearly it is closed under finite meet. Also for all $C \in \mathcal{G}$, $C \wedge D \neq \underline{0}$. If $C \wedge D = \underline{0}$, then $C(y) = 0$ for all $y \neq x$. Also $C(x) = 1$. Therefore $C = x_1$, a contradiction, since $x_1 \notin \mathcal{G}$]

Since, every fuzzy filter is contained in an ultra fuzzy filter (Theorem. 2.2.3), the collection $\{C \wedge D : C \in \mathcal{G}\}$ generates a fuzzy filter on X which is contained in an ultra fuzzy filter $\mathcal{F}$ on X. Also $\mathcal{F}$ does not contain x_1, since $\mathcal{F}$ contains D

which is the complement of x_1 in L^X. By Remark 2.2.14, $U(x,1) \cup \mathcal{F}$ is an ultra fuzzy topology on X.

Then $T \subseteq U(x,1) \cup \mathcal{F}$, for, if $A \in T$ and $A(x) \neq 1$, then $A \in U(x,1)$. If $A \in T$ and $A(x) = 1$, then $x_1 \leq A$ and $A \in G \subseteq \mathcal{F}$. Thus $T \subseteq U(x,1) \cup \mathcal{F}$. Hence every fuzzy topology on X is contained in an ultra fuzzy topology on X.

Conversely, assume that every fuzzy topology on X is contained in an ultra fuzzy topology on X. Now let R be a t-irreducible subset of L such that $1 \notin R$. Then for $x \in X$, $U(x,R) = \{A \in L^X : A(x) \notin R\}$ is a fuzzy topology on X (Remark 2.2.14). By assumption there exists an ultra fuzzy topology $\mathcal{U}$ on X such that $U(x,R) \subseteq \mathcal{U}$. Then $M = \{l \in L : x_l \notin \mathcal{U}\}$ is minimal t-irreducible, by Remark. 2.2.14 and $M \subseteq R$. For,

$$\begin{aligned} l \in M &\implies x_l \notin \mathcal{U} \\ &\implies x_l \notin U(x,R) \\ &\implies x_l(x) \in R \\ &\implies l \in R. \end{aligned}$$

If R is a t-irreducible subset of L such that $1 \in R$ and $R = \{1\}$, then R is minimal t-irreducible. Otherwise, that is, if R is a t-irreducible subset of L containing 1 and $R \neq \{1\}$, then $R - \{1\}$ is t-irreducible (Remark 2.2.15). Then $R - \{1\}$ contains a minimal t-irreducible subset of L by the just previous argument. □

2.3 Principal ultra fuzzy topologies

In [2], Babusundar S. showed that if $1 \in L$ is a t-irreducible element, then ultra fuzzy topologies on X are of the form $U(x,1) \cup \mathcal{F}$ where $U(x,1) = \{A \in L^X : A(x) \neq 1\}$ and $\mathcal{F}$ is an ultra fuzzy filter on X not containing the fuzzy point x_1. According to the nature of ultra fuzzy filters, we sort a new class of ultra fuzzy topologies, namely, the principal ultra fuzzy topologies and study some of its properties.

Definition 2.3.1. [2] Let A be a nonzero fuzzy subset of X. Then the subset $P(A)$ of L^X defined by $P(A) = \{B \in L^X : B \geq A\}$ is a fuzzy filter on X, called the principal fuzzy filter at A.

Definition 2.3.2. Let L be a complete and distributive lattice with 1 as a t-irreducible element. Then an ultra fuzzy topology of the form $U(x, 1) \cup \mathcal{F}$ where $U(x, 1) = \{A \in L^X : A(x) \neq 1\}$ and $\mathcal{F}$ is a principal ultra fuzzy filter not containing the fuzzy point x_1, is called a principal ultra fuzzy topology.

Theorem 2.3.3. [2] *A principal fuzzy filter on X at a fuzzy set A is an ultra fuzzy filter if and only if A is a fuzzy point x_l for some $x \in X$ such that l is an atom of L.*

Remark 2.3.4. By Theorem 2.3.3, a principal ultra fuzzy filter $\mathcal{F}$ on X is of the form $\mathcal{F} = P(x_l)$ where $x \in X$ and l is an atom of L. Consequently principal ultra fuzzy topologies on X are of the form $U(x, 1) \cup P(y_m)$ where $x \neq y$, 1 is a t-irreducible element and m is an atom of L.

Example 2.3.5. Let X be a set containing at least two elements and $L = \{0, 1/2, 1\}$. Then 1/2 is an atom and 1/2 and 1 are t- irreducible elements of L. Therefore ultra fuzzy topologies on X are of the form $U(x, 1/2)$ and $U(x, 1) \cup \mathcal{F}$ where $x \in X$ and $\mathcal{F}$ is an ultra fuzzy filter on X not containing the fuzzy point x_1. The principal ultra fuzzy filters on X not containing x_1 are of the form $P(y_{1/2}), y \neq x$ where $P(y_{1/2}) = \{A \in L^X : A \geq y_{1/2}\}$. Thus the principal ultra fuzzy topologies on X are $U(x, 1) \cup P(y_{1/2}),\ x \neq y$.

Theorem 2.3.6. *Let X be a set containing at least two elements. Then principal ultra fuzzy topologies on X exist if and only if L is a lattice with 1 as a t-irreducible element and contains atleast one atom.*

Proof : Let $\mathcal{U}$ be a principal ultra fuzzy topology on X. Then by Definition 2.3.2, $\mathcal{U} = U(x, 1) \cup \mathcal{F}$ where $x \in X$ and $\mathcal{F}$ is a principal ultra fuzzy filter on X not containing the fuzzy point x_1. Then $\{m \in L : x_m \notin \mathcal{U}\}$ is minimal t-irreducible, by Remark 2.2.14. This implies $\{1\}$ is a minimal t-irreducible subset which implies 1 is a t-irreducible element. Also since $\mathcal{F}$ is a principal ultra fuzzy filter, by Theorem 2.3.3, there exists an atom l of L such that $\mathcal{F} = P(x_l)$. The converse follows from Remark 2.3.4. □

Theorem 2.3.7. *When L is a Boolean lattice, the principal ultra fuzzy topologies on X are of the form $\wp(X-\{x\})\cup\mathcal{F}$ where $\mathcal{F}$ is an ultra filter on X not containing $\{x\}$.*

Proof : 1 is a t-irreducible element in a Boolean lattice L if and only if $L = \{0, 1\}$ (Theorem 2.3.3 of [2]) and when $L = \{0, 1\}$, the ultra fuzzy topologies on X coincides with the ultra topologies on X. □

Remark 2.3.8. When L is a finite chain, principal ultra fuzzy topologies exist on X by (Remark 2.3.4).

Definition 2.3.9. [*27*] Let (X, δ) be a fuzzy topological space and $C, D \in L^X$. Then C, D are said to be separated, if $\bar{C} \wedge D = C \wedge \bar{D} = \underline{0}$. A is said to be connected, if there does not exist separated sets $C, D \in L^X - \{\underline{0}\}$ such that $A = C \vee D$.

Definition 2.3.10. A fuzzy topological space is said to be totally disconnected if and only if the only connected sets are the fuzzy points.

Theorem 2.3.11. *Let L be a lattice with 1 as a t-irreducible element and contains at least one atom. If X contains at least three elements, no principal ultra fuzzy topology on X is totally disconnected. That is, there exists a connected fuzzy subset of X other than a fuzzy point.*

Proof : Let $U(x,1) \cup P(y_m), x \neq y$ and m an atom of L is a principal ultra fuzzy topology on X. Then the fuzzy subset A defined by $A(x) = 1$, $A(y) = 1$ and $A(z) = 0$ for all $z \neq x, y$ is connected. For, since 1 is a t-irreducible element, 1 cannot be written as the join of two elements of L, both are different from 1. Hence A cannot be expressed as the join of two separated fuzzy sets C, D such that $A = C \vee D$. □

Definition 2.3.12. *[27]* A fuzzy topological space (X, δ) is extremally disconnected if for every $A \in \delta$, $\bar{A} \in \delta$.

Theorem 2.3.13. *Every principal ultra fuzzy topology is extremally disconnected, provided L is a complemented lattice.*

Proof : Let δ be a principal ultra fuzzy topology on X. Then $\delta = U(x,1) \cup P(y_m), y \neq x$ and m an atom of L. Also let $V \in \delta$.

Case(a) : $x_1 \leq V$.

In this case, since $x_1 \not\leq V'$, V' is open so that V is closed. Therefore $V = \bar{V} \in \delta$.

Case(b) : $x_1 \not\leq V$.

If L is complemented, L^X is also complemented. So by Theorem 2.2.5, either $V \in P(y_m)$ or $V' \in P(y_m)$. Now,

$$\begin{aligned} V' \in P(y_m) &\implies V' \text{ open} \\ &\implies V \text{ closed} \\ &\implies V = \bar{V} \in \delta. \end{aligned}$$

$$\begin{aligned} V \in P(y_m) &\implies V \geq y_m \\ &\implies \bar{V} \geq V \geq y_m \\ &\implies \bar{V} \in P(y_m) \\ &\implies \bar{V} \in \delta. \end{aligned}$$

Thus in all cases $\bar{V} \in \delta$ and hence δ is extremally disconnected. □

Definition 2.3.14. [*27*] Let L be a lattice. An element $l \in L$ is called a molecule (or join irreducible) if $l > 0$ and for all $a, b \in L,\ l = a \vee b \Longrightarrow l = a$ or $l = b$. The set of all molecules in L is denoted by $M(L)$.

Remark 2.3.15. [27] *The set of all the molecules in L^X denoted by $M(L^X)$ is given by the set $M(L^X) = \{\, x_l \in Pt(L^X) : l \in M(L) \,\}$.*

Definition 2.3.16. [*27*] A fuzzy topological space (X, δ) is called T_0, if for every two distinguished molecules $x_\lambda, y_\gamma \in L^X$ there exist $U \in Q(x_\lambda)$ such that y_γ is not quasi-coincident with U or $V \in Q(y_\gamma)$ such that x_λ is not quasi-coincident with V.

Definition 2.3.17. [*27*] A fuzzy topological space (X, δ) is called T_1, if for every two distinguished molecules $e, d \in L^X$ such that $e \not\leq d$, there exists $U \in Q(e)$ such that d is not quasi-coincident with U.

Theorem 2.3.18. [*27*] *A fuzzy topological space (X, δ) is T_1 if and only if every fuzzy point in L^X is closed.*

Theorem 2.3.19. *Every principal ultra fuzzy topology is T_0.*

Proof : For any two distinct molecules in L^X, atleast one of them is open in a principal ultra fuzzy topology. □

Theorem 2.3.20. *No principal ultra fuzzy topology is T_1.*

Proof : If $y \neq x$, y_1 is not closed in U $= U(x,1) \cup P(y_m)$, since $y_1' \notin P(y_m)$ and $y_1'(x) = 1$ so that $y_1' \notin U(x,1)$. So by Theorem 2.3.18 U is not T_1. □

Definition 2.3.21. A fuzzy topological space is said to be a door space, if every fuzzy set is either open or closed.

Theorem 2.3.22. *Every ultra fuzzy topology of the form* $U(x,1) \cup \mathcal{F}$ *where* $x \in X$ *and* $\mathcal{F}$ *is an ultra fuzzy filter not containing the fuzzy point* x_1 *form a door space.*

Proof : Let A be a fuzzy subset of X. If A contains x_1, then A' is open so that A is closed. If A does not contain x_1, then A is open. □

2.4 Fuzzy topologies and pre-order relations

In [1], Andima S. J. & Thron W. J. defined a pre-order relation on X corresponding to each topology on X and investigated relationships between them. In this section, we conduct an analogous study in the case of fuzzy topological space.

Definition 2.4.1. For a fuzzy topology T on X, define a relation ρT on Pt (L^X) by $(x_l, y_m) \in \rho T$ if and only if every open set containing y_m contains x_l.

Theorem 2.4.2. *For a fuzzy topology* T *on* X*,* ρT *is a pre-order relation on* *Pt*(L^X).

Proof : (i) Obviously $(x_l, x_l) \in \rho T$ for all $x_l \in \mathrm{Pt}(L^X)$.
(ii) If $(x_l, y_m), (y_m, z_n) \in \rho T$, then every open set containing z_n, contains y_m and every open set containing y_m, contains x_l. That is, every open set containing z_n, contains x_l so that $(x_l, z_n) \in \rho T$. Therefore ρT is transitive. □

Theorem 2.4.3. *Let T_1, T_2 be fuzzy topologies on X. Then $T_1 \subseteq T_2 \Longrightarrow \rho T_1 \supseteq \rho T_2$.*

Proof :

$$\begin{aligned}(x_l, y_m) \in \rho T_2 &\Longrightarrow \text{ every } T_2 \text{ open set containing } y_m \text{ contains } x_l \\ &\Longrightarrow \text{ every } T_1 \text{ open set containing } y_m \text{ contains } x_l \\ &\Longrightarrow (x_l, y_m) \in \rho T_1.\end{aligned}$$

□

Definition 2.4.4. Let T be a fuzzy topology on X and let $x_l \leq y_m \Longleftrightarrow (x_l, y_m) \in \rho T$. Then for each $x_l \in Pt(L^X)$, the closure of x_l is $\{\bar{x_l}\} = \bigvee\{y_m \in Pt(L^X) : x_l \leq y_m\}$, the kernel of x_l is $\{\hat{x_l}\} = \bigvee\{y_m \in Pt(L^X) : y_m \leq x_l\}$.

Remark 2.4.5. For a given pre-order R on $Pt(L^X)$, there are in general, many fuzzy topologies T for which $\rho(T) = R$, but all of these fuzzy topologies have exactly the same point closures and kernals. Therefore given a pre-order R on $Pt(L^X)$, it makes sense to write $\{\bar{x_l}\}$ or $\{\hat{x_l}\}$ even though no fuzzy topology is specified. When needed for clarity, such notations as $\{\hat{x_l}\}^T$ or $\{\hat{x_l}\}^R$ will be used. For any $G \in T$, $G = \bigvee\{\{\hat{x_l}\} : x_l \leq G\}$ and for any $x_l \in Pt(L^X)$, $\{\hat{x_l}\} = \bigwedge\{G : x_l \leq G \in T\}$.

Definition 2.4.6. Let R be a pre-order on $Pt(L^X)$. Then the smallest fuzzy topology on X, in which all fuzzy sets $\{\bar{x_l}\} = \bigvee\{y_m \in Pt(L^X) : x_l R y_m\}, x_l \in Pt(L^X)$ are closed is called the point closure fuzzy topology of R and is denoted by $\mu(R)$. The smallest fuzzy topology on X, in which all fuzzy sets $\{\hat{x_l}\} = \bigvee\{y_m \in Pt(L^X) : y_m R x_l\}, x_l \in Pt(L^X)$ are open is called the kernel fuzzy topology of R and is denoted by $\nu(R)$.

Remark 2.4.7. The collection $\{\{\hat{x_l}\} : x_l \in Pt(L^X)\}$ is a base for the kernel fuzzy topology $\nu(R)$ (see Definition 2.1.14).

Theorem 2.4.8. *A fuzzy topology T on X has pre-order relation R on $Pt(L^X)$ if and only if $\mu(R) \leq T \leq \nu(R)$. In particular, $\rho(\mu(R)) = \rho(\nu(R)) = R$.*

Proof : Assume that T has pre-order relation R. For each $x_l \in Pt(L^X)$, $\{\bar{x}_l\}^R = \{\bar{x}_l\}^T$, which is closed in (X, T). Therefore, since $\mu(R)$, by definition, the smallest fuzzy topology in which the sets $\{\bar{x}_l\}^R$ are closed, $\mu(R) \leq T$. For any $G \in T$,

$$\begin{aligned} G &= \vee\{\{\hat{x}_l\}^T : x_l \leq G\} \\ &= \vee\{\{\hat{x}_l\}^R : x_l \leq G\} \in \nu(R) \end{aligned}$$

so that $T \leq \nu(R)$.

Conversely, suppose T is a fuzzy topology on X such that $\mu(R) \leq T \leq \nu(R)$. Then $\rho(\mu(R)) \supseteq \rho(T) \supseteq \rho(\nu(R))$.......................(1)
Since for each $x_l \in Pt(L^X)$, $\{\bar{x}_l\}^R$ is closed in $\mu(R)$, $\{\bar{x}_l\}^{\mu(R)} \subseteq \{\bar{x}_l\}^R$. Therefore,

$$\begin{aligned} (x_l, y_m) \in \rho(\mu(R)) &\implies y_m \in \{\bar{x}_l\}^{\mu(R)} \\ &\implies y_m \in \{\bar{x}_l\}^R \\ &\implies (x_l, y_m) \in R \end{aligned}$$

so that $R \supseteq \rho(\mu(R))$.....................(2)
Let $(x_l, y_m) \in R$ and let G be any open fuzzy set in $\nu(R)$ containing y_m. Since $\{\{\hat{x}_l\}^R : x_l \in Pt(L^X)\}$ forms a base for $\nu(R)$, there exists $z_n \in Pt(L^X)$ such that $y_m \in \{\hat{z_n}\}^R \subseteq G$. Then $(x_l, y_m) \in R$ and $(y_m, z_n) \in R \implies (x_l, z_n) \in R \implies x_l \in \{\hat{z_n}\}^R \subseteq G$. Therefore $(x_l, y_m) \in \rho(\nu(R))$ and $\rho(\nu(R)) \supseteq R$ ....(3)
Combining (1),(2) and (3) we get, $R \supseteq \rho(\mu(R)) \supseteq \rho(T) \supseteq \rho(\nu(R)) \supseteq R \implies \rho(\mu(R)) = \rho(T) = \rho(\nu(R)) = R$. □

Theorem 2.4.9. *Let R be a pre-order relation on $Pt(L^X)$. If for all $x_l \in Pt(L^X)$, $\{\hat{x}_l\}'$ is the join of a finite number of point closures, then $\mu(R) = \nu(R)$.*

Proof : Assume that for each $x_l \in Pt(L^X), \{\hat{x_l}\}'$ is the join of a finite number of point closures. Then $\{\hat{x_l}\}'$ is closed in $\mu(R)$ and hence $\{\hat{x_l}\}$ is open in $\mu(R)$. That is, $\nu(R) \subseteq \mu(R)$, since $\nu(R)$ is the smallest fuzzy topology in which $\{\hat{x_l}\}$ is open. But by Theorem 2.4.8, $\mu(R) \subseteq \nu(R)$ and hence $\mu(R) = \nu(R)$. □

Theorem 2.4.10. *Let X be a nonempty set and L a complete distributive lattice with 1 as a t-irreducible element. Then for any pre-order R on $Pt(L^X)$, $\nu(R) \subseteq \bigwedge\{T(x_1, y_m) : (y_m, x_1) \in R\}$ where $T(x_1, y_m)$ is the principal ultra fuzzy topology $U(x, 1) \cup p(y_m), x \neq y$.*

Proof : Since $\rho(\nu(R)) = R$, $(y_m, x_1) \in R$ if and only if $\nu(R) \subseteq T(x_1, y_m)$. Therefore $\nu(R) \subseteq T(x_1, y_m)$ for every $(y_m, x_1) \in R, x \neq y$. Hence $\nu(R) \subseteq \bigwedge\{T(x_1, y_m) : (y_m, x_1) \in R, x \neq y\}$. □

CHAPTER

3

Automorphisms of the Lattice of Fuzzy Topologies

3.1 Introduction

In 1958, Juris Hartmanis [16] determined the automorphisms of the lattice $LT(X)$ of all topologies on a fixed set X as follows : for $p \in S(X)$ and $\tau \in LT(X)$, define the mapping A_p by $A_p(\tau) = \{p(U) : U \in \tau\}$. Then $A_p(\tau)$ is a topology on X and A_p is an automorphism of $LT(X)$. If X is infinite or X contains atmost two elements, the set of all automorphisms of $LT(X)$ is precisely $\{A_p : p \in S(X)\}$. Otherwise, the set of all automorphisms of $LT(X)$ is $\{A_p : p \in S(X)\} \cup \{B_p : p \in S(X)\}$ where $B_p : LT(X) \longrightarrow LT(X)$ is defined by $B_p(\tau) = \{X - p(U) : U \in \tau\}$ for $\tau \in LT(X)$. From this result, we can conclude that, if X is an infinite set and P is any topological property, then the set of topologies in $LT(X)$ possessing the property P may be identified simply from the lattice structure of $LT(X)$, since the only automorphisms of $LT(X)$ for infinite X are those which simply permute elements of X. Therefore any automorphism of $LT(X)$ must map all the topologies in $LT(X)$ onto their homeomorphic images. Thus the topological properties of elements of $LT(X)$ must be determined by the position of the topologies in $LT(X)$ (see [24]). In this chapter, we try to investigate the group of automorphisms of the lattice $LFT(X, L)$ where L is a complete, distributive lattice.

Let X be any nonempty set and L be a complete and distributive lattice.

Definition 3.1.1. [*2*] A t-homomorphism from a lattice L into a lattice M is a function $f : L \longrightarrow M$ such that
(i) h is a homomorphism
(ii) $h(0) = 0$ and $h(1) = 1$
(iii) $h(\bigvee k_i) = \bigvee h(k_i)$ where $\{k_i : i \in I\}$ is an arbitrary subset of L.

Remark 3.1.2. Obviously every t-homomorphism is a homomorphism. But the converse need not be true.

Example 3.1.3. Let $L = \{1, 2, 3\}$ and $M = \{1, 2, 3, 4, 5\}$ be lattices under usual order " $\leq$ ". Define $f : L \longrightarrow M$ by $f(1) = 2$, $f(2) = 3$, $f(3) = 4$. Then f is a homomorphism, but not a t-homomorphism, since $f(0) \neq 1$ and $f(1) \neq 5$.

Example 3.1.4. Let X be an infinite set and L be a collection of subsets of $\wp(X)$ which are closed under finite union and M be a collection of subsets of $\wp(X)$ which are closed under arbitrary union. Then L and M are lattices under set inclusion and $M \subseteq L$. The empty set ϕ is the smallest element " 0 " and $\wp(X)$ is the largest element " 1 " of both L and M. If $f : M \longrightarrow L$ is the inclusion map, then f is a homomorphism such that $f(0) = 0$ and $f(1) = 1$, but is not a t-homomorphism. For every $x \in X$, the collection $\{\{x\}\}$ is in M. Also

$\bigvee_{x\in X}\{\{x\}\}$ in M is equal to $\wp(X)$, where as $\bigvee_{x\in X} f(\{\{x\}\})$ is the collection of all finite subsets of X. Thus $f(\bigvee_{x\in X}\{\{x\}\}) = \wp(X) \neq$ the collection of all finite subsets of X which is equal $\bigvee_{x\in X} f(\{\{x\}\})$.

3.2 Automorphisms of the lattice $LFT(X, L)$

Let X and Y be nonempty sets and L, M be complete, distributive lattices. For functions $f : X \longrightarrow Y$ and $h : M \longrightarrow L$, Babusundar [2] defined the induced function $H : M^Y \longrightarrow L^X$ as follows : for $D \in M^Y$ and $x \in X$, $H(D)(x) = h(D(f(x)))$. He observed that H is a t-isomorphism if and only if f is a bijection and h is a t-isomorphism. He also defined the induced function $H' : LFT(Y, M) \longrightarrow LFT(X, L)$ by $H'(\delta) = \{H(U) : U \in \delta\}$ for δ in $LFT(Y, M)$. We use the notations $H_{f^{-1},h}$ for H and $H^{\star}_{f^{-1},h}$ for H' and proceed further.

As a particular case of the above result, that is, when $X = Y$ and $L = M$, we obtain the following result.

Theorem 3.2.1. [2] *Let X be any nonempty set and L be a complete and distributive lattice. If $p : X \longrightarrow X$ is a bijection and $g : L \longrightarrow L$ is an isomorphism, then, $H_{p,g} : L^X \longrightarrow L^X$ defined by $H_{p,g}(C)(x) = g(C(p^{-1}(x)))$; $C \in L^X$, $x \in X$ is an automorphism of L^X. Further, if δ is a fuzzy topology on X, then the collection $H^{\star}_{p,g}(\delta) = \{H_{p,g}(A) : A \in \delta\}$ is also a fuzzy topology on X and $H^{*}_{p,g}$ is an automorphism of $LFT(X, L)$.*

Theorem 3.2.2. *Let X be a finite set and L be a finite F-lattice. Let $p : X \longrightarrow X$ be a bijection and $g : L \longrightarrow L$ be an isomorphism. Define $H_{p,g} : L^X \longrightarrow L^X$ by $H_{p,g}(C)(x) = g(C(p^{-1}(x)))$; $C \in L^X, x \in X$. For $\delta \in LFT(X, L)$, define $F^{*}_{p,g}$ by $F^{*}_{p,g}(\delta) = \{comp(H_{p,g}(C)) : C \in \delta\}$ where $comp(H_{p,g}(C))$ denotes the pseudo-complement of $H_{p,g}(C)$ in L^X. Then $F^{*}_{p,g}$ is an automorphism of $LFT(X, L)$.*

Proof : We have $\underline{0} = comp(\underline{1}) = comp(H_{p,g}(\underline{1}))$ and $1 = comp(\underline{0}) = comp(H_{p,g}(\underline{0}))$. Since $\underline{0}\,, \underline{1} \in \delta$, it follows that $\underline{0}\,, \underline{1} \in F^*_{p,g}(\delta)$.
Let $f_1, f_2 \in F^*_{p,g}(\delta)$. Then $f_1 = comp(H_{p,g}(C))$ and $f_2 = comp(H_{p,g}(D))$ for some $C, D \in \delta$. We have $C, D \in \delta \Longrightarrow C \wedge D \in \delta$. Now,

$$\begin{aligned} f_1 \vee f_2 &= comp(H_{p,g}(C)) \vee comp(H_{p,g}(D)) \\ &= comp[H_{p,g}(C) \wedge H_{p,g}(D)] \\ &= comp[H_{p,g}(C \wedge D)] \in F^*_{p,g}(\delta). \end{aligned}$$

Similarly, $f_1 \wedge f_2 \in F^*_{p,g}(\delta)$.
Thus $F^*_{p,g}(\delta)$ is a fuzzy topology on X.
For $\delta_1, \delta_2 \in LFT(X, L)$, let $F^*_{p,g}(\delta_1) = F^*_{p,g}(\delta_2)$. This implies,

$$\begin{aligned} \{comp(H_{p,g}(C)) : C \in \delta_1\} = \{comp(H_{p,g}(D)) : D \in \delta_2\} \Longrightarrow &\ \{C : C \in \delta_1\} \\ &= \{D : D \in \delta_2\} \\ \Longrightarrow &\ \delta_1 = \delta_2. \end{aligned}$$

Therefore $F^*_{p,g}$ is one to one.
For $\tau \in LFT(X, L)$, consider the collection $\delta = \{H^{-1}_{p,g}(comp(C)) : C \in \tau\}$. Then δ is a fuzzy topology on X and

$$\begin{aligned} F^*_{p,g}(\delta) &= \{comp(H_{p,g}(H^{-1}_{p,g}(comp(C)))) : C \in \tau\} \\ &= \{comp(comp(C)) : C \in \tau\} \\ &= \{C : C \in \tau\} \\ &= \tau. \end{aligned}$$

Therefore $F^*_{p,g}$ is onto. Also,

$$\begin{aligned}
\delta_1 \subseteq \delta_2 &\iff \{C : C \in \delta_1\} \subseteq \{D : D \in \delta_2\} \\
&\iff \{H_{p,g}(C) : C \in \delta_1\} \subseteq \{H_{p,g}(D) : D \in \delta_2\} \\
&\iff \{comp(H_{p,g}(C)) : C \in \delta_1\} \subseteq \{comp(H_{p,g}(D)) : D \in \delta_2\} \\
&\iff F^*_{p,g}(\delta_1) \subseteq F^*_{p,g}(\delta_2).
\end{aligned}$$

Hence $F^*_{p,g}$ is an automorphism of $LFT(X, L)$. $\square$

Example 3.2.3. Let $X = \{a, b\}$, $L = \{0, 1/2, 1\}$. Then the lattice $L^X = \{a^0b^0, a^1b^1, a^{1/2}b^{1/2}, a^0b^{1/2}, a^0b^1, a^{1/2}b^0, a^{1/2}b^1, a^1b^0, a^1b^{1/2}\}$ where a^ib^j ; $i, j = 0, 1/2, 1$ is the map $a \longrightarrow i$ and $b \longrightarrow j$. Let $S(X)$ denote the group of bijections on X and $A(L)$ denote the group of automorphisms of L. Then $S(X) = \{p_1, p_2\}$ where p_1 is the identity map on X and p_2 is the map on X which sends $a \longrightarrow b$ and $b \longrightarrow a$. $A(L)$ consists only one member g which is the identity map on L. Thus by Theorem 3.2.1, and Theorem 3.2.2, $H^*_{p_1,g}, H^*_{p_2,g}, F^*_{p_1,g}$ and $F^*_{p_2,g}$ are the automorphisms of the lattice $LFT(X, L)$.

Example 3.2.4. Let $X = \{x, y\}$, $L = \{0, a, b, 1\}$ where a and b are not comparable. Then the lattice $L^X = \{x^0y^0, x^ay^a, x^by^b, x^0y^1, x^0y^a, x^0y^b, x^1y^0, x^1y^a, x^1y^b, x^ay^b, x^by^a, x^ay^1, x^by^1, x^ay^0, x^by^0, x^1y^1\}$ where x^iy^j ; $i, j = 0, a, b, 1$ is the map on X which sends $x \longrightarrow i$ and $y \longrightarrow j$. Here, $S(X) = \{p_1, p_2\}$ where p_1 is the identity map on X and p_2 is the map which sends $x \longrightarrow y$ and $y \longrightarrow x$. $A(L) = \{g_1, g_2\}$ where g_1 is the identity map on L and g_2 is the map which sends $0 \longrightarrow 0$, $a \longrightarrow b$, $b \longrightarrow a$, $1 \longrightarrow 1$. Thus by Theorem 3.2.1, and Theorem 3.2.2, $H^*_{p_1,g_1}, H^*_{p_1,g_2}, H^*_{p_2,g_1}, H^*_{p_2,g_2}, F^*_{p_1,g_1}, F^*_{p_1,g_2}, F^*_{p_2,g_1}$ and $F^*_{p_2,g_2}$ are automorphisms of $LFT(X, L)$.

Remark 3.2.5. When $g : L \longrightarrow L$ is the identity map, then $H_{p,g}$ is denoted by H_p, $H^*_{p,g}$ is denoted by H^*_p and $F^*_{p,g}$ is denoted by F^*_p.

Example 3.2.6. When X contains only one element, we can identify $L^X = \{\underline{0}\,, \underline{1/2}, \underline{1}\}$ with $L = \{0, 1/2, 1\}$. Then the only fuzzy topologies on X are the discrete fuzzy topology L^X and the indiscrete fuzzy topology $\{\underline{0}\,, \underline{1}\}$. Thus $LFT(X,L) = \{\{\underline{0}\,, \underline{1}\}, L^X\}$ whose only automorphism is the identity map which corresponds to H_p^* where p is the identity map on X.

Remark 3.2.7. When $L = \{0,1\}$, $LFT(X,L)$ coincides with $LT(X)$, $H_{p,g}^*$ coincides with A_p and $F_{p,g}^*$ coincides with B_p where A_p and B_p are as defined in the beginning of this chapter. Note that we are identifying the subsets of X as characteristic functions.

3.3 Automorphisms of $LFT(X,L)$ when $L = \{0, 1/2, 1\}$

In this section, we determine the set of all automorphisms of the lattice $LFT(X,L)$ of all fuzzy topologies on X, when the membership lattice $L = \{0, 1/2, 1\}$ with usual order. Note that when $L = \{0, 1/2, 1\}$, $A(L)$, the group of automorphisms of L contains only one element, the identity map of L. Therefore as in the Remark 3.2.5, $H_{p,g} = H_p$, $H_{p,g}^* = H_p^*$ and $F_{p,g}^* = F_p^*$ (refer section 3.2). The pseudo-complement on L is defined by $\lambda' = 1 - \lambda$ for $\lambda \in L$.

Definition 3.3.1. Let X be a finite set and $L = \{0, 1/2, 1\}$. For a bijection p on X, define the mappings $H_p, F_p : L^X \longrightarrow L^X$ by $H_p(C)(x) = C(p^{-1}(x))\,;\, x \in X, C \in L^X$ and $F_p(C) =$ pseudo-complement of $H_p(C)$ in L^X. [Note that H_p can also be defined by $H_p(C) = \bigvee\{(p(x))_l : x_l \in C\}\,; C \in L^X\,]$. Also define the mappings $H_p^*\,, F_p^* : LFT(X,L) \longrightarrow LFT(X,L)$ by $H_p^*(\tau) = \{H_p(C) : C \in \tau\}$ and $F_p^*(\tau) = \{F_p(C) : C \in \tau\}$. Then H_p^* and F_p^* are automorphisms of $LFT(X,L)$ (refer Theorems 3.2.1 and 3.2.2).

Notation 3.3.2. Let X be any non-empty set and L be the lattice $\{0, 1/2, 1\}$.

An atom of the lattice $LFT(X, L)$ is of the form $\{\underline{0}\,, C, \underline{1}\}$ where $\underline{0} \neq C \neq \underline{1} \in L^X$. Let us denote the pseudo-complement of the fuzzy point $x_{1/2}$ by $x^{1/2}$ and that of x_1 by x^0. We shall denote the atoms of the type $\{\underline{0}\,, x_{1/2}, \underline{1}\}$ by $I_{x_{1/2}}$, atoms of the type $\{\underline{0}\,, x^{1/2}, \underline{1}\}$ by $\bar{I}_{x_{1/2}}$, atoms of the type $\{\underline{0}\,, x_1, \underline{1}\}$ by I_{x_1} and atoms of the type $\{\underline{0}\,, x^0, \underline{1}\}$ by $\bar{I}_{x_1}$. Also let $\eta = \{I_{x_{1/2}} : x \in X\} = \{\{\underline{0}\,, x_{1/2}, \underline{1}\} : x \in X\}$ and $\xi = \{\bar{I}_{x_{1/2}} : x \in X\} = \{\{\underline{0}\,, x^{1/2}, \underline{1}\} : x \in X\}$.

Remark 3.3.3. Note that an automorphism of a lattice maps atoms to atoms and dual atoms to dual atoms.

Lemma 3.3.4. *An automorphism of the lattice $LFT(X, L)$ maps a fuzzy topology to a fuzzy topology having the same cardinality.*

Proof : When T is a fuzzy topology consisting of a finite number n of open sets, then T is larger than precisely $n-2$ atoms. Therefore the image of T under an automorphism should also be larger than precisely $n - 2$ atoms. Hence the image of T consists of n open sets. When T is a fuzzy topology whose cardinality is infinite, say, α, then it contains α atoms. Hence the image of T also contains α atoms. Thus the cardinality of the image is also α. $\square$

Lemma 3.3.5. *The join of any atom from $\eta \cup \xi$ with any atom of $LFT(X, L)$ consists at most* 5 *open sets.*

Proof : For any $\underline{0} \neq B \neq \underline{1} \in L^X$,

$$\begin{aligned}\{\underline{0}\,, x_{1/2}, \underline{1}\} \vee \{\underline{0}, B, \underline{1}\} &= \{\underline{0}\,, x_{1/2}, B, \underline{1}\} \text{ if } B(x) = 1/2 \text{ or } 1, \\ &= \{\underline{0}\,, x_{1/2}, B, B \vee x_{1/2}, \underline{1}\} \text{ otherwise.}\end{aligned}$$

$$\begin{aligned}\{\underline{0}\,, x^{1/2}, \underline{1}\} \vee \{\underline{0}\,, B, \underline{1}\} &= \{\underline{0}\,, x^{1/2}, B, \underline{1}\} \text{ if } B(x) = 0 \text{ or } 1/2,\\ &= \{\underline{0}\,, x^{1/2}, B, B \wedge x^{1/2}, \underline{1}\} \text{ if } B(x) = 1.\end{aligned}$$

$\square$

Definition 3.3.6. Let a and b are elements of a complete lattice, then a and b are said to be complementary if $a \wedge b = 0$ and $a \vee b = 1$.

Lemma 3.3.7. *For every atom $u \notin \eta \cup \xi$, there is an atom v of $LFT(X, L)$ such that $u \vee v$ consists of 6 open sets.*

Proof : Let $u \notin \eta \cup \xi$. Then $u = \{\underline{0}\,, C, \underline{1}\}$ where $\underline{0} \neq C \neq \underline{1} \in L^X$ is such that $C \neq x_{1/2}$ and $C \neq x^{1/2}$ for any $x \in X$.

Now we prove that there exists an atom $v = \{\underline{0}\,, D, \underline{1}\}$ where $D \in L^X$ such that C and D are not comparable and not complementary.
Since $C \neq \underline{0}\,, C \neq \underline{1}\,, C \neq x_{1/2}$ and $C \neq x^{1/2}$, for any $x \in X,$ there exists $x, y \in X$ such that $x_{1/2} < C < y^{1/2}$.
If $x_{1/2} < C$, then there exists $z_1 \in X$ such that $x_{1/2}(z_1) < C(z_1)$. Therefore $C(z_1) \neq 0$ and $C(z_1) = 1/2$ or 1. If $z_1 = x$, then $1/2 = x_{1/2}(x) < C(x)$ and therefore $C(x) = C(z_1) = 1$.
If $C < y^{1/2}$, then there exists $z_2 \in X$ such that $C(z_2) < y^{1/2}(z_2)$. Therefore $C(z_2) \neq 1$ and therefore $C(z_2) = 0$ or $1/2$.
If $z_2 = y$, $C(z_2) = C(y) < y^{1/2}(y) = 1/2$. Therefore $C(z_2) = C(y) = 0$.
Now define $D \in L^X$ such that

$$\begin{aligned} D(w) &= x_{1/2}(z_1) \text{ if}\, w = z_1,\\ &= y^{1/2}(z_2) \text{ if } w = z_2,\\ &= C(w) \text{ otherwise.}\end{aligned}$$

Then $D(z_1) = x_{1/2}(z_1) < C(z_1)$. Therefore $C \not\leq D$ and $D(z_2) = y^{1/2}(z_2) > C(z_2)$. Therefore $D \not\leq C$. Thus C and D are not comparable.

Now we are going to show that C and D are not complementary.

Claim : $x_{1/2} \leq D$. ....**(1)**

For otherwise, $D(x) = 0$. But $D(x) = x_{1/2}(x) = x_{1/2}(z_1)$ or $D(x) = y^{1/2}(x) = y^{1/2}(z_2)$ or $D(x) = C(x)$ by definition of D. But $x_{1/2} < C$. Therefore $C(x) \neq 0$ and therefore $D(x) \neq C(x)$. Therefore $D(x) = x_{1/2}(x) = x_{1/2}(z_1)$ or $y^{1/2}(x) = y^{1/2}(z_2)$. $D(x) = 0$ and $x_{1/2}(x) = 1/2$. Therefore $D(x) \neq x_{1/2}(x)$. If $D(x) = y^{1/2}(x) = y^{1/2}(z_2) \geq 1/2$, a contradiction, since $D(x) = 0$. Therefore $x_{1/2} \leq D$. But $x_{1/2} < C$ and therefore $C \wedge D \neq \underline{0}$.

Claim : $D \leq y^{1/2}$. ....**(2)**

For otherwise $D(y) = 1$. But $D(y) = x_{1/2}(y) = x_{1/2}(z_1)$ or $D(y) = y^{1/2}(y) = y^{1/2}(z_2)$ or $D(y) = C(y)$, by definition of D. $C(y) \leq y^{1/2}(y) = 1/2$. Therefore $C(y) \neq 1$. Therefore $D(y) \neq C(y)$. If $D(y) = x_{1/2}(y) = x_{1/2}(z_1) \leq 1/2$, a contradiction. If $D(y) = y^{1/2}(y) = y^{1/2}(z_2)$, then $y^{1/2}(y) = 1/2$. Therefore $D(y) = 1/2$, a contradiction, since $D(y) = 1$. Therefore $D \leq y^{1/2}$. Also $C < y^{1/2}$. Therefore $C \vee D \leq y^{1/2}$ and hence $C \vee D \neq \underline{1}$.

Hence C and D are neither comparable nor complementary. Therefore $\{\underline{0}, C, \underline{1}\} \vee \{\underline{0}, D, \underline{1}\}$ contain 6 open sets. $\square$

Lemma 3.3.8. *An automorphism of the lattice $LFT(X, L)$ maps $\eta \cup \xi$ onto itself.*

Proof : Let $\alpha \in \eta \cup \xi$ and let A be an automorphism of $LFT(X, L)$. Clearly $A(\alpha)$ is an atom. To prove $A(\alpha) \in \eta \cup \xi$. Otherwise, there exists an atom β such that $A(\alpha) \vee \beta$ consists 6 open sets. Also there exists $\beta' \in LFT(X, L)$ such that $A(\beta') = \beta$. Clearly β' is an atom. Now $A(\alpha) \vee \beta = A(\alpha) \vee A(\beta') = A(\alpha \vee \beta')$. Therefore $\alpha \vee \beta'$ consists 6 open sets, a contradiction.

Also if $\alpha \in \eta \cup \xi$, consider the automorphism A^{-1} and let $A^{-1}(\alpha) = \beta$. Then $A(\beta) = \alpha$. Also $\beta \in \eta \cup \xi$, for otherwise, we can similarly get a contradiction. Thus $A(\eta \cup \xi) = \eta \cup \xi$. $\square$

Theorem 3.3.9. *Every automorphism of $LFT(X, L)$ maps either (η onto η and ξ onto ξ) or (η onto ξ and ξ onto η).*

Proof : The join of any two distinct atoms in η consists 5 open sets, because $\{\underline{0}\,, x_{1/2}, \underline{1}\} \vee \{\underline{0}\,, y_{1/2}, \underline{1}\} = \{\underline{0}\,, x_{1/2}, y_{1/2}, x_{1/2} \vee y_{1/2}, \underline{1}\}$ if $x \neq y$. The join of any two distinct atoms in ξ consists 5 open sets, because $\{\underline{0}\,, x^{1/2}, \underline{1}\} \vee \{\underline{0}\,, y^{1/2}, \underline{1}\} = \{\underline{0}\,, x^{1/2}, y^{1/2}, x^{1/2} \wedge y^{1/2}, \underline{1}\}$ if $x \neq y$. The join of an element from ξ with an element from η always consists 4 open sets. For, $\{\underline{0}\,, x_{1/2}, \underline{1}\} \vee \{\underline{0}\,, y^{1/2}, \underline{1}\} = \{\underline{0}\,, x_{1/2}, y^{1/2}, \underline{1}\}$ since $x_{1/2} \leq y^{1/2}$ holds for all $x, y \in X$.

Let A be an automorphism of $LFT(X, L)$. If $\alpha, \beta \in \eta$, and $A(\alpha) \in \eta$, then $A(\beta) \in \eta$. Otherwise, $A(\alpha) \in \eta$ and $A(\beta) \in \xi$. Therefore $A(\alpha) \vee A(\beta)$ has 4 open sets. But $A(\alpha) \vee A(\beta) = A(\alpha \vee \beta)$. Therefore $\alpha \vee \beta$ has 4 open sets, a contradiction.

Similarly, if $\alpha, \beta \in \eta$ and $A(\alpha) \in \xi$, then $A(\beta) \in \xi$. Thus either every element of η is mapped to η or every element of η is mapped to ξ. Similarly we can prove that either every element of ξ is mapped to ξ or every element of ξ is mapped to η. $\square$

Theorem 3.3.10. *When X is finite, the set of all automorphisms of the lattice $LFT(X, L)$ is precisely given by $\{H_p^* : p \in S(X)\} \bigcup \{F_p^* : p \in S(X)\}$.*

Proof : Let A be an automorphism of $LFT(X, L)$. We want to prove that $A = H_p^*$ or $A = F_p^*$ for some $p \in S(X)$. By the Theorem 3.3.9, A maps either (η onto η and ξ onto ξ) or (η onto ξ and ξ onto η).

Case (a) : When A maps η onto η and ξ onto ξ.

We will show that $A = H_p^*$ for some $p \in S(X)$. For $I_{x_{1/2}} \in \eta$, let $A(I_{x_{1/2}}) = I_{y_{1/2}}$ for some $y \in X$. This y is unique. Define $p : X \longrightarrow X$ by $p(x) = y$. We show that $p \in S(X)$. Let $x, x' \in X$ such that $x \neq x'$. This implies $I_{x_{1/2}} \neq I_{x'_{1/2}}$. Let $A(I_{x_{1/2}}) = I_{y_{1/2}}$ and $A(I_{x'_{1/2}}) = I_{y'_{1/2}}$. Then $p(x') = y'$. Now,

$$\begin{aligned} x \neq x' &\Longrightarrow I_{x_{1/2}} \neq I_{x'_{1/2}} \\ &\Longrightarrow A(I_{x_{1/2}}) \neq A(I_{x'_{1/2}}) \\ &\Longrightarrow I_{y_{1/2}} \neq I_{y'_{1/2}} \\ &\Longrightarrow y \neq y' \\ &\Longrightarrow p(x) \neq p(x') \end{aligned}$$

Therefore p is one to one. Let $y \in X$ and consider $I_{y_{1/2}}$. Since A is onto and A maps η onto η, there exists $x \in X$ such that $A(I_{x_{1/2}}) = I_{y_{1/2}}$. Then $y = p(x)$ and therefore p is onto. Thus $p \in S(X)$. Now we prove that $A = H_p^*$ on η.

$$\begin{aligned} \text{Now for } \ I_{x_{1/2}} \in \eta, \quad A(I_{x_{1/2}}) &= I_{y_{1/2}} \\ &= \{\underline{0}\,, y_{1/2}, \underline{1}\} \\ &= \{H_p(\underline{0}\,, H_p(x_{1/2}), H_p(\underline{1})\} \\ &= H_p^*(\{\underline{0}\,, x_{1/2}, \underline{1}\}) \\ &= H_p^*(I_{x_{1/2}}). \end{aligned}$$

Thus $A = H_p^*$ on η.

Now we will prove $A = H_p^*$ on ξ.

Claim : $A(I_{x_{1/2}}) = I_{y_{1/2}} \Longrightarrow A(\bar{I}_{x_{1/2}}) = \bar{I}_{y_{1/2}}$. ....**(1)**

If $\underline{0} \neq B \neq \underline{1} \in L^X$ is not comparable with $x_{1/2}$, then $B(x) = 0$. Also if $\underline{0} \neq B \neq \underline{1} \in L^X$ is not comparable with $x^{1/2}$, then $B(x) = 1$. Note that there exists no complement for $x_{1/2}$ or $x^{1/2}$ in L^X when $L = \{0, 1/2, 1\}$. Therefore there exists no $\underline{0} \neq B \neq \underline{1} \in L^X$ such that the join of the atom $\{\underline{0}\,, B, \underline{1}\}$ with the atom $I_{x_{1/2}}$ and that the join of the atom $\{\underline{0}\,, B, \underline{1}\}$ with $I_{x^{1/2}}$ contain more than 4 open sets. But if $x \neq y$, we can find an atom in $LFT(X, L)$ whose join with $I_{x_{1/2}}$ and $\bar{I}_{y_{1/2}}$ contain more than 4 open sets. For a given $I_{x_{1/2}}$ the above two properties characterize $\bar{I}_{x_{1/2}}$ in ξ. Also these two properties are preserved by automorphisms. Therefore it follows that, $A(I_{x_{1/2}}) = I_{y_{1/2}} \Longrightarrow A(\bar{I}_{x_{1/2}}) = \bar{I}_{y_{1/2}}$. Now,

$$
\begin{aligned}
A(\bar{I}_{x_{1/2}}) &= \bar{I}_{y_{1/2}} \\
&= \{\underline{0}\,, y^{1/2}, \underline{1}\} \\
&= \{H_p(\underline{0})\,, H_p(x^{1/2}), H_p(\underline{1})\} \\
&= H_p^*(\{\underline{0}\,, x^{1/2}, \underline{1}\}) \\
&= H_p^*(\bar{I}_{x_{1/2}}).
\end{aligned}
$$

Thus $A = H_p^*$ on ξ and hence $A = H_p^*$ on $\eta \cup \xi$.

Claim : $A(I_{x_{1/2}}) = I_{y_{1/2}} \Longrightarrow A(I_{x_1}) = I_{y_1}$. ....**(2)**

Let $J_1 = \{\underline{0}\,, x_{1/2}, \underline{1}\}$ and $J_2 = \{\underline{0}\,, x_1, \underline{1}\}$. Since, $x_{1/2} \leq x_1, J_1 \vee J_2$ contains only 4 open sets. Also for every atom $\{\underline{0}\,, B, \underline{1}\}$ such that $\underline{0} \neq B \neq \underline{1}\,,$ if $J_2 \vee \{\underline{0}\,, B, \underline{1}\}$ contains 6 open sets, then $J_2 \vee \{\underline{0}\,, B, \underline{1}\}$ contains J_1. For a given $I_{x_{1/2}}$, the above two properties characterize I_{x_1}. Also these two properties are preserved by automorphisms. Therefore it follows that $A(I_{x_{1/2}}) = I_{y_{1/2}} \Longrightarrow A(I_{x_1}) = A(I_{y_1})$.

Now take an atom $\{\underline{0}\,, C, \underline{1}\}, \underline{0} \neq C \neq \underline{1}$ which is not in $\eta \cup \xi$. To show that $A(\{\underline{0}\,, C, \underline{1}\}) = H_p^*(\{\underline{0}\,, C, \underline{1}\})$. Let $A(\{\underline{0}\,, C, \underline{1}\}) = \{\underline{0}\,, C', \underline{1}\}$ where $\underline{0} \neq C' \neq \underline{1} \in L^X$. To show that $C' = H_p(C)$ where $H_p(C) = \bigvee\{(p(x))_l : x_l \in C\}$. For this, it suffices to prove that $x_l \in C \Longleftrightarrow (p(x))_l \in C'$ for all $x \in X$ and $l \in L$, $l \neq 0$. For $x_l \in C$, let $A(I_{x_l}) = I_{(p(x))_l}$. Then $\{\underline{0}\,, x_l, \underline{1}\} \vee \{\underline{0}\,, C, \underline{1}\}$ contains 4 elements.

Therefore $A(I_{x_l} \vee \{\underline{0}\,, C, \underline{1}\}) = A(I_{x_l}) \vee A(\{\underline{0}\,, C, \underline{1}\}) = I_{p(x)_l} \vee \{\underline{0}\,, C', \underline{1}\}$ contain 4 elements. This implies $(p(x))_l \in C'$. For, when $l = 1/2$, $x_{1/2}$ is not complemented in L^X. When $l = 1, x_1 \in C \Longrightarrow x_{1/2} \in C \Longrightarrow (p(x))_{1/2} \in C'$. Therefore C' is not the lattice complement of $(p(x))_l$. Thus $(p(x))_l \in C'$.
By a similar argument, we can show that $(p(x))_l \in C' \Longrightarrow x_l \in C$. Hence $x_l \in C \Longleftrightarrow (p(x))_l \in C'$.

Thus $A = H_p^*$ on all atoms of $LFT(X,L)$ and since $LFT(X,L)$ is atomistic, it follows that $A = H_p^*$ on $LFT(X,L)$.

Case (b) : When A maps η to ξ and ξ to η.

We will show that $A = F_p^*$ for some $p \in S(X)$. For $I_{x_{1/2}} \in \eta$, let $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}}$ for some $y \in X$. This y is unique. Define $p : X \longrightarrow X$ by $p(x) = y$. We show that $p \in S(X)$. Let $x, x' \in X$ such that $x \neq x'$. This implies $I_{x_{1/2}} \neq I_{x'_{1/2}}$. Let $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}}$ and $A(I_{x'_{1/2}}) = \bar{I}_{y'_{1/2}}$. Then $p(x') = y'$. Now,

$$\begin{aligned}
x \neq x' &\Longrightarrow I_{x_{1/2}} \neq I_{x'_{1/2}} \\
&\Longrightarrow A(I_{x_{1/2}}) \neq A(I_{x'_{1/2}}) \\
&\Longrightarrow \bar{I}_{y_{1/2}} \neq \bar{I}_{y'_{1/2}} \\
&\Longrightarrow y \neq y' \\
&\Longrightarrow p(x) \neq p(x').
\end{aligned}$$

Therefore p is one to one. Let $y \in X$ and consider $\bar{I}_{y_{1/2}}$. Since A maps η onto ξ onto, there exists $x \in X$ such that $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}}$. Then $y = p(x)$ and therefore

p is onto. Thus $p \in S(X)$.

$$\begin{aligned}
\text{Now for, } I_{x_{1/2}} \in \eta, \quad A(I_{x_{1/2}}) &= \bar{I}_{y_{1/2}} \\
&= \{\underline{0}\,, y^{1/2}, \underline{1}\} \\
&= \{F_p(\underline{1}), F_p(x_{1/2}), F_p(\underline{0})\} \\
&= F_p^*(\{\underline{1}\,, x_{1/2}, \underline{0}\}) \\
&= F_p^*(I_{x_{1/2}}).
\end{aligned}$$

Thus $A = F_p^*$ on η.

Now we will prove $A = F_p^*$ on ξ. Let $x \in X$. Then,

$$\begin{aligned}
F_p^*(\bar{I}_{x_{1/2}}) &= F_p^*(\{\underline{0}\,, x^{1/2}, \underline{1}\}) \\
&= \{F_p(\underline{0}), F_p(x^{1/2}), F_p(\underline{0})\} \\
&= \{\underline{0}\,, y_{1/2}, \underline{1}\} \\
&= I_{y_{1/2}} \text{ where } p(x) = y.
\end{aligned}$$

Therefore it suffices to show that $A(\bar{I}_{x_{1/2}}) = I_{y_{1/2}}$.

Claim : $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}} \Longrightarrow A(\bar{I}_{x_{1/2}}) = I_{y_{1/2}}$. ....**(3)**

This can be proved by a similar argument used in the proof of the claim(1). Thus we get $A = F_p^*$ on ξ and hence $A = F_p^*$ on $\eta \cup \xi$.

Claim : $A(I_{x_{1/2}}) = \bar{I}_{y_{1/2}} \Longrightarrow A(I_{x_1}) = \bar{I}_{y_1}$. .... **(4)**

$I_{x_{1/2}} \vee I_{x_1}$ contains 4 open sets and for every atom $\{\underline{0}\,, B, \underline{1}\}$ $\underline{0} \neq B \neq \underline{1}$, if the join $\{\underline{0}\,, B, \underline{1}\} \vee I_{x_1}$ contains 6 open sets, then $\{\underline{0}\,, B, \underline{1}\} \vee I_{x_1}$ contains $I_{x_{1/2}}$. For a given $I_{x_{1/2}}$, I_{x_1} is characterized by these two properties. These two properties are preserved by automorphisms. Also the join $\bar{I}_{y_{1/2}} \vee \bar{I}_{y_1} = \{\underline{0}\,, y^{1/2}, \underline{1}\} \vee \{\underline{0}\,, y^0, \underline{1}\}$ contains only 4 open sets, since $y^0 < y^{1/2}$. Also for every atom $\{\underline{0}\,, B, \underline{1}\}$;

$\underline{0} \neq B \neq \underline{1}$, if the join $\{\underline{0}\,, B, \underline{1}\} \vee \bar{I}_{y_1}$ contains 6 open sets, then $\{\underline{0}\,, B, \underline{1}\} \vee \bar{I}_{y_1}$ contains $\bar{I}_{y_{1/2}}$. For, we have $\{\underline{0}\,, B, \underline{1}\} \vee \{\underline{0}\,, y^0, \underline{1}\} = \{\underline{0}\,, \underline{1}, B, y^0, B \vee y^0, B \wedge y^0\}$. If $B(y) = 1, B \vee y^0 = \underline{1}$ and if $B(y) = 0, B \leq y^0$ and hence $B \wedge y^0 = B$. Therefore if $\{\underline{0}\,, B, \underline{1}\} \vee \bar{I}_{y_1}$ contain 6 open sets, then, $B(y) = 1/2$. But then, $B \vee y^0 = y^{1/2}$. Therefore $\bar{I}_{y_{1/2}} = \{\underline{0}\,, y^{1/2}, \underline{1}\}$ is contained in $\{\underline{0}\,, B, \underline{1}\} \vee \bar{I}_{y_1}$. Also for a given $\bar{I}_{y_{1/2}}$, $\bar{I}_{y_1}$ is characterized by these two properties. Thus we obtain the claim.

Now take an atom $\{\underline{0}\,, D, \underline{1}\} \notin \eta \cup \xi$. To show that $A(\{\underline{0}\,, D, \underline{1}\}) = F_p^*(\{\underline{0}\,, D, \underline{1}\})$ for some $p \in S(X)$. Let $A(\{\underline{0}, D, \underline{1}\}) = \{\underline{0}\,, D', \underline{1}\}$. To show that $D' = \text{comp}(p(D))$.

For this, it is enough to show that

$D(x) = 1 \Longleftrightarrow D'(p(x)) = 0,$

$D(x) = 0 \Longleftrightarrow D'(p(x)) = 1$ and

$D(x) = 1/2 \Longleftrightarrow D'(p(x)) = 1/2.$

$D(x) = 1 \Longleftrightarrow x_1 \in D \Longleftrightarrow \{\underline{0}\,, x_1, \underline{1}\} \vee \{\underline{0}\,, D, \underline{1}\}$ contains exactly 4 open sets $\Longleftrightarrow A(\{\underline{0}\,, x_1, \underline{1}\} \vee \{\underline{0}\,, D, \underline{1}\}) = A(\{\underline{0}\,, x_1, \underline{1}\}) \vee A(\{\underline{0}\,, D, \underline{1}\}) = \{\underline{0}\,, y^0, \underline{1}\} \vee \{\underline{0}\,, D', \underline{1}\}$ also contains 4 open sets $\Longleftrightarrow D' \leq y^0 \Longleftrightarrow D'(y) = 0 \Longleftrightarrow D'(p(x)) = 0$.

By a similar kind of argument, $D(x) = 0 \Longleftrightarrow D \leq x^0 \Longleftrightarrow y_1 \in D' \Longleftrightarrow D'(y) = 1 \Longleftrightarrow D'(p(x)) = 1$.

If $D(x) = 1/2$, then $x_{1/2} \in D \Longleftrightarrow D' \leq y^{1/2} \Longleftrightarrow D'(y) \leq 1/2$.

Also, $D \leq x^{1/2} \Longleftrightarrow y_{1/2} \in D' \Longleftrightarrow D'(y) \geq 1/2$. That is, $D'(p(x)) = 1/2$.

Thus we have proved $A = F_p^*$ on all atoms of $LFT(X,L)$. Since $LFT(X,L)$ is atomistic, it follows that $A = F_p^*$ on $LFT(X,L)$. □

Remark 3.3.11. If X is infinite, there is no automorphism of $LFT(X,L)$ which maps η to ξ and ξ to η. Suppose A maps η to ξ and ξ to η. Since automorphisms preserves order, we have, $A(\bigvee\{I_{x_{1/2}} : x \in X\}) = \bigvee\{A(I_{x_{1/2}}) : x \in X\}$. That is, $A(\bigvee_{x\in X}\{\underline{0}\,, x_{1/2}, \underline{1}\}) = \bigvee_{x\in X}\{\underline{0}\,, x^{1/2}, \underline{1}\}$. But this is not possible, because when X is infinite, the cardinality of L.H.S. is $2^{|X|}$ while the cardinality of R.H.S. is $|X|$ where $|X|$ denote the cardinality of the set X. Thus we obtain, the following result.

Theorem 3.3.12. *When X is infinite and L is the lattice $\{0, 1/2, 1\}$, the set of all automorphisms of the lattice $LFT(X, L)$ is precisely $\{H_p^* : p \in S(X)\}$.*

Remark 3.3.13. In view of the Theorem 3.3.10 and Theorem 3.3.12 we shall make the following conjectures.

Conjecture 3.3.14. *Let X be a finite set and L be a finite F-lattice. Let $S(X)$ denote the set of all bijections on X and $A(L)$ denote the set of all automorphisms of L. For $p \in S(X)$ and $g \in A(L)$, define the mappings $H_{p,g}\,, F_{p,g} : L^X \longrightarrow L^X$ by $H_{p,g}(C)(x) = g(C(p^{-1}(x)))$ and $F_{p,g}(C) =$ pseudo-complement of $H_{p,g}(C)$ in L^X where $C \in L^X$ and $x \in X$. Also define the mappings $H^*_{p,g}$ and $F^*_{p,g}$: $LFT(X, L) \longrightarrow LFT(X, L)$ by $H^*_{p,g}(\delta) = \{H_{p,g}(A) : A \in \delta\}$ and $F^*_{p,g}(\delta) = \{F_{p,g}(A) : A \in \delta\}$ for $\delta \in LFT(X, L)$. Then the set of all automorphisms of the lattice $LFT(X, L)$ is precisely given by the set $\{H^*_{p,g} : p \in S(X), g \in A(L)\} \bigcup \{F^*_{p,g} : p \in S(X), g \in A(L)\}$.*

Conjecture 3.3.15. *Let X be an infinite set and L be a finite F-lattice. Let $S(X)$ denote the set of all bijections on X and $A(L)$ denote the set of all automorphisms of L. For $p \in S(X)$ and $g \in A(L)$, define the mapping $H_{p,g} : L^X \longrightarrow L^X$ by $H_{p,g}(C)(x) = g(C(p^{-1}(x)))$. Also define the mapping $H^*_{p,g} : LFT(X, L) \longrightarrow LFT(X, L)$ by $H^*_{p,g}(\delta) = \{H_{p,g}(A) : A \in \delta\}$ for $\delta \in LFT(X, L)$. Then the set of automorphisms of $LFT(X, L)$ is precisely given by the set $\{H^*_{p,g} : p \in S(X), g \in A(L)\}$.*

CHAPTER

4

Some Problems Related to Fuzzy Čech Closure Operators

4.1 Introduction

Let X be a nonempty set and L be an F-lattice (see Definition 2.1.10). To every fuzzy topology δ on X, we can associate a fuzzy closure operator ψ defined by $\psi(A)$ is the smallest closed set in L^X containing A for $A \in L^X$. Then the function $\psi : L^X \longrightarrow L^X$ has the following properties : (i) $\psi(\underline{0}) = \underline{0}$, (ii) $A \leq \psi(A)$ for all $A \in L^X$, (iii) $\psi(A \vee B) = \psi(A) \vee \psi(B)$ for all $A, B \in L^X$, (iv) $\psi(\psi(A)) = \psi(A)$ for all $A \in L^X$.

If we drop the last condition, we get more general fuzzy closure operators namely, the fuzzy Čech closure operators. The main results in this section are generalization of that in sec.1 of chapter 1.

Definition 4.1.1. Let X be a nonempty set and L be an F-lattice. A fuzzy Čech closure operator on X is a mapping $\psi : L^X \longrightarrow L^X$ satisfying the conditions :

(i) $\psi(\underline{0}) = \underline{0}$

(ii) $A \leq \psi(A)$ for all $A \in L^X$

(iii) $\psi(A \vee B) = \psi(A) \vee \psi(B)$ for all $A, B \in L^X$.

For brevity, we call ψ a fuzzy closure operator on X. The pair (L^X, ψ) or (X, ψ) is called a fuzzy closure space.

Definition 4.1.2. A fuzzy subset A of X is said to be closed, if $\psi(A) = A$ and open, if $\psi(A') = A'$ where A' denotes the pseudo-complement of A in L^X. Thus the closed sets of (X, ψ) are exactly the fixed points of ψ.

Example 4.1.3. Let$X$ be any set containing at least two elements and L be a finite chain having more than three elements. For each $A \in L^X$, let $\psi(A)$ be defined by

$$\psi(A)(x) = \begin{cases} A(x) & \text{if } A(x) = 0 \text{ or } 1, \\ \text{the immediate successor of A(x)}, & \text{otherwise.} \end{cases}$$

Then ψ is a fuzzy closure operator on X which is not fuzzy topological. For example, let $X = \{x, y\}$ and $L = \{0,\ 1/4,\ 3/4,\ 1\}$. Let $A, B, C \in L^X$ be defined by $A(x) = 1/4, A(y) = 0$, $B(x) = 3/4, B(y) = 0$ and $C(x) = 1, C(y) = 0$. Then $\psi(A) = B$, $\psi(B) = C$. Therefore $\psi(\psi(A)) \neq \psi(A)$ so that ψ is not fuzzy topological.

This example shows that in a fuzzy closure space (X, ψ), the closure of a fuzzy set need not be closed.

Definition 4.1.4. The set of all open sets in (X, ψ) is a fuzzy topology on X, called the fuzzy topology associated with ψ. To every fuzzy topology T on X, we can associate a fuzzy closure operator ψ on X, defined by $\psi(A) = cl(A)$ where $cl(A)$ denotes the closure of A in (X, T). We say that ψ is the closure operator associated with T. Note that a fuzzy closure operator need not be the closure operator associated with the fuzzy topology associated with it.

Definition 4.1.5. A fuzzy closure operator ψ on X is said to be fuzzy topological, if $\psi(\psi(A)) = \psi(A)$ for all $A \in L^X$.

Example 4.1.6. Let $\psi : L^X \longrightarrow L^X$ be defined by $\psi(A) = A$ for all $A \in L^X$. Then ψ is a fuzzy closure operator on X called the discrete fuzzy closure operator and is denoted by D. This fuzzy closure operator is fuzzy topological and defines the discrete fuzzy topology on X.

Example 4.1.7. Let $\psi : L^X \longrightarrow L^X$ be defined by

$$\psi(A) = \begin{cases} \underline{0} & \text{if } A = \underline{0}, \\ \underline{1} & \text{otherwise.} \end{cases}$$

Then ψ is a fuzzy closure operator on X called the indiscrete fuzzy closure operator and is denoted by I. This fuzzy closure operator is fuzzy topological and defines the indiscrete fuzzy topology on X.

Remark 4.1.8. A fuzzy closure operator on X is not uniquely determined by the collection of all open sets. That is, two different fuzzy closure operators on a set X can have the same associated fuzzy topology. But a fuzzy topological closure operator is uniquely determined by the associated fuzzy topology. In this sense, fuzzy closure operators can be considered as generalization of fuzzy topologies. Note that the fuzzy closure operator associated with a fuzzy topology is fuzzy topological.

Example 4.1.9. Let $X = \{x, y, z\}$ and $L = \{0, 1\}$. Let ψ_1 be the indiscrete fuzzy closure operator I on X (Definition 4.1.7) and ψ_2 be the fuzzy closure operator on X defined by $\psi_2(\chi_{\{x\}}) = \chi_{\{x,y\}}$, $\psi_2(\chi_{\{y\}}) = \chi_{\{y,z\}}$ and $\psi_2(\chi_{\{z\}}) = \chi_{\{z,x\}}$.

Then $\psi_1 \neq \psi_2$, but both have indiscrete fuzzy topology as the associated fuzzy topology.

Theorem 4.1.10. *Let ψ be a fuzzy closure operator on X and let δ be the associated fuzzy topology of ψ. If θ is the fuzzy closure operator associated with δ, then $\theta \leq \psi$.*

Proof : For $A \in L^X$,

$$\begin{aligned}
\theta(A) &= \bigwedge\{B \in L^X : \psi(B) = B \text{ and } B \geq A\} \\
&\geq \bigwedge\{B \in L^X : \psi(B) = B \text{ and } \psi(B) \geq \psi(A)\} \\
&= \bigwedge\{B \in L^X : B \geq \psi(A)\} \\
&= \psi(A)
\end{aligned}$$

Therefore $\theta \leq \psi$. □

Theorem 4.1.11. *If a fuzzy closure operator ψ is fuzzy topological, then the fuzzy closure operator corresponding to the associated fuzzy topology of ψ is ψ itself.*

Proof : If θ is the fuzzy closure operator corresponding to the associated fuzzy topology of ψ, then $\theta \leq \psi$ (Theorem 4.1.10). On the other hand, for $A \in Ł^X$, $\theta(A)$ is the smallest closed set containing A. But if ψ is fuzzy topological, $\psi(\psi(A)) = \psi(A)$ so that $\psi(A)$ is a closed set containing A. Therefore $\theta(A) \leq \psi(A)$ for all $A \in L^X$. That is, $\psi \leq \theta$ and hence $\theta = \psi$. □

Definition 4.1.12. Let (X, ψ) be a fuzzy closure space. A fuzzy subset B of X is said to be dense in a fuzzy subset A of X, if $B \leq A$ and $A \leq \psi(B)$. A fuzzy subset A of (X, ψ) is said to be dense in (X, ψ), if $\psi(A) = \underline{1}$.

Remark 4.1.13. The relation $\{(A, B) : A, B$ fuzzy subsets of (X, ψ) such that A is dense in B $\}$ completely determines the fuzzy closure structure of the fuzzy closure space (X, ψ). Indeed, $\psi(A)$ is the join of all fuzzy sets in which A is dense.

Theorem 4.1.14. *A fuzzy closure space (X, ψ) is fuzzy topological if and only if the relation $\{(A, B) : A, B$ fuzzy subsets of (X, ψ) such that A is dense in B $\}$ is transitive.*

Proof : If the relation is transitive, then $\psi(A) = \psi(\psi(A))$ for each $A \in L^X$, because A is dense in $\psi(A)$, $\psi(A)$ is dense in $\psi(\psi(A))$ and the transitivity implies that A is dense in $\psi(\psi(A))$, that is, $\psi(\psi(A)) \leq \psi(A)$.

Conversely, if (X, ψ) is fuzzy topological, and $A \leq B \leq C$, $B \leq \psi(A)$ and $C \leq \psi(B)$ then $\psi(A) = \psi(\psi(A))$ which establishes the transitivity. $\square$

Corollary 4.1.15. *If X is a fuzzy topological space and A, B are fuzzy subsets of X such that A is dense in B and B is dense in X, then A is dense in X.*

Remark 4.1.16. The following theorem shows that in a fuzzy topological fuzzy closure space the closure of a set is completely determined by Q-neighbourhoods (see Definition 2.1.26) of fuzzy points of the space.

Theorem 4.1.17. *Let (X, ψ) be a fuzzy topological fuzzy closure space and $A \in L^X$. Then a fuzzy point $x_l \in \psi(A)$ if and only if each Q- neighbourhood of x_l is quasi-coincident with A.*

Proof : Suppose $x_l \in \psi(A)$. Let $U \in Q(x_l)$ (ie, $x_l \not\leq U'$). If A is not quasi-coincident with U (ie, if $A \leq U'$), then $x_l \in \psi(A) \leq \psi(U')) = U'$. That is, $x_l \in U'$ contradicts with $U \in Q(x_l)$. Therefore A is not quasi-coincident with U.

Conversely, suppose every $U \in Q(x_l)$ is quasi-coincident with A. If $x_l \not\leq \psi(A)$, then $(\psi(A))' \in Q(x_l)$. That is $(\psi(A))'$ is a Q-neighbourhood of x_l. Therefore $(\psi(A))'$ is quasi-coincident with A (by assumption). That is, A is quasi-coincident with $(\psi(A))'$ which implies $A \not\leq \psi(A)$, a contradiction. Therefore $x_l \leq \psi(A)$. $\square$

Definition 4.1.18. Let ψ_1 and ψ_2 be two fuzzy closure operators on a set X. Then ψ_1 is said to be " coarser than " ψ_2 if $\psi_2(A) \leq \psi_1(A)$ for every $A \in L^X$ and is denoted by $\psi_1 \leq \psi_2$.

Remark 4.1.19. The relation $\leq$ defined above is a partial order on the set of all fuzzy closure operators on X. We denote the set of all fuzzy closure operators on X by $LFC(X, L)$. The least element of $LFC(X, L)$ is the indiscrete fuzzy closure operator I and the greatest element is the discrete fuzzy closure operator D.

Remark 4.1.20. $LFC(X, L)$ is a complete lattice. (See [19]).

Remark 4.1.21. If A, B are fuzzy subsets of a fuzzy closure space (X, ψ) such that $A \leq B$, then $\psi(A) \leq \psi(B)$.

Definition 4.1.22. The fuzzy closure operator ψ associated with a fuzzy topology T is given by

$$\psi(A) = \begin{cases} A & \text{if } A' \in T, \\ \bigwedge\{B : A \leq B, B' \in T\} & \text{otherwise.} \end{cases}$$

Theorem 4.1.23. *The associated fuzzy topology of a fuzzy closure operator ψ is the discrete fuzzy topology if and only if ψ is the discrete fuzzy closure operator.*

Proof : Let T be the associated fuzzy topology of the fuzzy closure operator ψ. If T is the discrete fuzzy topology, the associated fuzzy closure operator of T is the discrete fuzzy closure operator D. Then by Theorem 4.1.10, $D \leq \psi$. This implies $\psi = D$. The converse is obvious. □

Definition 4.1.24. A fuzzy closure operator on X other than the indiscrete fuzzy closure operator I is called an infra fuzzy closure operator if the only fuzzy closure operator on X strictly smaller than it is I. Note that the infra fuzzy closure operators on X are precisely the atoms of the lattice $LFC(X, L)$.

Definition 4.1.25. A fuzzy closure operator on X other than the discrete fuzzy closure operator D is called an ultra fuzzy closure operator if the only fuzzy closure operator on X strictly larger than it is D. The ultra fuzzy closure operators on X are precisely the dual atoms of the lattice $LFC(X, L)$.

Definition 4.1.26. Let $y^m, m \neq 1 \in L$ denote the fuzzy subset of X defined by

$$y^m(x) = \begin{cases} 1 & \text{if } x \neq y, \\ m & \text{otherwise.} \end{cases}$$

Remark 4.1.27. In [19], Johnson T. P. proved that when $L = [0, 1]$, the lattice $LFT(X, L)$ has no infra fuzzy closure operators. But in the following theorem, we prove that if L contains atoms and dual atoms, then $LFT(X, L)$ may contains infra fuzzy closure operators.

Theorem 4.1.28. *Let $x, y \in X$ and $l, m \in L$ with l is an atom and m is a dual atom. If either $x \neq y$ or $l \leq m$, then $\psi_{(x_l, y^m)} : L^X \to L^X$ defined by*

$$\psi_{(x_l,y^m)}(A) = \begin{cases} \underline{0} & \text{if } A = \underline{0}\ , \\ y^m & \text{if } A = x_l, \\ \underline{1} & \text{otherwise.} \end{cases}$$

is an infra fuzzy closure operator.

Proof : If ψ is a fuzzy closure operator on X such that $\psi < \psi_{(x_l,y^m)}$, then $\psi(x_l)$ will be strictly larger than y^m and hence equal to $\underline{1}$. Also $\psi(A) = \underline{1}$ for all $A \in L^X$ other than $\underline{0}$ and x_l. Hence $\psi = I$ and therefore $\psi_{(x_l,y^m)}$ is an infra fuzzy closure operator. □

Theorem 4.1.29. *Let X be a nonempty set and L be a complete lattice such that any non-zero element in L contains an atom and any element of L other than 1 is contained in a dual atom, then every infra fuzzy closure operator on X is of the form $\psi_{(x_l,y^m)}$.*

Proof : Let ψ be any fuzzy closure operator on X other than I. Then there exists a nonzero $A \in L^X$ such that $\psi(A) \neq \underline{1}$. Since A is nonzero, there exists an $x \in X$ such that $A(x) \neq 0$ and also there exists an atom l of L such that $l \leq A(x)$. Since $\psi(A) \neq \underline{1}$, there exists $y \in X$ such that $\psi(A)(y) \neq 1$. Then there exists dual atom $m \in L$ such that $\psi(A)(y) \leq m$ so that $\psi(A) \leq y^m$. Then $\psi(x_l) \leq y^m$ and can be verified that $\psi_{(x_l,y^m)} \leq \psi$. That is, $\psi_{(x_l,y^m)}(A) \geq \psi(A)$ for all $A \in L^X$. $[\psi_{(x_l,y^m)}(x_l) = y^m \geq \psi(x_l)$. Also $\psi_{(x_l,y^m)}(A) = \underline{1} \geq \psi(A)$ for all other $A \neq \underline{0} \in L^X]$. In particular, if ψ is an infra fuzzy closure operator on X, it follows that $\psi = \psi_{(x_l,y^m)}$. □

Remark 4.1.30. Combining the theorems 4.1.28 & 4.1.29, we have the following theorem.

Theorem 4.1.31. *Let X be a nonempty set and L be a complete lattice such that any non-zero element in L contains an atom and any element of L other than 1 is contained in a dual atom, then a fuzzy closure operator on X is an infra fuzzy closure operator if and only if it is of the form $\psi_{(x_l,y^m)}$ for some $x, y \in X$ and $l, m \in L$ with l is an atom and m is a dual atom.*

Remark 4.1.32. If the associated fuzzy topology of a fuzzy closure operator is the indiscrete topology, then that fuzzy closure operator need not be the indiscrete fuzzy closure operator.

Example 4.1.33. Let $x, y \in X$ and $l, m \in L$ such that l an atom and m a dual atom. If either $x \neq y$ or $l \leq m$, then the fuzzy closure operator $\psi_{(x_l,y^m)}$ defined as in the Theorem 4.1.29 is an infra fuzzy closure operator. But the associated fuzzy topology of $\psi_{(x_l,y^m)}$ is the indiscrete fuzzy topology.

Remark 4.1.34. The fuzzy closure operator ψ associated with the infra fuzzy topology $\{\underline{0}, B, \underline{1}\}$, $\underline{0} \neq B \neq \underline{1} \in L^X$ is given by

$$\psi(A) = \begin{cases} \underline{0} & \text{if } A = \underline{0}\ , \\ B' & \text{if } A = B', \\ \underline{1} & \text{otherwise.} \end{cases}$$

Remark 4.1.35. The fuzzy closure operator associated with an infra fuzzy topology $\{\underline{0}, B, \underline{1}\}$, $\underline{0} \neq B \neq \underline{1} \in L^X$ is not necessarily an infra fuzzy closure operator. For example, take $X = \{a, b, c\}$, $L = \{0, 1\}$ and consider the crisp infra fuzzy topology $T = \{\underline{0}, \chi_{\{b,c\}}, \underline{1}\}$. Then the fuzzy closure operator associated with T is given by,

$$\psi(A) = \begin{cases} \underline{0} & \text{if } A = \underline{0}\ , \\ A & \text{if } A = \chi_{\{a\}}, \\ \underline{1} & \text{otherwise.} \end{cases}$$

Let $\psi' : L^X \longrightarrow L^X$ by,

$$\psi'(A) = \begin{cases} \underline{0} & \text{if } A = \underline{0}\ , \\ \chi_{\{a,b\}} & \text{if } A = \chi_{\{a\}}, \\ \underline{1} & \text{otherwise.} \end{cases}$$

Then $I < \psi' < \psi$. Therefore ψ is not an infra fuzzy closure operator.

Theorem 4.1.36. *The fuzzy closure operator ψ associated with an ultra fuzzy topology U is an ultra fuzzy closure operator.*

Proof : Let $\psi^\star$ be a fuzzy closure operator strictly larger than ψ. Since $\psi^\star > \psi$, there exists $A \neq \underline{0} \in L^X$ such that $\psi^\star(A) \leq \psi(A)$ but, $\psi^\star(A) \neq \psi(A)$. This implies $A' \notin U$. (Otherwise, if $A' \in U$ then $\psi(A) = A \Longrightarrow \psi^\star(A) < A$, a contradiction). Then $\psi(A) > A$, but $\psi^\star(A) = A$. This implies A' is not open in (x, ψ), but open in $(X, \psi^\star)$. Also every open set in (x, ψ) is open in $(x, \psi^\star)$. Thus the associated fuzzy topology of $\psi^\star$ is strictly larger than the ultra fuzzy topology U and hence is discrete. Therefore $\psi^\star$ is the discrete fuzzy closure operator and hence ψ is an ultra fuzzy closure operator. $\square$

4.2 More about fuzzy closure operators

The notion of neighbourhood of a subset in a closure space X is closely related to an interior operator on X. In this section we extend the notion of interior operator (Sec.35 of [10]) to the fuzzy case.

Definition 4.2.1. Let (X, ψ) be a fuzzy closure space. For each $A \in L^X$, define $int_\psi(A) = (\psi(A'))'$. Then $int_\psi(A)$ is called the interior of A in (X, ψ) or ψ-interior of A.

Remark 4.2.2. In any fuzzy closure space (X, ψ)

(i) $\text{int}_\psi(\underline{1}) = \underline{1}$

(ii) $\text{int}_\psi(A) \leq A$ for all $A \in L^X$

(iii) $\text{int}_\psi(A \wedge B) = \text{int}_\psi(A) \wedge \text{int}_\psi(B)$ for all $A, B \in L^X$.

Definition 4.2.3. The fuzzy interior operator on a set X is a function $int : L^X \longrightarrow L^X$ such that

(i) $int(\underline{1}) = \underline{1}$

(ii) $int(A) \leq A$ for all $A \in L^X$

(iii) $int(A \wedge B) = int(A) \wedge int(B)$ for all $A, B \in L^X$.

Remark 4.2.4. Let int be a fuzzy interior operator on X. Define $\psi : L^X \longrightarrow L^X$ by $\psi(A) = (int(A'))'$, $A \in L^X$. Then ψ is a fuzzy closure operator on X and $int_\psi = int$.

Remark 4.2.5. Thus open sets of the closure space (X, ψ) are fixed elements of the fuzzy interior operator. If ψ is a fuzzy closure operator on X and int_ψ is the corresponding fuzzy interior operator, then $int_\psi(A) = (\psi(A'))'$ and $\psi(A) = (int_\psi(A'))'$. Thus a fuzzy closure operator on a set is uniquely determined by the fuzzy interior operator and vice versa.

Definition 4.2.6. A neighbourhood of a subset A in a fuzzy closure space (X, ψ) is any fuzzy subset U of X such that $A \leq int(U)$.

Definition 4.2.7. A neighbourhood of a fuzzy point $x_l \in L^X$ is any fuzzy subset $U \in L^X$ such that $x_l \leq int(U)$.

Definition 4.2.8. The neighbourhood system of a fuzzy point $x_l \in L^X$ (fuzzy set $A \in L^X$) is the collection of all neighbourhoods of the fuzzy point x_l (fuzzy set A).

Definition 4.2.9. Let (X, ψ) be a fuzzy closure space. A fuzzy subset $A \in L^X$ is open, if it is a neighbourhood of itself. That is, if $A \leq int(A)$.

Remark 4.2.10. We have $int(A) \leq A$ holds always. Therefore a fuzzy subset $A \in L^X$ is open if and only if $int(A) = A$.

Remark 4.2.11. In a fuzzy closure space (X, ψ), $int(A)$ need not be open. For example, let $X = \{x, y\}$, $L = \{0, 1/4, 3/4, 1\}$ with usual order. The pseudo-complement of $A \in L^X$ is defined by $A'(x) = 1 - x$ and ψ be the indiscrete fuzzy closure operator I on X. Let $A \in L^X$ be such that $A(x) = 1/4, A(y) = 3/4$. Then, $int(A)$ is not open in (X, ψ) because, $int(A) = [\psi(A')]' \neq A$.

Remark 4.2.12. Let $\mathcal{F}$ be a neighbourhood system of a fuzzy subset A (or a fuzzy point x_l) in a fuzzy closure space (X, ψ). Then $\mathcal{F}$ is a fuzzy filter on X.

4.3 Automorphisms of the lattice $LFC(X, L)$

In the following discussions, the set X and the lattice L are finite and further L is distributive with $M(L) = L - \{0\}$ (see Definition 2.3.14). Then the set of all fuzzy points of L^X (or elements of Pt (L^X)) are molecules of L^X. (see Remark 2.3.15). For our convenience, we use the notation D to denote Pt (L^X).

Lemma 4.3.1. *[8] If λ is a molecule in a distributive lattice L, then $\lambda \in \bigvee \lambda_i$, $\lambda_i \in L \Longrightarrow \lambda \leq \lambda_i$ for some i.*

Lemma 4.3.2. *Let ψ be a fuzzy closure operator on X. Define a relation $\rho\psi$ on D, by $(x_l, y_m) \in \rho\psi$ if and only if $y_m \leq \psi(x_l)$. Then $\rho\psi$ is a reflexive relation on D.*

Proof : Since $x_l \leq \psi(x_l)$ for all $x_l \in D$, the result follows. □

Lemma 4.3.3. *Let R be a reflexive relation on D. Define a mapping $\nu R : L^X \longrightarrow L^X$ by $\nu R(A) = \bigvee\{y_m : x_l R y_m$ for some $x_l \leq A\}$, $A \in L^X$. Then νR is a closure operator on X.*

Proof : (i) $\nu R(\underline{0}) = \underline{0}$

(ii) Since R is a reflexive relation, $x_l R x_l$ for all $x_l \in D$. Therefore for $A \in L^X$, $x_l \leq A$ implies $x_l \leq \nu R(A)$. Hence $A \leq \nu R(A)$ for all $A \in L^X$.

(iii) For $A, B \in L^X$ and for $y_m \in Pt(L^X)$,

$$\begin{aligned} y_m \leq \nu R(A \vee B) &\Longleftrightarrow x_l R y_m \text{ for some } x_l \leq A \vee B \\ &\Longleftrightarrow x_l R y_m \text{ for some } (x_l \leq A \text{ or } x_l \leq B), \\ &\qquad \text{by Lemma 4.3.1} \\ &\Longleftrightarrow (x_l R y_m \text{ for some } x_l \leq A) \\ &\qquad \text{or } (x_l R y_m \text{ for some } x_l \leq B) \\ &\Longleftrightarrow (y_m \leq \nu R(A)) \text{ or } (y_m \leq \nu R(B)) \\ &\Longleftrightarrow y_m \leq \nu R(A) \vee \nu R(B). \end{aligned}$$

Hence $\nu R(A \vee B) = \nu R(A) \vee \nu R(B)$.

□

Notation 4.3.4. Denote the lattice of all reflexive relations on D under set inclusion by $LR(D)$.

Lemma 4.3.5. *Let $\psi \in LFC(X, L)$. Then the mapping ρ defined by $\rho(\psi) = \rho\psi$ is a dual isomorphism of $LFC(X, L)$ onto $LR(D)$.*

Proof : Let R be a reflexive relation on D. Then for the fuzzy closure operator νR on X, $\rho(\nu R) = R$ so that ρ is onto. Also for $\psi_1, \psi_2 \in LFC(X, L)$,

$$\begin{aligned} \psi_1 \leq \psi_2 &\iff \psi_2(A) \leq \psi_1(A) \text{ for all } A \in L^X \\ &\iff \psi_2(x_l) \leq \psi_1(x_l) \text{ for all } x_l \in D, \text{ since } X \text{ is finite.} \\ &\iff \rho\psi_2 \leq \rho\psi_1. \end{aligned}$$

The fact that ρ is one to one also follows from this. □

Lemma 4.3.6. *Let R be a reflexive relation on D. Then the mapping ν defined by $\nu(R) = \nu R$ is a dual isomorphism of $LR(D)$ onto $LFC(X, L)$.*

Proof : Let $\psi \in LFC(X, L)$. Then for the reflexive relation $\rho\psi$ on D, $\nu(\rho\psi) = \psi$ so that ν is onto.
Also for $R_1, R_2 \in LR(D)$,

$$\begin{aligned} R_1 \subseteq R_2 &\iff \{(x_l, y_m) : x_l R_1 y_m\} \subseteq \{(x_l, y_m) : x_l R_2 y_m\} \\ &\iff \{y_m : x_l R_1 y_m \text{ for some } x_l \leq A\} \\ &\qquad \subseteq \{y_m : x_l R_2 y_m \text{ for some } x_l \leq A\} \text{ for all } A \in L^X \\ &\iff \bigvee \{y_m : x_l R_1 y_m \text{ for some } x_l \leq A\} \\ &\qquad \leq \bigvee \{y_m : x_l R_2 y_m \text{ for some } x_l \leq A\} \text{ for all } A \in L^X \\ &\iff \nu R_1(A) \leq \nu R_2(A) \text{ for all } A \in L^X \\ &\iff \nu R_2 \leq \nu R_1 \\ &\iff \nu(R_2) \leq \nu(R_1). \end{aligned}$$

The fact that ν is one to one also follows from this. $\square$

Lemma 4.3.7. *Let $R \in LR(D)$ and $\Delta = \{(x_l, x_l) : x_l \in D\}$. Then $R - \Delta \subseteq \wp((D \times D) - \Delta)$ and the correspondence $R \longrightarrow R - \Delta$ is an isomorphism of $LR(D)$ onto $\wp((D \times D) - \Delta)$.*

Proof : Proof is obvious. $\square$

Lemma 4.3.8. *Let $G \subseteq \wp((D \times D) - \Delta)$. Then $G \cup \Delta$ is a reflexive relation on D and the correspondence $G \longrightarrow G \cup \Delta$ is an isomorphism of $\wp((D \times D) - \Delta)$ onto $LR(D)$.*

Proof : Proof is obvious. $\square$

Theorem 4.3.9. *Let X be a non-empty set, L be a finite distributive lattice and p be a bijection on $(D \times D) - \Delta$. For $\psi \in LFC(X, L)$, let $R_{p,\psi} = p(\rho\psi - \Delta) \cup \Delta$. Then $R_{p,\psi} \in LR(D)$. Further, let $T_p\psi = \nu R_{p,\psi}$. Then $T_p\psi \in LFC(X, L)$ and the mapping T_p defined by $T_p(\psi) = T_p\psi$ for $\psi \in LFC(X, L)$ is an automorphism of $LFC(X, L)$.*

Proof : For $\psi \in LFC(X, L)$, the relation $\rho\psi = \{(x_l, y_m) : x_l, y_m \in D$ and $y_m \leq \psi(x_l)\} \in LR(D)$, by Lemma 4.3.2. Then $\rho\psi - \Delta = \{(x_l, y_m) : y_m \leq \psi(x_l), x_l \neq y_m\} \in \wp((D \times D) - \Delta)$. Now for $p \in S((D \times D) - \Delta)$, $p(\rho\psi - \Delta) = \{p(x_l, y_m) : y_m \leq \psi(x_l), x_l \neq y_m\} \in \wp((D \times D) - \Delta)$. Then $R_{p,\psi} = p(\rho\psi - \Delta) \cup \Delta \in LR(D)$, by Lemma 4.3.8 and $T_p\psi = \nu R_{p,\psi} \in LFC(X, L)$, by Lemma 4.3.3 . Also for

$\psi_1, \psi_2 \in LFC(X, L)$,

$$\begin{aligned}
\psi_1 \leq \psi_2 &\iff \rho\psi_2 \leq \rho\psi_1, \text{ by Lemma 4.3.5} \\
&\iff \rho\psi_2 - \Delta \leq \rho\psi_1 - \Delta, \text{ by Lemma 4.3.7} \\
&\iff p(\rho\psi_2 - \Delta) \leq p(\rho\psi_1 - \Delta) \\
&\iff R_{p,\psi_2} \leq R_{p,\psi_1}, \text{ by Lemma 4.3.8} \\
&\iff T_p\psi_1 \leq T_P\psi_2, \text{ by lemma 4.3.6} \\
&\iff T_p(\psi_1) \leq T_p(\psi_2).
\end{aligned}$$

Further, since the correspondences $\psi \longrightarrow \rho\psi$, $\rho\psi \longrightarrow \rho\psi - \Delta$, $\rho\psi - \Delta \longrightarrow p(\rho\psi - \Delta)$, $p(\rho\psi - \Delta) \longrightarrow p(\rho\psi - \Delta) \cup \Delta$ ($= R_{p,\psi}$) and $R_{p,\psi} \longrightarrow T_p\psi$ are bijections, it follows that, $T_p : \psi \longrightarrow T_p\psi$ is a bijection. Hence T_p is an automorphism of the lattice $LFC(X, L)$. □

Definition 4.3.10. For $p \in S(X)$ and $A \in L^X$, define $p(A)$ by $p(A)(x) = A(p^{-1}(x)), x \in X$. Then $p(A) \in L^X$.

Theorem 4.3.11. *Let X be a nonempty set and L be a complete, distributive lattice. For a fuzzy closure operator ψ on X and $p \in S(X)$, define $T'_p\psi$ by $T'_p\psi(A) = p^{-1}(\psi(p(A))), A \in L^X$. Then $T'_p\psi \in LFC(X, L)$ and the mapping $T'_p : LFC(X, L) \longrightarrow LFC(X, L)$ defined by $T'_p(\psi) = T'_p\psi$ is an automorphism of $LFC(X, L)$.*

Proof : For $p \in S(X)$ and $\psi_1, \psi_2 \in LFC(X, L)$,

$$\begin{aligned}
T'_p(\psi_1) = T'_p(\psi_2) &\implies T'_p\psi_1 = T'_p\psi_2 \\
&\implies T'_p\psi_1(A) = T'_p\psi_2(A) \ \forall A \in L^X \\
&\implies p^{-1}(T'_p\psi_1(A)) = p^{-1}(T'_p\psi_2(A)) \ \forall A \in L^X \\
&\implies \psi_1(p(A)) = \psi_2(p(A)) \ \forall A \in L^X \\
&\implies \psi_1 = \psi_2.
\end{aligned}$$

Therefore T_p' is one to one.

Now let $\psi \in LFC(X, L)$. Define ϕ by $\phi(A) = p(\psi(p^{-1}(A)))$ for $A \in L^X$. Then $\phi \in LFC(X, L)$, for

(i)
$$\begin{aligned}\phi(\underline{0}) &= p(\psi(p^{-1}(\underline{0}))) \\ &= p(\psi(\underline{0})) \\ &= p(\underline{0}) \\ &= \underline{0}\end{aligned}$$

(ii) $p^{-1}(A) \subseteq \psi(p^{-1}(A))$. Therefore $A \subseteq p(\psi(p^{-1}(A)))$. That is, $A \subseteq \phi(A)$ for all $A \in L^X$.

$$\begin{aligned}\text{(iii)}\quad \phi(A \vee B) &= p(\psi(p^{-1}(A \vee B))) \\ &= p(\psi(p^{-1}(A) \vee p^{-1}(B))) \\ &= p(\psi(p^{-1}(A))) \vee p(\psi(p^{-1}(B))) \\ &= \phi(A) \vee \phi(B).\end{aligned}$$

Now,

$$\begin{aligned}T_p'(\phi)(A) &= p^{-1}(\phi(p(A))) \\ &= p^{-1}(p(\psi(p^{-1}(p(A))))) \\ &= \psi(A) \text{ for all } A \in L^X.\end{aligned}$$

Therefore $T_p'(\phi) = \psi$ and hence T_p' is onto. Further, for $\psi_1, \psi_2 \in LFC(X, L)$,

$$\begin{aligned}\psi_1 \geq \psi_2 &\iff \psi_1(A) \subseteq \psi_2(A) \ \forall\ A \in L^X \\ &\iff \psi_1(p(A)) \subseteq \psi_2(p(A)) \ \forall\ A \in L^X,\ p \in S(X) \\ &\iff p^{-1}(\psi_1(p(A))) \subseteq p^{-1}(\psi_2(p(A))) \ \forall\ A \in L^X \\ &\iff T_p'\psi_1(A) \subseteq T_p'\psi_2(A) \ \forall\ A \in L^X \\ &\iff T_p'\psi_1 \geq T_p'\psi_2 \\ &\iff T_p'(\psi_1) \geq T_p'(\psi_2).\end{aligned}$$

Hence T_p' is an automorphism of $LFC(X, L)$. □

Definition 4.3.12. Let (X, ψ), (Y, ϕ) be fuzzy closure spaces. A one to one function θ from X onto Y is called a fuzzy closure isomorphism, if $\theta(\psi(A)) = \phi(\theta(A))$ for all $A \in L^X$.

Definition 4.3.13. A fuzzy closure space (X, ψ) is said to be completely homogeneous, if every bijection on X is a fuzzy closure isomorphism. That is, $p(\psi(A) = \psi(p(A))$ for all $p \in S(X)$ and for all $A \in L^X$.

Remark 4.3.14. The closure spaces (X, I) where I is the indiscrete fuzzy closure operator and (X, D) where D is the discrete fuzzy closure operator are completely homogeneous fuzzy closure spaces.

Remark 4.3.15. If a fuzzy closure operator ψ on X is left fixed by every automorphism of the lattice $LFC(X, L)$, then the fuzzy closure space (X, ψ) is completely homogeneous. For,

Let (X, ψ) be a closure space such that ψ is left fixed by every automorphism of the lattice $LFC(X, L)$. For $p \in S(X)$ and $\phi \in LFC(X, L)$, define $T_p'\phi$ by $T_p'\phi(A) = p^{-1}(\phi(p(A)))$ for $A \in L^X$. Then by Theorem 4.3.11, $T_p'\phi \in LFC(X, L)$ and the mapping T_p' defined by $T_p'(\phi) = T_p'\phi$ is an automorphism of $LFC(X, L)$. Then by assumption, $T_p'(\psi) = \psi$ for all $\psi \in LFC(X, L))$. That is, $\psi(A) = p^{-1}(\psi(p(A)))$ for all $A \in L^X$. This implies $p(\psi(A) = \psi(p(A))$ for all $A \in L^X$. Thus every $p \in S(X)$ is a fuzzy closure isomorphism and hence (X, ψ) is completely homogeneous.

4.4 Quasi-discrete fuzzy closure operators

In this section, we extend the notion of quasi-discrete closure operators (Sec.26.A of [10]) to the fuzzy case.

Let X be any nonempty set and L be a distributive lattice with $M(L) = L - \{0\}$. Let D be the set as denoted in the beginning of the Section 4.3. Then the molecules of L^X are the elements of D.

Definition 4.4.1. *The quasi-discrete modification of a fuzzy closure operator U on a set X is a function $V : L^X \to L^X$ defined by*

$$\begin{aligned} V(A) &= \bigvee\{y_m : y_m \leq U(x_l) \text{ for some } x_l \leq A\},\ A \in L^X \\ &= \bigvee\{U(x_l) : x_l \leq A\}. \end{aligned}$$

Remark 4.4.2. The quasi-discrete modification V of a fuzzy closure operator U on a set X is a fuzzy closure operator. For,

(i) $V(\underline{0}) = \underline{0}$

(ii) $x_l \leq U(x_l)$ for all $x_l \in L^X$. Therefore if $x_l \leq A$, then $x_l \leq V(A)$ so that $A \leq V(A)$ for all $A \in L^X$.

(iii) For $A, B \in L^X$ and for $y_m \in Pt(L^X)$,

$$\begin{aligned} y_m \leq V(A \vee B) &\iff y_m \leq U(x_l) \text{ for some } x_l \leq A \vee B \\ &\iff y_m \leq U(x_l) \text{ for some } (x_l \leq B \text{ or } x_l \leq B), \\ &\qquad \text{by Lemma 4.3.1} \\ &\iff (y_m \leq U(x_l) \text{ for some } x_l \leq A) \\ &\qquad \text{or } (y_m \leq U(x_l) \text{ for some } x_l \leq B) \\ &\iff (y_m \leq V(A)) \text{ or } (y_m \leq V(B)) \\ &\iff y_m \leq (V(A) \vee V(B)). \end{aligned}$$

Hence $V(A \vee B) = V(A) \vee V(B)$.

Definition 4.4.3. *A fuzzy closure operator U is called a quasi-discrete fuzzy closure operator, if U coincides with its quasi-discrete modification. That is, if $U(A) = \bigvee\{U(x_l) : x_l \leq A\}$ for all $A \in L^X$.*

Remark 4.4.4. The quasi-discrete modification of a fuzzy closure operator U is the coarsest quasi-discrete fuzzy closure operator finer than U.

Example 4.4.5. The discrete and indiscrete fuzzy closure operators are quasi-discrete.

Lemma 4.4.6. *Let X be a finite set and L be a finite lattice. Then every fuzzy closure operators on X are quasi-discrete.*

Proof : Let U be a fuzzy closure operator on X and V be its quasi-discrete modification. Then,

$$\begin{aligned} V(A) &= \bigvee_{x_l \leq A} \{U(x_l)\} \\ &= U(\bigvee_{x_l \leq A} (x_l)) \\ &= U(A) \end{aligned}$$

□

Definition 4.4.7. *Let R be a reflexive relation on D. Then the fuzzy closure operator ψ_R defined by $\psi_R(A) = \bigvee\{y_m : x_l R y_m$ for some $x_l \leq A\}$, $A \in L^X$ is called the fuzzy closure operator associated with R.*

Remark 4.4.8. Let ψ_{R_1}, ψ_{R_2} be fuzzy closure operators associated with the reflexive relations R_1 and R_2 respectively. Then $R_1 \subseteq R_2 \Longleftrightarrow \psi_{R_1} \geq \psi_{R_2}$.

Lemma 4.4.9. *Suppose that U is a fuzzy closure operator on X and V its quasi-discrete modification. Consider the relation $R = \{(x_l, y_m) \in D \times D : y_m \leq U(x_l)\}$. Then R is a reflexive relation on D and V is the fuzzy closure operator associated with R.*

Proof : Since $x_l \leq U(x_l)$ for all $x_l \in D$, R is reflexive. Further,

$$\begin{aligned}\psi_R(A) &= \bigvee\{y_m : x_l R y_m \text{ for some } x_l \leq A\}\\ &= \bigvee\{y_m : y_m \leq U(x_l) \text{ for some } x_l \leq A\}\\ &= \bigvee\{U(x_l) : x_l \leq A\}\\ &= V(A).\end{aligned}$$

□

Theorem 4.4.10. *A fuzzy closure operator U is associated with a reflexive relation (see Definition 4.4.7) on D if and only if U is quasi-discrete.*

Proof : If U is associated with a reflexive relation R, then

$$\begin{aligned}U(A) &= \bigvee\{y_m : x_l R y_m \text{ for some } x_l \leq A\}\\ &= \bigvee\{y_m : y_m \leq U(x_l) \text{ for some } x_l \leq A\}\\ &= \bigvee\{U(x_l) : x_l \leq A\}.\end{aligned}$$

Therefore U is quasi-discrete.

Conversely, if U is quasi-discrete, by lemma 4.4.9, U is associated with the reflexive relation $R = \{(x_l, y_m) \in D \times D : y_m \leq U(x_l)\}$. □

Theorem 4.4.11. *Suppose that $\Re$ is the set of all reflexive relations on D and U_R is the fuzzy closure operator associated with each $R \in \Re$. Then the mapping $R \longrightarrow U_R$ is a dual isomorphism from the set of all reflexive relations on D to the set of all quasi-discrete fuzzy closure operators on X.*

Proof : Obviously the mapping is a bijection. So by Remark 4.4.8, the theorem follows. □

Remark 4.4.12. Thus the study of quasi-discrete fuzzy closure operators on the set X is reduced to the study of reflexive relations on D.

Notation 4.4.13. The quasi-discrete modification of a fuzzy closure operator ψ on a set X is denoted by $q\psi$.

Definition 4.4.14. [*10*] Let $(X, \leq)$ be a partially ordered set and let Y be a subset of $(X, \leq)$. Then the lower modification of an $x \in X$ in Y is the greatest element of Y less than or equal to x. That is, the element y of Y with the following property : $x \geq y$, and if $y_1 \in Y$ and $x \geq y_1$, then $y \geq y_1$.

Remark 4.4.15. The quasi-discrete modification $q\psi$ of a fuzzy closure operator ψ on a set X is the lower modification of ψ in the set of all quasi-discrete fuzzy closure operators on X and $q\psi$ exists for each $\psi \in LFC(X, L)$.

Definition 4.4.16. [*10*] A mapping $f : X \longrightarrow X$ is said to be idempotent if $f \circ f = f$.

Lemma 4.4.17. *Let Y be the subset of $LFC(X, L)$ consisting of all quasi-discrete fuzzy closure operators on X. Then,*

(1) There exists an order preserving idempotent mapping f of $LFC(X, L)$ into itself, such that Range $\{f\} = Y$ and $\psi \geq f(\psi)$ for each $\psi \in LFC(X, L)$ and

(2) $f(\psi) = sup_Y\{\phi : \phi \in Y, \psi \geq \phi\}$ for each ψ. That is,

(3) f is uniquely determined by Y and $sup_Y\{\phi_\alpha\} = sup_{LFC(X,L)}\{\phi_\alpha\}$ whenever $\{\phi_\alpha\}$ is a nonempty family in Y and

(4) $inf_Y\{f(\psi_\alpha)\} = f(inf_{LFC(X,L)}\{\psi_\alpha\})$.

Proof : From the remark 4.4.15, we obtain the condition : For each $\psi \in LFC(X, L)$, there exists the lower modification (ie, the quasi-discrete modification) of ψ in Y...(*)

So in order to prove the condition (1), it is enough to show that the conditions (1) and (*) are equivalent.
Suppose condition (1) holds. If $f(\psi)$ denotes the lower modification of ψ in Y, then clearly the mapping $f : \psi \longrightarrow f(\psi)$ fulfils the condition (*) and furthermore (2) holds.

Conversely, suppose the condition (*) holds. We shall prove that $f(\psi)$ is the lower modification of $\psi \in LFC(X, L)$ in Y. Given $\psi \in LFC(X, L)$, if $\phi \leq \psi, \phi \in Y$, then $f(\phi) \leq f(\psi)$ (since f is order preserving) and $f(\phi) = \phi$ because $\phi = f(\chi)$ for some $\chi \in LFC(X, L)$ (since Range$\{f\} = Y$), hence $f(f(\chi)) = f(\chi)$ (since f is idempotent) and finally $f(\phi) = \phi$. Thus $\phi \leq f(\psi)$ which shows that $f(\psi)$ is indeed the lower modification of ψ in Y.

The mapping f is uniquely determined by Y because the quasi-discrete modification of a fuzzy closure operator is unique. Now suppose that f is a mapping satisfying the condition (1). Let $\{\phi_\alpha\}$ be a nonempty family in Y. We know that Y is the set of all $\phi \in LFC(X, L)$ such that $f(\phi) = \phi$. If ϕ is the supremum of $\{\phi_\alpha\}$ in Y, then ϕ is the supremum of $\{\phi_\alpha\}$ in $LFC(X, L)$, for, indeed ϕ is an upper bound of $\{\phi_\alpha\}$ in $LFC(X, L)$ and if ψ is any upper bound of $\{\phi_\alpha\}$ in $LFC(X, L)$, then $f(\psi) \geq f(\phi_\alpha) = \phi_\alpha$ for each α, and hence $f(\psi)$ is a upper bound of $\{\phi_\alpha\}$ in Y which implies $f(\psi) \geq \phi$ and thus $\psi \geq \phi$. If ψ is the supremum of $\{\phi_\alpha\}$ in $LFC(X, L)$, then $f(\psi) \geq f(\phi_\alpha) = \phi_\alpha$ for each α and hence $f(\psi) \geq \psi$ which implies $f(\psi) = \psi$. Thus $\psi \in Y$ and hence ψ is the supremum of $\{\phi_\alpha\}$ in Y.

Finally, let ψ is the infimum of a family $\{\psi_\alpha\}$ in $LFC(X, L)$ and let $\phi = f(\psi)$. We shall prove that ϕ is the infimum of $\{f(\psi_\alpha)\}$ in Y. Evidently ϕ is a lower bound of $\{f(\psi_\alpha)\}$ in Y and if χ is any lower bound of $\{f(\psi_\alpha)\}$ in Y, then $\psi \geq \chi$ and hence $\phi = f(\psi) \geq f(\chi) = \chi$, that is, $\phi \geq \chi$. $\square$

Remark 4.4.18. Given a set X, the poset of all quasi-discrete fuzzy closure operators on X will be denoted by $q[LFC(X,L)]$. Recall that (Notation 4.4.13) the quasi-discrete modification is denoted by q and hence $q[LFC(X,L)]$ is the set of all $q\psi$ such that $\psi \in LFC(X,L)$, that is, the set of all quasi-discrete fuzzy closure operators on X.

Definition 4.4.19. [*10*] Let $(X,\leq)$ be a partially ordered set. A subset Y of $(X,\leq)$ is said to be completely join-stable in X, if $\sup\{Y_\alpha\} \in Y$ for each nonempty family $\{Y_\alpha\}$ in Y such that the supremum exists.

Definition 4.4.20. [*10*] A mapping of a partially ordered set into another partially ordered set is said to be completely join preserving, if it preserves suprema of nonempty families.

Theorem 4.4.21. *Let X be a set. Then the poset $q[LFC(X,L)]$ is a complete lattice which is completely join stable and the mapping $q : LFC(X,L) \longrightarrow q[LFC(X,L)]$ is surjective and completely join preserving.*

Proof : The quasi-discrete modification $q\psi$ of $\psi \in LFC(X,L)$ is the lower modification of ψ in the set of all quasi-discrete closure operators on X and $q\psi$ exists for each $\psi \in LFC(X,L)$. Therefore the theorem follows from the Lemma 4.4.17. □

4.5 Fuzzy topological modification of a fuzzy closure operator

In this section, we extend Čech's notion of topological modification of a closure operator (Sec.16.B of [10]) to the fuzzy case. We assume that X is a nonempty set and L is a complete, distributive and pseudo-complemented lattice. In [27], Liu defined the continuity of a function in fuzzy topological spaces using open (closed) sets and in [10], Čech defined the continuity of a function in Čech closure spaces using open (closed) sets. So the following definition is valid.

Definition 4.5.1. Let (X, ψ) and (Y, ϕ) be fuzzy closure spaces. Then the mapping $f : (X, \psi) \longrightarrow (Y, \phi)$ is said to be continuous if and only if the inverse image of each open (closed) subset of (Y, ϕ) is an open (closed) subset of (X, ψ).

Remark 4.5.2. The proof of the following three theorems are on the same line of the corresponding theorems of Čech [10], hence we state it without proof.

Theorem 4.5.3. *A fuzzy closure operator ψ on a set X is coarser than a fuzzy closure operator ϕ on X if and only if the identity mapping of (X, ϕ) onto (X, ψ) is continuous.*

Theorem 4.5.4. *Let $(X, \psi), (Y, \phi), (Z, \chi)$ be fuzzy closure spaces. If $f : (X, \psi) \rightarrow (Y, \phi)$ and $g : (Y, \phi) \longrightarrow (Z, \chi)$ are continuous, then their composite $g \circ f : (X, \psi) \longrightarrow (Z, \chi)$ is continuous.*

Theorem 4.5.5. *For a mapping of a fuzzy closure space (X, ψ) into a fuzzy topological space (Y, δ) to be continuous, the following condition is necessary and sufficient : The inverse image of every open (closed) fuzzy subset of (Y, δ) is an open (closed) fuzzy subset of (X, ψ).*

Remark 4.5.6. Let (X, ψ) be a fuzzy closure space and let Θ be the collection of all open subsets of (X, ψ). Then there exists exactly one fuzzy topological fuzzy closure operator ϕ on X such that the collection Θ is the set of all ϕ-open subsets and this fuzzy closure operator ϕ is the coarsest fuzzy closure operator in the collection $\mathcal{C}$ of all fuzzy closure operators χ on X such that Θ is the collection of all χ-open subsets. In particular ϕ is coarser than ψ. If $A \in L^X$, then $\phi(A) = \bigwedge\{F : A \leq F, F \text{ closed in } (X, \psi)\}$ ....(*)
Indeed, since ϕ is a fuzzy topological fuzzy closure operator, $\phi(A)$ is closed in

(X, ϕ), $(\phi(A))'$ is open in (X, ϕ) and by our assumption, $(\phi(A))'$ is open in (X, ψ) and hence $\phi(A)$ is closed in (X, ψ). In particular (*) holds.

Definition 4.5.7. A fuzzy topological modification or simply the fuzzy T-modification of a fuzzy closure operator ψ on a set X denoted by $\tau\psi$ is defined to be the closure operator ϕ defined by (*). The fuzzy topological modification of a fuzzy closure space (X, ψ) is defined to be the space (X, ϕ) where ϕ is the fuzzy topological modification of ψ and is denoted by $\tau(X, \psi)$.

Theorem 4.5.8. *Each of the following conditions is necessary and sufficient for a fuzzy closure operator ϕ to be the fuzzy topological modification of a fuzzy closure operator ψ on a set X.*

(a) The fuzzy closure operator ϕ is fuzzy topological and the collections of all ψ-open sets and of all ϕ-open sets coincide.

(b) The fuzzy closure operator ϕ is fuzzy topological and the collections of all ψ-closed sets and of all ϕ-closed sets coincide.

Proof : By the remark 4.5.6, condition (a) is necessary and sufficient. Since a fuzzy set is closed if and only if its pseudo-complement is open, conditions (a) and (b) are equivalent. □

Theorem 4.5.9. *The fuzzy topological modification of a fuzzy closure operator ψ on a set X is the finest fuzzy topological fuzzy closure operator on X coarser than ψ. Stated in other words, the following condition is necessary and sufficient for a fuzzy closure operator ϕ on a set X to be the fuzzy topological modification of a fuzzy closure operator ψ on X : ϕ is a fuzzy topological fuzzy closure operator on X coarser than ψ and if χ is any fuzzy topological fuzzy closure operator on X coarser than ψ, then χ is coarser than ϕ.*

Proof : Obviously $\tau\psi$ is a fuzzy topological fuzzy closure operator coarser than ψ. If χ is any fuzzy closure operator on X coarser than ψ, then every χ-open set is ψ-open and hence $\tau\psi$-open, for, $A\ is\,\chi$-open $\Longrightarrow \chi(A') = A'$. Now $\chi \leq \psi \Longrightarrow \chi(A) \geq \psi(A)$ for every $A \in L^X$. Therefore $\psi(A') \leq \chi(A') = A'$. But $A' \leq \psi(A')$. Hence $\psi(A') = A'$ so that A is ψ-open. If χ is in addition fuzzy topological, then this implies that χ is coarser than $\tau\psi$. Thus the condition is necessary.

Conversely, let ϕ be a fuzzy closure operator fulfilling the condition. We must prove $\tau\psi = \phi$. Since $\tau\psi$ is a fuzzy topological fuzzy closure operator coarser than ψ, by assumption, $\tau\psi$ is coarser than ϕ. Thus we have $\tau\psi \leq \phi \leq \psi$. Since the open sets of (X,ψ) and $(X,\tau\psi)$ are identical, the open sets of ϕ and $\tau\psi$ are identical, which implies $\phi = \tau\psi$. □

Theorem 4.5.10. *Let ψ be a fuzzy closure operator on X. In order that ϕ be the fuzzy topological modification of ψ, the following condition is necessary and sufficient : ϕ be a fuzzy topological fuzzy closure operator on X and each mapping f of (X,ψ) into a fuzzy topological space Q be continuous if and only if the mapping f is continuous as a mapping of (X,ϕ) into Q.*

Proof : First assume the condition. Since the identity mapping $I : (X,\psi) \longrightarrow (X,\tau\psi)$ is continuous, by the condition, the mapping $I : (X,\phi) \longrightarrow (X,\tau\psi)$ is also continuous which implies (by Theorem 4.5.3) that ϕ is finer than $\tau\psi$. Since $I : (X,\phi) \longrightarrow (X,\phi)$ is continuous, by the condition, the mapping $I : (X,\psi) \longrightarrow (X,\phi)$ is also continuous which implies (by Theorem 4.5.4) that ϕ is coarser than ψ. Thus ϕ is a fuzzy topological fuzzy closure operator coarser than ψ, and finer than $\tau\psi$. So (by Theorem 4.5.9), necessarily $\phi = \tau\psi$.

Conversely, let $\phi = \tau\psi$. If f is a continuous mapping of $(X,\tau\psi)$ into a fuzzy closure space Q (not necessarily fuzzy topological), then $f : (X,\psi) \longrightarrow Q$ is also continuous as the composition of the continuous mappings $I : (X, \psi) \longrightarrow (X, \tau\psi)$ and $f : (X, \tau\psi) \longrightarrow Q$. Conversely if $f : (X, \psi) \longrightarrow Q$ is continuous, then the inverse image of every open subset of Q is ψ-open (Definition 4.5.1) and hence $\tau\psi$ open (Theorem 4.5.8) which implies $f : (X, \tau\psi) \longrightarrow Q$ is continuous provided that Q is fuzzy topological (Theorem 4.5.5). □

Definition 4.5.11. [*10*] Let $(X, \leq)$ be a partially ordered set and let Y be a subset of $(X, \leq)$. Then the upper modification of an $x \in X$ in Y is the least element of Y greater than or equal to x. That is, the element $y \geq x$ of Y with the following property : if $y_1 \in Y$ and $x \leq y_1$, then $y \leq y_1$.

Remark 4.5.12. The fuzzy topological modification $\tau\psi$ of a fuzzy closure operator ψ on a set X is the upper modification of ψ in the set of all fuzzy topological closure operators on X and $\tau\psi$ exists for each $\psi \in LFC(X, L)$.

Lemma 4.5.13. *Let Y be the subset of $LFC(X, L)$ consisting of all fuzzy topological closure operators on X. Then,*

(1) There exists an order preserving idempotent mapping f of $LFC(X, L)$ into itself, such that Range$\{f\} = Y$ and $\psi \leq f(\psi)$ for each $\psi \in LFC(X, L)$ and

(2) $f(\psi) = inf_Y\{\phi : \phi \in Y, \psi \leq \phi\}$ for each ψ. That is,

(3) f is uniquely determined by Y and $inf_Y\{\phi_\alpha\} = inf_{LFC(X,L)}\{\phi_\alpha\}$ whenever $\{\phi_\alpha\}$ is a nonempty family in Y and

(4) $sup_Y\{f(\psi_\alpha)\} = f(sup_{LFC(X,L)}\{\psi_\alpha\})$.

Proof : From the remark 4.5.6, we obtain the condition : For each $\psi \in LFC(X, L)$, there exists the upper modification (ie, the fuzzy topological modification) of ψ in Y...(*)

So in order to prove the condition (1), it is enough to show that the conditions (1) and (*) are equivalent.
Suppose condition (1) holds. If $f(\psi)$ denotes the upper modification of ψ in Y, then clearly the mapping $f : \psi \longrightarrow f(\psi)$ fulfils the condition (*) and furthermore (2) holds.

Conversely, suppose the condition (*) holds. We shall prove that $f(\psi)$ is the upper modification of $\psi \in LFC(X, L)$ in Y. Given $\psi \in LFC(X, L)$, if $\psi \leq \phi, \phi \in Y$, then $f(\psi) \leq f(\phi)$, (since f is order preserving) and $f(\phi) = \phi$ because $\phi = f(\chi)$ for some $\chi \in LFC(X, L)$, (since Range$\{f\} = Y$). Hence $f(f(\chi)) = f(\chi)$ (since f is idempotent) and finally $f(\phi) = \phi$. Thus $f(\psi) \leq \phi$ which shows that $f(\psi)$ is the upper modification of ψ in Y.

The mapping f is uniquely determined by Y, because the fuzzy topological modification of a fuzzy closure operator is unique. Now suppose that f is a mapping satisfying the condition (1). Let $\{\phi_\alpha\}$ be a nonempty family in Y. We know that Y is the set of all $\phi \in LFC(X, L)$ such that $f(\phi) = \phi$. If ϕ is the infimum of $\{\phi_\alpha\}$ in Y, then ϕ is the infimum of $\{\phi_\alpha\}$ in $LFC(X, L)$, for, indeed ϕ is a lower bound of $\{\phi_\alpha\}$ in $LFC(X, L)$ and if ψ is any lower bound of $\{\phi_\alpha\}$ in $LFC(X, L)$, then $f(\psi) \leq f(\phi_\alpha) = \phi_\alpha$ for each α, and hence $f(\psi)$ is a lower bound of $\{\phi_\alpha\}$ in Y which implies $f(\psi) \leq \phi$ and thus $\psi \leq \phi$. If ψ is the infimum of $\{\phi_\alpha\}$ in $LFC(X, L)$, then $f(\psi) \leq f(\phi_\alpha) = \phi_\alpha$ for each α and hence $f(\psi) \leq \psi$ which implies $f(\psi) = \psi$. Thus $\psi \in Y$ and hence ψ is the infimum of $\{\phi_\alpha\}$ in Y.

Finally, let ψ is the supremum of a family $\{\psi_\alpha\}$ in $LFC(X, L)$ and let $\phi = f(\psi)$. We shall prove that ϕ is the supremum of $\{f(\psi_\alpha)\}$ in Y. Evidently ϕ is an upper bound of $\{f(\psi_\alpha)\}$ in Y and if χ is any upper bound of $\{f(\psi_\alpha)\}$ in Y, then $\psi \leq \chi$ and hence $\phi = f(\psi) \leq f(\chi) = \chi$, that is, $\phi \leq \chi$. □

Remark 4.5.14. Given a set X, the partially ordered set of all fuzzy topological fuzzy closure operators on X will be denoted by $\tau[LFC(X,L)]$. Recall that (Definition 4.5.7) the fuzzy topological modification is denoted by τ and hence $\tau[LFC(X,L)]$ is the set of all $\tau\psi$ such that $\psi \in LFC(X,L)$, that is, the set of all fuzzy topological fuzzy closure operators on X.

Definition 4.5.15. Let $(X, \leq)$ be a partially ordered set. A subset Y of $(X, \leq)$ is said to be completely meet-stable in X, if $inf\{y_\alpha\} \in Y$ for each nonempty family $\{y_\alpha\}$ such that the infimum exists.

Definition 4.5.16. A mapping of a partially ordered set into another partially ordered set is said to be completely meet preserving, if it preserves infima of nonempty families.

Theorem 4.5.17. *Let X be a set. Then the poset $\tau[LFC(X,L)]$ is a complete lattice which is completely meet stable and the mapping $\tau : LFC(X,L) \longrightarrow \tau[LFC(X,L)]$ is surjective and completely meet preserving.*

Proof : The fuzzy topological modification $\tau\psi$ of $\psi \in LFC(X,L)$ is the upper modification of ψ in the set of all fuzzy topological fuzzy closure operators on X and $\tau\psi$ exists for each $\psi \in LFC(X,L)$. Therefore the theorem follows from the Lemma 4.5.13. $\square$

4.6 Fuzzy closure operator induced by a fuzzy pseudo metric

In Sec.18.A of [10], Edward Čech described the closure operator induced by a pseudo metric. In this section we extend some of his results to the fuzzy case.

Let X be a nonempty set and L be a complete, distributive and pseudo-complemented lattice with $M(L) = L - \{0\}$. Then the set of all fuzzy points in L^X are molecules of L^X and we denote this set by D.

Definition 4.6.1. [32] Let $d : D \times D \rightarrow [0, \infty)$ satisfying the conditions :

(i) $d(x_l, x_m) = 0$ where $l \leq m$.

(ii) $d(x_l, y_m) = d(y_m, x_l)$

(iii) $d(x_l, z_n) \leq d(x_l, y_m) + d(y_m, z_n)$

(iv) if $d(x_l, y_m) < r$ where $r > 0$, then there exists $n > l$ such that $d(x_n, y_m) < r$, where $x, y, z \in X$ and $l, m, n \in L - \{0\}$. Then d is called a fuzzy pseudo metric on D and (X, d) is a fuzzy pseudo metric space . Moreover, if d satisfies the condition :

(v) if $d(x_l, y_m) = 0$, then $x = y$ and $l \leq m$, then d is called a fuzzy metric on D and (X, d) is a fuzzy metric space.

Definition 4.6.2. [32] Let $Q^r_{x_a} = \{y_b : d(y_b, x_a) < r\}$ and $B(x_a, r) = \bigvee\{y_b : y_b \in Q^r_{x_a}\}$. Then $B(x_a, r)$ is called a sphere with center x_a and radius r.

Definition 4.6.3. [32] $A \in L^X$ is said to be open if and only if for every $x_a \in A$, there exists $B(x_a, r) \leq A$.

Definition 4.6.4. Let (X, d) be a fuzzy pseudo metric space. The number $d(x_l, y_m)$ is called the distance from x_l to y_m in (X, d) under d. If $A, B \in L^X$, then $d(A, B)$ is defined to be

$$d(A, B) \quad = \quad \begin{cases} \infty & \text{if } A = \underline{0} \text{ or } B = \underline{0}, \\ inf\{d(x_l, y_m) : x_l \in A \text{ and } y_m \in B\} & \text{otherwise.} \end{cases}$$

The distance $d(x_l, A)$ from a fuzzy point x_l to a fuzzy set A is defined to be $inf\{d(x_l, y_m) : y_m \in A\}$. A mapping $f : (X, d) \longrightarrow (X_1, d_1)$ is said to be distance preserving, if $d_1(f(x_l), f(y_m)) = d(x_l, y_m)$ for each $x_l, y_m \in D$. Two fuzzy pseudo metric spaces are said to be isomorphic, if there exists a distance preserving bijective mapping of one onto the other. With every fuzzy pseudo metric there is associated a fuzzy closure operator which will be described as follows : Let d be a fuzzy pseudo metric for a set X. For $A \in L^X$, the relation

$$\psi(A) \quad = \quad \begin{cases} A & \text{if } A = \underline{0}, \\ \bigvee\{x_l : d(x_l, A) = 0\} & \text{otherwise.} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

is a fuzzy closure operator on X, for, obviously ψ is a mapping from L^X into L^X and,

(i) $\psi(\underline{0}) = \underline{0}$

(ii) $x_l \in A \Longrightarrow d(x_l, A) = 0$ yields $A \leq \psi(A)$

(iii) since $d(x_l, A \vee B) = min\{d(x_l, A), d(x_l, B)\}$, we get $\psi(A \vee B) = \psi(A) \vee \psi(B)$.

Definition 4.6.5. If d is a fuzzy pseudo metric on a set X, then the fuzzy closure operator ψ defined by (*) is said to be the fuzzy closure operator induced by d.

Remark 4.6.6. Every fuzzy pseudo metric space (X, d) will be considered as a fuzzy closure space (X, ψ) where ψ is the fuzzy closure operator induced by d. For example, if we say that f is a continuous mapping of a fuzzy pseudo metric space (X, d_1) into another one (X, d_2), it is to be understood that the mapping $f : (X, \psi_1) \longrightarrow (X, \psi_2)$ is continuous where ψ_i is the fuzzy closure operators induced by d_i. Similarly we shall speak about closed or open subsets of a fuzzy pseudo metric space.

Definition 4.6.7. A fuzzy closure operator ψ (or a fuzzy closure space (X, ψ)) is said to be fuzzy pseudo metrizable, if ψ is induced by a fuzzy pseudo metric. For convenience, two pseudo metrics on the same set will be called fuzzy topologically equivalent, if they induce the same fuzzy closure operators.

Example 4.6.8. Let X be a set. The relation $d : (x_l, y_m) \longrightarrow 0$ for every $(x_l, y_m) \in D \times D$ is a fuzzy pseudo metric on D inducing the indiscrete fuzzy closure operator on X. Conversely, if a fuzzy pseudo metric induces the indiscrete fuzzy closure operator on X, then necessarily $d(x_l, y_m) = 0$ for each $(x_l, y_m) \in D \times D$ because $d(x_l, y_m) \neq 0$ implies x_l not belongs to the closure of y_m.

Example 4.6.9. Given a set X, consider d which assigns to a pair (x_l, y_m), the element 0 if $x_l = y_m$ and 1 if $x_l \neq y_m$. Then d is a fuzzy pseudo metric on X inducing the indiscrete fuzzy closure operator on X.

Example 4.6.10. If d is a fuzzy pseudo metric on the set D and r is a positive real, then the relation $(x_l, y_m) \longrightarrow r.d(x_l, y_m)$ denoted by $r.d$ is a fuzzy pseudo metric on X inducing the same fuzzy closure operator as d. Since clearly $d = r.d$ if and only if $d(x_l, y_m) = 0$ for each (x_l, y_m), we obtain that the indiscrete fuzzy closure operator on X is the only fuzzy closure operator on X induced by exactly one fuzzy pseudo metric.

Theorem 4.6.11. *A mapping f of a fuzzy pseudo metric space (X, d) into another one (X_1, d_1) is continuous at a fuzzy point x_l of L^X if and only if the following conditions is fulfilled : For each positive real r, there exists a positive real s such that $d(x_l, y_m) < s \Longrightarrow d_1(f(x_l), f(y_m)) < r$.*

Proof : The implication $d(x_l, y_m) < s \Longrightarrow d_1(f(x_l), f(y_m)) < r$ is equivalent to this assertion : the image under f of the open s-sphere about x_l in (X, d) is contained in the open r-sphere about $f(x_l)$ in (X_1, d_1). Since open spheres form local bases, the statement follows. □

Definition 4.6.12. A Lipschitz continuous mapping or simply a Lipschitz mapping of a fuzzy pseudo metric space (X, d) into another one (X_1, d_1) is a mapping $f : (X, d) \Longrightarrow (X_1, d_1)$ such that there exists a non-negtive K, called a Lipschitz bound of f, with $K.d(x_l, y_m) \geq d_1(f(x_l), f(y_m))$ for each $(x_l, y_m) \in D \times D$.

Theorem 4.6.13. *Every Lipschitz continuous mapping is continuous.*

Proof : Let f be a Lipschitz continuous mapping of (X, d) into (X_1, d_1) and let always $K.d(x_l, y_m) \geq d_1(f(x_l), f(y_m))$ where K is a positive real. Given $r > 0$ put $s = r.K^{-1}$ and apply Theorem 4.6.11. □

Theorem 4.6.14. *If (X, d) is a fuzzy pseudo metric space, x_l, y_m, z_n, t_p are fuzzy points of L^X and A is a nonzero fuzzy subset of L^X, then*

(i) $|\, d(x_l, A) - d(y_m, A)\,| \leq d(x_l, y_m)$.

(ii) $|\, d(x_l, y_m) - d(z_n, t_p)\,| \leq d(x_l, z_n) + d(y_m, t_p)$.

Proof : (i) If $z_n \in A$, then $d(x_l, A) \leq d(x_l, z_n)$ and by the triangle inequality $d(x_l, A) \leq d(x_l, y_m) + d(y_m, z_n)$. Taking the g.l.b. of $d(y_m, z_n)$ for $z_n \in A$, we obtain $d(x_l, A) \leq d(x_l, y_m) + (y_m, A)$ which implies $(d(x_l, A) - d(y_m, A)) \leq d(x_l, y_m)$. The same inequality holds with x_l and y_m interchanged.

(ii) Formula (ii) follows by a double application of the triangle inequality : $d(x_l, y_m) \leq d(x_l, z_n) + d(z_n, y_m) \leq d(x_l, z_n) + d(z_n, t_p) + d(t_p, y_m)$ and consequently $(d(x_l, y_m) - d(z_n, t_p)) \leq d(x_l, z_n) + d(y_m, t_p)$. Now the conclusion follows as in the proof of (a).

□

Theorem 4.6.15. *If (X, d) is a fuzzy pseudo metric space, then*

(a) The function $\{\, x_l \longrightarrow d(x_l, A)\}$: $(X, d) \longrightarrow R$ is continuous for each nonzero fuzzy set A in X.

(b) (X, d) is a fuzzy topological space. That is, the fuzzy closure operator ψ induced by d is fuzzy topological.

(c) Every open (closed) sphere in (X, d) *is an open (closed) subset of* (X, d).

Proof : (a) It follows from (i) of Theorem 4.6.14 that each function of (a) is a Lipschitz continuous function. Therefore each mapping of (a) is continuous by Theorem 4.6.13.

(b) To prove (b), it is sufficient to show that $\psi(A)$ is closed for each nonzero fuzzy subset A of L^X where ψ is the fuzzy closure operator induced by d. If f is the function of (a) corresponding to A, then clearly $\psi(A) = f^{-1}(\{0\})$. Since f is continuous and $\{0\}$ is a closed subset of R, $\psi(A)$ is a closed subset of (X, d), by Definition 4.5.1.

(c) The open (closed) r-sphere about a fuzzy point x_l is clearly the inverse image of the open interval $(-r, r)$ (the closed interval $[-r, r]$) of R under the function $\{y_m \longrightarrow d(y_m, x_l)\} : (X, d) \longrightarrow R$ which is continuous by (a) because $d(y_m, x_l) = d(y_m, \{x_l\})$. Now the statement follows by Definition 4.5.1. $\square$

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