Ph.D. THESIS **MATHEMATICS**

A STUDY ON CYCLES IN GRAPHS

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DECLARATION

I hereby declare that the thesis, entitled "**A STUDY ON CYCLES IN GRAPHS**" is based on the original work done by me under the supervision of **Dr. Raji Pilakkat**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

List of Symbols

G	Graph
$V(G)$	Vertex set of G
$E(G)$	Edge set of G
$n(G)$	Order of G
$m(G)$	Size of G
$\Delta(G)$	Maximum degree of G
$\delta(G)$	Minimum degree of G
$\cong$	Isomorphic
$\overline{G}$	Complement of G
$d_G(v)$	Degree of the vertex v in G
$\kappa(G)$	Vertex connectivity of G
P_n	Path on n vertices
C_n	Cycle on n vertices
K_n	Complete graph
$K_{m,n}$	Complete bipartite graph
$L_{m,n}$	Lollipop graph
W_n	Wheel graph
$W_n^{(m)}$	Wind mill graph

H_n	Helm graph
S_n	Shell graph
B_N	Bow graph
BF	Butterfly graph
F_n	Friendship graph
$C_{n,m}$	Tadpole graph or dragon graph
$g(G)$	Girth of G
$c(G)$	Circumference of G
$G_1 \circ G_2$	Corona of G_1 and G_2
$\langle D \rangle$	Subgraph of G induced by D
$CN[G, x]$	Cycle neighbor polynomial of G
$CN^*[G, z]$	Modified cycle neighbor polynomial of G
$[G]_{cyn}$	Class of all graphs which are cyn equivalent to G
$S(G)$	Subdivision graph of G
G^2	Square graph of G
G^k	k^{th} power of graph G
$S'(G)$	Splitting graph of G
$M(G)$	Middle graph of G
$T_1(G)$	Semi total (line) graph of G
$T_2(G)$	Semi total (point) graph of G
$T(G)$	Total graph of G
$L(G)$	Line graph of G
$G^\dagger$	Derived graph of G
$\gamma(G)$	Domination number of G
$\Gamma(G)$	Upper domination number of G

$\beta_0(G)$	Maximum cardinality of a vertex independent set
$\gamma_H(G)$	Hausdorff domination number of G
$\gamma_{cH}(G)$	Connected Hausdorff domination number of G
$\gamma_{T_1}(G)$	T_1 domination number of G
$\gamma_{T_0}(G)$	T_0 domination number of G
$\gamma_{cT_0}(G)$	Connected T_0 domination number of G
$i(G)$	Independent domination number of G
$\gamma_t(G)$	Total domination number of G
$\gamma_{cy}(G)$	Cycle domination number of G
$\gamma_{cl}(G)$	Clique domination number of G
$\gamma_g(G)$	Global domination number of G
$diam(G)$	Diameter of G
$d(u, v)$	Distance between the vertices u and v
$mcc(G)$	Number of maximal cyclic components in G
$cd(u, v)$	Cyclic distance between the vertices u and v
$ce(v)$	Cyclic eccentricity of a vertex v
$crad(G)$	Cyclic radius of G
$cdiam(G)$	Cyclic diameter of G
$CC(G)$	Cyclic centre of G
$CP(G)$	Cyclic periphery of G
$R(G)$	Reduced graph of a graph G
$CD(G)$	Cyclic distance matrix of G
$cd - spectrum$	Cyclic distance spectrum

Introduction

Graph Theory is a comparitively young but rapidly developing branch of Mathematics. Within a few decades, graph theory had an explosive growth together with the fast developments in allied subjects like Computer Science and Bioinformatics. Graph theory can be used as a mathematical tool for designing and analysing communication networks, social network systems etc. It has wide range of applications in almost all branches of science, engineering, social sciences and even in linguistics.

Domination is a flourishing area of graph theory. O. Ore and C. Berge introduced the concept of domination. In his book called Theory of graphs [38] O. Ore used the terms dominating sets and domination number for the first time. A detailed survey on domination can be found in [28] and [29]. Another important concern of research in graph theory is graph polynomials. These are polynomials assigned to graphs. In 1878, J.J Silvester introduced the first graph polynomial called edge difference polynomial [52]. Since then many graph polynomials were introduced and studied extensively. Also distance is an important concept of graph theory which is the basis of many symmetry concepts in graphs. In addition to the usual distance concept between vertices of a graph, several other distance concepts are also defined and studied in graph theory. Detour distance [20], superior distance [34], signal distance [35] and steiner distance [9] are a few to mention.

In the present work, a new cycle based univariate graph polynomial called cycle neighbor polynomial and an improvised version called modified cycle neighbor polynomial of the same are introduced and studied. Also some cycle related dominations viz, Hausdorff domination and T_1 domination and a generalisation of T_1 domination called T_0 domination are introduced and some of their properties are studied. Finally a new distance concept related to cycles called cyclic distance in graphs is also introduced and discussed.

0.1 An overview of the thesis

As the title “A STUDY ON CYCLES IN GRAPHS” suggests, this thesis is a study of results involving cycles of the given graph. In this thesis a univariate graph polynomial, some particular dominations and a distance concept related to cycles are introduced and studied. All the graphs considered in this thesis are finite, simple and undirected.

Apart from the introductory chapter, the thesis comprises of six chapters in which the work is presented. The **first chapter** consists of a brief introduction, a summary of the thesis and the preliminaries. In the section preliminaries, the notations and terminologies used in the upcoming chapters are detailed. Basic ideas and definitions needed for the development of the thesis are provided. Some basic concepts of graph theory, domination in graphs, distance in graphs and graph polynomials are also discussed briefly.

In the **second chapter**, a new univariate graph polynomial called Cycle Neighbor Polynomial of a graph is introduced. The cycle neighbor polynomial of a graph G of order n is denoted by $CN[G,x]$ and is defined as,

$$CN[G,x] = \Sigma_{k=0}^{c(G)} c_k(G)x^k$$

where $c(G)$ is the circumference of G, $c_0(G)$ is the number of cycle neighbor free vertices (vertices which do not belong to any cycle of G) and $c_k(G)$ is the number of cycles of length k, $3 \leq g(G) \leq k \leq c(G) \leq n$ in G, where $g(G)$ is the girth of G. Many graph properties like girth, circumference, number of cycles of different lengths, whether the graph is Hamiltonian, pancyclic or weakly pancyclic, unicyclic, acyclic or not, a bipartite graph etc, can be directly obtained from the polynomial expression. Some general properties of this polynomial are obtained. The cycle neighbor polynomial of some graphs are computed. Connected graphs which contain maximum and minimum number of terms in its cycle neighbor polynomial and cycles and trees whose complements also have the same cycle neighbor polynomial as the original graph are characterized. Some graph modifications which do not change this polynomial of a graph are also discussed. An attempt is made to establish a kind of equivalence relation between any connected graph G and planar graphs in terms of number of cycles of different lengths in the graph G. In addition to this a unique planar graph viz, unique almost path like structure is assigned to every connected graph using cycle neighbor equivalence. Also the cycle neighbor roots of a graph is introduced and graphs having zero as the only cycle neighbor root is characterized. Location of roots of (r)-pancyclic graphs, weakly (r)-pancyclic graphs and unicyclic graphs are obtained.

The study of cycle neighbor polynomials is continued in the **third chapter**. Motivated from the definition of generalised cycles in graphs, a modified version of cycle neighbor polynomial of a graph is introduced. A brief comparitive study of cycle neighbor polynomials and modified cycle neighbor polynomials of graphs is carried out. The first section is concluded with the result that in terms of completeness property of graph polynomials, modified cycle neighbor polynomial of a graph is stronger than cycle neighbor polynomial of the graph. Modified cycle neighbor polynomial of some graph operations are obtained in the next section. Cycle neighbor polynomial of graph operations establishes that many properties of the resulting graph like, bipartite propetrty of subdivision graph of a simple graph, pancyclicity of square of path graphs, bipartite propetrty of splitting graphs of paths and stars, near bipartite property of semi total line graph and semi total point graph of paths, pancyclicity of total graph of paths, hamiltonicity and in particular pancyclicity of k^{th} power G^k of graphs of diameter k for $k = 2, 3, 4, ...$, the property that middle graph of a star is a split graph etc., can be directly obtained using the tool of cycle neighbor polynomial.

The fact that every non isolated vertex of a Hausdorff graph [48] belongs to a cycle of the graph, motivated us to introduce a new domination concept. In the **fourth chapter**, Hausdorff domination is introduced by imposing the condition on the dominating set that the graph induced by the dominating set is a Hausdorff graph. A necessary and sufficient condition for a dominating set to be Hausdorff dominating is obtained. A relation between Hausdorff domination number and independent domination number is established. Also it is proved that the span of every non independent Hausdorff dominating set contains a cycle of length greater than or equal to four. Connected Hausdorff domination is also defined and some of its properties are studied. Finally, some relation between Hausdorff domination number (or connected Hausdorff domination number) and other domination parameters like connected domination number, total domination number, global domination number etc are obtained.

In **chapter five**, two generalisations of of Hausdorff domination viz T_1 domination and T_0 domination are introduced and discussed. Graphs having T_1 domination number equal to $1, 2, 3$ and n are characterized. As in the case of Hausdorff domination it is obsevred that every non independent T_1 dominating set contains a cycle. It is proved that a dominating set is T_0 dominating if and only if the graph induced by the set is K_2 free. Some properties of T_0 dominating set and connected T_0 dominating set are also obtained.

In the **sixth chapter**, a new distance concept called cyclic distance in graphs is introduced. Maximal cycle neighbor sets and maximal cyclic components of a graph are defined and then cyclic distance between vertices of a graph is introduced using these concepts. A vertex similarity measure called cyclic similar vertices is defined using the notion of cyclic distance. Some properties of this new distance are obtained and it is proved that cyclic distance induces a pseudo metric on the set of vertices of a graph. Cyclic radius, cyclic diameter, cyclic center, cyclic periphery etc., of a connected graph with respect to cyclic distance are defined and discussed analogue to radius, diameter, center and periphery of a graph with respect to the classical distance between vertices. Also it is observed that for an acyclic graph, the notions of cyclic distance and the classical distance between vertices coincide.

The study of cyclic distance in graphs is extended in **chapter seven**. Corresponding to every simple graph which is not a tree, a new graph called reduced graph of the graph is obtained by contracting each of the maximal cyclic components to a single vertex in the original graph. A characterization property for the reduced graph to be a tree is obtained. Some interesting properties of reduced graph of a graph are also dealt with. In the reduced graph of a graph the order and size of the graph is diminished in some manner. As a graph with minimum order and size can be studied easily, the introduction of the concept of reduced graph of a graph enables us to study large complicated graphs in a simplified way. Also a new graph matrix viz, cyclic distance matrix of a graph is introduced and discussed. It is proved that the determinant of cyclic distance matrix of a graph whose reduced graph is free of cyclic flowers with more than two maximal cyclic components is independent of the structure of the graph but it depends only on the number of maximal cyclic components in the graph.

In the epilogue, some directions for future work are mentioned. A list of presented and published papers and a bibliography are also provided at the end of the thesis.

Chapter 1

Preliminaries

It is believed that Swiss mathematician Leonard Euler introduced the basic idea of graphs in eighteenth century. His attempts and solution to the popular Konigsberg bridge problem is considered to be the origin of graph theory. Graph theory, a branch of mathematics is the study of connection between things. These things are formally referred to as nodes or vertices and the connections are referred as links or edges. The reason for the growth of graph theory is its applicability in almost every discipline like Sociology, Psychology, Anthropology, Architecture, Biology, Chemistry, Computer Science, Theoretical Physics, Communication networks etc,. The fundamental reason for such a fast growth of graph theory is, any problem arising from real-world situation where relationships between pairs of elements in the system exist, can be modelled into a graph. Then using some existing results of graph theory or by finding some new ones we can find solution to the problem.

1.1 Basic terminology

This section explores the basics of graph theory that is needed for the development of subsequent chapters. It includes the basic definitions and notations that may appear in the forthcoming chapters, the concept of domination in graph theory, major distance concepts in graphs and a brief literature review of graph polynomials. We adopt the basic definitions and notations as in [7], [3] and [28].

Basic terminology

"A **graph** G [7] is an ordered pair (V, E) consisting of the disjoint sets V of vertices and E of edges, together with an incidence function $\psi(G) : E \to V \times V$ which associates each edge of G with an unordered pair of vertices of G". "A graph having finite number of vertices and edges is called a **finite graph** [7]". "The number of vertices and number of edges of a finite graph G are called the **order** [3] and **size** [3] of G respectively". "Two or more edges having same end vertices are called **multiple edges** [3] and an edge with identical end vertices is called a **loop** [7]". "A graph is a **simple graph** [7] if it has no multiple edges or loops".

"The end vertices of an edge are said to the **incident** [7] with the edge". "Two vertices are **adjacent** [7] if they are incident with a common edge and they are called neighboring vertices. Similarly two edges are **adjacent** [7] if they are incident to a common end vertex".

"The **adjacency matrix** [7] of a graph G of order n is an $n \times n$ matrix $A(G)$ $= [a_{ij}]$ where a_{ij}, the ij^{th} entry of the matrix is 1 or 0 according as the pair of vertices v_i and v_j are adjacent or not adjacent in G".

"A **complete graph** [7] on n vertices is a simple graph in which every pair of vertices are adjacent". It is denoted by K_n. "A graph is **bipartite** [7] if its vertex set can be partitioned into two subsets A and B so that any edge of G has one end vertex in A and the other end in B". "Whenever each vertex of A is joined to every vertex of B in a bipartite graph, it is called a **complete bipartite graph** [7]". If $|A| = m$ and $|B| = n$ or viceversa it is denoted by $K_{m,n}$.

"The **complement** [7] of a simple graph G, denoted by $\overline{G}$ is the simple graph with vertex set $V(G)$ itself and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. A selfcomplementary graph is one which is isomorphic to its complement".

"Two graphs G and H are **isomorphic** [7], if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$; such a pair (θ, ϕ) of mappings is called an isomorphism between G and H and is denoted as $G \cong H$".

"For a graph G, the graph H is said to be a **subgraph** [7] of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ψ_H is restriction of ψ_G to $E(H)$. Let $G(V, E)$ be any graph and V' be a nonempty subset of V". Then "the subgraph of G whose vertex set is V' and edge set is the set of all edges in G whose both ends are in V' is called a **subgraph of G induced by** V' [7]". It is denoted by $\langle V' \rangle$.

"For a vertex v in G, the **degree** [7] of v is the number of edges incident with v". It is denoted as $d_G(v)$ or $d(v)$. If G is a simple graph, d(v) is the number of **neighbors of v** [28] in G. "The set of vertices adjacent to v is called the open neighborhood of v denoted by $N(v)$". "The set $N(v) \cup \{v\}$ is called the closed neighborhood of v denoted by $N[v]$". The maximum and minimum degrees of vertices in G are denoted respectively by $\Delta(G)$ and $\delta(G)$ [3]. "A vertex of degree zero is called an **isolated vertex** [39]". "A vertex of degree one is called a **pendant vertex** or an **end vertex** [3]" . "A vertex adjacent to a pendant vertex is called a **support vertex** [39]".

"A **walk** [7] is an alternating sequence $v_0e_1v_1e_2...v_{i-1}e_iv_i...v_n$ of vertices and edges in which the vertices v_{i-1} and v_i are the end points of the edge e_i". "The **length** of a walk is the number of edges in the walk". "A **path** [7] is a walk having all the vertices distinct". "A path on n vertices is denoted by P_n. A path with u and v as end vertices is called a u-v path. A **trail** [7] is a walk where all the edges are distinct. A **closed trail** in which all the vertices are distinct is called a **cycle** [7]". A cycle of length n is denoted by C_n.

"In a graph G which has at least one cycle, the length of a longest cycle is called its **circumference** [7] and the length of a shortest cycle is its **girth** [7]". "A **chord** of a cycle C is an edge not in C whose end points lie in C. A graph G is **chordal** if every cycle of length at least four in G has a chord".

"A graph G is **connected** [7] if for each pair of vertices u and v in $V(G)$, there is a u-v path in G". "A **disconnected graph** [7] is a graph which is not connected". "A graph is **acyclic** if it contains no cycles". "A connected acyclic graph is called a **tree** [7]". **Components** [3] of a graph G are the maximal connected subgraphs of G". "A **cut edge** or **bridge** (or a **cut vertex**) [55] of a graph is an edge (or vertex) whose deletion increases the number of components". "A **nonseparable graph** [7] is connected, nontrivial and has no cut vertices".

"A **block** [7] of a graph is maximal nonseparable subgraph". If G is nonseparable then G itself is called a block.

"The **distance** [7] between two vertices u and v of a connected graph G, denoted by $d(u, v)$, is the length of the shortest u-v path in G". "The eccentricity $e(v)$ of a vertex v is $max\{d(u, v) : v \in V(G)\}$". "Maximum of the eccentricities of the vertices of G is called the **diameter** [55] of G and the minimum of the eccentricities of its vertices is called the **radius** [55] of G". The abbreviations $diam(G)$ and $rad(G)$ are used to denote the diameter and radius respectively of a graph G. "The **center** [55] of a graph G, $C(G)$ is the subgraph induced by the vertices of minimum eccentricity".

"For a graph G of order n, the **distance matrix** [4] of G is a matrix of order n, denoted by $D(G)$ and is defined as $D(G) = [d_{i,j}]$ where $d_{i,j} = d(v_i, v_j)$ is the distance between v_i and v_j".

"Assigning colors to the vertices of a graph is called a **vertex coloring**" [55]. If no two adjacent vertices receive the same color, then such a coloring is called a proper vertex coloring [55]. "The minimum number of colors required for a proper vertex coloring of a graph G is called its **chromatic tumber** [55] , and it is denoted by $\chi(G)$".

Operations on graphs

Some graph operations used in this thesis are the following. "The **union** [55] of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$".

"The **corona** of two graphs [26] K and H is formed from one copy of K and

$|V(K)|$ copies of H where the i^{th} vertex of K is adjacent to every vertex in the i^{th} copy of H". It is denoted by KoH

"The **line graph** [28] $L(G)$ of a graph G is the graph with vertex set $E(G)$ in which two vertices are adjacent if they are adjacent edges in G".

"Edge contraction [55] is an operation which removes an edge e from G and simultaneously merging the two vertices that it previously joined". The resulting graph is denoted by G/e.

"For a graph G, adding an edge e which is in $\overline{G}$ but not in G is called edge addition [28] and is denoted by $G + e$. Similarly removing an edge e from G is called edge deletion [28] it is denoted by $G - e$".

"Vertex identification of two graphs G and H with disjoint vertex sets is denoted by $G.H$ and it is obtained by identifying a vertex of G with a vertex of H".

1.2 Domination in Graph Theory

Domination in graph theory is one of the major research area of graph theory. The concept of dominating set, domination number and different types of dominations are briefly discussed here.

"Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is called a dominating set [28] if every vertex in G is either in D or is adjacent to an element of D". "The minimum cardinality of all dominating sets in G is called the domination number [28] and is denoted by $\gamma(G)$". Different types of dominating sets have been studied by imposing conditions on the dominating sets. A detailed survey can be found in [28] and [29]. "A dominating set D is called an independent dominating set [11] if $\langle D\rangle$ is the empty graph". "A dominating set D is called a connected dominating set [44] if $\langle D\rangle$ is connected". "D is called total dominating [10] if $\langle D\rangle$ has no isolated vertices". "D is global dominating [44] if it is a dominating set of $\overline{G}$, the complement of G". "D is cycle dominating [36] if $\langle D\rangle$ is a cycle and D is a dominating clique [37], if $\langle D\rangle$ is a complete graph". The corresponding minimum cardinality of independent dominating set, connected dominating set, total dominating set, global dominating set, cycle dominating set and clique dominating set are respectively called independent domination number, connected domination number, total domination number, global domination number, cycle domination number and clique domination number and are denoted respectively by $i(G)$, $\gamma_c(G)$, $\gamma_t(G)$, $\gamma_g(G)$, $\gamma_{cy}(G)$ and $\gamma_{cl}(G)$ [28]. The maximum size of an independent set of vertices in a graph G is called independence number and is denoted by $\beta_0(G)$ [28].

1.3 Graph Polynomials

The polynomials associated with graphs which encode the number of subgraphs of the graph with given properties are called graph polynomials. First polynomial in graph theory was introduced by J.J. Sylvester in 1878 [52] and further studied by J. Petersen in [40]. It is a multivariate polynomial depending on the ordering of the vertices of the graph. Since then, several graph polynomials such as chromatic polynomial [43], Tutte polynomial [14], characteristic polynomial [1], matching polynomial [17], independence polynomial [24], interlace polynomial

[41], clique polynomial [25] etc. have been introduced and studied extensively.

Applications of graph polynomials arise in many areas outside graph theory as well. For example, matching polynomial [17] and Hossoya polynomial [30] have many applications in Statistical Physics and Theoretical Chemistry. In the past few decades, many graph polynomials have been studied and plenty of theoretical and practical approaches have been developed.

1.4 Distance in graphs

Distance is one of the most basic concepts of graph-theoretic subjects. For a connected graph $G(V, E)$ and for $u, v \in V(G)$, the classical distance between u and v is the length of a shortest path connecting u and v. Many generalizations of distance parameter can be seen in Graph Theory. The following are some among them. In 1989, Chartrand et al introduced a natural and nice generalization called Steiner distance [9] of the concept of classical graph distance. Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G. Then the Steiner distance $d_G(S)$ among the vertices of S is the minimum size among all connected subgraphs whose vertex sets contain S. In 1993, Chartrand et al introduced detour distance [20] in graphs. For a connected graph $G(V, E)$, the detour distance $D(u, v)$ between u and v is the length of a longest path P between u and v in G. For a simple connected graph G and for two vertices u and v of G, let $D_{u,v} = N[u] \cup N[v]$. K M Kathiresan and G Marimuthu defined a $D_{u,v}$ - walk as a $u - v$ walk in G that contains every vertex of $D_{u,v}$. The superior distance [34] $d_D(u, v)$ from u to v is the length of a shortest $D_{u,v}$-walk.

Chapter 2

Cycle Neighbor Polynomial of Graphs

2.1 Introduction

[1] [2] There are plenty of graph polynomials in the literature of graph theory. These are studied because some of them are generating functions of some graph properties, some count the number of occurrences of certain graph features and some others make an attempt to find complete graph invariants. In this chapter a new univariate graph polynomial called cycle neighbor polynomial of a graph is introduced. This polynomial is defined based on number of cycles of different lengths in the given graph G. Cycle neighbor polynomial is essentially a generating function for the number of cycles of various lengths in the graph G.

[1] Part of this chapter has been published in Malaya Journal of Mathematik, Vol. S, No. 1, 27-31, 2020

[2] Part of this chapter has been published in Communications in Mathematics and Applications, Vol.11, No. 4, 549-558, 2020

The motivation behind the definition of this polynomial is that many graph properties like girth, circumference, number of cycles of different lengths, whether the graph is Hamiltonian [7], whether it is pancyclic [6] or weakly pancyclic [8], whether it is unicyclic [26], whether it is acyclic or not, whether it is a bipartite graph etc, can be directly obtained from the polynomial expression. A kind of equivalence relation viz, cycle neighbor equivalence of graphs which is independent of the order and structure of the graph but depending on the number of cycles of different lengths is also introduced. This relation partitions the class of all finite, simple graphs with no specific direction to the edges into sets having the same cycle neighbor polynomial. So that corresponding to every graph, we can find a planar graph which is cycle neighbor equivalent to the given graph. Also this chapter covers a brief study on the roots of cycle neighbor polynomial of graphs.

2.2 Cycle neighbor polynomial of graphs

In this section a new univariate graph polynomial called cycle neighbor polynomial of a graph is introduced and some properties of this polynomial are observed.

Definition 2.2.1. *Let $G(V,E)$ be any graph. A vertex of the graph G is said to be a cycle neighbor free vertex if it does not belong to any cycle of length greater than or equal to three in the graph G.*

Definition 2.2.2. *Let G be a simple graph of order n. The Cycle Neighbor Polynomial of G denoted by $CN[G,x]$ is defined as,*

$$CN[G,x] = \Sigma_{k=0}^{c(G)} c_k(G)x^k$$

where $c_0(G)$ is the number of cycle neighbor free vertices in G, $c(G)$ is the circumference of G and $c_k(G)$ is the number of cycles of length k in the graph G, for $3 \leq g(G) \leq k \leq c(G) \leq n$ and $g(G)$ is the girth of G.

If we label the vertices of the graph, the distinct cycles of different lengths can be distinguished easily. For example, consider the graph G in Figure 2.1. G contains only one cycle neighbor free vertex, which is labelled as i. There are six 3-cycles, which are $abca$, $cdec$, $chfc$, $cgfc$, $fghf$ and $chgc$. There are four 4-cycles, which are $cfghc$, $cfhgc$, $chfgc$ and $jklmj$ and there is only one 6-cycle, which is $nopqrsn$. Therefore $c_0(G) = 1$, $c_3(G) = 6$, $c_4(G) = 4$ and $c_6(G) = 1$. Hence $CN[G, x] = x^6 + 4x^4 + 6x^3 + 1$.

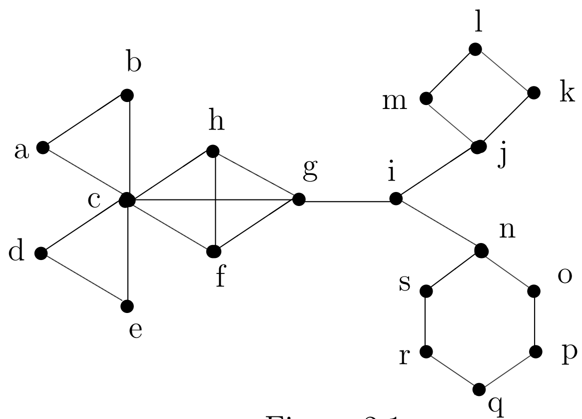


Figure 2.1

Observation 2.2.3. *1. For any simple graph G, $c_1(G) = c_2(G) = 0$ in $CN[G, x]$.*

2. If G_1 and G_2 are isomorphic graphs, then $CN[G_1, x] = CN[G_2, x]$.

3. If H is an induced subgraph of G, then $deg(CN[G, x]) \geq deg(CN[H, x])$

4. $c_0(G)$, the constant term in the cycle neighbor polynomial of G is the number of cycle neighbor free vertices in G.

5. *If a graph G contains no cycle neighbor free vertices, then zero is a root of its cycle neighbor polynomial.*

Proposition 2.2.4, gives some intersting properties of cycle neighbor polynomial of a graph which follows directly from the definition.

Proposition 2.2.4. *Let $G(V, E)$ be any graph of order n. If $CN[G, x]$ is a nonconstant polynomial then,*

1. *The lowest exponent of x of the nonconstant term in the cycle neighbor polynomial is the girth of G and the highest exponent is the circumference of G.*
2. *If $c_k(G) \neq 0$ for all k where $3 \leq k \leq n$ then G is pancyclic.*
3. *The degree of the cycle neighbor polynomial of G is n if and only if G is Hamiltonian.*
4. *The set of all exponents of x of the nonconstant terms in the cycle neighbor polynomial is the cycle spectrum $CS(G)$ [42] of G.*

Theorem 2.2.5. *Let G be a nontrivial graph of order n. Then $CN[G, x]$ is a constant polynomial if and only if G is a forest.*

Proof. Suppose if possible, $CN[G, x]$ be a nonconstant polynomial of degree m for some $m \geq 3$ and $c_m(G) \neq 0$. That is, G has at least one cycle of length m. Hence it is not a forest.

Conversely, if G is a forest, then $CN[G, x] = c_0(G)$, the number of cycle neighbor free vertices, a constant polynomial. □

Corollary 2.2.6 follows from the fact that the maximum number of edges in any acyclic graph of order n is $n-1$.

Corollary 2.2.6. *Let $G(V,E)$ be a graph of order n and size m. If $m \geq n$, then $CN[G,x]$ is a nonconstant polynomial.*

Proposition 2.2.7. *Let G be any graph. Then deg$(CN[G,x]) \leq n-1$ if and only if G is non hamiltonian.*

Proposition 2.2.8. *The cycle neighbor polynomial of any graph G of order n contains at most $n-2$ terms*

Proof. The general expression for cycle neighbor polynomial of G is $CN[G,x] = c_0(G) + c_3(G)x^3 + c_4(G)x^4 + ... + c_{c(G)}(G)x^{c(G)}$. If G is non hamiltonian, then $c(G) < n$ and when G is Hamiltonian, $c(G) = n$ and $c_0(G) = 0$ □

According to Whiteny's Theorem [7] a graph G of order $n \geq 3$ is two connected if and only if any two vertices of G are connected by at least two internally disjoint paths. Therefore no cycle neighbor free vertices can be found in any two connected graph. Hence we have Theorem 2.2.9;

Theorem 2.2.9. *Let G be any graph of order $n \geq 3$. If G is two connected, then $CN[G,x]$ is a polynomial of degree greater than or equal to 3*

Remark 2.2.10. *The converse of the conclusion of Theorem 2.2.9 need not be true. That is there are graphs for which deg$(CN[G,x]) \geq 3$ but G is not two connected. The cycle neighbor polynomial of the graph in Figure 2.2 is $x^5 + x^6$, whose degree is six, even though it is not two connected.*

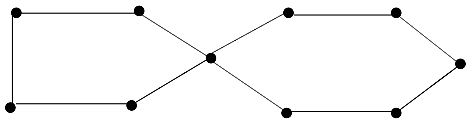


Figure 2.2

A graph G is bipartite if and only if it has no odd cycles [7]. Hence the cycle neighbor polynomial of bipartite graphs contains no odd powers of x. Hence we have the result;

Theorem 2.2.11. *A graph is bipartite if and only if $CN[G, x]$ of G is an even polynomial*

A graph G is 2-colorable if and only if it is bipartite [7]. Hence it follows that;

Corollary 2.2.12. *Let G be any graph. Then $CN[G, x]$ is an even polynomial if and only if $\chi(G) = 2$, where $\chi(G)$ is the chromatic number of G.*

Let G and H be two disjoint graphs with circumferences $c(G)$ and $c(H)$ respectively. Since there are no edges between G and H, a cycle of length k, $0 \leq k \leq p$ where $p = max\{c(G), c(H)\}$ in $G \cup H$ is either a k-cycle in G or a k-cycle in H. So that we have;

Proposition 2.2.13. *Let G and H be any two graphs and let $G \cup H$ be the disjoint union of G and H. Then $CN[G \cup H, x] = CN[G, x] + CN[H, x]$*

Corollary 2.2.14. *If a graph G has n components $G_1, G_2, ..., G_n$ then $CN[G, x] = CN[G_1, x] + CN[G_2, x] + ... + CN[G_n, x]$*

Next we find the cycle neighbor polynomial of some particular graphs and some graph classes.

Proposition 2.2.15. *Let G be a unicyclic graph of order n. If the length of the cycle is m, $3 \leq m \leq n$ then the cycle neighbor polynomial of G is $CN[G, x] = x^m + (n - m)$*

Proof. Since G has only one cycle say C_m of length m and the remaining vertices $V(G) - V(C_m)$ are cycle neighbor free, the result follows. □

Corollary 2.2.16. *Let $G \cong C_n$, a cycle on n vertices. Then $CN[G, x] = x^n$.*

Definition 2.2.17. *[19] "A Tadpole graph (or dragon graph) $C_{n,m}$, $n \geq 3$, $m \geq 1$ is obtained by joining a cycle C_n, $n \geq 3$ to a path P_m on m vertices with a bridge."*

Corollary 2.2.18. *Let $G \cong C_{n,m}$, then $CN[G, x] = x^n + m$.*

Proposition 2.2.19. *For a wheel graph $W_n \cong C_{n-1} + K_1$, $n \geq 4$*

$$CN[W_n, x] = (n-1)\sum_{k=3}^{n} x^k + x^{n-1}$$

Proof. In $W_n = C_{n-1} + K_1$, let $v \in V(K_1)$ be the central vertex of W_n. Then v is adjacent to every vertex of C_{n-1} and vertices of C_{n-1} has only two neighbors other than v. It can be easily verified that the number of cycles of length k, in W_n is $n - 1$ for $3 \leq k \leq n$, $k \neq n - 1$ and there are n cycles of length $n - 1$. Therefore, $CN[W_n, x] = (n-1)\sum_{k=3}^{n} x^k + x^{n-1}$. □

Definition 2.2.20. *[19] "A Helm graph H_n, $n > 3$ is obtained from a wheel graph W_n by attaching a pendant edge at each vertex on the rim of the wheel W_n"*

Corollary 2.2.21. $CN[H_n, x] = CN[W_n, x] + (n - 1)$*, $n \geq 4$*

Proposition 2.2.22. *For any complete graph K_n, $n \geq 3$*

$$CN[K_n, x] = \frac{n!}{2}\left[\frac{x^3}{3(n-3)!} + \frac{x^4}{4(n-4)!} + ... + \frac{x^{n-2}}{(n-2)2!} + \frac{x^{n-1}}{(n-1)} + \frac{x^n}{n}\right]$$

Proof. In K_n every vertex is adjacent to every other vertex. Hence to get the number of k-cycles of length k, $3 \leq k \leq n$ in K_n, choose k vertices out of n in $\binom{n}{k}$ ways and multiply it with the number of permutations $(k!)$ of these k vertices and divide it by $2k$, in order to avoid the repetition of the cycle count with which each cycle is represented by the permutation. That is in $CN[K_n, x]$, $c_k(G) = \binom{n}{k}\frac{k!}{2k} = \frac{n!}{2k(n-k)!}$

Therefore, $CN[K_n, x] = \frac{n!}{2}[\frac{x^3}{3(n-3)!} + \frac{x^4}{4(n-4)!} + ... + \frac{x^{n-2}}{(n-2)2!} + \frac{x^{n-1}}{(n-1)} + \frac{x^n}{n}]$ □

Definition 2.2.23. *[19] "A Lollipop graph $L_{n,m}$, $n \geq 3$, $m \geq 1$ is obtained by joining a complete graph K_n, $n \geq 3$ to a path P_m on m vertices with a bridge."*

Corollary 2.2.24. $CN[L_{n,m}, x] = CN[K_n, x] + m$

Definition 2.2.25. *[19] "A Windmill graph $W_n^{(m)}$, is the graph obtained by taking m copies of the complete graph K_n, $n \geq 3$ with a common vertex." $W_3^{(m)}$ is also called the friendship graph and it is denoted by F_m*

Corollary 2.2.26. $CN[W_n^{(m))}, x] = mCN[K_n, x]$

Corollary 2.2.27. $CN[F_m, x] = mx^3$

Proposition 2.2.28. *For a complete bipartite graph $K_{m,n}$, $m \geq 2$, $n \geq 2$ $CN[K_{m,n}, x] = \Sigma_{k=2}^{min\{m,n\}} \frac{k!^2}{2k}\binom{m}{k}\binom{n}{k}x^{2k}$. In particular, $CN[K_{n,n}, x] = \Sigma_{k=2}^{n} \frac{k!^2}{2k}\binom{n}{k}^2 x^{2k}$*

Proof. A complete bipartite graph $K_{m,n}$ contains no odd cycles. The number of cycles of length $2k$, $k = 2, 3, ..., min\{m, n\}$ in $K_{m,n}$ is given by $\frac{2mn(m-1)(n-1)...(m-k+1)(n-k+1)}{4k} = \frac{k!^2}{2k}\binom{m}{k}\binom{n}{k}$ after deleting duplication of cycle count. When $m = n$, number of cycles of length $2k$ in $K_{n,n}$ is $\frac{k!^2}{2k}\binom{n}{k}^2$. Hence, $CN[K_{m,n}, x]$ $= \Sigma_{k=2}^{min\{m,n\}} \frac{k!^2}{2k}\binom{m}{k}\binom{n}{k}x^{2k}$ and, $CN[K_{n,n}, x] = \Sigma_{k=2}^{n} \frac{k!^2}{2k}\binom{n}{k}^2 x^{2k}$. □

Definition 2.2.29. *[7] "A Shell graph $S_n \cong P_{n-1} + K_1$, $n \geq 3$ which can also be defined as the graph obtained from the cycle C_n, $n \geq 4$ by adding the edges corresponding to the $n-3$ concurrent chords of the cycle. The vertex at which all chords are concurrent is called the apex of the shell."*

Proposition 2.2.30. $CN[S_n, x] = (n-2)x^3 + (n-3)x^4 + (n-4)x^5 + ... + 2x^{n-1} + x^n$

Proof. Let $v_1, v_2, ..., v_{n-1}$ be the vertices of P_{n-1} and let v be the vertex of K_1. Every cycle of length k, $3 \leq k \leq n$ contains the consecutive vertices $v_i, v_{i+1}, ..., v_{i+(k-2)}$, $1 \leq i \leq n-k$ of P_{n-1}. Hence the number of such cycles of length k is $n-(k-1)$, $3 \leq k \leq n$. Therefore it follows that, $CN[S_n, x] = (n-2)x^3 + (n-3)x^4 + (n-4)x^5 + ... + 2x^{n-1} + x^n$. □

Definition 2.2.31. *[32] "A bow graph is a double shell with same apex in which each shell has any order."*

Corollary 2.2.32. *Let B_N be a bow graph of order $N \geq 5$, which includes shells S_n and S_m such that $N = m + n - 1$ then*

$$CN[B_N, x] = CN[S_n, x] + CN[S_m, x]$$

Proof. B_N includes the shells S_n and S_m with the same apex v, so that v is a cut vertex of B_N. Hence there are no cycles in B_N which have edges with one end in S_n and other end in S_m. Hence the result. □

Definition 2.2.33. *[55] "A butterfly graph BF is a bow graph with exactly two pendant edges at the apex."*

Corollary 2.2.34. *If BF is a butterfly graph with $N \geq 7$ vertices, then*

$$CN[BF, x] = CN[B_{N-2}, x] + 2$$

2.3 Graphs with a given order and having maximum and minimum number of terms in its cycle neighbor polynomial

The cycle neighbor polynomial of any graph of order n has at most $n-2$ terms and at least one term. In this section we characterize connected graphs having maximum and minimum number of terms in its cycle neighbor polynomial.

Theorem 2.3.1. *Let G be a connected graph of order n, $n \geq 4$. Then $CN[G,x]$ has exactly $n-2$ terms if and only if G is pancyclic or $G \cong H_{n-1,1}$, where $H_{n-1,1}$ is a graph consisting of a pancyclic graph H on $n-1$ vertices and a copy of K_1 connected to any one of the vertices of H by a bridge.*

Proof. If G is pancyclic or $G \cong H_{n-1,1}$, then it is clear that $CN[G,x]$ has exactly $(n-2)$ terms.

Now assume that for the graph G, $CN[G,x]$ has exactly $(n-2)$ terms. Suppose if possible, G is neither pancyclic nor $G \cong H_{n-1,1}$ but $CN[G,x]$ contains $(n-2)$ terms. Since G is not pancyclic, G does not contain a cycle of length l for some l for which $3 \leq l \leq n$

Claim: $l \neq n$

Suppose if possible $l = n$. Then G must contain cycles of all lengths k, $3 \leq k \leq n-1$. Otherwise, the number of terms in $CN[G,x]$ will be less than $(n-2)$, contradicting our assumption. Therefore G must be a connected graph of order n and contains cycles of all lengths k, $3 \leq k \leq n-1$. Hence $G \cong H_{n-1,1}$, another contradiction to the assumption that $G \ncong H_{n-1,1}$. Therefore $l \neq n$.

So let $l = k$, $3 \leq k \leq n-1$. When G contain no cycles of length k, in order for $CN[G, x]$ to have $(n-2)$ terms, G must contain a Hamilton cycle and at least one cycle neighbor free vertex, a contradiction. □

Since a Lollipop graph $L_{n-1,1}$, $n \geq 4$ is obtained by attaching complete graphs K_1 and K_{n-1} by a bridge, we have the following Corollary.

Corollary 2.3.2. *The Lollipop graph on $n \geq 4$ vertices contains $n-2$ terms in its cycle neighbor polynomial.*

Definition 2.3.3. *[5] "A cactus graph is a connected graph in which no two cycles have an edge in common."*

Definition 2.3.4. *A cactus graph G in which the length of every cycle in G is k and every vertex belongs to at least one cycle of G is called k-cycle neighbor graph .*

The graph in Figure 2.3 is a 3-cycle neighbor graph.

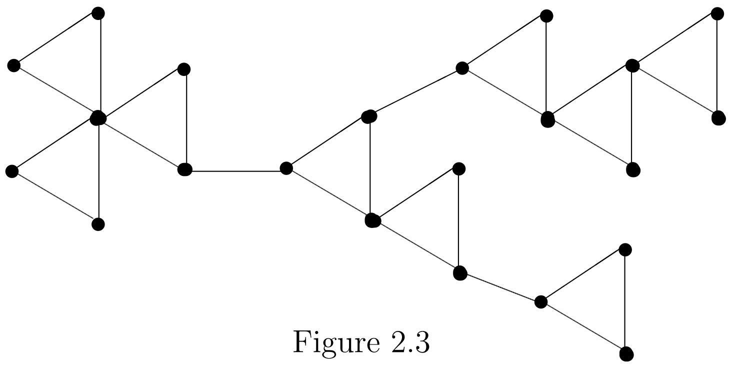


Figure 2.3

Theorem 2.3.5. *Let G be a connected graph of order n. Then the cycle neighbor polynomial of G has exactly one term if and only if one of the following conditions holds.*

1. *G is a tree*

2. *G is k-cycle neighbor graph for some k, $3 \leq k \leq n$.*

Proof. If (1) holds, then trivially, $CN[G,x] = n$. And when (2) holds, every vertex of G belongs to atleast one cycle of G, hence G contains no cycle neighbor free vertices and the lengths of all cycles in G are k, therefore $CN[G,x] = c_k(G)x^k$, where $c_k(G)$ is the number of k-cycles in G.

Conversely, suppose that $CN[G,x] = c_k(G)x^k$, $k \neq 1$ or 2. If $k = 0$, then $CN[G,x] = c_0(G)$, and since G is connected, it is a tree. If $k \neq 0$, then $3 \leq k \leq n$. Hence G has $c_k(G)$ cycles of length k. Also since G is connected, each of these $c_k(G)$ cycles are connected to m other k-cycles either by a common vertex or by a bridge, where $1 \leq m \leq (c_k(G) - 1)$. But no pair of these cycles have an edge in common. Otherwise, these two cycles will then form a new cycle of length greater than k, which contradicts $CN[G,x] = c_k(G)x^k$. □

Corollary 2.3.6 is a direct consequence of Theorem 2.3.5.

Corollary 2.3.6. *If G is not connected, the cycle neighbor polynomial of G contains exactly one term if and only if*

1. *G is a forest*

2. *Each component of G is k-cycle neighbor graph for the same value of k, $k = 3, 4, 5, ...$*

2.4 Cycles and trees having the same cycle neighbor polynomial as their complements

In this section we prove that among all connected acyclic graphs, only paths on n vertices, $n = 2, 3$ or 4 and among all cycles C_n, only C_5 have the same cycle neighbor polynomial as their complements.

Theorem 2.4.1. *Let T be any tree on $n \geq 2$ vertices and let $\overline{T}$ be the complement of T. Then $CN[T, x]$ and $CN[\overline{T}, x]$ are the same if and only if $T \cong P_n$, path on n vertices, where $n = 2, 3$ or 4*

Proof. When $T \cong P_n$, $n = 2, 3$ or 4, $\overline{T}$ is also acyclic with the same order. Hence $CN[T, x] = CN[\overline{T}, x]$.

If T is a tree on $n > 4$ vertices, we have the following cases.

Case (i) T is a path.

Suppose that $T \cong P_n$, $n \geq 5$. Then T has exactly two pendant vertices. For $n \geq 5$, the support vertices of the pendant vertices are distinct and nonadjacent. These support vertices together with the pendant vertices will form a cycle of length 4 in $\overline{T}$.

Case (ii) T is not a path.

In this case, T has three or more pendant vertices. These pendant vertices will be adjacent in $\overline{T}$ and will form a cycle in $\overline{T}$. Hence in both cases $\overline{T}$ is not acyclic and $CN[T, x] \neq CN[\overline{T}, x]$. □

Theorem 2.4.2. *Let C_n be any cycle, $n \geq 3$. Then $CN[C_n, x] = CN[\overline{C_n}, x]$ if* *and only if* $n = 5$

Proof. For $n = 5$, C_n is self complementary. Since isomorphic graphs have the same cycle neighbor polynomial, $CN[C_5, x] = CN[\overline{C_5}, x]$. For $n = 3$ and 4, $\overline{C_n}$ is acyclic, hence in this case, $CN[C_n, x] \neq CN[\overline{C_n}, x]$. When $n > 5$, $\overline{C_n}$ contains triangles whereas C_n does not. Therefore in this case also we have $CN[C_n, x] \neq CN[\overline{C_n}, x]$. □

2.5 Some graph modifications which do not affect the cycle neighbor polynomial

In this section we consider some graph modifications like edge removal, edge addition, edge contraction and a special kind of vertex identification under which the cycle neighbor polynomial of a graph will be unaffected.

Theorem 2.5.1. *Let G be any graph. Then $CN[G, x] = CN[G \setminus e, x]$ if and only if e is a cut edge of G.*

Proof. Suppose that $CN[G, x] = CN[G \setminus e, x]$. If $e = uv$ is not a cut edge, then there are one or more internally disjoint paths joining u and v other than e. Hence it is clear that e belongs to a cycle of G and the removal of e from G will affect at least one of the coefficients $c_k(G)$ where $3 \leq k \leq c(G)$. Therefore $CN[G, x] \neq CN[G \setminus e, x]$.

Conversely if e is a cut edge of G, both G and $G \setminus e$ will have the same number of cycles of different lengths and the same number of cycle neighbor free vertices. Therefore $CN[G, x] = CN[G \setminus e, x]$. □

Theorem 2.5.2. *For any edge e in $\overline{G}$, $CN[G,x] = CN[G+e,x]$ if and only if e is an edge joining different components of G.*

Proof. If e is an edge joining different components of a graph G, then e is a cut edge of $G+e$ and hence by Theorem 2.5.1, $CN[G+e,x] = CN[G,x]$

Conversely let $CN[G,x] = CN[G+e,x]$. Suppose if possible, e is not an edge joining different components of G. Then the ends of e lie in the component say G_1 of G. Since G_1 is a connected subgraph of G, the edge e in the subgraph $G_1 + e$ of $G+e$ is either an edge of a cycle in $G_1 + e$ or a chord of a cycle in $G_1 + e$. In both cases, the number of cycles in G and $G+e$ are different which contradicts $CN[G,x] = CN[G+e,x]$. □

Definition 2.5.3. *[55] "Edge contraction is an operation which removes an edge e from G and simultaneously merging the two vertices that it previously joined. The resulting graph is denoted by G/e."*

Theorem 2.5.4. *Let G be any triangle free graph, then $CN[G,x] = CN[G/e,x]$ if and only if e is a cut edge of G and both end points of e are not cycle neighbor free vertices.*

Proof. If e is a cut edge of G such that both end points of e are not cycle neighbor free vertices of G then both G and G/e have the same number of cycle neighbor free vetices and the same number of cycles of different lengths k for all possible values of k.

Conversely let $CN[G,x] = CN[G/e,x]$. If e is not a cut edge, then e belongs to at least one cycle of G. Also since G is triangle free, the length of one or more cycles in G will be diminished in G/e, a contradiction to the assumption that $CN[G, x] = CN[G/e, x]$. □

Theorem 2.5.5. *Suppose that G and H are graphs with disjoint vertex sets and let $G.H$ be a graph obtained by identifying a vertex of G with a vertex of H. Then $CN[G.H, x] = CN[G \cup H, x]$ if and only if both the vertices $v_1 \in V(G)$ and $v_2 \in V(H)$ which are being identified in $G.H$ belong to some cycles of G and H respectively.*

Proof. Let G and H be any two vertex disjoint graphs and let $v_1 \in V(G)$ and $v_2 \in V(H)$. Consider the following cases.

Case (i) Both v_1 and v_2 are cycle neighbor free vertices in G and H respectively. Then clearly $CN[G.H, x] = CN[G \cup H, x] - 1$

Case (ii) One of $v_1 \in V(G)$ or $v_2 \in V(H)$ is a cycle neighbor free vertex. Then also $CN[G.H, x] = CN[G \cup H, x] - 1$

Case (iii) Let v_1 belong to a cycle of G and v_2 belong to a cycle of H. Then the identification of the vertices v_1 in G and v_2 in H will not affect the number of cycles in G and H and therefore $CN[G.H, x] = CN[G \cup H, x]$. This completes the proof. □

Recall that a Barbell graph [21] is a graph obtained by taking two copies of C_n, $n \geq 3$ and joining them with one edge. This idea can be generalized by joining any two graphs G and H with one edge. This family of graphs (infinitely many) is called the Type 1 barbell-like graphs . Joining any two graphs G and H by merging a vertex $u_i \in V(G)$ with a vertex $v_j \in V(H)$ results in a Type 2 barbell-like graph. These ideas are very important in chemistry because many compounds are formed by making a single bond between atoms of molecules or by fusing atoms of different molecules together. Theorem 2.5.2 implies that Type 1 barbell-like graphs obtained from G and H will have the same cycle neighbor polynomial as $G \cup H$ and Theorem 2.5.5 gives a necessary and sufficient condition for a Type 2 barbell-like graph obtained from two graphs G and H to have the same cycle neighbor polynomial as that of $G \cup H$.

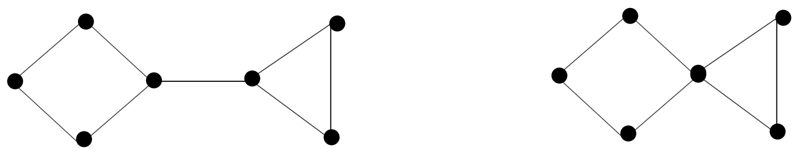
2.6 Cycle neighbor equivalence of graphs

In this section cycle neighbor eqivalence or in short, cyn-equivalence of graphs is introduced, which establishes a kind of equivalence between graphs having the same cycle neighbor polynomial. In wireless sensor networks, cactus network [5] is very important. Using cyn-equivalence, we can find a cactus network, which gives an extension to ring architecture corresponding to any graph. Also this equivalence can be used to represent a cactus network in a condensed form with a simple graph of very small diameter as compared to that of the given network especially when the network is very lengthy.

Definition 2.6.1. *Let $\mathcal{G}$ be the class of all finite undirected simple graphs. Two graphs G and H in $\mathcal{G}$ are said to be cycle neighbor equivalent (or cyn-equivalent) if $CN[G, z] = CN[H, z]$. That is both G and H have the same cycle neighbor polynomial. It is denoted by $G \ \widetilde{cyn} \ H$*

Clearly $\widetilde{cyn}$ is an equivalence relation on $\mathcal{G}$. The collection of all graphs in $\mathcal{G}$, which are cyn-equivalent to a graph G is denoted by $[G]_{cyn}$. That is
$[G]_{cyn} = \{H \in \mathcal{G} : CN[H, z] = CN[G, z]\}$. Isomorphic graphs are cyn-equivalent.

But there are non isomorphic graphs which are also cyn-equivalent. For example G and H in Figure 2.4 are cyn-equivalent.



G Figure 2.4 H

Proposition 2.6.2. *All acyclic graphs on the same number of vertices are cyn-equivalent. In particular, all trees on the same number of vertices are cyn-equivalent.*

Proposition 2.6.3. *Let G be any nontrivial acyclic graph on $n \geq 2$ vertices and let $\overline{G}$ be the complement of G. Then $\overline{G} \in [G]_{cyn}$ if and only if one of the following statements holds*

1. $n = 2$
2. $n = 3$ *but G is not isomorphic to* $\overline{K_3}$
3. $G \cong P_4$ *or* $2K_2$

Proof. From Proposition 2.6.2, it follows that all acyclic graphs on the same number of vertices are cyn-equivalent. Hence $\overline{G} \in [G]_{cyn}$ if and only if $\overline{G}$ is acyclic.

When $n = 2$, either $G \cong K_2$ or $G \cong \overline{K_2}$. In both cases, $\overline{G}$ is acyclic.

When $n = 3$, $G \cong P_3$ or $G \cong K_2 \cup K_1$ or $G \cong \overline{K_3}$ and $\overline{G}$ is acyclic if G is not isomorphic to $\overline{K_3}$.

When $n = 4$, $G \cong P_4$ or $G \cong 2K_2$ or $G \cong P_3 \cup K_1$ or $G \cong K_2 \cup \overline{K_2}$ or $G \cong \overline{K_4}$. Here $\overline{G}$ is acyclic only if $G \cong P_4$ or $2K_2$.

Now let $n \geq 5$. If G contains only one component, then G is a tree. Hence it follows from Theorem 2.4.1, that $\overline{G}$ contains a cycle. Otherwise, G contains at least two components, hence $\overline{G}$ cannot be acyclic. □

Let C_n be a cycle on $n \geq 3$ vertices. When $n \leq 4$, $\overline{C_n}$ is acyclic. When $n = 5$, $C_n \cong \overline{C_n}$. And when $n \geq 6$, $\overline{C_n}$ contains a triangle. Hence it follows that;

Proposition 2.6.4. *Let C_n be any cycle, $n \geq 3$. Then $\overline{C_n} \in [C_n]_{cyn}$ if and only if $n = 5$*

If $G \cong C_n$, a cycle on $n \geq 3$ vertices, then $L(G)$, the line graph of G is isomorphic to G. Hence we have ;

Proposition 2.6.5. *If $G \cong C_n$, a cycle on $n \geq 3$ vertices, then $L(G) \in [G]_{cyn}$.*

A cut edge contribute nothing to the length of any cycle in a graph whereas, the addition or removal of any edge other than a cut edge will alter the cycle neighbor polynomial of a graph.

Proposition 2.6.6. *Let G be any graph. Then $G \setminus e \in [G]_{cyn}$ if and only if e is a cut edge of G.*

Proposition 2.6.7. *Let G be any graph with at least two components, then $G + e \in [G]_{cyn}$ if and only if the end points of e lie in different components of G.*

Proposition 2.6.8 follows from Theorem 2.5.4.

Proposition 2.6.8. *Let G be any triangle free graph. Then G and G/e are cyn-equivalent if and only if e is a cut edge of G and both end points of e are not cycle neighbor free vertices.*

Even when two unicyclic graphs of the same order are not isomorphic, the length of the unique cycles in those graphs can be the the same. Hence Proposition 2.6.9 follows.

Proposition 2.6.9. *Let G_1 and G_2 be two connected, disjoint unicyclic graphs of the same order. Then G_1 and G_2 are cyn-equivalent if and only if G_1 and G_2 both contain the same number of cycle neighbor free vertices.*

Definition 2.6.10. *A graph $G \in \mathcal{G}$ is cycle neighbor unique (or cyn-unique) if $CN[G, z] = CN[H, z]$ implies that H is isomorphic to G.*

Theorem 2.6.11. *All cycles C_n are cyn-unique for $n \geq 3$*

Proof. Let G be any graph of order n such that $G \; \widetilde{cyn} \; C_n$. Then $CN[G, z] = z^n$. Hence G has exactly one cycle of length n and every vertex of G belongs to that cycle. Therefore, $G \cong C_n$ $\square$

Corollary 2.6.12. *Complete graph K_n is cyn-unique if and only if $n = 1$ or 3*

Proof. It is obvious that K_1 is cyn-unique. When $n = 3$, $K_n \cong C_n$. Hence Theorem 2.6.11 applies. When $n = 2$, K_2 and $\overline{K_2}$ have the same cycle neighbor polynomial. When $n > 3$, consider any cycle neighbor graph H, which contain as many cycles of length k for $3 \leq k \leq n$, as in K_n. Then $CN[K_n, z] = CN[H, z]$. Hence the result. $\square$

Remark 2.6.13. *The only cyn-unique graphs are the complete graphs K_1 and K_3 and all cycles C_n, $n \geq 3$. Also it follows from Proposition 2.6.7 that there are no cyn-unique graphs with more than one component.*

Remark 2.6.14. *Since in a cactus graph, no cycle has a chord, it follows that every cactus graph is planar. Hence we have;*

Theorem 2.6.15. *Every non acyclic graph is cyn-equivalent to a cactus graph.*

Proof. Let G be a non acyclic graph. If G itself is a cactus graph, then there is nothing to prove. So let G has a cycle which contains a chord. Without loss of generality, let H be a subgraph of G on $n \geq 4$ vertices which is a chordal graph with a unique chord. Then this subgraph H of G is cyn-equivalent to a cactus graph containing three cycles whose lengths are the same as the lengths of the three different cycles in H. Similarly to every block of G, which is not a cut edge, we can find cactus graphs having the same number of cycles of different lengths as in that block. Then by connecting all these cactus graphs corresponding to different such blocks either by identifying any two vertices, one from each such cactus graph or by connecting them by bridges we get a cactus graph having the same number of cycles of different lengths as in the original graph. Hence the proof. $\square$

An immediate consequence of Theorem 2.6.15 is ;

Corollary 2.6.16. *Every non acyclic graph is cyn-equivalent to a planar graph.*

Remark 2.6.17. *It follows from Theorem 2.6.15 that a complete graph K_n, $n \geq 3$ is cyn-equivalent to a cactus graph. So that when n is large, it can be* *seen that K_n is cyn-equivalent to cactus graph of large order and arbitrarily large diameter. Also this is true for all graphs $K_n \setminus F$, $n \geq 4$ where $K_n \setminus F$ is any subgraph of K_n for which $V(K_n \setminus F) = V(K_n)$ and $E(K_n \setminus F) = E(K_n) \setminus F$, where F is a nonempty subset of $E(K_n)$ whose elements are chords of K_n. Hence we have the following result :*

Proposition 2.6.18. *Cyn-equivalence of any two graphs G and H is independent of the order, diameter and connectivity of the graphs G and H.*

Remark 2.6.19. *From Proposition 2.2.22, for any Complete graph K_n, $n \geq 3$ $CN[K_n, z] = \frac{n!}{2}[\frac{z^3}{3(n-3)!} + \frac{z^4}{4(n-4)!} + ... + \frac{z^{n-2}}{(n-2)2!} + \frac{z^{n-1}}{(n-1)} + \frac{z^n}{n}]$. It follows from Proposition 2.2.22 and Proposition 2.6.18 that when the circumference of a cactus graph G is n, and the number of cycles of different lengths k, $3 \leq k \leq n$ in G is equal to $\frac{n!}{2k(n-k)!}$, then G is cyn-equivalent to a simple graph containing some pendant vertices attached to the vertices of K_n, with the number of pendant vertices in that simple graph is equal to the number of cycle neighbor free vertices in the cactus graph.*

Definition 2.6.20. *A graph G is called an almost path if $V(G)$ can be partitioned into $U_1, U_2, ..., U_m, W$, where $\langle U_i \rangle$ is a cycle on $|U_i|$ vertices for $i = 1, 2, ..., m$ and $\langle W \rangle$ is a path on $|W|$ vertices with the connectiivity among $U_1, U_2, ..., U_m$ and W is as follows;*
(i) for each U_i at least one vertex of U_i is adjacent to exactly one vertex of any one of the sets U_j or W, $j = 1, 2, ..., m$ with $j \neq i$ or at most two vertices of U_i, each are adjacent to exactly one vertex belonging to two different sets among U_j or W, $j = 1, 2, ..., m$, $j \neq i$ and
(ii) at least one end vertex of W is adjacent to exactly one vertex of any one of

the sets U_j, $j = 1, 2, ..., m$ *or both the end vertices of* W *each of them are adjacent to exactly one vertex belonging to two different sets among* U_j *for* $j = 1, 2, ..., m$.

Simply an almost path graph G can be considered as a path, in which the nodes are either cycles or cycle neighbor free vertices and all the cycle neighbor free vertices are clustered together to form an induced path in G. .

Definition 2.6.21. *An almost path graph* G *is said to have a unique almost path like structure (UAPLS) if* $V(G) = W \cup U_1 \cup U_2 \cup ... \cup U_m$ *as in Definition 2.6.20 and* $\langle W \rangle$ *and* $\langle U_i \rangle$ *are arranged in a line such that* $\langle W \rangle$ *is on the extreme left and* $\langle U_i \rangle$ *are next to* $\langle W \rangle$ *which are arranged in such a way that the lengths of the cycles* $\langle U_i \rangle$ *are monotonic increasing from left to right.*

Theorem 2.6.22. *In terms of cyn-equivalence, there is a surjective mapping from the set of all graphs to the set of all UAPLSs.*

Remark 2.6.23. *Consider the star graph* $K_{n,1}$ *on* $n \geq 3$ *vertices. The corresponding UAPLS is a path on* $n + 1$ *vertices. The diameter of* $K_{n,1}$ *is* 2. *It is interesting to observe that when* n *is large,* $K_{n,1}$ *which has diameter two is cyn-equivalent to a path having very large diameter. Another particular case is that of a complete graph* K_n *on* $n \geq 3$ *vertices. The diameter of* K_n *is one. But as* n *increases, the diameter of its UAPLS increases tremendously. The following table gives the relation between the order of* K_n *and the order of its UAPLS for* $3 \leq n \leq 10$.

Order of K_n	Order of the UAPLS of K_n
n = 3	3
n = 4	24
n = 5	150
n = 6	960
n = 7	6825
n = 8	54768
n = 9	493164
n = 10	5061600

2.7 Cycle neighbor roots of a graph

In this section cycle neighbor roots of a graph is introduced and cycle neighbor roots and location of roots of some particular graphs are studied.

Definition 2.7.1. *A zero of the cycle neighbor polynomial $CN[G, z]$ of G is called a cycle neighbor root of G.*

Observation 2.7.2. *Zero is a cycle neighbor root of a graph G, if and only if G contains no cycle neighbor free vertices. For such graphs, the multiplicity of zero as a root of $CN[G, z]$ has an interesting interpretation. Multiplicity of 0 as a cycle neighbor root of a graph is the girth $g(G)$ of G. Hence $g(G)$ is greater than or equal to three. Also from the definition it is clear that cyn equivalent graphs has the same set of cycle neighbor roots.*

From Definition 2.3.4, we know that, a connected graph G is a k-cycle neighbor graph, if every vertex of G belongs to at least one k-cycle in G and every edge is either a cut edge or an edge of a k-cycle in G. Theorem 2.7.3 gives a characterization of k- cycle neighbor graphs in terms of cycle neighbor roots.

Theorem 2.7.3. *Let G be a connected graph of order n. Then zero is the only cycle neighbor root of G if and only if G is a k-cycle neighbor graph, where $3 \leq k \leq n$.*

Proof. Suppose that zero is the only cycle neighbor root of G. Then $CN[G, z] = c_k(G)z^k$, $3 \leq k \leq n$, where $c_k(G)$ is the number of cycles of length k in G. Since G is connected, it is a collection of connected cycles of same length k with no common edges. Hence G is a k-cycle neighbor graph.

Converse is obvious. □

Corollary 2.7.4. *Let G be any graph. Then zero is the only cycle neighbor root of G if and only if every component of G is a k-cycle neighbor graph.*

Since all the coefficients of cycle neighbor polynomial of any graph G is non-negative, we have the result;

Theorem 2.7.5. *The cycle neighbor polynomial of any graph has no zeros in the interval* $(0, \infty)$

Theorem 2.7.6. *Let G be a connected graph of order $n \geq 3$, whose cycle neighbor polynomial has degree n and zero is a cycle neighbor root with multiplicity 3. Then G is (r)-pancyclic , $r \geq 1$ [8], if and only if the remaining $n - 3$ cycle neighbor roots of G are distinct and non real when n is odd, non real except the root -1 when n is even and these roots together with $z = 1$ forms the vertices of a regular $(n - 2)$-gon inscribed in the unit circle in the complex plane.*

Proof. Suppose that G is (r)-pancyclic. Then,

$$\begin{aligned} CN[G,z] &= rz^3 + rz^4 + rz^5 + ... + rz^n \\ &= rz^3(1 + z + z^2 + ... + z^{n-3}) \end{aligned}$$

Since, $\Sigma_{i=0}^{n-3} z^i = \frac{z^{n-2}-1}{z-1}$, the roots of $\Sigma_{i=0}^{n-3} z^i$ are the $(n-2)$th roots of unity, other than $z = 1$. Therefore, $z = exp^{\frac{2\Pi ik}{n-2}}$, $k = 1, 2, ..., n-3$, which are distinct, complex conjugate pairs when n is odd and a set of complex conjugate pairs together with -1, when n is even and these roots lie on the unit circle.

Conversely, it is clear that if $deg(CN[G; z]) = n$ with zero as a cycle neighbor root with multiplicity 3 and the remaining $n-3$ cycle neighbor roots of G are distinct, non real when n is odd, non real except the root -1 when n is even and these roots together with $z = 1$ forms the vertices of a regular polygon of $n-2$ sides inscribed in $|z| = 1$, the general expression for the cycle neighbor polynomial of G is $rz^3(1 + z + z^2 + ... + z^{n-3}) = rz^3 + rz^4 + rz^5 + ... + rz^n$, which is the cycle neighbor polynomial of a (r)-pancyclic graph. □

Remark 2.7.7. *It is still an open problem whether there exists (k)-pancyclic graphs for* $k \geq 3$ *[8]. Theorem 2.7.6 holds for all (r)-pancyclic graphs with* $r = 1$ *and* $r = 2$ *and it holds for (r)-pancyclic graphs with* $r \geq 3$ *if such graphs exist.*

Definition 2.7.8. *A connected graph* G *of order* n *is said to be weakly (r)-pancyclic [8],* $r = 1, 2, 3, ...$ *if* G *contains exactly* r *k-cycles for all* $k, l \leq k \leq n$*, where* $4 \leq l \leq n-1$*.*

Corollary 2.7.9. *Let* G *be a connected graph which is weakly (r)-pancyclic of* *order n. Then all the non zero cycle neighbor roots of G are distinct, non real when n is odd, non real except the root -1 when n is even and all these non zero roots lie on the unit circle.*

Theorem 2.7.10. *Let G be a connected graph of order n such that the cycle neighbor polynomial of G has exactly two terms. Then $CN[G, z]$ has a nonzero real root if and only if one of the following conditions hold.*

1. *G contains at least one cycle neighbor free vertex and all cycles in G are of the same length.*

2. *The length of any cycle in G is either m or k, where, $3 \leq k < m \leq n$ and $(m - k)$ is odd.*

Proof. Let G be a connected graph of order n such that the cycle neighbor polynomial of G contains exactly two terms. Then either
$CN[G, z] = c_0(G) + c_k(G)z^k$, $3 \leq k \leq n - 1$ or $CN[G, z] = c_k(G)z^k + c_m(G)z^m$, $3 \leq k < m \leq n$ When $CN[G, z] = c_0(G) + c_k(G)z^k$, $z = (\frac{-c_0(G)}{c_k(G)})^{\frac{1}{k}}$ and when

$$\begin{aligned} CN[G, z] &= c_k(G)z^k + c_m(G)z^m \\ &= z^k(c_k(G) + c_m(G)z^{m-k}), \end{aligned}$$

the nonzero roots of $CN[G, z]$ are given by $z = (\frac{-c_k(G)}{c_m(G)})^{\frac{1}{m-k}}$. Since a negative number has a real nth root if and only if n is odd it follows that there is a real root for $CN[G, z]$ in the above cases if and only if k is odd in the first case and $m - k$ is odd in the second case.

Conversely assume that one of the conditions (1) or (2) in the statement of the theorem hold. Then either $CN[G,z] = c_0(G) + c_k(G)z^k$, or $CN[G,z] = c_k(G)z^k + c_m(G)z^m$. Without loss of generality, let us assume that $k < m$. Then $z = (\frac{-c_0(G)}{c_k(G)})^{\frac{1}{k}}$ with odd value of k or $z = (\frac{-c_k(G)}{c_m(G)})^{\frac{1}{m-k}}$ with $m-k$ as odd. Hence $CN[G,z]$ has a nonzero real root. $\square$

Since the cycle neighbor polynomial of a unicyclic graph G on n vertices is $CN[G,z] = z^k + (n-k)$, $k \leq n$ [1], and since the kth roots of $-(n-k)$ lie on a circle of radius $n-k$ in the complex plane, we have the following result;

Proposition 2.7.11. *Let G be a unicyclic graph on n vertices. Then all the cycle neighbor roots of G has absolute value $|n-k|$, which is the length of the cycle in G and $3 \leq k \leq n$. In particular, all these roots lie on a circle of radius $|n-k|$ in the complex plane.*

Conclusion

A new univariate graph polynomial called cycle neighbor polynomial of a graph is introduced in this chapter. Cycle neighbor polynomial of a graph directly encodes the number of cycles of different lengths and the number of cycle neighbor free vertices in the graph. Eventhough it is hard to find the number of cycles of different lengths in a general graph, the concept of cycle neighbor polynomial of a graph is interesting and important because it reveals many graph properties of the underlying graph. The concept of cycle neighbor equivalence and the UAPLS enables us to find a unique planar graph associated with every non acyclic graph. It is observed that cycle neighbor polynomial of a graph cannot have any positive real roots. Graphs having zero as the only cycle neighbor root is characterized. Location of roots of some graphs are also obtained.

Chapter 3

Modified Cycle Neighbor Polynomial of a Graph

3.1 Introduction

[1] [2] Motivated from the interpretation of simple cycles of lengths one and two [56], an improvisation of cycle neighbor polynomial of a graph is introduced in this chapter. The advantage of this definition is that this polynomial distinguishes more graph classes than that of cycle neighbor polynomial. Also we study the cycle neighbor polynomial of some graph operations, graph modifications and that of graphs derived from the given graph with respect to this modified polynomial. It helps us to view some interesting properties of the resulting graph through cycle neighbor polynomial in a vivid manner.

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[2]Part of this chapter has been published in Malaya Journal of Mathematik, Vol. 8, No. 4, 1703–1707, 2020

3.2 Modified cycle neighbor polynomial of graphs

We improve the definition of cycle neighbor polynomial of a graph by taking into account the isolated vertices, non isolated cycle neighbor free vertices and bridges which were not considered in the original cycle neighbor polynomial.

Definition 3.2.1. *Modified cycle neighbor polynmial of a graph G of order n is denoted by $CN^*[G;z]$ and it is defined as*

$$CN^*[G;z] = \Sigma_{k=0}^{c(G)} c_k(G) z^k,$$

where $c_0(G)$ is the number of isolated vertices, $c_1(G)$ is the number of non isolated cycle neighbor free vertices, $c_2(G)$ is the number of cut edges and $c_k(G)$ is the number of cycles of length k in G with $3 \leq g(G) \leq k \leq c(G) \leq n$.

The zeros of modified cycle neighbor polynomial of G are the roots of $CN^*[G;z]$.

Proposition 3.2.2 and Proposition 3.2.3 are direct implications of definition of modified cycle neighbor polynomial.

Proposition 3.2.2.

1. *Let G be any graph. Then the cycle neighbor polynomial $CN[G;z]$ and modified cycle neighbor polynomial $CN^*[G;z]$ of G are the same if and only if G contains no bridges and no non isolated cycle neighbor free vertices. For such graphs, both the set of all cycle neighbor roots and modified cycle neighbor roots are the same.*

2. *For a graph G, $CN^*[G;z]$ is a constant polynomial if and only if $G \cong \overline{K_n}$, the empty graph on $n = 1, 2, 3, ...$ vertices.*

3. *Let G be a connected graph of order $n \geq 3$, then $CN^*[G;z]$ contains exactly one term if and only if G is a k-cycle neighbor graph containing no cut edges.*

4. *Let $G(V,E)$ be an acyclic graph with $CN^*[G;z] = a_0 + a_1z + a_2z^2$. Then $a_0 + a_1 = |V(G)|$ and $a_2 = |E(G)|$. Moreover, when G is a connected acyclic graph, then $a_0 = 0$, $a_1 = |V(G)|$ and $a_2 = |E(G)| = a_1 - 1$.*

Proposition 3.2.3. 1. *The degree of $CN^*[G;z]$ of a graph G is two if and only if G is a forest. In particular, degree of $CN^*[G;z]$ of a connected graph G is two if and only if G is a tree.*

2. *No polynomial of degree one can be the modified cycle neighbor polynomial of a graph.*

Proof. 1. Since trees and forests are acyclic (1) follows.

2. Suppose $P(z)$ is a polynomial of degree one say $P(z) = a_0 + a_1z$, $a_1 \neq 0$, which is the modified cycle neighbor polynomial of a graph G. Then G contains a_0 isolated vertices and a_1 non isolated cycle neighbor free vertices. Since $a_1 \neq 0$, the induced subgraph of these non isolated cycle neighbor free vertices is a non trivial forest. Hence it contains at least one bridge and therefore $deg(P(z)) \geq 2$, a contradiction. □

If G is a tree of order of n and size $n-1$, then by Proposition 3.2.2, $CN^*[G;z]$ $= nz + (n-1)z^2$. Therefore we have :

Corollary 3.2.4. *Let G be a connected acyclic graph of order n. Then the zeros of modified cycle neighbor polynomial of G are 0 and $\frac{-n}{n-1}$.*

Corollary 3.2.5. *Let G be a graph of order n, which is a forest not containing any isolated vertices. Let $G_1, G_2, ..., G_k$ be the components of G of order $n_1, n_2, ..., n_k$ respectively such that $n_1 + n_2 + ... + n_k = n$. Then the set of zeros of modified cycle neighbor polynomial of G is $\{0, \frac{-n}{n-k}\}$.*

Theorem 3.2.6. *Let G be a connected graph of order n, $n \geq 4$. Then the modified cycle neighbor polynomial of G contains maximum number of terms if and only if $G \cong H_{n-1,1}$, where $H_{n-1,1}$ is a graph consisting of a pancyclic graph H on $n-1$ vertices and a copy of K_1 connected to any one of the vertices of H by a bridge.*

Proof. Let the modified cycle neighbor polynomial of G be $CN^*[G; z] = a_0 + a_1 z + a_2 z^2 + ... + a_k z^k$, where k is the circumference $c(G)$ of G. Since G is connected, $a_0 = 0$. Also when $a_1 \neq 0$, then $a_n = 0$ since circumference of G is less than or equal to $n-1$ whenever G contains cycle neighbor free vertices. Hence the number of terms in $CN^*[G; z]$ is less than or equal to $n-1$. Thus for a graph with its modified cycle neighbor polynomial containing maximum number of terms, we have $a_1 \neq 0$, $a_2 \neq 0$, ... , $a_{n-1} \neq 0$, and $a_n = 0$. But this is possible only when $a_1 = 1$ and $a_2 = 1$. Otherwise $a_1 \geq 2$, $a_2 \geq 2$ and then $k = c(G) \leq n-2$, so that $a_{n-1} = 0$. Hence G contains a subgraph H containing cycles of all lengths k, for $3 \leq k \leq n-1$ and a cycle neighbor free vertex attached to H by a bridge. That is $G \cong H_{n-1,1}$.

Conversely, when $G \cong H_{n-1,1}$, $CN^*[G; z]$ contains $n-1$ terms, which is the maximum possible number of terms in the modified cycle neighbor polynomial of any graph G. This completes the proof. □

Corollary 3.2.7. *Let G be a connected graph of order $n \geq 3$. Then the number* *of terms in modified cycle neighbor polynomial of G is less than or equal to $n-1$.*

Note that when G is a connected graph of order $n < 3$, that is when G is isomorphic to K_1 or K_2, the modified cycle neighbor polynomial of G contains exactly n terms. And the number of terms in the modified cycle neighbor polynomial of a connected graph G is minimum, that is $CN^*[G; z]$ contains exactly one term if and only if $G \cong K_1$ or G is a k-cycle neighbor graph with each edge of G belongs to a k-cycle in G. Empty graphs $\overline{K_n}$ and graphs G with all of its components are k-cycle neighbor graphs without cut edges are examples of disconnected graphs whose modified cycle neighbor polynomial contains exactly one term.

Similar to the cycle neighbor equivalence and cycle neighbor uniqueness defined in 2.6.1 and 2.6.10, cycle neighbor equivalence and cycle neighbor uniqueness with respect to modified cycle neighbor polynomial of graphs can also be defined.

Definition 3.2.8. *Two graphs G and H are said to be cycle neighbor equivalent with respect to modified cycle neighbor polynomial if $CN^*[G; z] = CN^*[H; z]$ and G and H are called cycle neighbor unique with respect to modified cycle neighbor polynomial if $CN^*[G; z] = CN^*[H; z]$ then $G \cong H$.*

We use the abbereviations *cyn**-equivalence and *cyn**-uniqueness respectively to denote cycle neighbor equivalence and cycle neighbor uniqueness of graphs with respect to modified cycle neighbor polynomial.

Theorem 3.2.9. *Let T be a tree of order n and let $\overline{T}$ be the complement of T. Then $CN^*[T; z] = CN^*[\overline{T}; z]$ if only if $T \cong P_n$ where $n = 1$ or 4.*

Proof. Consider a tree T. First let T be a path P_n. Then for $n = 2$ and 3, $\overline{P}_n$ contains isolated vertices while P_n does not and $P_4 \cong \overline{P}_4$. Therefore $CN^*[P_4; z]$ $= CN^*[\overline{P}_4; z]$. For $n \geq 5$, P_n is acyclic and $\overline{P}_n$ contains cycles. Now let T is not a path. Then the order of T is greater than or equal to four. Since T is acyclic and it is not a path, there are more than two pendant vertices in T. These pendant vertices will form a cycle in $\overline{T}$. Hence $\overline{T}$ is not acyclic. Therefore in this case, $CN^*[T; z] \neq CN^*[\overline{T}; z]$. Conversely when $T \cong P_n$ with $n = 1$ or 4, $CN^*[T; z] = CN^*[\overline{T}; z]$. Hence the proof. □

Remark 3.2.10. *The only acyclic graphs G such that $CN^*[G, z] = CN^*[\overline{G}, z]$ are paths P_n, with $n = 1$ or 4.*

Theorem 3.2.11. *Let G be a graph of order n. If G is isomorphic to any of the following graphs,*

1. *C_n, a cycle on n vertices.*
2. *$\overline{K}_n$, empty graph on n vertices.*
3. *P_n, a path on n vertices, where $n = 1, 2$ or 3.*
4. *K_n, a complete graph on n vertices, where $n = 1, 2$ or 3.*
5. *H, where H is a graph containing exactly two cycles joined by a bridge between them.*

then G is cyn*-unique.

Proof. Suppose that G be a graph which satisfies one of the conditions in the statement of the theorem. Let us consider each case one by one.

Case (1) When $G \cong C_n$, G contains no bridges or cycle neighbor free vertices. Hence by Proposition 3.2.2, $CN^*[G; z] = CN[G; z] = z^k$, Where k is the length of the cycle in G. Hence by the same reasoning as in the case of cyn-uniqueness of cycles C_n, $n \geq 3$ in Theorem 2.6.11, it follows that cycles C_n, $n \geq 3$ are *cyn**-unique.

Case (2) $G \cong \overline{K}_n$. Then $CN^*[G; z] = n$, a constant polynomial. If H is any graph other than G with $CN^*[H; z] = n$, it means that H contains n isolated vertices and no edges. That is $H \cong G$. Therefore $\overline{K}_n$ is *cyn**-unique.

Case (3) $G \cong P_n$, where $n = 1, 2$ or 3. Then by Proposition 3.2.2, $CN^*[G; z] = a_1 z + a_2 z^2$, with $a_2 = a_1 - 1$. Since there is a unique non isomorphic tree on $n \leq 3$ vertices, $CN^*[G; z] = CN^*[H; z] = a_1 z + a_2 z^2$ implies that $H \cong G$.

Case (4) $G \cong K_n$, where $n = 1, 2$ or 3.

When $n = 1$, $CN^*[G; z] = 1$, hence it is clear from case (2) that K_1 is *cyn**-unique.

The only simple graphs of order two are K_2 and $\overline{K}_2$. $\overline{K}_2$ contains isolated vertices while K_2 does not. Hence K_2 is also *cyn**-unique.

When $n = 3$, $K_3 \cong C_3$ hence by case (1), K_3 is *cyn**-unique.

Case (5) $G \cong H$. Then $CN^*[H; z] = z^2 + z^k + z^m$, where k and m are the lengths of the cycles in G with $k \geq 3$, $m \geq 3$ and $m + k = n$. Suppose if possible, H_1 is a graph of order n such that $CN^*[H_1; z] = CN^*[H; z]$ but $H_1 \ncong H$. Therefore H_1 contains exactly two cycles of lengths k and m and these cycles will be disjoint, otherwise they will have a vertex in common and therefore H_1 will contain a cycle neighbor free vertex contradicting our assumption that

$CN^*[H_1; z] = z^2 + z^k + z^m$. By case (1), cycles C_n are *cyn**-unique and since order of H_1 is $m + k = n$, one end point of the bridge in H_1 should be in the k-cycle and the other end is in the m-cycle of H_1. That is $H_1 \cong H$. Therefore H is *cyn**-unique. □

Remark 3.2.12. *Let G and H be two graphs. Whenever G is cyn*-equivalent to H, then G is cyn-equivalent to H. But two cyn-equivalent graphs need not be cyn*-equivalent. For example, the graphs G and H in figure 2.4 are cyn-equivalent but they are not cyn*-equivalent. On the other hand, every cyn-unique graph is cyn*-unique. But the converse need not be. For example, the graph G in Figure 3.1 is cyn*-unique but it is not cyn-unique.*

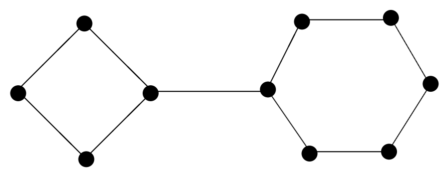

Figure 3.1 - G

The main difference between cycle neighbor polynomial of a graph and its modified cycle neighbor polynomial is that the cut edges in G are also taken into account in the modified cycle neighbor polynomial of a graph. As a result, all the graph modifications considered in section 2.5 except the one in Theorem 2.5.5, which do not affect the cycle neighbor polynomial of a graph will certainly alter the modified cycle neighbor polynomial of the graph. Hence these graph modifications will not produce *cyn**-equivalent graphs. But under the graph modification denoted by $G.H$ of disjoint graphs G and H, and defined as in Theorem 2.5.5, $G.H$ and $G \cup H$ are *cyn**-equivalent whenever the vertices in G and H which are identified in $G.H$ are not cycle neighbor free vertices of the graphs G and H.

A graph polynomial is complete [33] if it distinguishes all non isomorphic graphs. Formulation of a complete graph polynomial which can be easily computed is not yet succeeded mainly due to two reasons. The first one is there are so many indistinguishable non isomorphic graphs. And the second reason is that such a graph polynomial is too hard to compute. The two univariate polynomials cycle neighbor polynomial and modified cycle neighbor polynomial of a graph introduced in chapters two and three respectively can be compared in terms of this 'completeness' property of graph polynomials. Every *cyn*- unique graph is cyn^*- unique. But as the converse of results need not hold always, the modified cycle neighbor polynomial of a graph distinguishes more non isomorphic graphs than that of cycle neighbor polynomial of the graph. As a consequence, modified cycle neighbor polynomial of a graph can be considered to be better than the cycle neighbor polynomial of the graph in terms of this completeness property of graph polynomials. So here after we will be considering the modified cycle neighbor polynomials of graphs, and call it cycle neighbor polynomial itself if there is no confusion.

3.3 Cycle neighbor polynomial of some graph operations

Many graph modification problems concern destroying or creating cycles. In this section we study the cycle neighbor polynomial of some graph operations, graph modifications and that of graphs derived from the given graph. It helps us to view some interesting properties of the resulting graph through cycle neighbor polynomial in a vivid manner.

First we consider the corona GoH of two graphs G and H. Let G be a connected graph of order $n \geq 2$ with k, $1 \leq k \leq n$ cycle neighbor free vertices. Note that no cycle will be added or deleted from the induced subgraph G of GoH. We obtain $CN^*[GoH; z]$ when H is a path, cycle or a star graph in terms of cycle neighbor polynomial of G.

Theorem 3.3.1. *Let P_m be a path on $m \geq 1$ vertices. Then*

$$CN^*[GoP_m; z] = \begin{cases} CN^*[G; z] + nz^2 + nz, & \text{if } m = 1; \\ CN^*[G; z] + nz^3 - kz, & \text{if } m = 2; \\ CN^*[G; z] + n\{\Sigma_{k=3}^{m+1}\{m - (k-2)\}z^k\} - kz, & \text{if } m \geq 3; \end{cases}$$

Proof. When $m = 1$, corresponding to the vertex in each of the n copies of P_1, n new edges and n new cycle neighbor free vertices will be introduced in GoP_1. When $m = 2$, corresponding to the edge in each of the n copies of P_2, n triangles will be introduced in GoP_2 and there are no cycle neighbor free vertices in GoP_2. When $m \geq 3$, together with all cycles of different lengths in G, $(m-1)$ triangles, $(m-2)$ 4-cycles,..., one m-cycle will be formed in GoP_m with a vertex of G common to all these cycles in GoP_m. Hence every vertex of G belong to at least one cycle of GoP_m and we have $CN^*[GoP_m; z] = CN^*[G; z] + n\{\Sigma_{k=3}^{m+1}\{m - (k-2)\}z^k\} - kz$. □

Theorem 3.3.2. *Let C_m be a cycle on $m \geq 3$ vertices. Then,*

$$CN^*[GoC_m; z] = CN^*[G; z] + n\{\Sigma_{l=3}^{m+1} z^l + z^m\} - kz$$

Proof. In GoC_m at each of the n vertices of G, there is a wheel graph on $m+1$ vertices with the central vertex as the vertex of G. Hence it follows from Proposition 2.2.19 that $CN^*[GoC_m; z] = CN^*[G; z] + n\{\Sigma_{l=3}^{m+1} z^l + z^m\} - kz$. $\square$

Theorem 3.3.3. *For $K_{m,1}$, $m \geq 3$,*

$$CN^*[GoK_{m,1}; z] = CN^*[G; z] + n\{mz^3 + \binom{m}{2} z^4\} - kz$$

Proof. Let H be a subgraph of $GoK_{m,1}$ induced by a vertex of G and a copy of $K_{m,1}$, $m \geq 3$. Then there are two vertices say u and v of degree $m+1$ and m vertices of degree two in H. Let $V(H) = A \cup B$, where $A = \{u, v\}$ and $B = \{v_1, v_2, ..., v_m\}$, the set of all vertices of degree two in H. Since v_i, $1 \leq i \leq m$ is adjacent to u and v only, there are exactly m triangles in H. There are $(m-1)$ 4-cycles through each vertex v in H. In fact there are $\frac{m(m-1)}{2}$ 4-cycles in H.

Finally note that the maximum length of any cycle in H is four. Since $V(H) = A \cup B$ and no two vertices in B are adjacent, the sequence of vertices which form any cycle in H will be either an alternating sequence of vertices from A and B respectively or a sequence of the form v_i, u, v, $1 \leq i \leq 3$. In the first case, since there are only two vertices in A, any alternating vertex sequence from A and B without repetition contain a maximum of four vertices. In the second case, $\{v_i, u, v\}$ induces a triangle in H. Hence there are no cycles of length greater than four in H. $\square$

Subdivision graph $S(G)$ [55] of the graph G is obtained by subdividing each edge of G exactly once by a new vertex. In the next result, we compare $CN^*[G; z]$ and $CN^*[S(G); z]$ of a graph G.

Theorem 3.3.4. *Let G be a connected graph of order $n \geq 2$ with $CN^*[G;z] = \Sigma_{k=1}^{c(G)} c_k z^k$. Then,*

$$CN^*[S(G);z] = (c_1 + c_2)z + 2c_2 z^2 + \Sigma_{k=3}^{c(G)} c_k z^{2k}.$$

Proof. The number of edges in G will be doubled in its subdivision graph $S(G)$ by the introduction of a new vertex on every edge of G. Hence corresponding to every bridge in G, there is a cycle neighbor free vertex in $S(G)$. Also the number of bridges and lengths of every cycle in G will be doubled in $S(G)$. □

It follows from $CN^*[S(G);z]$ that the subdivision graph of every simple graph G is bipartite. The fact that $g(S(G)) = 2g(G)$ and $c(S(G)) = 2c(G)$ is immediate from $CN^*[G;z]$, where $g(G)$ and $c(G)$ are respectively the girth and circumference of G.

Square of a graph G [27] is obtained by adding edges in G, which connect pairs of vertices of G at a distance two apart. It is denoted by G^2. Next we obtain $CN^*[G^2;z]$, when G is a path or a star graph.

Theorem 3.3.5. *Let P_n be a path on $n \geq 3$ vertices. Then,*

$$CN^*[P_n^2;z] = \Sigma_{k=3}^{n}\{n-(k-1)\}z^k.$$

Proof. Let the vertices of P_n be labelled as $v_1, v_2, ..., v_n$. Then for $1 \leq i \leq n-2$, each v_i is adjacent to v_{i+2} in P_n^2. Hence it follows that for $1 \leq i \leq n-2$, the graph induced by the set $\{v_i, v_{i+1}, v_{i+2}\}$ is a triangle in P_n^2 and no triangles are induced by $\{v_i, v_j, v_k\}$ if v_i, v_j, v_k does not form a set of consecutive vertices of $V(P_n)$. Therefore, there are exactly $n-2$ triangles. Also, since every vertex belongs to at least one triangle, there are no bridges or cycle neighbor free vertices in P_n^2. In general, for $1 \leq i \leq k-1$, $v_i v_{i+1} v_{i+3} v_{i+5} ... v_{i+(k-2)} v_{i+(k-1)} v_{i+(k-3)} v_{i+(k-5)} ... v_{i+2} v_i$, is a k-cycle for odd k in P_n^2 and

$v_i v_{i+1} v_{i+3} v_{i+5} ... v_{i+(k-1)} v_{i+(k-2)} v_{i+(k-4)} v_{i+(k-6)} ... v_{i+2} v_i$ is a k-cycle for even k where $3 \leq k \leq n$. Hence in P_n^2, there are $(n-3)$ 4-cycles, $(n-4)$ 5-cycles,..., $(n-(k-1))$ k-cycles,..., one n-cycle without duplication. □

The cycle neighbor polynomial $CN^*[P_n^2; z]$ of P_n^2 reveals that it is hamiltonian and pancyclic for $n \geq 3$.

Theorem 3.3.6. *Let G be a graph of diameter two. If order of G is n, Then,*

$$CN^*[G^2; z] = \frac{n!}{2}[\frac{z^3}{3(n-3)!} + \frac{z^4}{4(n-4)!} + ... + \frac{z^{n-2}}{(n-2)2!} + \frac{z^{n-1}}{(n-1)} + \frac{z^n}{n}].$$

Proof. Since $diam(G) = 2$, $d(v_i, v_j) \leq 2$, for every $v_i, v_j \in V(G)$. Hence in G^2, v_i is adjacent to v_j, for every i, j, $1 \leq i, j \leq n$, $i \neq j$. Therefore, $G^2 \cong K_n$. Hence the result follows from the expression for cycle neighbor polynomial of complete graphs (Proposition 2.2.22). □

Since $diam(K_{m,1}) = 2$, Corollary 3.3.7 is a direct consequence of Theorem 3.3.6.

Corollary 3.3.7. $CN^*[K_{m,1}^2; z] = \frac{(m+1)!}{2}[\frac{z^3}{3(m-2)!} + \frac{z^4}{4(m-3)!} + ... + \frac{z^{m-1}}{(m-1)2!} + \frac{z^m}{m} + \frac{z^{m+1}}{m+1}]$

In general, power of a graph G^k, $k = 2, 3, 4, ...$ is obtained by adding edges in G which connect pairs of vertices v_i, v_j if $d(v_i, v_j) \leq k$. Hence for all graphs with $diam(G) = k$, $G^k \cong K_n$, therefore we have Theorem 3.3.8.

Theorem 3.3.8. *Let G be a graph of order n with $diam(G) = k$, $k = 2, 3, 4, ...$ Then,*

$$CN^*[G^k; z] = CN^*[K_n, z].$$

The splitting graph $S^{'}(G)$ [47] of a graph G is obtained by adding new vertices $v^{'}$ to G, corresponding to each vertex v of G and then joining the vertex $v^{'}$ to all vertices of G adjacent to v in G. Now we find $CN^*[S^{'}(G); z]$ when G is a path or a star graph.

Theorem 3.3.9. *Let P_n be a path on $n \geq 2$ vertices. Then*

$$CN^*[S^{'}(P_n); z] = \begin{cases} 3z^2 + 4z, & \text{if } n = 2; \\ \Sigma_{k=3}^{n-1}\{n - (k-1)\}z^{2k}, & \text{if } n \geq 3; \end{cases}$$

Proof. Let the vertices of P_n be labelled as $v_1, v_2, ..., v_n$, with v_1 and v_2 as the pendant vertices. Let $v_i^{'}$ be the vertex in $S^{'}(P_n)$ corresponding to v_i,
$1 \leq i \leq n$. Then $v_1^{'}$ and $v_n^{'}$ are the pendant vertices of $S^{'}(P_n)$. For $1 \leq i \leq n-k$, $v_i v_{i+1}^{'} v_{i+2} v_{i+3}^{'} v_{i+4} ... v_{i+(k-2)}^{'} v_{i+(k-1)} v_{i+k} v_{i+(k-1)}^{'} ... v_{i+2} v_{i+1} v_i$ is a $2k$-cycle in $S^{'}(P_n)$, when k is odd and $v_i v_{i+1}^{'} v_{i+2} v_{i+3}^{'} v_{i+4} ... v_{i+(k-1)}^{'} v_{i+k} v_{i+(k-1)} v_{i+(k-2)}^{'} ... v_{i+2}^{'} v_{i+1} v_i$ is a $2k$-cycle in $S^{'}(P_n)$, when k is even. Hence there are $(n-2)$ 4-cycles, $(n-3)$ 6-cycles,...,two $2(n-2)$-cycles and one $2(n-1)$ cycle in $S^{'}(P_n)$. Hence the proof. □

Theorem 3.3.10. *Let $K_{m,1}^{'}$ be the splitting graph of $K_{m,1}$, $m \geq 2$. Then*

$$CN^*[K_{m,1}^{'}; z] = \binom{m}{2} z^4 + mz^2 + mz.$$

Proof. Let $V(K_{m,1}^{'}) = A \cup B$ with $A = \{v, v^{'}\}$, where v and $v^{'}$ are the central vertex of $K_{m,1}$ and its corresponding vertex in $K_{m,1}^{'}$ respectively and
$B = \{u_1, u_2, ..., u_m, u_1^{'}, u_2^{'}, ..., u_m^{'}\}$ where u_i and $u_i^{'}$, $1 \leq i \leq m$ are the pendant

vertices of $K_{m,1}$ and its corresponding vertex in $K^{'}_{m,1}$ respectively. Then A and B are independent sets. Hence the vertices of any cycle in $K^{'}_{m,1}$ is an alternating sequence of vertices from A and B. Since $|A| = 2$, the length of any cycle in $K^{'}_{m,1}$ is four and there are $(m-1)$ 4-cycles through u_1, $(m-2)$ 4-cycles through u_2 without repetition and so on. Hence there are $\binom{m}{2}$ 4-cycles in $K^{'}_{m,1}$. Also there are m pendant vertices and m pendant edges corresponding to $u^{'}_i$, $1 \leq i \leq m$. Hence the proof. □

Since there are no odd cycles in both $P^{'}_n$ and $K^{'}_{m,1}$, it follow that the splitting graphs of paths and that of star graphs are bipartite.

Duplication of a vertex v of a graph G is the graph $G^{'}$ obtained by adding a vertex $v^{'}$ in G with $N(v^{'}) = N(v)$. Here we consider $CN^*[G^{'}; z]$ of G, when G is a path, cycle or a star graph.

The duplication of a pendant vertex of a path P_n, $n \geq 2$ adds a new pendant vertex in $P^{'}_n$. Therefore;

Proposition 3.3.11. *Let $P^{'}_n$ be the graph obtained by the duplication of a pendant vertex of P_n, $n \geq 2$. Then $CN^*[P^{'}_n; z] = CN^*[P_n; z] + z^2 + z$.*

Theorem 3.3.12. *Let $P^{'}_n$ be the graph obtained by the duplication of a non pendant vertex of P_n, $n \geq 2$ then*

$$CN^*[P^{'}_n; z] = CN^*[P_n; z] + z^4 - 2z^2 - 3z.$$

Proof. The subgraph of $P^{'}_n$ induced by the duplication of a non pendant vertex vertex of P_n, its corresponding vertex and their neighbors is a 4-cycle in $P^{'}_n$ and consequently, the number of cycle neighbor free vertices of P_n will be reduced by three and number of bridges of P_n will be reduced by two in $P_n^{'}$. □

Theorem 3.3.13. *Let $C_n^{'}$ be the graph obtained by the duplication of a vertex of the cycle C_n, $n \geq 3$ then*

$$CN^*[C_n^{'}; z] = CN^*[C_n; z] + z^n + z^4.$$

Proof. Let v be any vertex of C_n and $v^{'}$ be the duplication of v in $C_n^{'}$. Then $\{v, v^{'}\} \cup N(v)$ induces a 4-cycle and $\{v^{'}\} \cup V(C_n) \setminus \{v\}$ induces an n-cycle in $C_n^{'}$. Therefore there are two n-cycles and a 4-cycle in $C_n^{'}$. □

Theorem 3.3.14. *Let $K_{m,1}^{'}$ be the graph obtained by the duplication of the central vertex of $K_{m,1}$, $m \geq 2$. Then*

$$CN^*[K_{m,1}^{'}; z] = \tbinom{m}{2} z^4 + mz^2 + mz.$$

Proof. Let $V(K_{m,1}^{'}) = A \cup B$ with $A = \{v, v^{'}\}$, where v and $v^{'}$ are the central vertex of $K_{m,1}$ and duplication of v in $K_{m,1}^{'}$ respectively and $B = \{u_1, u_2, ..., u_m\}$, where u_i, $1 \leq i \leq m$ are the pendant vertices of $K_{m,1}$. Since A and B form a partition of $V(K_{m,1}^{'})$ and since A and B are independent sets, as in the case of splitting graph of $K_{m,1}$ there are $\frac{m(m-1)}{2}$ 4-cycles in $K_{m,1}^{'}$. Also since both v and $v^{'}$ are adjacent to all the vertices of B, There are no cycle neighbor free vertices or bridges in $K_{m,1}^{'}$. Hence the result. □

It follows that the graph obtained by the duplication of the central vertex of $K_{m,1}$, $m \geq 2$ is bipartite. It is obvious that if $G^{'}$ is the graph obtained by the duplication of any one of the pendant vertices of $K_{m,1}$, then $CN^*[G^{'}; z] = CN^*[K_{m,1}; z] + z^2 + z$

Duplication of of a vertex $w \in V(G)$ of a graph G by an edge [54] produces a new graph $G^{'}$ by adding an edge $e^{'} = u^{'}v^{'}$ to G such that $N(v^{'}) = \{w, u^{'}\}$ and $N(u^{'}) = \{w, v^{'}\}$. In the next result we obtain $CN^*[G^{'}; z]$ of a graph G.

Theorem 3.3.15. *Let G be a connected graph of order $n \geq 2$ which contains k, $0 \leq k \leq n$ cycle neighbor free vertices and let $G^{'}$ be the graph obtained by the duplication of a vertex $w \in V(G)$ by an edge. Then*

$$CN^*[G^{'}; z] = \begin{cases} CN^*[G; z] + z^3 - z, & \textit{if } w \textit{ is a cycle neighbor free vertex} \\ CN^*[G; z] + z^3, & \textit{otherwise} \end{cases}$$

Proof. Duplication of of a vertex $w \in V(G)$ of a graph G by an enge $e = uv$ produces a triangle $wuvw$ in $G^{'}$. Therefore, the number of cycle neighbor free vertices will be reduced by one if w is a cycle neighbor free vertex. □

Let $G^{'}$ be the graph obtained by duplication of each vertex of G by edges. Then clearly $G^{'} \cong GoP_2$ hence it follows from Theorem 3.3.1 that
$CN^*[G^{'}; z] = CN^*[G; z] + nz^3 - kz$, where k is the number of cycle neighbor free vertices in G.

The middle graph $M(G)$ (also known as the semi total (line) graph $T_1(G)$) [46] of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if they are adjacent edges of G or one is a vertex and the other is an edge incident with it. Next we obtain $CN^*[M(G); z]$ when $G \cong P_n$.

Theorem 3.3.16. *Let $M(P_n)$ be the middle graph of the path P_n, $n \geq 2$. Then,*

$$CN^*[M(P_n); z] = \begin{cases} 2z^2 + 3z, & \textit{if } n = 2 \\ (n-2)z^3 + 2z^2 + 2z, & \textit{if } n > 2 \end{cases}$$

Proof. Let $V(M(P_n)) = A \cup B$ where $A = \{v_1, v_2, ..., v_n\}$, is the set of vertices of P_n and $B = \{u_1, u_2, ..., u_{n-1}\}$ be the vertices of $M(P_n)$ corresponding to the edges of P_n. The subgraph of $M(P_n)$ induced by B is P_{n-1} and for $2 \leq i \leq n-1$, v_i is adjacent to u_{i-1} and u_i. Hence the graph induced by $\{u_{i-1}, v_i, u_i\}$ is a triangle for $2 \leq i \leq n-1$. Also u_1v_1 and v_nu_{n-1} are bridges of $M(P_n)$. Hence the result. □

If $V(G)$ of a graph G can be partitioned into an independent set and an acyclic set, then G is said to be a near-bipartite graph [2]. From the proof of Theorem 3.3.16, it is clear that middle graph of path P_n, $n \geq 2$ is near bipartite.

Theorem 3.3.17. *Let $M(K_{m,1})$ be the middle graph of $K_{m,1}$, $m \geq 3$. Then,*

$$CN^*[M(K_{m,1}); z] = \tfrac{(m+1)!}{2}[\tfrac{z^3}{3(m-2)!} + \tfrac{z^4}{4(m-3)!} + ... + \tfrac{z^{m-1}}{(m-1)2!} + \tfrac{z^m}{m} + \tfrac{z^{m+1}}{m+1}] + mz^2 + mz.$$

Proof. Let $V(M(K_{m,1})) = \{v, v_1, v_2, ..., v_m, u_1, u_2, ..., u_m\}$, where $\{v, v_1, v_2, ..., v_m\}$ is $V(K_{m,1})$ with v as the central vertex and $\{u_1, u_2, ..., u_m\}$ corresponds to the edges of $K_{m,1}$. Since every edge in $K_{m,1}$ are adjacent and are incident with v, the subgraph of $M(K_{m,1})$ induced by $\{v, u_1, u_2, ..., u_m\}$ is K_m and in $M(K_{m,1})$, $|N(v_i)| = 1$ for $1 \leq i \leq m$. Hence the result follows from Proposition 2.2.22. □

A split graph [18] is a graph whose vertices can be partitioned into two subsets, such that one subset induces a clique, and the other induces an independent set. A graph is called a cograph or complement reducible graph [13] if it contains no induced P_4 and a graph is called trivially perfect [22] if it is a cograph and contains no induced C_4. Since $CN^*[M(K_{m,1}); z] = CN^*[K_m; z] + mz^2 + mz$, it is obvious that middle graph of every star graph is a split graph. Also since the graph induced by any four vertices of $M(K_{m,1})$ contains a triangle, middle graph of every star graph is a cograph and is trivially perfect too.

The semi total (point) graph $T_2(G)$ [46] of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if they are adjacent vertices of G or one is a vertex and the other is an edge incident with it.

Theorem 3.3.18. *Let P_n be a path on $n \geq 2$ vertices. Then*

$$CN^*[T_2(P_n); z] = (n-2)z^3.$$

Proof. Let $V(T_2(P_n)) = A \cup B$, where $A = \{v_1, v_2, ..., v_n\} = V(P_n)$ and $B = \{u_1, u_2, ..., u_{n-1}\}$ be the vertices of $T_2(P_n)$ corresponding to the edges of P_n. Then for $1 \leq i \leq n-1$, $\langle \{v-i, u_i, v_{i+1}\} \rangle$ is a triangle in $T_2(P_n)$ and every vertex of $T_2(P_n)$ is in at least one triangle. Also since u_i, u_j, $1 \leq i, j \leq n-1$, $i \neq j$ are non adjacent and $N(u_i) = \{v_i, v_i + 1\}$ for $1 \leq i \leq n-1$ there are no cycles of length greater than three. □

Theorem 3.3.19. $CN^*[T_2(K_{n-1,1}); z] = (n-2)z^3$*, where* $n \geq 3$.

Proof. Let $V(T_2(K_{n-1,1})) = A \cup B$, where $A = \{v_1, v_2, ..., v_n\} = V(K_{n-1,1})$ with v_n as the central vertex and $B = \{u_1, u_2, ..., u_{n-1}\}$ be the vertices of $T_2(P_n)$ corresponding to the edges of $K_{n-1,1}$ such that u_i is incident with v_i and v_n of $K_{n-1,1}$ for $1 \leq i \leq n-1$. Then for $1 \leq i \leq n-1$, $\langle \{v_i, u_i, v_n\} \rangle$ is a triangle in $T_2(K_{n-1,1})$ and every vertex of $T_2(K_{n-1,1})$ is in at least one triangle. Also since u_i, u_j, $1 \leq i, j \leq n-1$, $i \neq j$ are non adjacent and $N(u_i) = \{v_i, v_n\}$ for $1 \leq i \leq n-1$ there are no cycles of length greater than three as in the case of $T_2(P_n)$. □

As in the case of middlle graph of P_n, the semi total (point) graph of P_n is also near bipartite. And from the proof of expression for $CN^*[T_2(K_{n-1,1});z]$, it is clear that $T_2(K_{n-1,1})$, is totally perfect. Also it is trivial from the expressions of $CN^*[T_2(P_n);z]$ and $CN^*[T_2(K_{n-1,1});z]$ that $T_2(P_n)$ and $T_2(K_{n-1,1})$ are cyn^*-equivalent.

The total graph $T(G)$ [53] of a graph G is a graph whose vertex set is $V(T(G)) = V(G) \cup E(G)$ and two distinct vertices x and y of $T(G)$ are adjacent if x and y are adjacent vertices of G or adjacent edges of G or x is a vertex incident with edge y. Now we find $CN^*[T(G);z]$ when $G \cong P_n$.

Theorem 3.3.20. *Let $T(P_n)$ be the total graph of path P_n, $n \geq 2$. Then,*

$$CN^*[T(P_n);z] = \Sigma_{k=3}^{2n-1}(2n-k)z^k.$$

Proof. Let $V(T(P_n)) = \{v, v_1, v_2, ..., v_n, u_1, u_2, ..., u_{n-1}\}$, where $\{v, v_1, v_2, ..., v_n\}$ $= V(T(P_n))$ and u_i, $1 \leq i \leq n-1$ are the vertices of $T(P_n)$ corresponding to the edges of P_n. Then for $2 \leq i \leq n-1$ and $2 \leq j \leq n-2$, $|N(v_i)| = |N(u_j)| = 4$, $|N(u_1)| = |N(u_{n-1}| = 3$ and $|N(v_1)| = |N(v_n)| = 2$. Let $3 \leq k \leq n$. When k is odd, for, $1 \leq i \leq n - \lfloor \frac{k}{2} \rfloor$, $v_i v_{i+1} v_{i+2} ... v_{i+\frac{k-1}{2}} u_{i+\frac{k-3}{2}} u_{i+\frac{k-5}{2}} ... u_i v_i$ is a k-cycle and for for, $1 \leq i \leq n - \lceil \frac{k}{2} \rceil$, $u_i u_{i+1} u_{i+2} ... u_{i+\frac{k+1}{2}} v_{i+\frac{k+1}{2}} v_{i+\frac{k-1}{2}} ... v_{i+1} u_i$ is a k-cycle. And when k is even, for, $1 \leq i \leq n - \frac{k}{2}$, $v_i v_{i+1} v_{i+2} ... v_{i+\frac{k}{2}} u_{i+\frac{k}{2}} u_{i+\frac{k-2}{2}} ... u_i v_i$ and $u_i u_{i+1} u_{i+2} ... u_{i+\frac{k-2}{2}} v_{i+\frac{k}{2}} v_{i+\frac{k-2}{2}} ... v_i u_i$ is a k-cycle in $T(P_n)$. Hence in both cases, there are $n - \frac{k-1}{2} + n - \frac{k+1}{2} = n - \frac{k}{2} + n - \frac{k}{2} = 2n - k$ k-cycles in in $T(P_n)$. Hence the proof. □

It is obvious from the expression for $CN^*[T(P_n);z]$ that total graph of path P_n, is pancyclic for $n \geq 2$.

Derived graph of a simple graph G denoted by $G^\dagger$ was introduced by Jog et al in their paper [51]. For a simple graph $G(V,E)$, its derived graph $G^\dagger$ is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if their distance in G is two.

It is clear from the definition of derived graph $G^\dagger$ of G that for a path P_n, $n \geq 2$, $P_2^\dagger \cong \overline{K_2}$, $P_n^\dagger \cong P_{\lfloor \frac{n}{2} \rfloor} \cup P_{\lceil \frac{n}{2} \rceil}$, when n is odd and $P_n^\dagger \cong P_{\frac{n}{2}} \cup P_{\frac{n}{2}}$ when n is even. Hence we have;

Proposition 3.3.21. *Let $P_n^\dagger$ be the derived graph of path P_n, $n \geq 2$. Then,*

$$CN^*[P_n^\dagger; z] = \begin{cases} 2, & \text{if } n = 2; \\ z^2 + 2z + 1, & \text{if } n = 3; \\ (n-2)z^2 + nz, & \text{if } n \geq 3; \end{cases}$$

Similarly for a cycle C_n, $n \geq 3$, $C_3^\dagger \cong \overline{K_3}$, $C_4^\dagger \cong P_2 \cup P_2$, $C_n^\dagger \cong C_n$ if n is odd and $n \geq 5$ and $C_n^\dagger \cong C_{\frac{n}{2}} \cup C_{\frac{n}{2}}$ if n is even and $n \geq 6$ we have the result;

Proposition 3.3.22. *Let $C_n^\dagger$ be the derived graph of cycle C_n, $n \geq 3$. Then,*

$$CN^*[C_n^\dagger; z] = \begin{cases} 3, & \text{if } n = 3; \\ 2z^2 + 4z + 1, & \text{if } n = 4; \\ z^n, & \text{if } n \geq 5 \text{ and } n \text{ is odd}; \\ 2z^{\frac{n}{2}}, & \text{if } n \geq 6 \text{ and } n \text{ is even}; \end{cases}$$

$CN^*[C_n^\dagger; z] = 2z^{\frac{n}{2}}$ for $n \geq 6$ and n is even, implies that $C_n^\dagger$ is disconnected, otherwise the two cycles in $C_n^\dagger$ will have a vertex in common and hence n cannot be even.

Conclusion

In this chapter an improvised version of cycle neighbor polynomial of a graph is introduced, motivated from the interpretation of simple cycles of lengths one and two in a graph. A comparitive study of cycle neighbor polynomials and modified cycle neighbor polynomials of graphs based on the completeness property of graph polynomials we arrive at the conclusion that, modified cycle neighbor polynomial is stronger than cycle neighbor polynomial of the graph. Also cycle neighbor polynomial of graph operations establishes that many properties of the resulting graph like, bipartite propetrty of subdivision graph of a simple graph, pancyclicity of square graph of P_n, bipartite propetrty of splitting graphs of paths and stars, near bipartite property of semi total line graph of P_n and semi total point graph of P_n, pancyclicity of total graph of P_n, hamiltonicity and in particular pancyclicity of k^{th} power G^k of graphs of diameter k for $k = 2, 3, 4, ...,$ middle graph of stars are split graph etc., can be directly observed using the tool of cycle neighbor polynomial.

Chapter 4

Hausdorff Domination

4.1 Introduction

[1] Different types of dominating sets are studied in the literature of graph theory by imposing different conditions on the dominating sets. One such domination called Hausdorff domination is introduced in this chapter.

A simple graph G is said to be a Hausdorff Graph [48], if for any two distinct vertices u and v of G, one of the following conditions holds

1. At least one of u and v is isolated.
2. There exists two nonadjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v.

Hausdorff domination is defined by imposing the Hausdorff graph property on

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the subgraph induced by the dominating set. An interesting property of such a dominating set is that every independent dominating set is Hausdorff dominating and every non independent Hausdorff dominating set contains a cycle. When a graph contains a Hausdorff dominating set which is not independent dominating and the graph induced by that dominating set does not contain any isolated vertices, then every vertex in that dominating set will be dominated by at least two vertices other than itself. Hence such a Hausdorff dominating set can be considered as strong total dominating set, since every vertex in a total dominating set [10] is dominated by at least one vertex other than that vertex.

4.2 Hausdorff domination

In this section, Hausdorff domination is introduced and a characterization property for a dominating set to be Hausdorff dominating is obtained.

Definition 4.2.1. *A dominating set $D \subseteq V$ is said to be Hausdorff dominating, if $\langle D \rangle$ is Hausdorff.*

Minimum cardinality of a Hausdorff dominating set is called the Hausdorff domination number and is denoted by $\gamma_H(G)$. Such a Hausdorff dominating set with cardinality $\gamma_H(G)$ is referred to as a γ_H -set.

For any graph G, $\gamma(G) \leq \gamma_H(G)$

Theorem 4.2.2. *Let $G = (V, E)$ be any graph. A dominating set $D \subseteq V$ is a Hausdorff dominating set if and only if one of the following statements hold.*

1. *$\langle D \rangle$ is an empty graph*

2. *If* $\langle D\rangle$ *is triangle free and if* $v \in D$ *is not an isolated vertex in* $\langle D\rangle$*, then the degree* $d_{\langle D\rangle}(v)$ *is greater than or equal to 2.*

3. *If* $\langle D\rangle$ *contains* K_3 *as a proper subgraph, then* $d_{\langle D\rangle}(v) \geq 3$ *for at least two vertices of* K_3 *and for all other vertices which are non isolated in* $\langle D\rangle$ *have degree* ≥ 2*.*

Proof. Assume that $D \subseteq V$ is a Hausdorff dominating set of G. If for any two distinct vertices u and v of $\langle D\rangle$, both u and v are isolated, then $\langle D\rangle$ is an empty graph hence there is nothing to prove.

Suppose that $\langle D\rangle$ contains at least one nontrivial connected component. Such components cannot have a vertex of degree one, since then $\langle D\rangle$ cannot be Hausdorff. Hence for every vertex v in any nontrivial connected component of $\langle D\rangle$, $d_{\langle D\rangle}(v) \geq 2$.

If $\langle D\rangle$ contains K_3 and $d_{\langle D\rangle}(v) < 3$ for at least two vertices of K_3 then those vertices in pairs will not have two non adjacent edges incident with them. Hence $\langle D\rangle$ cannot be Hausdorff. On the other hand if $d_{\langle D\rangle}(v) = 2$ only for one vertex or $d_{\langle D\rangle}(v) \geq 3$ for all vertex in K_3 then there are nonadjacent edges incident with every pair of vertices in K_3. Hence in $\langle D\rangle$, for every non isolated vertex v, $d_{\langle D\rangle}(v) \geq 2$ and $d_{\langle D\rangle}(v) \geq 3$ for at least two vertices in every K_3 which is an induced subgraph of $\langle D\rangle$.

Conversely, assume that D is a dominating set for which one of the three stated conditions hold. Then it is proved that D is a Hausdorff dominating set. If $\langle D\rangle$ is the empty graph, then clearly D is a Hausdorff dominating set.

Suppose that (2) holds. Let (u, v) be a pair of distinct vertices in $\langle D\rangle$. If one of them is an isolated vertex or if u and v belong to different components of $\langle D\rangle$, then there is nothing to prove. If both of them are non isolated and belongs to the same component of $\langle D\rangle$, then there arise the following cases.

(i) u and v are adjacent. Then since $d_{\langle D\rangle}(u)$ and $d_{\langle D\rangle}(v)$ are greater than or equal 2, there exists u_1, v_1 in $\langle D\rangle$, such that u_1 is adjacent to u, v_1 is adjacent to v and the edges uu_1 and vv_1 are non adjacent. Here $u_1 \neq v_1$, otherwise $\{u, u_1(= v_1), v\}$ will form the vertices of K_3 in the triangle free graph $\langle D\rangle$.

(ii) u and v are non adjacent. Then they are joined by at least one path of length two or greater than two. If the u-v path is of length 2, there exists a vertex w such that uwv is a u-v path and since, $d_{\langle D\rangle}(u) \geq 2$, there exists a vertex $x \neq w$ adjacent to u. So that xu and wv are non adjacent edges incident with u and v respectively. If the length of the u-v path is greater than 2, then there exists at least two vertices $u_1 \neq v_1$ such that, $uu_1...v_1v$ is is a u-v path in $\langle D\rangle$ and u_1u and v_1v are non adjacent edges incident with u and v respectively.

Now let (3) hold. Consider two adjacent vertices u, v in $\langle D\rangle$. If $\{u, v\}$ does not belongs to the vertex set of any K_3 in $\langle D\rangle$ then by the above reasoning, non adjacent edges incident with u and v can be found. Otherwise, there exists $w \in D$ such that $\langle\{u, v, w\}\rangle$ is K_3. Then either $d_{\langle D\rangle}(u) \geq 3$ or $d_{\langle D\rangle}(v) \geq 3$ or both $d_{\langle D\rangle}(u)$ and $d_{\langle D\rangle}(v) \geq 3$. Without loss of generality assume that $d_{\langle D\rangle}(u) =$ 2 or 3 and $d_{\langle D\rangle}(v) \geq 3$ then $\exists$ a vertex x different from u and w in D adjacent to v in $\langle D\rangle$. Thus in this case the edges e_1 and e_2 are non adjacent, where $e_1 = wu$ is incident with u and $e_2 = xv$ is incident with v. Hence $\langle D\rangle$ is Hausdorff. $\square$

Corollary 4.2.3 follows directly from Theorem 4.2.2.

Corollary 4.2.3. *If $D \subseteq V$ is a Hausdorff dominating set of a graph $G(V, E)$ then $\langle D \rangle$ has no vertices of degree one. In other words, $\langle D \rangle$ is free of pendant vertices.*

Theorem 4.2.4. *For any graph G on n vertices $\gamma_H(G) = 1$ if and only if*

$$\triangle(G) = n - 1$$

Proof. If $\gamma_H(G) = 1$, then there exists a vertex v of G which is adjacent to all other vertices of G. Therefore $d(v) = n - 1$ and hence $\triangle(G) = n - 1$

Conversely, if $\triangle(G) = n - 1$, then G has a vertex v which dominate every vertex of G and $\langle v \rangle$ is Hausdorff. Hence $\gamma_H(G) = 1$ □

Since all graphs G mentioned in Corollary 4.2.5 have $\triangle(G) = n-1$, it follows immediately from Theorem 4.2.4

Corollary 4.2.5.
1. *For any complete graph K_n, $\gamma_H(K_n) = 1, \forall n \geqslant 1$*
2. *For any star graph $K_{1,n}$, $\gamma_H(K_{1,n}) = 1, n \geqslant 1$*
3. *For any wheel graph $W_{n+1} = C_n + K_1, \gamma_H(W_{n+1}) = 1, n \geqslant 3$*

4.3 Hausdorff domination and independent domination

In this section we see an inclusion relation between independent dominating sets and Hausdorff dominating sets.

Theorem 4.3.1. *Every independent dominating set is Hausdorff dominating*

Proof. Let $G = (V, E)$ be any graph, let $D \subseteq V$ be an independent dominating set of G. Then $\langle D \rangle$ is the empty graph. Hence it follows from Theorem 4.2.2 that $\langle D \rangle$ is Hausdorff. □

Corollary 4.3.2. *For any graph G, $\gamma_H(G) \leq i(G)$*

The inequality in Corollary 4.3.2 may be strict. It will be proved in Theorem 4.3.6 that $\gamma_H(K_{m,n}) = 4$ if $m \geq 4$ and $n \geq 4$. Therefore, for all complete bipartite graphs $K_{m,n}$ with $m \geq 5$ and $n \geq 5$ $\gamma_H(K_{m,n}) < i(K_{m,n})$, since $i(K_{m,n}) = min\{m, n\}$.

By considering the Hausdorff domination number $\gamma_H(G)$, the domination chain can be extended as follows;

Proposition 4.3.3. *The domination chain $\gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$ [7] can be extended as $\gamma(G) \leq \gamma_H(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$.*

Remark 4.3.4. *The converse of the conclusion of Theorem 4.3.1 need not be true. For example in Figure 4.1, $\{a, b, c, d\}$ is both independent and Hausdorff dominating while $\{e, f, g, h\}$ is Hausdorff dominating but not independent.*

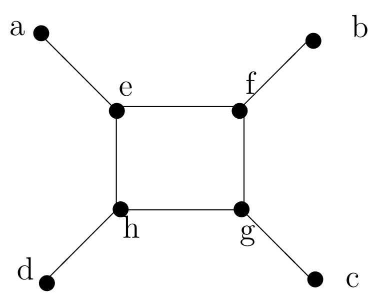


Figure 4.1

Theorem 4.3.5. *1. $\gamma_H(P_n) = \lceil \frac{n}{3} \rceil$, for any path P_n on n vertices.*

2. $\gamma_H(C_n) = \lceil \frac{n}{3} \rceil$*, for any cycle* C_n *on* n *vertices.*

Proof. For any path P_n on n vertices, a dominating set D can be Hausdorff dominating if and only if $\langle D \rangle$ is an empty graph. Otherwise, $\langle D \rangle$ will contain two or more pendant vertices and hence by corollary 4.2.3, it cannot be Hausdorff. Therefore $\lceil \frac{n}{3} \rceil = \gamma(P_n) \leq \gamma_H(P_n) \leq i(P_n) = \lceil \frac{n}{3} \rceil$

For any cycle C_n on $n \geq 4$ vertices the set of all vertices constitute a Hausdorff dominating set. By the same reasoning as in the case of paths P_n, any dominating set D of C_n of cardinality $< n$, will be Hausdorff dominating if and only if $\langle D \rangle$ is an empty graph. Hence $\lceil \frac{n}{3} \rceil = \gamma(C_n) \leq \gamma_H(C_n) \leq i(C_n) = \lceil \frac{n}{3} \rceil$ □

Theorem 4.3.6. *For any complete bipartite graph* $K_{m,n}$*,*

$$\gamma_H(K_{m,n}) = \begin{cases} 1, & \text{if either } m \text{ or } n = 1; \\ 2, & \text{if } m \geq 2,\ n \geq 2 \text{ and at least one of m or n is } 2; \\ 3, & \text{if } m \geq 3,\ n \geq 3 \text{ and at least one of m or n is } 3; \\ 4, & \text{if } m \geq 4 \text{ and } n \geq 4. \end{cases}$$

Proof. $\gamma_H(K_{m,n}) = 1$ if either m or n $= 1$ is a particular case of Theorem 4.2.4.

Since $\gamma(K_{m,n}) = 2$ for $m \geq 2$ $n \geq 2$ and $i(K_{m,n}) = 2$ if $m \geq 2$, $n \geq 2$ and at least one of m or n is 2 then since $\gamma(K_{m,n}) \leq \gamma_H(K_{m,n}) \leq i(K_{m,n})$, $\gamma_H(K_{m,n}) = 2$ if $m \geq 2$, $n \geq 2$ and at least one of m or n is 2

When $m \geq 3$, $n \geq 3$ and at least one of m or n is 3 then since $K_{m,n}$ does not have a vertex of degree $m + n - 1$, by Theorem 4.2.4, $\gamma_H(K_{m,n}) \neq 1$. Suppose if possible, D is a γ_H-set of $K_{m,n}$ of cardinality 2 then either $\langle D \rangle$ is K_2 or an empty graph on 2 vertices. In the first case $\langle D \rangle$ is not Hausdorff and in the second case,

D is not dominating. Hence $\gamma_H(K_{m,n}) = 3 = min\{m, n\}$ if $m \geq 3$, $n \geq 3$ and at least one of m or n is 3.

If $m \geq 4$ and $n \geq 4$, then four vertices, two each from the bipartite sets will form a Hausdorff dominating set. So $\gamma_H(K_{m,n}) \leq 4$. Suppose, if possible, $\gamma_H(K_{m,n}) < 4$. Then by the above reasoning, $\gamma_H(K_{m,n})$ cannot be 1 or 2. If D is a γ_H-set of cardinality 3, then either $\langle D \rangle$ is P_3 or $K_2 \cup K_1$ or an empty graph on three vertices. If $\langle D \rangle$ is P_3 or $K_2 \cup K_1$ then it is not Hausdorff and D is not dominating if $\langle D \rangle$ an empty graph on three vertices. Hence $\gamma_H(K_{m,n})$ cannot be three. □

Theorem 4.3.7. *The graph induced by a Hausdorff dominating set which is not independent, contains a cycle C_m, $m \geq 4$.*

Proof. Let $D \subseteq V$ be any Hausdorff dominating set. Suppose that D is a non independent dominating set, then $\langle D \rangle$ is not an empty graph. Let $v \in D$. If v is a non isolated vertex in $\langle D \rangle$, it is a vertex of a connected component say H of $\langle D \rangle$. Since $\langle D \rangle$ is Hausdorff, the subgraph H also should be Hausdorff. Then by Theorem 4.2.2, $d_H(v) \geq 2$, $\forall v \in V(H)$. So that H cannot be a tree. Hence H is not acyclic and contains a cycle C_m for $m \geq 3$. Now if $d_H(v) = 2$ $\forall v \in V(H)$, then H is a cycle C_m with m vertices. and since H is Hausdorff, $m \geq 4$. If H contains K_3, by Theorem 4.2.2, $d_H(v) \geq 3$ for at least two vertices of K_3. Let u_1 and u_2 be the vertices adjacent to the vertices of K_3 of degree > 2 in H. Consider the following cases.

Case 1: $u_1 = u_2$, then u_1 together with the vertices of K_3 will form a cycle of length 4

Case 2: $u_1 \neq u_2$, and if u_1 and u_2 are adjacent. In this case, two internally disjoint paths can be found from u_1 to u_2, one along the vertices of K_3 and the other along the edge u_1u_2. Adjoining these two paths from u_1 to u_2 a cycle of length 5 will be obtained.

Case 3: $u_1 \neq u_2$, and u_1 and u_2 are not adjacent in H. Since $d_H(u_1)$ and $d_H(u_2)$ are greater than or equal to 2, if at least one of u_1 or u_2 is adjacent to the third vertex of K_3 under consideration, then there is a cycle of length 4 in H. If u_1 and u_2 are joined by a path not along the vertices of K_3 then also a cycle of length greater 4 can be obtained by adjoining these two internally disjoint u_1-u_2 paths.

Case 4: u_1 and u_2 are not connected through any path other than that along the vertices of K_3. In this case, suppose if possible the other end blocks in the direction opposite to that of K_3 from u_1 and u_2 do not contain any cycle of length greater than or equal to 4. Then these blocks are either a triangle or a pendant edge. In both cases H cannot be Hausdorff. Hence both these blocks should contain a cycle of length greater than or equal to 4.

Now let H be triangle free. Let $u, v \in V(H)$. As H is a connected Hausdorff graph of order > 2, every vertex in H has degree ≥ 2 and in fact the order of H is ≥ 4. Let e be any edge in H with end points say v_1 and v_2, which is not a cut edge of H. Since H is not a tree such an edge will exist. As H is Hausdorff and $d(v) \geq 2$ for all $v \in H$, a path from v_1 to v_2 not through e can be found. Then since H is triangle free this path together with e will form a cycle of length ≥ 4. Hence the proof. □

Since trees are acyclic, Corollary 4.3.8 follows from Theorem 4.3.7

Corollary 4.3.8. *For any tree T, the Hausdorff dominating set and independent dominating set are the same. Hence $\gamma_H(T) = i(T)$, for any tree T.*

Definition 4.3.9. *Let $G(V, E)$ be a graph. A set $S \subseteq V(G)$ is called a cycle neighbor set of G, if for any two vertices u and v in S, there is a cycle in G, which contains both u and v.*

There are at least two internally disjoint paths joining any two vertices of a cycle neighbor set in G.

From the definition of cycle neighbor sets it is clear that for any cycle neighbor set C of a graph G, cardinality of C is either zero or greater than or equal to three.

Remark 4.3.10. *In general the Hausdorff dominating set of a graph G, which is not independent is either a cycle neighbor set or a union of cycle neighbor sets in G.*

Theorem 4.3.11. *For any graph G of order $n \geq 2$, $3 \leq \gamma_H(G) + \gamma_H(\overline{G}) \leq n+1$*

Proof. Let G be any graph of order $n \geq 2$. If $\gamma_H(G) = 1$, then by Theorem 4.2.4, there exists a vertex v of degree $n-1$ in G. Hence v is an isolated vertex in $\overline{G}$. Hence $\gamma_H(\overline{G}) \geq 2$. Similarly if $\gamma_H(\overline{G}) = 1$, then $\gamma_H(G) \geq 2$. In this case, $\gamma_H(G) + \gamma_H(\overline{G}) \geq 3$ Also the lower bound is obvious when $\gamma_H(G) \geq 2$.

Now an upper bound is obtained by proceeding as follows. Since

$i(G) \leq n - \Delta(G)$, and since $\gamma_H(G) \leq i(G)$,

$$\gamma_H(G) + \gamma_H(\overline{G}) \leq i(G) + i(\overline{G})$$

$$\begin{aligned}
&\leq n - \Delta(G) + n - \Delta(\overline{G}) \\
&= 2n - [\Delta(G) + \Delta(\overline{G})] \\
&\leq 2n - [\Delta(G) + \delta(\overline{G})] \\
&= 2n - (n-1) \\
&= n + 1
\end{aligned}$$

Therefore, $3 \leq \gamma_H(G) + \gamma_H(\overline{G}) \leq n + 1$. □

Remark 4.3.12. *The bounds are sharp. For the graph* $G = K_{1,n-1}$, $n \geq 5$ $\gamma_H(G) = 1$, $\gamma_H(\overline{G}) = 2$, *and for* $G = K_n, \gamma_H(G) = 1, \gamma_H(\overline{G}) = n$ *we get* $\gamma_H(G) + \gamma_H(\overline{G}) = n + 1$

In Theorem 4.3.13, we will prove that, for all connected triangle free graphs with at least two vertices, the Hausdorff domination number of its complement will be two.

Theorem 4.3.13. *If* G *is a connected triangle free graph of order greater than or equal to two, then* $\gamma_H(\overline{G}) = 2$

Proof. Since G is a connected graph of order ≥ 2, it contains an edge say uv. If order of G is two, then G is isomorphic to K_2 and $\overline{G}$ is isomorphic to an empty graph on two vertices. Therefore, $\gamma_H(\overline{G}) = 2$. If order of G greater than two, then no vertex of G is adjacent to both u and v, because G is triangle free. Therefore every vertex in G which are adjacent to u are dominated by v in $\overline{G}$ and those vertices adjacent to v in G are dominated by u in $\overline{G}$ and all vertices which are non adjacent to both u and v are dominated by both u and v in $\overline{G}$. So $\{u, v\}$ forms an independent dominating set of $\overline{G}$. Therefore it is also a Hausdorff dominating set of $\overline{G}$. Hence $\gamma_H(\overline{G}) \leq 2$. Now let if possible $\gamma_H(\overline{G}) = 1$, then G would have an isolated vertex, a contradiction. Which proves $\gamma_H(\overline{G}) = 2$ □

Theorem 4.3.14. *For any graph G on n vertices $\gamma_H(G) = n$ if and only if $G = \overline{K_n}$*

Proof. Let $\gamma_H(G) = n$. Then for any γ_H-set D, $|D| = n$. ie., every vertex of G belongs to every γ_H-set. Hence $\langle D \rangle = G$. By Corollary 4.3.2, $\gamma_H(G) \leq i(G)$. Also since $O(G) = n$, $i(G) = n$. Hence $G = \overline{K_n}$, an empty graph on n vertices.

The converse is obvious. □

Theorem 4.3.15. *If $i(G) \leq 4$ for a graph G, then $\gamma_H(G) = i(G)$. Moreover, if $i(G) \leq 3$, then every γ_H-set is an i-set and viceversa.*

Proof. By Corollary 4.3.2, whatever be $i(G)$, $\gamma_H(G) \leq i(G)$.

Consider the following cases.

Case (i) $i(G) = 1$, then clearly $\gamma_H(G) = 1$ and every i-set will form a γ_H-set and viceversa.

Case (ii) $i(G) = 2$. Then $\gamma_H(G) = 2$. Otherwise $\gamma_H(G) = 1$. In this case, a singleton subset of V dominates all the vertices of G. Therefore, $i(G) = 1$, a contradiction. As every γ_H-set of cardinality 2 is independent, every γ_H-set is also an i-set.

Case (iii) $i(G) = 3$. Then $\gamma_H(G)$ can't be 1 or 2, as in these cases, every γ_H-set is also an independent dominating set of cardinality less than $i(G)$. Therefore, $\gamma_H(G) = 3$. Now let D be a γ_H-set of cardinality 3 such that $\langle D \rangle$ is not independent. Then $\langle D \rangle$ is either K_3 or $K_1 \cup K_2$. In both cases $\langle D \rangle$ cannot be Hausdorff. Hence $\langle D \rangle$ must be independent, implying that i-set and γ_H-set are the same when $i(G)$=3.

Case (iv) $i(G) = 4$. Then $\gamma_H(G) = 1$ or $\gamma_H(G) = 2$ or $\gamma_H(G) = 3$ or $\gamma_H(G) = 4$. By cases (i), (ii) and (iii), $\gamma_H(G) = 1$ or 2 or 3 will imply that every γ_H-set is independent dominating contradicting $i(G) = 4$. Hence $\gamma_H(G)$ cannot be less than 4 when $i(G)$=4. Hence $\gamma_H(G) = i(G) = 4$. □

Remark 4.3.16. *It follows from Theorem 4.3.15, for a graph G, $i(G) = 4$ implies $\gamma_H(G) = 4$. But in this case, there may exist γ_H-set which is different from an i-set.*

For example, the graph in Figure 4.1 has $A \subset V$, $B \subset V$, where, $A = \{a, b, c, d\}$ forms an i-set which is also Hausdorff dominating. But $B = \{e, f, g, h\}$ is a γ_H-set but not independent even though $|A| = |B|$

Remark 4.3.17. *The conclusion of Theorem 4.3.15 need not be true for $i(G) \geq 5$. The graph in Figure 4.2 has $\gamma_H(G) = 4 < i(G) = 6$*

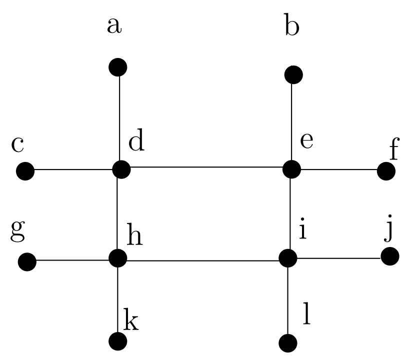


Figure 4.2

4.4 Connected Hausdorff domination

In this section the additional property of connectedness is imposed on Hausdorff dominating sets and some properties of the resulting dominating sets are studied.

Definition 4.4.1. *A dominating set $D \subseteq V$ is called a connected Hausdorff dominating set, if $\langle D \rangle$ is both connected and Hausdorff. The minimum cardinality of a connected Hausdorff dominating set is denoted by $\gamma_{cH}(G)$, any connected Hausdorff dominating set with cardinality $\gamma_{cH}(G)$ is called a γ_{cH}-set and $\gamma_{cH}(G)$ is called the connected Hausdorff domination number of G.*

Proposition 4.4.2. *For any graph G, $\gamma_H(G) \leq \gamma_{cH}(G)$*

Proof. Since every connected Hausdorff dominating set is Hausdorff dominating, it follows that $\gamma_H(G) \leq \gamma_{cH}(G)$. □

Note that no subgraph of a tree contains a cycle. From Theorem 4.3.7, we know that every non independent Hausdorff dominating set contains a cycle of length greater than or equal to four. Therefore a tree T can have a connected Hausdorff dominating set if and only if $\gamma_{cH}(T) = 1$. Hence we have Proposition 4.4.3

Proposition 4.4.3. *No tree other than the star graph has a connected Hausdorff dominating set.*

Observation 4.4.4.

1. *For any cycle C_n, $n \geq 4$ $\gamma_{cH}(C_n) = n$ and $\gamma_H(C_n) = \lceil n/3 \rceil$. Hence $\gamma_H(C_n) < \gamma_{cH}(C_n)$.*

2. *If a graph G has a spanning cycle $C_n, n \geq 4$, it contains a connected Hausdorff dominating set. In particular, Every Hamiltonian graph with more*

than four vertices contains a connected Hausdorff dominating set since in this case, $V(G)$ itself is a connected Hausdorff dominating set. But the condition is not sufficient, that is, the existence of a connected Hausdorff dominating set in a graph G need not imply that G is Hamiltonian. For example, the graph G in Figure 4.3 has a connected Hausdorff dominating set. But G is not Hamiltonian.

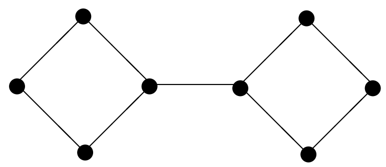


Figure 4.3

Theorem 4.4.5. *For any nontrivial connected Hausdorff dominating set D of any graph G there exists a subset V_1 of D such that $\langle V_1 \rangle$ contains a cycle C_m, on m vertices for $m \geqslant 4$*

Proof. Proof follows directly from Theorem 4.3.7 □

Corollary 4.4.6. *For any nontrivial connected Hausdorff dominating set $D \subseteq V$ of G, $|D| \geq 4$. In particular $\gamma_{cH}(G) \geq 4$*

Corollary 4.4.7. *If G is triangle free and has a nontrivial connected Hausdorff dominating set then the girth $g(G) \geq 4$.*

Remark 4.4.8. *For the wheel graph $G = W_{n+1} = K_1 + C_n$, $n \geq 4$, the vertex v of degree n is a Hausdorff dominating set. Also the graph induced by the vertices of C_n is connected Hausdorff, for $n \geq 4$. For this graph, both $D = \{v\}$ and $V - D$ are connected Hausdorff dominating sets and $\gamma_H(G) = \gamma_{cH}(G) = 1$*

Theorem 4.4.9. *Unicyclic graphs can have a connected Hausdorff dominating set if and only if $G \cong C_m$ for $m \geq 4$ or G contains a cycle C_m for $m \geq 4$ with one or more pendant vertices adjacent to all or some of the vertices of C_m*

Proof. If G is any of the two types of graphs as mentioned in the statement, then the vertices of C_m, $m \geq 4$ forms a connected Hausdorff dominating set of G with $\gamma_{cH}(G) = m$

Conversely, let G be any uni-cyclic graph with a connected Hausdorff dominating set D. Then by Theorem 4.4.7, $\langle D \rangle$ contains at least one cycle C_m where $m \geq 4$. ie., the unique cycle of G is in fact contained in the graph induced by every connected Hausdorff dominating set. If $G = C_m$, then there is nothing to prove. On the other hand, let $v \in V - V(C_m)$. Since G has a connected Hausdorff dominating set, G itself is connected. Therefore there exists a path from v to every vertex of C_m. It is claimed that there do not exist two internally disjoint paths from v to the vertices of C_m. Otherwise G contains more than one cycle. Hence there exists exactly one vertex u in C_m and exactly one path from v to u such that $d(u, v)$ is minimum, where $u \in V(C_m)$

In fact v is a pendant vertex of G, which is adjacent to a vertex of C_m. If v is not a pendant vertex,then a pendant vertex say u in $V(G)$ and a unique path containing v, joining u to the nearest vertex say w in C_m can be found. Since G has a connected Hausdorff dominating set D, in order to dominate all the vertices in this u-w path they should belong to D. Hence either the pendant vertex u or a support vertex of u belong to $D \Rightarrow \langle D \rangle$ is not Hausdorff. Now if v is a pendant vertex, but it is not adjacent to any vertex of C_m, then either v is dominated by a support vertex u, where $u \in D$ or $v \in D$. In both cases $\langle D \rangle$

contains a pendant vertex and hence is not Hausdorff. Hence if $v \in V - V(C_m)$ then it should be adjacent to a vertex of C_m □

Theorem 4.4.10. *If D is a γ_{cH}-set of a connected graph G, then both endpoints of every cut edge of G belongs to D.*

Proof. Suppose, if possible, any or both endpoints of a cut edge do not belong to the γ_{cH}-set D. Then the graph induced by D is disconnected, a contradiction to D is a γ_{cH}-set of G. □

4.5 Relation between Hausdorff dominating sets and other domination parameters

From the very definition, Every nontrivial connected Hausdorff dominating set is connected dominating and total dominating. Therefore, $\gamma_c(G) \leq \gamma_{cH}(G)$ and $\gamma_t(G) \leq \gamma_{cH}(G)$

Since every cycle of length greater than three is Hausdorff [48], all cycle dominating sets [36] of cardinality greater than three is Hausdorff dominating. Thus we have;

Proposition 4.5.1. *Every cycle dominating set D with $|D| \geq 4$ is Hausdorff dominating and connected Hausdorff dominating. In particular $\gamma_{cy}(G) = \gamma_{cH}(G)$.*

But a Hausdorff dominating set containing a cycle need not be cyle dominating. Figure 4.4 illustrates this. In this figure, the set of all vertices on the 6-cycle together with the pendant vertices which are at a distance 2 from the nearest vertex of the 6-cycle forms a Hausdorff dominating set but it is not cycle dominating. The graph G in Figure 4.4 has $\gamma(G) = 7$, $\gamma_H(G) = 9$, and $i(G) = 11$. But G does not have a cycle domiating set or a connected Hausdorff dominating set. Hence it follows that if a graph G has a cycle dominating set of cardinaloty greater than three, then the same will be a connected Hausdorff dominating set of G.

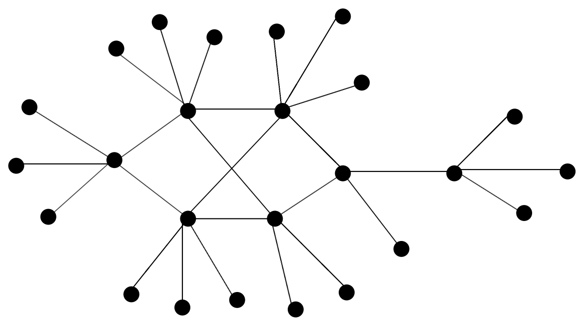


Figure 4.4

Theorem 4.5.2. *If G is the corona $C_m o K_1$, $m \geq 4$ then $i(G) = \gamma_H(G) = \gamma_t(G) = \gamma_c(G) = \gamma_{cy}(G) = \gamma_{cH}(G) = m$*

Proof. In $C_m o K_1$, $m \geq 4$ the pendant vertices will form a γ_H-set. This set is also independent dominating. Since, $\gamma_H(G) \leq i(G)$ by Corollary 2.3.2, $i(G) = \gamma_H(G) = m$.

Clearly vertices of C_m, $m \geq 3$ will form a total dominating, connected dominating,cycle dominating and connected Hausdorff dominating set. So $\gamma_t(G) = \gamma_c(G) = \gamma_{cy}(G) = \gamma_{cH}(G) = m$ □

Theorem 4.5.3. *Every clique dominating set [37] of a graph G with clique domination number $\gamma_{cl}(G) \geq 4$ is a connected Hausdorff dominating set.*

Proof. Since every complete graph K_n is Hausdorff for $n \geq 4$, every dominating clique is a connected Hausdorff dominating set. $\square$

Corollary 4.5.4. *Let G be a graph having a dominating clique. If $\gamma_{cl}(G) \geq 4$, then $\gamma_{cH}(G) \leq \gamma_{cl}(G)$*

Remark 4.5.5. *If G has a dominating clique and if $\gamma(G) \geq 2$, then $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq \gamma_{cl}(G)$ [28]*

Therefore if G has a dominating clique with $\gamma_{cl}(G) \geq 4$ and if $\gamma(G) \geq 2$, then the above domination chain can be extended as $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq \gamma_{cH}(G) \leq \gamma_{cl}(G)$. If $\gamma(G) = 1$, Then $\gamma(G) = \gamma_t(G) = \gamma_c(G) = \gamma_H(G) = \gamma_{cH}(G) = \gamma_{cl}(G)$

The conclusion of Corollary 4.5.4 need not hold if $\gamma_{cl}(G) < 4$. Figures 4.5 and 4.6 are examples of graphs for which $\gamma_{cl}(G) = 3 < \gamma_{cH}(G) = 4$ and $\gamma_{cl}(G) = 2 < \gamma_{cH}(G) = 4$ respectively.

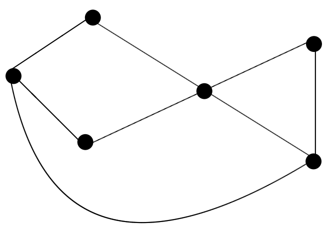


Figure 4.5

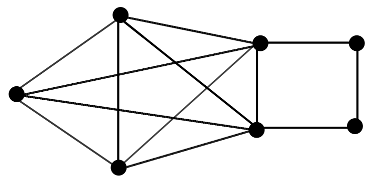


Figure 4.6

The corona $K_p o K_1$ has $\gamma = \gamma_t = \gamma_c = \gamma_{cH} = \gamma_{cl} = p$ if $p \geq 4$

In 1989, Sampathkumar E. introduced the notion of global dominating set in his paper [45]. Theorem 4.5.6 gives a relation between this type of dominating set and γ_H-set.

Theorem 4.5.6. *If D is a γ_H-set of the graph G such that $\langle D \rangle$ contains an isolated vertex, and if $diam(G) \geq 5$, then D is a global dominating set.*

Proof. Consider any graph G with $diam(G) \geq 5$. Let D be a γ_H-set of G such that $\langle D \rangle$ contains at least one isolated vertex. It is asserted that that D is a global dominating set of G. As $diam(G) \geq 5$, D must contain more than one vertices otherwise $diam(G) = 2$. Since $\langle D \rangle$ contains an isolated vertex it will dominate all the vertices of D in $\overline{G}$. Now it is claimed that for every vertex $v \in V \setminus D$, $|N[v] \cap D| < |D|$. Otherwise, there exists a vertex $v \in V \setminus D$ such that $|N[v] \cap D| = |D|$. Then for any two vertices u_1, u_2 of G, there arise the following cases.

Case(i) If u_1, u_2 are in D, then $u_1 v u_2$ is a path connecting u_1 and u_2. Hence $d(u_1, u_2) \leq 2$

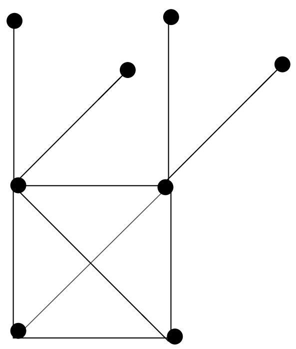
Case(ii) Let u_1, u_2 are in $V \setminus D$, then there exists $u_1^{'}$, $u_2^{'}$ in D such that u_1 is adjacent to $u_1^{'}$ and u_2 is adjacent to $u_2^{'}$. So that $u_1 u_1^{'} v u_2^{'} u_2$ is a path joining u_1 and u_2, when $v \neq u_1$ and $v \neq u_2$. If $v = u_1$ or $v = u_2$ then $u_1 = v u_2^{'} u_2$, $u_1 u_1^{'} v = u_2$ respectively form $u_1 - u_2$ paths and hence $d(u_1, u_2) \leq 4$

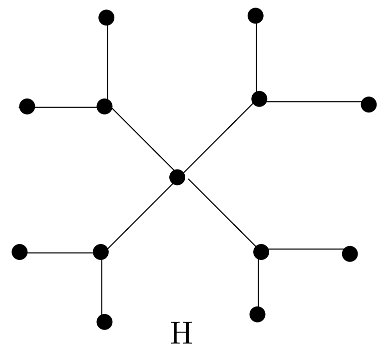
Case(iii) If $u_1 \in D$ and $u_2 \in V \setminus D$ and if $u_2^{'} \in D$ dominates u_2 then $u_1 v u_2^{'} u_2$ is a path joining u_1 and u_2. Therefore $d(u_1, u_2) \leq 3$

So that the distance between any pair of vertices is at most four, a contradiction to $diam(G) \geq 5$. Hence if $v \in V \setminus D$, then $|N[v] \cap D| < |D|$. So there exists a vertex u in D which is not in $N[v] \cap D$ and dominates v in $\overline{G}$. Thus D is a dominating set of $\overline{G}$. Hence the theorem. □

Corollary 4.5.7. *If D is a γ_H-set of G containing an isolated vertex, and if* *diam*$(G) \geq 5$, *then* $\gamma_g(G) \leq \gamma_H(G)$

Remark 4.5.8. *Theorem 4.5.6 need not hold for graphs with diameter* ≤ 4. *For example, complete graphs* $K_n, n \geq 2$ *has diameter* 1. $\gamma_H(K_n) = 1$ *and* $\gamma_g(K_n) = n$. *For* $K_{1,3}$, *diameter* $= 2$, $\gamma_H(K_{1,3}) = 1$ *and* $\gamma_g(K_{1,3}) = 2$ *In Figure 4.7, diameter of the graph* G *is* 3, $\gamma_H(G) = 3$ *and* $\gamma_g(G) = 4$ *and in Figure 4.8, diameter of* H *is* 4, $\gamma_H(H) = 4$ *and* $\gamma_g(H) = 5$


G
Figure 4.7


H
Figure 4.8

Conclusion

In this chapter, Hausdorff domination number and connected Hausdorff domination number are introduced. Necessary and sufficient condition for a dominating set to be Hausdorff dominating is obtained. The relation between Hausdorff domination and independent domination are discussed. Also an attempt is made to compare Hausdorff domination with other domination parameters. It is proved that whenever a Hausdorff dominating set $D \subseteq V$ of a graph G is not an independent dominating set, then it contains a cycle neighbor set of G. Hence whenever the graph induced by a Hausdorff dominating set does not contain any isolated vertices, every vertex in the Hausdorff dominating set will be dominated by at least two vertices other than itself.

Chapter 5

T_1 Domination and T_0 Domination

5.1 Introduction

[1] In this Chapter a generalisation of Hausdorff domination called T_1 domination and a generalisation of T_1 domination viz, T_0 domination are introduced. So that every Hausdorff dominating set is T_1 dominating and every T_1 dominating set is T_0 dominating.

A simple graph G is said to be T_1 [50], if for any two distinct vertices u and v of G, one of the following holds

[1]Part of this chapter has been published in International Journal of Research in Advent Technology, Vol. 7, No. 2, 120-125, 2019

1. At least one of u and v is isolated.

2. There exists edges e_1 and e_2 such that e_1 is incident with u but not with v and e_2 is incident with v but not with u.

A simple graph G is said to be T_0 [49], if for any two distinct vertices u and v of G, one of the following holds

1. At least one of u and v is isolated.

2. There exists an edges e such that either e is incident with u but not with v or e is incident with v but not with u.

5.2 T_1 domination

T_1 Domination is defined as follows.

Definition 5.2.1. *Let $G(V, E)$ be any graph. A dominating set $D \subseteq V$ is said to be T_1 dominating, if $\langle D\rangle$ is a T_1 graph. The minimum cardinality of all T_1 dominating sets is called the T_1 domination number and is denoted by $\gamma_{T_1}(G)$. A T_1 dominating set with cardinality $\gamma_{T_1}(G)$ is called a γ_{T_1} -set.*

Theorem 5.2.2 characterizes a T_1 dominating set.

Theorem 5.2.2. *Let $G = (V, E)$ be any simple graph. A dominating set $D \subseteq V$ is a T_1 dominating set if and only if for every vertex $u \in D$, $d_{\langle D\rangle}(u) \neq 1$.*

Proof. Let $D \subseteq V$ be T_1 dominating. Then for every pair u, v in D, either

(i) u or v or both are isolated in $\langle D\rangle$ or

(ii) there are edges $e_1 \neq e_2$ in $\langle D\rangle$ such that e_1 is incident with u but not with v and e_2 is incident with v but not with u.

If (i) holds for $u \in D$, then $d_{\langle D\rangle}(u) = 0$.

If u is not isolated in $\langle D\rangle$, then $d_{\langle D\rangle}(u) \geq 1$. Suppose if possible, $d_{\langle D\rangle}(u) = 1$. Then there is a vertex $v \in D$ which is adjacent to u in $\langle D\rangle$. Hence uv is the only edge which is incident with u, which is incident with v also, a contradiction to $\langle D\rangle$ is T_1.

Now let $d_{\langle D\rangle}(u) \geq 2$, for every $u \in D$ for which $d_{\langle D\rangle}(u) \neq 0$. Let u and v be any two vertices in D with $d_{\langle D\rangle}(u) \geq 2$ and $d_{\langle D\rangle}(v) \geq 2$. If u and v are non adjacent in $\langle D\rangle$, then clearly there are distinct, non adjacent edges incident with them. If u and v are adjacent in $\langle D\rangle$, then since, both $d_{\langle D\rangle}(u)$ and $d_{\langle D\rangle}(v)$ are greater than one, there are vertices w_1 and w_2 (or $w_1 = w_2 = w$) adjacent to u and v respectively. So that the edge uw_1 (or uw) is incident with u but not with v and the edge vw_2 (or vw) is incident with v but not with u. Hence for every vertex $u \in D$, $d_{\langle D\rangle}(u) \neq 1$.

Converse is obvious. □

Remark 5.2.3. *A dominating set $D \subseteq V$ of a graph $G(V, E)$ is T_1 dominating if and only if $\langle D\rangle$ has no pendant vertices.*

Corollary 5.2.4. *Every independent dominating set is T_1 dominating*

Proof. Let $G = (V, E)$ be any graph, let $D \subseteq V$ be an independent dominating set of G. Then $\langle D\rangle$ is the empty graph. Therefore, for every vertex $u \in D$, $d_{\langle D\rangle}(u) \neq 1$. Hence $\langle D\rangle$ is a T_1 graph. □

Corollary 5.2.5. *For any graph G, $\gamma_{T_1}(G) \leq i(G)$*

Remark 5.2.6. *The converse of the conclusion of Corollary 5.2.4 need not be true. The* T_1 *domination number of the graph* G *in F Figure 5.1 is 3 and the independent domination number of* G *is 5.*

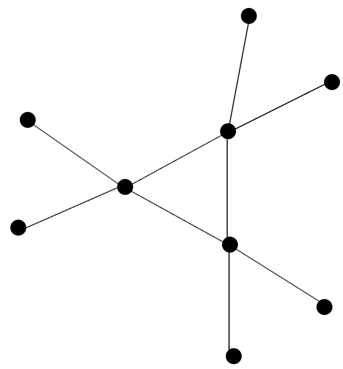


Figure 5.1

Theorem 5.2.7. *For any graph* $G(V, E)$*, the graph induced by every* T_1 *dominating set of* G*, which is not independent, contains a cycle of length greater than or equal to three.*

Proof. Let $D \subseteq V$ be a T_1 dominating set, which is not independent and let v be a non isolated vertex in $\langle D \rangle$. Then v is a vertex of a connected component H of $\langle D \rangle$. Since $\langle D \rangle$ is a T_1 graph, the component H is also T_1. So that H has no pendant vertices and hence it is not a tree. That is H is not acyclic and contains at least one cycle of length greater than or equal to three. □

Corollary 5.2.8. *Let* $G(V, E)$ *be any graph of order* $n \geq 3$ *and let* $D \subseteq V$ *be a* T_1 *dominaating set of* G*, which is not independent then* $|D| \geq 3$.

Corollary 5.2.9. *For any acyclic graph* $G(V, E)$*, the* T_1 *dominating set and independent dominating set are the same. Hence* $\gamma_{T_1}(G) = i(G)$.

Theorem 5.2.10. *For any graph* G *if* $i(G) \leq 3$*, then* $\gamma_{T_1}(G) = i(G)$*. Moreover if* $i(G) < 3$ *then every* γ_{T_1} *-set is an* i*-set and viceversa.*

Proof. Since every independent dominating set is a T_1 dominaating set and every non independent T_1 dominaating set contains at least three vertces, whenever $i(G) \leq 3$, we have $\gamma_{T_1}(G) = i(G)$ and for $i(G) < 3$, every γ_{T_1} -set is an i-set and viceversa. $\square$

Remark 5.2.11. *When $i(G) = 3$, every γ_{T_1} -set need not be an i-set. Figure 5.2 illustrates this.*

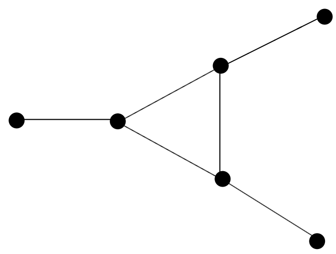


Figure 5.2

Since every T_1 dominating set is Hausdorff dominating, we have the following Proposition

Proposition 5.2.12. *Let $G = (V, E)$ be any graph on n vertices then,*

1. $\gamma_{T_1}(G) = 1$ *if and only if* $\triangle(G) = n - 1$
2. $\gamma_{T_1}(G) = 2$ *if and only if* $i(G) = 2$
3. $\gamma_{T_1}(G) = 3$ *if and only if* $i(G) = 3$
4. $\gamma_{T_1}(G) = n$ *if and only if* $G = \overline{K_n}$

5.3 T_0 domination

Using the definition of T_0 graph, T_0 domination is defined as follows.

Definition 5.3.1. *Let G be any graph with vertex set V. A dominating set $S \subseteq V$ is said to be T_0 dominating, if $\langle S \rangle$ is a T_0 graph. The minimum cardinality of all such T_0 dominating sets is called T_0 domination number and is denoted by $\gamma_{T_0}(G)$. Such a T_0 dominating set with cardinality $\gamma_{T_0}(G)$ is called a γ_{T_0} -set.*

Since every Hausdorff dominating set is T_1 dominating and every T_1 dominating set is T_0 dominating, for any graph G, $\gamma(G) \leq \gamma_{T_0}(G) \leq \gamma_{T_1}(G) \leq \gamma_H(G)$.

In [49] Seena V and Raji P proved that a graph G is T_0 if and only if K_2 is not a component of G.

A characterization of a T_0 dominating set follows directly from this result.

Proposition 5.3.2. *Let $G = (V, E)$ be any graph. A dominating set $S \subseteq V$ is a T_0 dominating set if and only if no component of $\langle S \rangle$ is K_2.*

Proposition 5.3.3. *For any graph G, every independent dominating set is T_0 dominating.*

Proof. Let $I \subseteq V$ be an independent dominating set of a graph $G = (V, E)$. Since K_2 is not a component of $\langle I \rangle$, $\langle I \rangle$ is a T_0 graph. □

Corollary 5.3.4. *For any graph G, $\gamma_{T_0}(G) \leq i(G)$.*

Remark 5.3.5. *The converse of the conclusion of Proposition 5.3.3 need not be true. For example the set of all darkened vertices shown in in Figure 5.3 is T_0 dominating but not independent. Here $\gamma_{T_0}(G)$=3 and $i(G)$=5.*

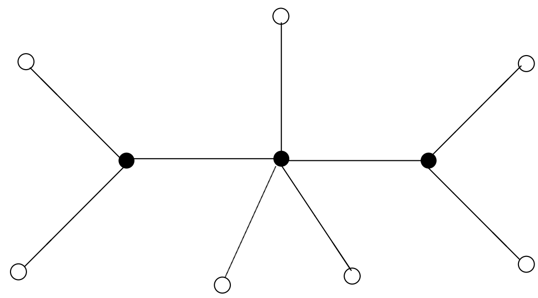


Figure 5.3 - G

Theorem 5.3.6. *For any positive integer k, there exist a graph G such that*

$$i(G) - \gamma_{T_0}(G) = k$$

Proof. Consider the path P_3. Let G be the graph obtained from P_3 by attaching exactly j pendant edges to each vertex of P_3, where $j \geqslant 2$. Then $\gamma_{T_0}(G) = 3$ and

$i(G) = 3 + (j - 1)$ when $j \geq 2$. Therefore, $i(G) - \gamma_{T_0}(G) = j - 1$. Since, $j \geqslant 2$, $i(G) - \gamma_{T_0}(G) = k$, $k = 1, 2, 3...$ □

Theorem 5.3.7 characterizes graphs for which $\gamma_{T_0}(G) = 1$, $\gamma_{T_0}(G) = 2$, $\gamma_{T_0}(G) = n - 1$ and $\gamma_{T_0}(G) = n$.

Theorem 5.3.7. *Let G be any graph on 'n' vertices. Then*

1. $\gamma_{T_0}(G) = 1$ *if and only if* $\triangle(G) = n - 1$
2. $\gamma_{T_0}(G) = 2$ *if and only if* $i(G) = 2$
3. $\gamma_{T_0}(G) = n$ *if and only if* $G = \overline{K_n}$
4. $\gamma_{T_0}(G) = n - 1$ *if and only if* $G \cong K_2$ *or* $G \cong K_2 \cup \overline{K_{n-2}}$.

Proof. (1) is obvious.

(2) Suppose that $\gamma_{T_0}(G) = 2$. Let $D \subseteq V(G)$ be a γ_{T_0} -set. Then $|D| = 2$ and $\langle D \rangle$ is empty, otherwise $\langle D \rangle$ is K_2 which is not T_0. Hence D is independent dominating. Also since $\gamma_{T_0}(G) \leq i(G)$, it follows that $i(G) = 2$

Conversely, let $i(G) = 2$. If $\gamma_{T_0}(G) \neq 2$, then by Corollary 5.3.4, $\gamma_{T_0}(G) = 1$. Then the γ_{T_0}-set is also independent dominating, contradicting $i(G) = 2$. Hence $\gamma_{T_0}(G) = 2$.

(3) Let $\gamma_{T_0}(G) = n$. Then every γ_{T_0}-set D contains every vertices of G and hence $\langle D \rangle = G$. Also since $\gamma_{T_0}(G) \leq i(G)$ and order of G is n, $i(G)$ must be n. Hence $G = \overline{K_n}$. The converse is obvious.

(4) If $G = K_2$ or $K_2 \cup \overline{K_{n-2}}$, then $\gamma_{T_0}(G) = n - 1$.

Conversely suppose that $\gamma_{T_0}(G) = n - 1$

Case (i) G is connected.

If $\Delta(G) = 0$, then since G is connected, $G \cong K_1$ and therefore $\gamma_{T_0}(G) = 1 = n$.

If $\Delta(G) = 1$, then since G is connected, G has exactly two vertices and $G \cong K_2$. Therefore $\gamma_{T_0}(G) = 1 = n - 1$.

If $\Delta(G) \geq 2$, then $i(G) \leq n - \Delta(G)$. Hence $\gamma_{T_0}(G) \leq n - \Delta(G) \leq n - 2$ by Corollary 5.3.4. Therefore the only connected graph with $\gamma_{T_0}(G) = n - 1$ is K_2.

Case (ii) G is disconnected.

When $\Delta(G) = 0$ or $\Delta(G) \geq 2$, then $\gamma_{T_0}(G) = n$ and less than or equal to $n - \Delta(G)$ respectively. Therefore, $\gamma_{T_0}(G) = n - 1$ if and only if $\Delta(G) = 1$. In this case, the only nontrivial connected component of G are K_2. Suppose that r

components of G are K_2. Then we have, $1 \leq r \leq \lceil \frac{n}{2} \rceil$ and $\gamma_{T_0}(G) = n - r$. Thus $\gamma_{T_0}(G) = n - 1$ if and only if $n - r = n - 1$. ie., if and only if $r = 1$. □

Corollary 5.3.8. *Let G be a graph of order n with $\Delta(G) > 0$. If G is distinct from any of the graphs $K_2 \cup nK_1$ where $n = 0, 1, 2...$ then $\gamma_{T_0}(G) \leq n-2$. Further equality holds for $G = P_4$ and C_4*

Theorem 5.3.9. *Let G be any nontrivial connected graph of order n, then*

$$\gamma_{T_0}(G) + \gamma_{T_0}(\overline{G}) \leq 2n - 1 \tag{5.1}$$

$$\gamma_{T_0}(G)\gamma_{T_0}(\overline{G}) \leq n(n-1) \tag{5.2}$$

Further equality holds if and only if $G \cong K_2$.

Proof. If $G = K_1$ then $\gamma_{T_0}(G) = \gamma_{T_0}(\overline{G}) = n$. There are no nontrivial graphs for which $\gamma_{T_0}(G) = \gamma_{T_0}\overline{(G)} = n$. Therefore, $\gamma_{T_0}(G) + \gamma_{T_0}(\overline{G}) \leq 2n - 1$ and $\gamma_{T_0}(G)\gamma_{T_0}(\overline{G}) \leq n(n-1)$.

Furthermore, equality holds in (5.1) and (5.2) if and only if either $\gamma_{T_0}(G) = n$ and $\gamma_{T_0}(\overline{G}) = n - 1$ or $\gamma_{T_0}(G) = n - 1$ and $\gamma_{T_0}(\overline{G}) = n$. By Theorem 5.3.7, this is true if and only if $G \cong K_2$ or $\overline{G} \cong K_2$. □

Proposition 5.3.10. *For any graph G, if $i(G) = 3$ then $\gamma_{T_0}(G) = 3$*

Proof. Let $i(G) = 3$ and let $S \subseteq V(G)$ be a T_0 dominating set with $|S| < 3$. Since a connected graph with two vertices is not T_0, S is an independent dominating set, a contradiction. □

Remark 5.3.11. *The converse of the conclusion of Proposition 5.3.10 need not be true. For example in figure 5.1, $\gamma_{T_0}(G)$=3 but $i(G)$=5.*

Theorem 5.3.12. *For complete bipartite graphs $K_{m,n}$*

$$\gamma_{T_0}(K_{m,n}) = \begin{cases} 1, & \text{if } m = 1 \text{ or } n = 1 \\ 2, & \text{if } m = 2,\ n \geq 2 \text{ or } m \geq 2,\ n = 2 \\ 3, & \text{if } m > 2,\ n > 2 \end{cases}$$

Proof. Case (i) If either m or n is one, then $\Delta(G) = m+n-1$. Hence $\gamma_{T_0}(G) = 1$ by Theorem 5.3.7.

Case (ii) If one of m or n is exactly 2, then $i(G) = 2$. Hence by Theorem 5.3.7, $\gamma_{T_0}(G) = 2$

Case (iii) Since $\gamma(K_{m,n}) = 2$, and since $\gamma(G) \leq \gamma_{T_0}(G)$ for any graph G, we have, $\gamma_{T_0}(K_{m,n}) \geq 2$. Let U, V be the two partite sets of $V(K_{m,n})$. If we take two vertices from the same partite set say U of $K_{m,n}$, then they will not dominate other vertices of U and if we take one vertex from U and other vertex from V then these two vertices dominate $K_{m,n}$ but the subgraph induced by these vertices is isomorphic to K_2, which is not a T_0 graph. Therefore, $\gamma_{T_0}(K_{m,n}) \geq 3$. the choice of any two vertices from one partite set and a third vertex from the other set will dominate the vertices of $K_{m,n}$ and the span of these vertices is P_3, which is T_0. Hence the theorem. □

Corollary 5.3.13. *For $K_{m,n}$, $\gamma_{T_0}(K_{m,n}) \leq 3$ for all values of m and n*

Remark 5.3.14. *If $G = K_{m,n}$; $m \geq 4$, $n \geq 4$ then, $\gamma(G) < \gamma_{T_0}(G) < i(G)$.*

Theorem 5.3.15. *If G is a connected graph of order ≥ 2, which contain no K_3 as an induced subgraph, then $\gamma_{T_0}(\overline{G}) = 2$*

Proof. Since G is a connected graph of order ≥ 2, it contains at least an edge say uv. If order of G is two, then G is isomorphic to K_2 and $\overline{G}$ is isomorphic to $\overline{K_2}$. Therefore, $\gamma_{T_0}(\overline{G}) = 2$. If order of G is greater than two, then no vertex of G is adjacent to both u and v, because G is triangle free. Therefore every vertex in G which are adjacent to u are dominated by v in $\overline{G}$ and those vertices adjacent to v in G are dominated by u in $\overline{G}$ and all vertices which are non adjacent to both u and v are dominated by both u and v in $\overline{G}$. So $\{u, v\}$ forms an independent dominating set of $\overline{G}$. Therefore it is also a T_0 dominating set of $\overline{G}$. Hence $\gamma_{T_0}(\overline{G}) \leq 2$. Now let if possible $\gamma_{T_0}(\overline{G}) = 1$, then G would have an isolated vertex, a contradiction. Which proves $\gamma_{T0}(\overline{G}) = 2$ □

Theorem 5.3.16. *Let $G(V, E)$ be any graph. Then for any T_0-dominating set $D \subseteq V$ of G, $\langle D \rangle$ can never be a matching of G.*

Proof. Suppose if possible, $D \subseteq V$ be a T_0-dominating set of G such that, $\langle D \rangle$ is a matching of G. Then $\langle D \rangle$ consists of disconnected edges. That is, $\langle D \rangle$ has K_2 as a component, a contradiction to D is a T_0-dominating set of G. □

5.4 Connected T_0 domination

Definition 5.4.1. *Let $G = (V, E)$ be any graph. A dominating set $S \subseteq V$ is called a connected T_0 dominating set, if $\langle S \rangle$ is both connected and T_0. The minimum cardinality of all connected T_0 dominating sets is denoted by $\gamma_{cT_0}(G)$ and is called the connected T_0 domination number of G. Any connected T_0 dominating set with cardinality $\gamma_{cT_0}(G)$ is called a γ_{cT_0}-set of G.*

Observation 5.4.2. *For any connected graph G, $\gamma_c(G) \leq \gamma_{cT_0}(G)$. This inequality is sharp for P_4*

Theorem 5.4.3. *Let G be any connected graph with $\gamma_c(G) \neq 2$. Then $\gamma_{cT_0}(G) = \gamma_c(G)$*

Proof. Since $\gamma_c(G) \neq 2$, the graph induced by any γ_c-set is not K_2 and hence the γ_c-set is connected T_0 dominating. Also since $\gamma_c(G) \leq \gamma_{cT_0}(G)$, it follows that $\gamma_{cT_0}(G) = \gamma_c(G)$. □

Theorem 5.4.4. *Let a and b be two positive integers with $a > 2$ and $b \geq 2a+2$. Then there is a graph G on b vertices with $\gamma(G) = \gamma_{T_0}(G) = \gamma_{cT_0}(G) = a$ and $i(G) \geq a+1$.*

Proof. Consider the path $P = (u_1, u_2, ..., u_a)$ on a vertices. Let $b = 2a+r$, $r \geq 2$. Let G be the graph obtained from P by attaching two or more pendant edges at u_1 and u_2 and one pendant edge at each u_i, $i \geq 3$. Let v_i, $i \geq 3$ be the pendant vertices attached to u_i, $i \geq 3$. Clearly $D = \{u_1, u_2, ..., u_a\}$ is a γ-set which is also a connected T_0 dominating set. Hence $\gamma(G) = \gamma_{T_0}(G) = \gamma_{cT_0}(G) = a$. Any independent dominating set of G of minimum cardinality will be one among the following. $\{u_i, v_3, v_4, ..., v_a\} \bigcup N(u_j)$ where u_i is the vertex of maximum degree among u_1 and u_2 and $N(u_j)$ is the open neighborhood of u_1 or u_2 with minimum cardinality such that $i \neq j$ or $\{u_1, u_3, v_4, u_5, v_6...\} \bigcup N(u_2)$ if $d(u_1) \geq d(u_2)$ and $|N(u_2)| \leq |N(u_1|$ or $\{u_2, v_3, u_4, v_5, u_6...\} \bigcup N(u_1)$ if $d(u_2) \geq d(u_1)$ and $|N(u_1)| \leq |N(u_2|$. In all these cases the cardinality of the i-set is $(a-1)+min\{d(u_1), d(u_2)\}$, where $min\{d(u_1), d(u_2)\} \geq 2$. Therefore $i(G) \geq a+1$. □

Theorem 5.4.5. *Let T be any tree of order n, $n \geq 4$. If T is not isomorphic to $K_{1,n-1}$ then $\gamma_{cT_0}(\overline{T}) = 3$.*

Proof. Since T is not isomorphic to $K_{1,n-1}$ (a star graph), $\Delta(T) \leq n-2$. Consider the following cases.

Case (i) T is not a path.

Since T is not a path, it has at least three pendant vertices say, v_1, v_2 and v_3. Therefore, $d_{\overline{T}}(v_i) = n - 2$, $i = 1, 2, 3$. Also since T is not a star, the support vertex of at least one of the v_i will be different from support vertices of the other two, for $i = 1, 2, 3$. Therefore, $\{v_1, v_2, v_3\}$ forms a dominating set of $\overline{T}$. In $\overline{T}$, $\langle\{v_1, v_2, v_3\}\rangle$, the graph induced by $\{v_1, v_2, v_3\}$ is either K_3 or P_3. Hence $\{v_1, v_2, v_3\}$ is a connected T_0 dominating set of $\overline{T}$. So that $\gamma_{cT_0}(\overline{T}) \leq 3$. Since T has no isolated vertices, $\gamma_{cT_0}(\overline{T})$ cannot be one. Also since there are no connected T_0 dominating sets of cardinality two, $\gamma_{cT_0}(\overline{T}) \geq 3$. Hence it follows that $\gamma_{cT_0}(\overline{T}) = 3$.

Case (ii) T is a path P_n on n vertices, $n \geq 4$.

Let v_1 and v_2 be the pendant vertices and v_3 be any one of the support vertices. Then $d_{\overline{T}}(v_i) = n - 2$, $i = 1, 2$ and $d_{\overline{T}}(v_3) = n - 3$. In $\overline{T}$, the subgraph induced by $\{v_1, v_2, v_3\}$ is P_3 and it forms a connected T_0 dominating set of $\overline{T}$. Hence by the same reasoning as in case (i), it follows that $\gamma_{cT_0}(\overline{T}) = 3$. □

Corollary 5.4.6. *Let T be a tree of order > 1, then $\overline{T}$ has a connected T_0 dominating set if and only if T is not a star.*

Proof. If T is a star on $n > 1$ vertices, $\overline{T}$ is disconnected. Hence $\overline{T}$ cannot have a connected T_0 dominating set.

Conversely, let T be a tree, which is not a star. Then $n \geq 4$. Therefore by Theorem 5.4.5, $\overline{T}$ has a connected T_0 dominating set. □

Proposition 5.4.7. *Let G be a bi-star $B(m,n)$ on p vertices, then* $\gamma_{cT_0}(G) = p - max\{m,n\}$

Theorem 5.4.8. *[44] For any tree T of order p, the connected domination number of $T = p - e$, where e is the number of pendant vertices in T*

Theorem 5.4.9. *Let T be any tree. Then the γ_c-set and γ_{T_0}-set are the same if and only if T is not a bi-star.*

Proof. Let T be a tree on p vertices. Assume that the γ_c-set and γ_{T_0}-set are the same. Suppose if possible, T is a bi-star. Then by Theorem 5.4.7, $\gamma_c(T) = p - e = 2$, where, e is the number of pendant vertices. So that the graph induced by the γ_c-set is K_2, which is not T_0. Hence by Proposition 5.4.7 and Theorem 5.4.8, if T is a bi-star, then $\gamma_c(T) \neq \gamma_{cT_0}(T)$

Conversely, if T is a bi-star, then there are only two support vertices for all the pendant vertices in T and the graph induced by them is K_2, which forms a connected dominating set of T but not connected Hausdorff dominating. □

Conclusion

In this chapter two generalisations of Hausdorff domination viz, T_1 domination and T_0 domination are discussed. Every T_0 dominating set is T_1 dominating and every T_1 dominating set is Hausdorff dominating. Also it is observed that every non independent T_1 dominating set contains at least one cycle neighbor set with three or more vertices. If the graph induced by a T_0 dominating set or a T_1 dominating set does not contain any isolated vertices, then these dominating sets will be total dominating too.

Chapter 6

Cyclic Distance in Graphs

6.1 Introduction

[1] Generally, graphs representing many real life situations are very complicated, large and contains plenty of cycles, circuits etc,. Some popular examples are social networking systems and electric circuits. The advantage of defining cyclic distance is that such graphs can be studied in a smaller frame using this notion. Cyclic distance reduces the distance between two vertices in a graph. This concept enables us to treat two distinct vertices as a single unit when they belong to a subgraph which is at least two connected. It is like viewing an object from a very far away place as in the case of satellite view of a very large area, study of astronomical bodies and objects from far away places, the technique used in cartography etc,.

[1] Part of this chapter is accepted to be published in South East Asian Journal of Mathematics and Mathematical Sciences

6.2 Cyclic distance in graphs

In this section using the notion of cycle neighbor sets introduced in Definition 4.3.9, maximal cyclic components of a graph are defined and in terms of which cyclic distance between two vertices of a graph is introduced.

Definition 6.2.1. *A cycle neighbor set C of a graph $G(V, E)$ is said to be a maximal cycle neighbor set, if it is not contained in any larger cycle neighbor set of G. That is for any vertex $u \in V \setminus C$, $C \cup \{u\}$ is not a cycle neighbor set.*

Definition 6.2.2. *The subgraph induced by a maximal cycle neighbor set of a graph G is called a maximal cyclic component of G.*

It is clear from the definition that a block of a graph G is a maximal cyclic component of G if and only if the block is not a bridge.

Definition 6.2.3. *If a vertex of a graph G does not belong to any non trivial cycle in G, then it is called a trivial maximal cyclic component of G.*

For an acyclic graph, every vertex is a trivial maximal cyclic component.

Definition 6.2.4. *The number of maximal cyclic components in a graph G is called the m-cyclic component number and is denoted by $mcc(G)$.*

For example, consider the graph G in Figure 6.1. For this graph G, m-cyclic component number $mcc(G) = 11$. For an acyclic graph G, $mcc(G)$ is equal to order of G.

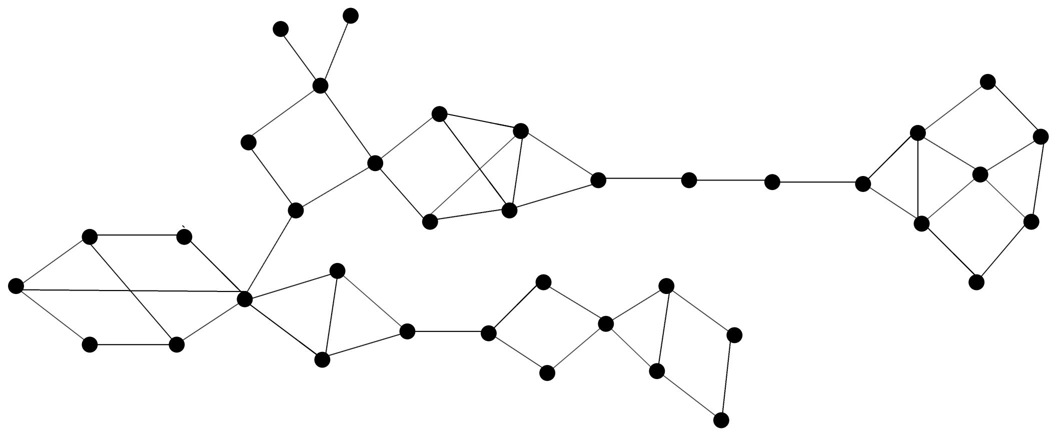


Figure 6.1 - G

Definition 6.2.5. *Two maximal cyclic components are said to be neighbors or neighboring maximal cyclic components, if they have either a vertex in common or they are connected by a bridge between them. Two maximal cyclic components are said to be disjoint if they have no vertices in common and two maximal cyclic components are said to be distinct if they have at most one vertex in common.*

Remark 6.2.6. *It is clear that when all the maximal cyclic components in a graph G are complete graphs, such that no two trivial maximal cyclic components are adjacent, then $mcc(G) = \theta_v(G)$, the (vertex) clique cover number [15] of G. Hence for all graphs G, for which no pair of trivial maximal cyclic components are neighbors, $mcc(G) \leq \theta_v(G)$.*

Proposition 6.2.7. *Let G be any graph. Then any two maximal cyclic components of G can have at most one vertex in common.*

Proof. Suppose that G_1 and G_2 are two maximal cyclic components of G. Suppose if possible, G_1 and G_2 have more than one common vertices. Let u and v be two common vertices of G_1 and G_2. Then there are two internally disjoint paths joining in G. Concatenating these two paths we get a cycle in G. Therefore,

$G_1 \cup G_2$ is a maximal cyclic component of G contradicting the maximality of G_1 and G_2. $\square$

Definition 6.2.8. *Let $G(V, E)$ be any connected graph. For any two vertices $u, v \in V(G)$, the cyclic distance between u and v is defined as the minimum number of maximal cyclic components to be traversed from the maximal cyclic component containing u to the maximal cyclic component containing v other than the one containing u. It is denoted by $cd_G(u, v)$ or simply $cd(u, v)$.*

It is clear from the definition that when the vertices u and v belong to the same maximal cyclic component of G, then $cd(u, v) = 0$. As an illustration consider the graph in Figure 6.2. Here the cyclic distance between u and v, $cd(u, v) = 3$ but $d(u, v) = 6$. It is obvious that, $cd(u, v) = cd(v, u)$. In general for any graph $G(V, E)$, $cd(u, v) \leq d(u, v)$ where $u, v \in V(G)$.

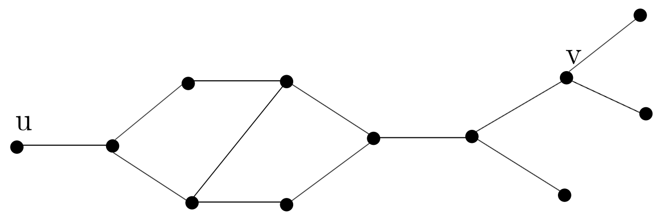


Figure 6.2

Definition 6.2.9. *A connected graph G is called a cyclic path graph if G contains no maximal cyclic components with three or more neighbors.*

That is every maximal cyclic components in a cyclic path graph has at most two neighboring maximal cyclic components.

Definition 6.2.10. *The length of a cyclic path graph is the maximum of the cyclic distances $cd(u, v)$ for all pairs u, v of vertices in that graph.*

It is clear from the definition that length of a cyclic path graph will be one less than the number of maximal cyclic components in G.

Definition 6.2.11. *Let $G(V, E)$ be any graph and and let M the set of all maximal cyclic components in G. Let $N \subseteq M$. Then the subgraph induced by N denoted by $G[N]$ is the graph which consists of all maximal cyclic components in N and all cut edges in G which have both end points in N.*

Definition 6.2.12. *Let $G(V, E)$ be a connected graph and $u, v \in V(G)$ and let $cd(u, v)$ be the cyclic distance between u and v. The cyclic path between u and v in G is a subgraph of G, which is a cyclic path graph of minimum length induced by the maximal cyclic components which contribute towards the cyclic distance $cd(u, v)$ between the vertices u and v.*

Hence if either u or v or both u and v belong to more than one maximal cyclic components, then a subgraph of G, which is a cyclic path graph with minimum length containing u and v is considered as the cyclic path between u and v if such a path exist otherwise the cyclic distance between any pair u, v of vertices belonging to two different components of G is taken to be infinity. It is illustrated in Figure 6.3. A,B,C,D and E are the maximal cyclic components in G. The cyclic path between the vertices u and v is the cyclic path graph induced by C and D.

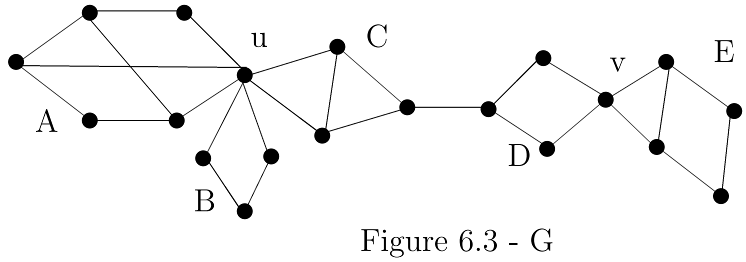


Figure 6.3 - G

Using the terminology of cyclic path between u and v, the cyclic distance $cd(u, v)$ between $u, v \in V(G)$ of a graph G can be defined as the length of the cyclic path joining u and v.

Let $G(V, E)$ be any acyclic graph. Then every vertex in G is a trivial maximal cyclic components so that for every $u, v \in V(G)$, $cd(u, v) = d(u, v)$. That is for an acyclic graph, the notions of cyclic distance and the classical distance between two vertices coincide.

Proposition 6.2.13. *Let $G(V, E)$ be any connected graph. Then for any two vertices $u, v \in V(G)$, the cyclic path between u and v is unique. In particular, the cyclic path between any two maximal cyclic components of G is unique.*

Proof. Assume that $u, v \in V(G)$ such that $cd(u, v) \geq 1$. Suppose if possible, the cyclic path between u and v is not unique. Then there are more than one cyclic paths between u and v in G. Without loss of generality, let there be two internally disjoint cyclic paths between u and v. Combining these two cyclic paths, we get a subgraph containing both u and v so that $cd(u, v) = 0$, a contradiction to $cd(u, v) \geq 1$. □

Definition 6.2.14. *Two vertices u and v of a graph G are said to be cyclic similar vertices if the cyclic distance $cd(u, v)$ between u and v is zero and a graph G is called a cycle neighbor graph, if for every two vertices u and v in $V(G)$, $cd(u, v) = 0$.*

Remark 6.2.15. *In [31], Jalsia M. P. and Raji Pilakkat defined a graph to be track connected, if for every pair of vertices u, v in G there exists two internally disjoint paths connecting u and v. The same concept is considered in the defini-*

tion of cycle neighbor graphs with another terminology, since it seems to be more appropriate here.

All hamiltonian graphs, in particular, cycles C_n, $n \geq 3$, complete graphs K_n, $n \neq 2$, pancyclic graphs [6], connected T_1 graphs [50], connected Hausdorff graphs [48] etc., are examples of cycle neighbor graphs. All subgraphs induced by a maximal cycle neighbor sets of a graph G are cycle neighbor graphs. For a cycle neighbor graph G, $mcc(G) = 1$

Theorem 6.2.16. *Let G be any connected graph. Then the following statements are equivalent.*

1. *G is a cycle neighbor graph.*
2. *The vertex connectivity $\kappa(G)$ is greater than or equal to two.*
3. *$V(G)$ is a cycle neighbor set.*

Proof. Suppose that G is a cycle neighbor graph. Then $cd(u, v) = 0$ for all vertices $u, v \in V(G)$. That is for every pair u, v of vertices in G, there is a cycle containing these vertices. Hence there are at least two internally disjoint paths joining every pair of vertices in G. Therefore, G is at least two-connected and hence $\kappa(G) \geq 2$.

Now let us prove that if (2) does not hold then (3) cannot hold. Suppose that, $\kappa(G) \leq 1$. Then since G is connected, there is at least one cut vertex say w in G and there are vertices u and v in G such that $u...w...v$ is the only path connecting u and v. Therefore u and v together does not belong to any cycle of G. Hence $V(G)$ is not a cycle neighbor set.

If $V(G)$ is a cycle neighbor set, then by definition, every vertex of G belongs to a cycle of G and therefore G is a cycle neighbor graph. Hence (1). □

If the graph G is k-connected, where $k \geq 2$, then G has no pendant vertices. Hence we have:

Corollary 6.2.17. *Let G be a connected graph. If G is a cycle neighbor graph, then G has no pendant vertices.*

The concept of cyclic distance helps to develop a topological structure on connected graphs.

Theorem 6.2.18. *Let $G(V, E)$ be any connected graph. Then the cyclic distance induces a pseudo metric on $V(G)$.*

Proof. Let $u, v \in V(G)$. From the definition of cyclic distance, it is clear that $cd(u, v) \geq 0$ for all $u, v \in V(G)$ and $cd(u, v) = cd(v, u)$. Now let u, v, w be any three vertices in G.

Claim: $cd(u, w) \leq cd(u, v) + cd(v, w)$

If G is acyclic or if all of u, v and w belong to the same cycle of G, then the inequality holds trivially. Now consider the following cases.

Case (i) Only two among u, v and w are in the same maximal cyclic component. Then the cyclic distance between two pairs will be the same and the cyclic distance between the other pair is zero.

Case (ii) All the vertices u, v and w are in distinct maximal cyclic components of G. By Proposition 6.2.13, the cyclic path between u and w is unique. Therefore, $cd(u,w) = cd(u,v) + cd(v,w)$ or $cd(u,w) < cd(u,v) + cd(v,w)$ according as v is in between the cyclic path connecting u and w or not. Hence the proof. □

6.3 Cyclic edge partition

In this section a partition of the edges of a connected graph based on maximal cyclic components is introduced.

Definition 6.3.1. *Let $G(V,E)$ be any graph. Then two edges e_1, e_2 in $E(G)$ are said to be cyclic edge related if either e_1 and e_2 belong to the same maximal cyclic components of G or both e_1 and e_2 are cut edges of G. It is denoted by*
$e_1 \;\widetilde{c}\; e_2$

Recall that an edge decomposition of a graph G is a collection of subgraphs of G with each edge of G belongs to exactly one subgraph, cyclic edgde partition of a graph G gives a decomposition of the graph G.

Definition 6.3.2. *Let G be a connected graph. Then G is called a cyclic tree if every pair of neighboring maximal cyclic components are connected by a bridge between them.*

That is a connected graph G is a cyclic tree if every pair of neighboring maximal cyclic components in G are connected through bridges. Hence a cyclic tree can be considered as a generalised tree in which the nodes are maximal cyclic components and neighboring maximal cyclic components are connected by bridges.

We observe the following properties of cyclic edge relation ' $\widetilde{c}$ '

- Clearly the relation ' $\widetilde{c}$ ' defines an equivalence relation on the set of edges of the graph G.

- Partition corresponding to the equivalence relation ' $\widetilde{c}$ ' constitute a collection of edge disjoint subgraphs of G. Moreover, edge induced subgraphs of the partition are different blocks of G.

- Let G be any graph and let the relation ' $\widetilde{c}$ ' partition the edge set $E(G)$ of G as $E(G) = E_1 \cup E_2 \cup, ..., \cup E_n$, then

$$n = \begin{cases} \text{mcc(G)}, & \text{if G has no cut edges;} \\ \text{mcc(G)} + 1, & \text{if G has cut edges.} \end{cases}$$

- Let $G(V, E)$ any graph and let $E(G) = E_1 \cup E_2 \cup, ..., \cup E_n$ be the cyclic edge partition with E_1 as the set of all cut edges in G. Then the cyclic distance between any two vertices u, v in G depends on the edges in E_1 if the cyclic path between u and v contains at least two adjacent edges in E_1.

- Let $G(V, E)$ be a graph with cyclic edge partition $E(G) = E_1 \cup E_2 \cup, ..., \cup E_n$ with E_1 as the set of all cut edges in G with respect to the cyclic edge relation as in Definition 6.3.1. and let $CN^*[G; z] = a_0 + a_1 z + a_2 z^2 + ... + a_k z^k$, be the modified cycle neighbor polynomial of G, where k is the circumference of G. Then $a_2 = |E_1|$.

- Let $G(V, E)$ a cyclic tree with cyclic edge partition $E(G) = E_1 \cup E_2 \cup, ..., \cup E_n$ and E_1 be the set of all cut edges in G, then $|E_1| = n - 2$.

Theorem 6.3.3. *For a graph $G(V, E)$, let $E(G) = E_1 \cup E_2 \cup, ..., \cup E_n$ be the cyclic edge partition of G. Then for any i, $1 \leq i \leq n$, $|E_i| = 2$ if and only if G*

contains exactly two cut edges. Moreover, there is at most one set E_i in $\{E_i\}$, $1 \leq i \leq n$, for which $|E_i| = 2$.

Proof. Suppose that $|E_i| = 2$, for some i, $1 \leq i \leq n$. By the definition of cyclic edge partition, $\langle E_i \rangle$, the subgraph induced by the edges in E_i is either a maximal cyclic components or the collection of all cut edges in G. For the edge induced subgraph $\langle E_i \rangle$ to be a maximal cyclic components, $|E_i| \geq 3$. Hence $|E_i| = 2$ if and only if E_i is the collection of all cut edges in G. That is if and only if the number of cut edges in G is exactly two.

Conversely suppose that G contains exactly two cut edges. Then for one set E_i, $1 \leq i \leq n$, $|E_i| = 2$ and for any other set E_j, $j \neq i$, $E_j \geq 3$, since $\langle E_j \rangle$ is a maximal cyclic components of G.

To prove that there is at most one set E_i in $\{E_i\}$, $1 \leq i \leq n$, for which $|E_i| = 2$, let us assume the contrary. Suppose if possible, there are more than one set E_i in the cyclic edge partition of G for which $|E_i| = 2$. Without loss of generality, let E_i, E_j with $i \neq j$ be in $\{E_i\}$, $1 \leq i \leq n$ with $|E_i| = |E_j| = 2$. Then the subgraphs induced by E_i and E_j are distinct pairs of cut edges in G. Hence $E_i \cup E_j$ is the collection of all cut edges in G, a contradiction to the assumption that $\{E_i\}$, $1 \leq i \leq n$, is a cyclic edge partition of G. □

Theorem 6.3.4. *Let $G(V, E)$ be any graph with cyclic edge partition $E(G) = E_1 \cup E_2 \cup, ..., \cup E_n$. Then $n = 1$ if and only if one of the following statements holds.*

1. *G is a cycle neighbor graph*

2. *G is a forest.*

Proof. Suppose that the cyclic edge partition of G contains exactly one element. Then every edge of G belong to the same set with respect to the relation ' $\widetilde{c}$ '. Hence either every edge of G belong to the same maximal cyclic component or every edge of G is a cut edge. In the first case, G is a cycle neighbor graph and in the second case, G is a forest.

Converse is obvious that for every cycle neighbor graph and for every forest, their cyclic edge partition contains exactly one element. $\square$

Corollary 6.3.5. *Let G be a disconnected graph with cyclic edge partition $E(G)$ $= E_1 \cup E_2 \cup, ..., \cup E_n$. Then $n = 1$ if and only if G is a forest.*

6.4 Cyclic radius and Cyclic diameter

In this section, cyclic radius, cyclic diameter, cyclic center, cyclic periphery etc., of a connected graph with respect to cyclic distance are defined analogue to radius, diameter, center and periphery of a graph with respect to the usual distance between vertices. Throughout this section only connected graphs are considered.

Definition 6.4.1. *Let G be any graph. The cyclic eccentricity of a vertex v is denoted by $ce_G(v)$ or simply $ce(v)$ and is defined as $ce(v) = max_{u \in V(G)} cd(u, v)$. Let $u, v \in V(G)$ then v is called a cyclic eccentric vertex of u if $ce(u) = cd(u, v)$.*

Definition 6.4.2. *For a graph G, cyclic diameter (denoted by $cdiam(G)$) and cyclic radius (denoted by $crad(G)$) are respectively defined as the largest and smallest cyclic eccentricities of the vertices in the graph G. That is, $cdiam(G)$ $= max_{v \in V(G)} ce(v)$ and $crad(G) = min_{v \in V(G)} ce(v)$*

It is clear from the definition that, for a connected graph G with radius $rad(G)$ and diameter $diam(G)$,

1. $ce(v) \leq e(v)$ for any vertex v in G, where $e(v)$ is the eccentricity of the vertex v in G.

2. $crad(G) \leq rad(G)$.

3. $cdiam(G) \leq diam(G)$.

4. $0 \leq crad(G) \leq cdiam(G) \leq 2crad(G)$.

Definition 6.4.3. *Cyclic center (denoted by $CC(G)$) and cyclic periphery (denoted by $CP(G)$) of a graph G are defined as the set of all vertices for which cyclic eccentricity is equal to the cyclic radius and the set of all vertices for which cyclic eccentricity is equal to the cyclic diameter respectively.*

Let us denote the number of vertices in the cyclic center and cyclic periphery of a graph G by $|CC(G)|$ and $|CP(G)|$ respectively.

Definition 6.4.4. *Let G be a connected graph. Then G is said to be cyclic self centered if $cdiam(G) = crad(G)$.*

Proposition 6.4.5. *For any graph G, $cdiam(G) = 0$ if and only if G is a cycle neighbor graph.*

Remark 6.4.6.

- *Let G be a connected graph of order n, which is not a tree. Then $0 \leq cdiam(G) \leq n-3$, since G contains at least one maximal cyclic components with three or more vertices. Also these bounds are sharp. For any cycle neighbor graph G, $cdiam(G) = 0$ and if G is any unicyclic graph* *of order $n \geq 4$ obtained by attaching two paths on $(n-3)-k$ and k, $0 \leq k \leq (n-3)$ vertices respectively to any two distinct vertices of K_3 by bridges, then $cdiam(G) = (n-3)$.*

- *Clearly a tree T of order n has maximum diameter when it is a path and in this case, $diam(T) = n-1$. Similarly cyclic diameter of a graph G with $mcc(G) = n$ will be maximum, when the graph induced by all the maximal cyclic components of G form a cyclic path graph and for this graph $cdiam(G) = n-1$. That is for a cyclic path graph G, $cdiam(G) = mcc(G)-1$. Hence for any graph G, $cdiam(G) \leq mcc(G)-1$.*

Definition 6.4.7. *A graph G is called a cyclic flower if there are at least two blocks in G which are not cut edges such that each block is a cycle neighbor graph of order greater than or equal to three and all these blocks has exactly one common vertex.*

That is a cyclic flower has at least two non trivial maximal cyclic components and there is a unique common vertex for all these maximal cyclic components.

Definition 6.4.8. *The common vertex of all the maximal cyclic component of a cyclic flower graph is called flower centric vertex.*

The simplest cyclic flower is the friendship graph F_2 which is constructed by joining two copies of the cycle C_3 with a common vertex. We can construct non isomorphic cyclic flowers with the same number of vertices and also with the same number of maximal cyclic components. For a cyclic flower G with $mcc(G) = n$, every maximal cycic component in G has $mcc(G)-1$ neighbors and in this case, $crad(G) = 0$ and $cdiam(G) = 1$. The graph in Figure 6.4 is an example of a cyclic flower with four maximal cyclic components.

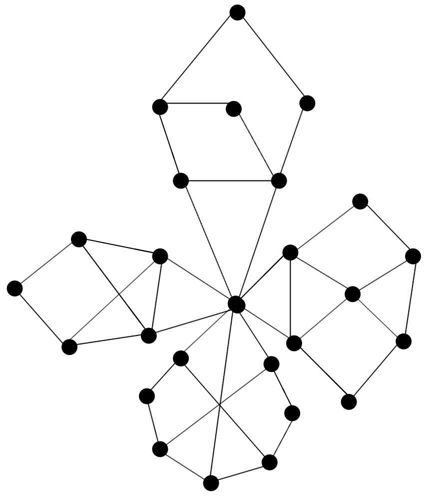


Figure 6.4

Definition 6.4.9. *If a path on k vertices is attached to the flower centric vertex of a cyclic flower (or to any one vertex of a cycle neighbor graph) by a bridge, then it is called cyclic flower with k-stem (or a cycle neighbor graph with k-stem).*

The graph in Figure 6.5 is a cyclic neighbor graph with 6-stem.

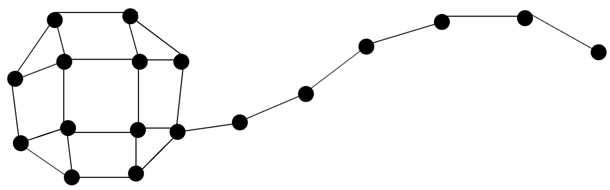


Figure 6.5

Theorem 6.4.10. *Let G be a connected graph. Then $cdiam(G) = 1$ if and only if one of the following conditions hold*

1. *G has two maximal cyclic components connected by a bridge*

2. *G is a cyclic flower*

3. *G is either a graph containing a cyclic flower and a cycle neighbor graph connected by a bridge between the flower centric vertex of the cyclic flower and any vertex of the cycle neighbor graph or G contains two cyclic flowers connected by a bridge through their flower centric vertices.*

Proof. If G is any one of the graphs as in the statement of the theorem, then clearly $cdiam(G) = 1$.

Now let $cdiam(G) = 1$. If there is only one maximal cyclic subgraph, then G is a cycle neighbor graph and hence $cdiam(G) = 0$. Therefore G has at least two maximal cyclic components say M_1 and M_2. Since G is connected, these maximal cyclic components M_1 and M_2 are connected either by a bridge between them or they have a vertex in common.

Consider the first possibility that G contains two maximal cyclic components M_1 and M_2 connected by a bridge. If there are exactly two maximal cyclic components in G then (1) holds. When there are more maximal cyclic components in G other than M_1 and M_2, then since G is connected, either these additional maximal cyclic components will have a vertex in common with $M_1 \cup M_2$ or they will be connected to $M_1 \cup M_2$ by bridges. But if the vertices in $M_1 \cup M_2$ which is shared by these additional maximal cyclic components are different from the connecting vertices of M_1 and M_2, then clearly $cdiam(G) > 1$. Therefore every extra maximal cyclic components in G which has a vertex in common with $M_1 \cup M_2$ will be sharing either the connecting vertex of M_1 to M_2 or that of M_2 to M_1. In this case, (3) holds. No additional maximal cyclic components can be connected to $M_1 \cup M_2$ by bridges, since then $cdiam(G)$ will be increased at least by one.

Now consider the second possibility that G contains a cyclic flower with two maximal cyclic components say M_1 and M_2. Then there arises the following cases.

Case (i) There are more maximal cyclic components in G which have a vertex in common with $M_1 \cup M_2$.

Then all these maximal cyclic components willbe sharing a a common vertex. Otherwise cyclic diameter will be more than one. Hence (2) holds.

Case (ii) If the maximal cyclic components in G other than M_1 and M_2 are connected by bridges to $M_1 \cup M_2$.

In this case the connecting edge is unique and its one end is the flower centric vertex of $M_1 \cup M_2$ and the other end is common for all the additional maximal cyclic components. Otherwise $cdiam(G) > 1$. Therefore (3) holds.

Case (i) and case (ii) can also occur together and in this case, G is a graph containing two cyclic flowers connected by a bridge through their flower centric vertices. This completes the proof. □

Let $u, v \in A$, where $A \subseteq V(G)$ is a maximal cycle neighbor set of a connected graph G. Then it is clear that $|cd(u,x) - cd(v,x)| \leq 1$ for any vertex $x \in V(G) \backslash A$. Also when the subgraphs induced by $A, B \subseteq V(G)$ are two neighboring maximal cyclic components of G, then $|cd(u,x) - cd(v,x)| \leq 1$ for all $u \in A$, $v \in B$ and $x \in V(G) \setminus A \cup B$. Hence it follows that;

Lemma 6.4.11. *Let the subgraphs induced by $A, B \subseteq V(G)$, which are two different maximal cycle neighbor sets of a connected graph G such that the corresponding maximal cyclic components are neighbors in G. Then,*

1. $|ce(u) - ce(v)| \leq 1$ *for all $u, v \in A$, where A is any of the maximal cycle neighbor sets in G.*

2. $|ce(u) - ce(v)| \leq 1$, *for all $u \in A$ and $v \in B$.*

Theorem 6.4.12. *Let G be a connected graph. If there is a positive integer k such that $crad(G) < k < cdiam(G)$ then there exists a vertex $v \in V(G)$ with $ce(v) = k$.*

Proof. Let $u, v \in V(G)$ such that $ce(u) = crad(G)$ and $ce(v) = cdiam(G)$. Consider the cyclic path between u and v. Let S and W be the set of all vertices in that cyclic path, with $ce(s) < k$ for every $s \in S$ and $ce(w) \geq k$ for every $w \in W$. From the definition of cyclic path it is clear that the vertices of S and W are connected through a vertex y in W, which is common to two neighboring maximal cyclic components or they are connected by a bridge with one end in S and the other end say y in W. In both cases $ce(y) \geq k$. Then by the Lemma 6.4.10, $|ce(w) - ce(y)| \leq 1$. Therefore we have, $ce(y) = k$. Hence the proof. $\square$

Theorem 6.4.13. *Let G be a connected graph. Then G is cyclic self centered if and only if one of the following statements hold;*

1. *G is a cycle neighbor graph.*

2. *G is a graph with $cdiam(G) = 1$ which is not a cyclic flower.*

Proof. G is a cyce neighbor graph if and only if $cdiam(G) = 0$. Therefore, cycle neighbor graphs are cyclic self centered.

Now Suppose that $cdiam(G) = 1$. Then by Theorem 6.4.9, G has two maximal cyclic components connected by a bridge or G is a cyclic flower or G is a graph containing a cyclic flower and a cycle neighbor graph connected by a bridge between the flower centric vertex of the cyclic flower and any vertex of the cycle neighbor graph or G contains two cyclic flowers connected by a bridge through their flower centric vertices. Among them when G is a cyclic flower, $crad(G) = 0$ and $cdiam(G) = 1$ and in all other cases, G is cyclic self centered.

Now let $crad(G) = cdiam(G) = k$ where $k \geq 2$. Then $ce(v) = k$ for all $v \in V(G)$. Let w be a cyclic eccentric vertex of v. Then $cd(v, w) = k$. By Proposition 4.2.13, there is a unique cyclic path connecting v and w. In that cyclic path there are $k+1$ maximal cyclic components say $G_0, G_1, G_2, ..., G_k$ such that $v \in V(G_0)$ and $w \in V(G_k)$. Let $u \in V(G_l)$ where $1 \leq l \leq k - 1$. From the definition of cyclic paths, it is clear that there is at least one vertex say u in $V(G_l)$ such that u does not belong to $V(G_{l-1})$ and u does not belong to $V(G_{l+1})$ and $ce(u) = k$. Correspondingly there is a cyclic eccentric vertex $y \in V(G)$ with $cd(u, y) = k$. Consider the cyclic path connecting v and y. Since the cyclic path between any two vertices of a connected graph is unique, either u lies interior to the cyclic path between v and y or v lies interior to the cyclic path between u and y. In the first case, it is clear that $cd(v, y) = cd(v, u) + cd(u, y) > k$ and in the second case, $cd(y, w) = cd(y, v) + cd(v, w) > k$, a contradiction to $cdiam(G) = k$. Hence $k < 2$ whenever $crad(G) = cdiam(G) = k$. Hence the proof. $\square$

Corollary 6.4.14. *If G is a cyclic self centered graph then $cdiam(G) \leq 1$.*

Definition 6.4.15. *A collection of m cyclic flowers and a collection of n cyclic similar graphs (where $m \geq 0, n \geq 0$ and both m and n are finite) are attached to the vertices of a cycle neighbor graph H through bridges then the resulting graph G is called a cyclic bouquet if $m + n \geq 2$.*

Definition 6.4.16. *The cycle neighbor graph H to which all the cyclic flowers and cycle neighbor graphs are attached in a cyclic bouquet is called the central cyclic component of the cyclic bouquet.*

Graph in Figure 6.6 is an example of a cyclic bouquet.

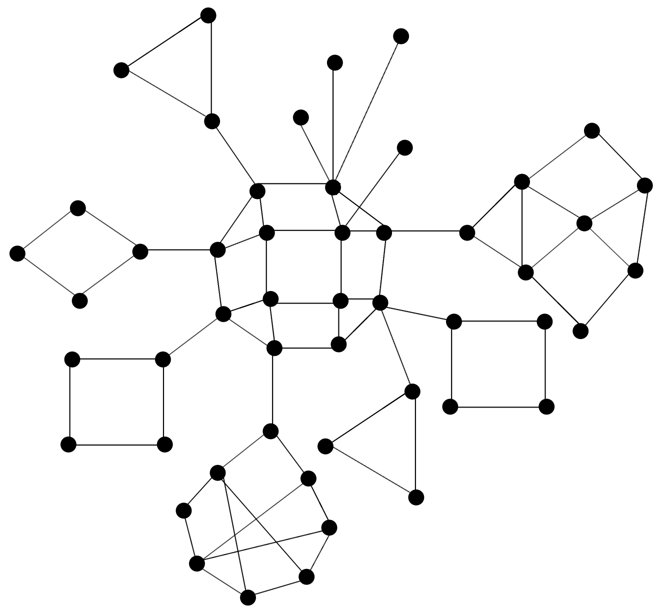


Figure 6.6

From the definition of a cyclic bouquet, it is clear that the cyclic diameter of any cyclic bouquet G is two. The number of vertices in the cyclic center $|CC(G)|$ is equal to the number of vertices in the central cyclic component H and cyclic periphery $CP(G)$ consists of all vertices in G other than those in the central cyclic component H. Hence we can costruct a cyclic bouquet G with the desired number of verices $k \geq 1$ but $k \neq 2$ in the cyclic center $CC(G)$ and $l \geq 2$ vertices in the cyclic periphery $CP(G)$.

Theorem 6.4.17. *Let G be a connected graph which is not a tree of order $n \geq 4$. Then $|CC(G)| = 1$ and $|CP(G)| = n - 1$ if and only if one of the following statements hold;*

1. *G is cyclic flower.*

2. *G is cyclic flower with 1-stem or G is cycle neighbor graph with 1-stem.*

3. *G is a cyclic bouquet whose central cyclic component is K_1.*

Proof. If G is any one of the graphs as in the statement of the theorem, then it is clear that $|CC(G)| = 1$ and $|CP(G)| = n - 1$.

To prove the converse, let G be a graph with $|CC(G)| = 1$ and $|CP(G)| = n - 1$. Let $CC(G) = \{u\} \subseteq V(G)$. Then $ce(u) < ce(v)$ for all $v \in V(G) \setminus \{u\}$. Also since every vertex in $V(G) \setminus \{u\}$ is a cyclic peripheral vertex, $cd(u, v) \leq 1$ for all $v \in V(G) \setminus \{u\}$. Otherwise, there exists some vertex $v \in V(G) \setminus \{u\}$ such that $cd(u, v) \geq 2$. Then we can find a vertex x in the cyclic path between u and v such that $cd(u, x) = 1$, which contradicts the fact that x is a cyclic peripheral vertex. Therefore, $cd(u, v) \leq 1$. Consider the following cases.

Case (i) $cd(u, v) = 1$, for all $v \in V(G) \setminus \{u\}$. Since G is not a tree, it is clear that G is a cyclic bouquet with K_1 as central cyclic component, which is not a star graph.

Case (ii) $cd(u, v) < 1$. In this case, $cd(u, v) = 0$, for all $v \in V(G) \setminus \{u\}$.

Hence either G is a cycle neighbor graph or u is a cut vertex which belongs to every maximal cyclic components of G. But when G is a cycle neighbor graph then, $|CC(G)| = |CP(G)| = n$. Therefore u is a cut vertex satisfying the above condition and in this case G is a cyclic flower.

Case (iii) $cd(u, v) \leq 1$. Let $cd(u, v) = 0$ for every $v \in A$ for a nonempty subset $A \subseteq V(G) \setminus \{u\}$ and $cd(u, v) = 1$ for all $v \in B$ for a nonempty subset $B \subseteq V(G) \setminus \{u\}$ with $A \cup B = V(G) \setminus \{u\}$. Then either $\langle A \cup \{u\} \rangle$, the graph induced by $A \cup \{u\}$ is a cycle neighbor graph or $\langle A \cup \{u\} \rangle$ is a cyclic flower with u as the flower centric vertex. In both cases, $|B| = 1$, otherwise $|CC(G)| \neq 1$. Therefore G is a cyclic flower with 1-stem or a cyclic similar graph with 1-stem $\square$

Corollary 6.4.18. *If $|CC(G)| = 1$ and $|CP(G)| = n - 1$ for a connected graph G of order $n \geq 4$, then $1 \leq cdiam(G) \leq 2$.*

Conclusion

In this chapter a new distance concept called cyclic distance in graphs is introduced. For an acyclic graph, the notions of cyclic distance and the classical distance between vertices coincide. Cyclic distance induces a pseudo metric on the set of vertices of a graph hence this distance concept can be used to develop a topological structure on connected graphs. With respect to this new distance concept, cyclic radius, cyclic diameter, cyclic center, cyclic periphery, etc., of a graph are introduced. Also cyclic self centered graphs, graphs G with $cdiam(G) = 1$ and graphs G of order $n \geq 4$ with $|CC(G)| = 1$ and $|CP(G)| = n - 1$ are characterized.

Chapter 7

Cyclic Distance Matrix of a Graph

7.1 Introduction

In this chapter, we deal with a method of condensing a graph by shrinking the maximal cyclic components. The graph obtained by this shrinking process is called reduced graph of the original graph. Here some properties of reduced graph of a graph are obtained. Also a new graph matrix called cyclic distance matrix of a graph is defined using the concept of cyclic distance and some of its properties are discussed. An interesting property of the determinant of cyclic distance matrix of a graph, whose reduced graph is free of cyclic flowers with three or more maximal cyclic components is also obtained .

7.2 Reduced graph of a graph

This section is devoted to the study of reduced graph of a graph, which is obtained by contracting all the maximal cyclic components and identifying each of them with single vertices. In this new graph, the vertices representing neighboring maximal cyclic components of the original graph are connected through bridges. So that corresponding to every simple graph G, which is not a tree, we can find a maximal cyclic components contracted graph, the reduced graph of G, which is different from the original graph. A simple real life situation where this concept matches can be considered as follows. The different organ systems of human body can be represented by a large complicated graph. Whenever a peripheral study of this graph is needed, we can use the reduced graph and when a detailed study of a particular organ is needed it is done by zooming in that particular node.

Definition 7.2.1. *Let G be any connected graph. The reduced graph of G, denoted by $R(G)$ is the graph obtained from G by contracting (or shrinking) each maximal cyclic component of G to a single vertex and joining two vertices of $R(G)$ by an edge if they correspond to neighboring maximal cyclic components of G.*

We can observe the following simple properties of the reduced graph $R(G)$ of a graph G.

Proposition 7.2.2.

- *The reduced graph of a tree T is the tree itself. That is, $R(T) \cong T$.*
- *The reduced graph of a cycle neighbor graph is K_1.*

- *The number of vertices in $R(G)$ of G is equal to the number of maximal cyclic components $mcc(G)$ of G.*

- *If the graph G is the disjoint union of the graphs $G_1, G_2, ..., G_n$ then the reduced graph $R(G)$ of G is given by $R(G) = R(G_1) \cup R(G_2) \cup ... \cup R(G_n)$, where $R(G_i)$ is the reduced graph of G_i, $i = 1, 2, ..., n$.*

- *When G is not a forest, $|V(R(G))| \leq |V(G)| - 2$.*

Let G be a graph which contains a cyclic flower with n maximal cyclic components, where $n \geq 3$. Then the reduced graph of G contains a clique on n vertices. In this case, $R(G)$ cannot be acyclic. Thus we have :

Theorem 7.2.3. *The reduced graph of graph G is a tree if and only if G contains no cyclic flowers with more than two maximal cyclic components.*

The following corollaries follow immediately from Theorem 7.2.3

Corollary 7.2.4. *For a graph G, $R(G)$ is a path if and only if G is a cyclic path graph.*

Corollary 7.2.5. *For a cyclic tree G, $R(G)$ is a tree.*

Corollary 7.2.6. *The number of bridges in the reduced graph $R(G)$ of a graph G is equal to the number of bridges in G + the number of cut vertices which are common to exactly two maximal cyclic components of G.*

Corollary 7.2.7. *The number of edges in the reduced graph $R(G)$ of a graph G, which are not bridges depends on the number of cyclic flowers in G with more than two maximal cyclic components of G.*

Corollary 7.2.8. *Every edge in the reduced graph $R(G)$ of a graph G is a cut edge if and only if G contains no cyclic flowers with more than two maximal cyclic components.*

Corollary 7.2.9. *Let $E(G) = E_1 \cup E_2 \cup, ..., \cup E_n$, $n \geq 2$ and $E(R(G)) = F_1 \cup F_2 \cup, ..., \cup F_m$, $m \geq 1$ be the cyclic edge partitions of G and $R(G)$ with E_1 and F_1 as the set of cut edges in G and $R(G)$ respectively. Then $|E_1| = |F_1|$ if and only if G contains no cyclic flowers with exactly two maximal cyclic components. In particular for a connected graph G, $E(R(G)) = F_1$ and $|E_1| = |F_1|$ if and only if G is a cyclic tree.*

Proof. Suppose that $|E_1| = |F_1|$. In addition to the cut edges in G, new cut edges will be added in $R(G)$, corresponding to the flower centric vertex of a cyclic flower with two maximal cyclic components. Since $|E_1| = |F_1|$, no such new cut edge is added in $R(G)$. That means every neighboring maximal cyclic components in G other than those cyclic flowers with more than two maximal cyclic components are connected through bridges. Hence by Corollary 7.2.8, G contains no cyclic flowers with more than two maximal cyclic components. So that there are no cyclic flowers in G. Converse is obvious. $\square$

Remark 7.2.10. *If the reduced graph $R(G)$ of G is not a tree, then $R(G)$ contains at least one complete graph K_n, $n \geq 3$, where n is determined by the number of maximal cyclic components in the cyclic flower of G.*
Since every complete graph K_n on $n \geq 5$ vertices is non planar, Corollary 7.2.11 follows.

Corollary 7.2.11. *If a graph G contains a cyclic flower with five or more maximal cyclic components, then $R(G)$ is non planar.*

Theorem 7.2.12. *The reduced graph $R(G)$ of a graph G is a bipartite if and only if $R(G)$ is a tree.*

Proof. Let $R(G)$ be a bipartite graph. Then $R(G)$ has no odd cycles. Suppose if possible, $R(G)$ is a not a tree. Then $R(G)$ contains a complete graph K_n for $n \geq 3$. Hence $R(G)$ contains odd cycles, a contradiction to $R(G)$ is bipartite.

Conversely if $R(G)$ is a tree, trivially it is bipartite. □

7.3 Cyclic distance matrix of a graph

All graphs considered in this section are connected unless otherwise specified. Let G is a graph containing $k \geq 2$ maximal cyclic components viz, $M_1, M_2, ..., M_k$ and let $M(G)$ denote the set of all maximal cyclic components. That is, $M(G) = \{M_1, M_2, ..., M_k\}$.

Definition 7.3.1. *Let G be a connected graph with $M(G) = \{M_1, M_2, ..., M_k\}$. Consider any two maximal cyclic components M_i, M_j, where $1 \leq i, j \leq k$ with set of vertices $\{v_1, v_2, ..., v_n\}$ and $\{u_1, u_2, ..., u_m\}$ in M_i and M_j respectively. Then the distance between the maximal cyclic components M_i and M_j is denoted by $dist(M_i, M_j)$ and is defined as*

$$dist(M_i, M_j) = max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \{cd(v_i, u_j)\}$$

That is $dist(M_i, M_j)$ is the maximum value among all cyclic distances $cd(v_i, u_j)$ for vertices v_i in the maximal cyclic components M_i and for vertices u_j in the maximal cyclic components M_j.

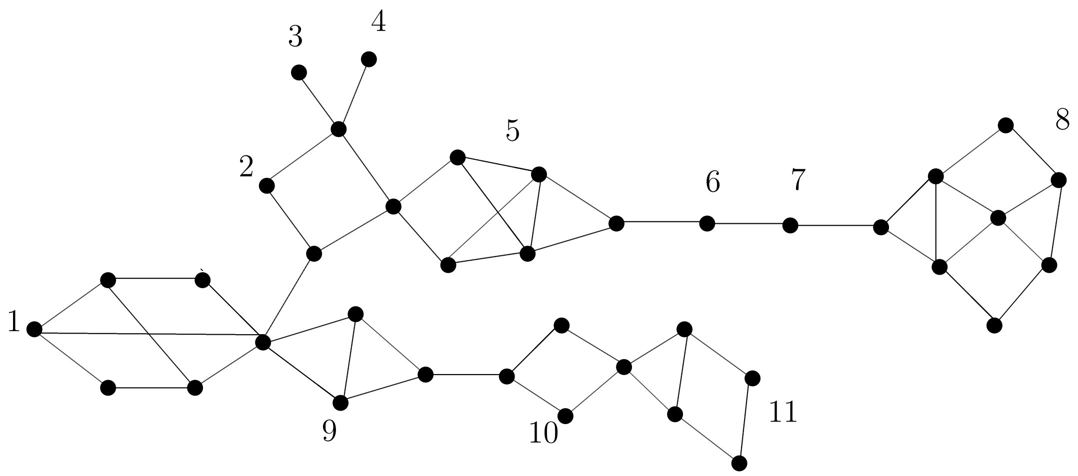
Theorem 7.3.2 follows directly from the definition of distance between maximal cyclic components.

Theorem 7.3.2. *Let G be a connected graph with $M(G) = \{M_1, M_2, ..., M_k\}$. Then $max_{1 \le i,j \le k} dist(M_i, M_j) = cdiam(G)$, the cyclic diameter of G.*

Definition 7.3.3. *The cyclic distance matrix of a connected graph G with $M(G) = \{M_1, M_2, ..., M_k\}$ is a $k \times k$ matrix denoted by $CD(G)$ and is defined as follows. The rows and columns of $CD(G)$ are indexed by the set $M(G)$. The (i, j)-th entry of $CD(G)$ is $dist(M_i, M_j)$, the distance between the maximal cyclic components M_i and M_j.*

In example 7.3.4 we consider the cyclic distance matrix of the graph G in Figure 7.1 with the set of maximal cyclic components $M(G) = \{1, 2, ..., 11\}$,

Example 7.3.4.



G

Figure 7.1

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} &
\begin{pmatrix}
0 & 1 & 2 & 2 & 2 & 3 & 4 & 5 & 1 & 2 & 3 \\
1 & 0 & 1 & 1 & 1 & 2 & 3 & 4 & 1 & 2 & 3 \\
2 & 1 & 0 & 2 & 2 & 3 & 4 & 5 & 2 & 3 & 4 \\
2 & 1 & 2 & 0 & 2 & 3 & 4 & 5 & 2 & 3 & 4 \\
2 & 1 & 2 & 2 & 0 & 1 & 2 & 3 & 2 & 3 & 4 \\
3 & 2 & 3 & 3 & 1 & 0 & 1 & 2 & 3 & 4 & 5 \\
4 & 3 & 4 & 4 & 2 & 1 & 0 & 1 & 4 & 5 & 6 \\
5 & 4 & 5 & 5 & 3 & 2 & 1 & 0 & 5 & 6 & 7 \\
1 & 1 & 2 & 2 & 2 & 3 & 4 & 5 & 0 & 1 & 2 \\
2 & 2 & 3 & 3 & 3 & 4 & 5 & 6 & 1 & 0 & 1 \\
3 & 3 & 4 & 4 & 4 & 5 & 6 & 7 & 2 & 1 & 0
\end{pmatrix}
\end{array}
$$

It is clear from CD(G) that $cdiam(G) = 7$, eventhough it is not trivial from the graph.

Proposition 7.3.5 gives some trivial properties of cyclic distance matrix of a graph G.

Proposition 7.3.5. *1. For any connected graph G, the cyclic distance matrix $CD(G)$ is a zero diagonal, symmetric matrix with nonnegative entries over Z, the set of integers. Hence trace of $CD(G) = 0$. Moreover, since $CD(G)$ is a real symmetric matrix, all its eigen values are real.*

2. The maximum value among all entries in the cyclic distance matrix of a graph G is $cdiam(G)$.

3. Let G be a graph with k maximal cyclic components. Then the entries

$CD(G)_{i,j}$ of $CD(G) \in \{0, 1, 2, ..., k-1\}$ for $1 \leq i, j \leq k$, since $cdiam(G) \leq k-1$.

4. *The cyclic distance matrix of a cycle neighbor graph is O, the zero matrix of order one.*

Since each maximal cyclic component of a tree is isomorphic to K_1, we have Proposition 7.3.6;

Proposition 7.3.6. *The cyclic distance matrix of a tree and its distance matrix [4] are the same.*

Proposition 7.3.7 follows directly from the definition of distance between maximal cyclic components.

Proposition 7.3.7. *The number of ones in the i-th row of the cyclic distance matrix of a graph is the number of neighboring maximal cyclic components of the i-th maximal cyclic components.*

Theorem 7.3.8. *The cyclic distance matrix of a graph G is a binary matrix if and only if one of the following statements hold*

1. *G has two maximal cyclic components connected by a bridge*
2. *G is a cyclic flower*
3. *G is either a graph containing a cyclic flower and a cycle neighbor graph connected by a bridge between the flower centric vertex of the cyclic flower and any vertex of the cycle neighbor graph or G contains two cyclic flowers connected by a bridge through their flower centric vertices.*

Proof. Suppose that cyclic distance matrix of a graph G is a binary matrix. Let $M(G) = \{M_1, M_2, ..., M_k\}$. Then $k \neq 1$. Otherwise, G will be cyclic similar. In that case, $CD(G)$ cannot be a binary matrix. Hence $k \geq 2$. Since G contains at least two maximal cyclic components and $CD(G)$ is a binary matrix, for all $i \neq j$ with $1 \leq i, j \leq k$, $dist(M_i, M_j) = 1$. Therefore all the maximal cyclic components in G are neighbors to each other. So that $cdiam(G) = 1$. Hence the proof follows from Theorem 6.4.9.

Conversely, if G is any one of the graphs as in the statement of the theorem, $cdiam(G) = 1$. Hence $dist(M_i, M_j)$ is either zero or one for every pair M_i, M_j where $1 \leq i, j \leq k$. Therefore $CD(G)$ is a binary matrix. □

Definition 7.3.9. *The set of eigen values, which are the roots of the characterestic polynomial det $(CD(G) - \lambda I)$ of the cyclic distance matrix $CD(G)$ of a graph G is called the cyclic distance spectrum of G. It is denoted by cd-spectrum of G.*

The cyclic distance matrix $CD(G)$ of a graph G is not unique. It depends on the labelling of the maximal cyclic components of G. A relabelling of the maximal cyclic components of G will result in permutation of the rows and columns simultaneously. Hence for any labeling the eigen values of the graph will be the same. Since $CD(G)$ is a symmetric matrix, the eigen values of $CD(G)$ are real. Also the sum of the eigen values of $CD(G)$ equal to trace of $CD(G) =$ zero, and determinant of $CD(G)$ equal to the product of the eigen values.

The cd-spectrum of the graph G in figure 7.1, obtained using R prgramme correct to seven decimal places, is given below.

$\{30.2986329, -0.4177651, -0.5719994, -0.6061719,$
$-0.8905015, -1.0595263, -2.0000000, -2.0000000,$
$-2.4402655, -6.3511823, -13.9612209\}$

Clearly sum of eigen values is zero and product of eigen values equal to 3584. Hence determinant of $CD(G) = 3584 \neq 0$. So that $CD(G)$ is nonsingular and hence rank of $CD(G) = 11$ it is also equl to $mcc(G)$.

Let G be a connected graph with $mcc(G) = k \geq 2$ and $cdiam(G) = 1$. Then $CD(G)$ is of the form,

$$\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & ... & k \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ . \\ k \end{array} & \begin{pmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & 1 & ... & 1 \\ 1 & 1 & 0 & ... & 1 \\ . & . & . & ... & . \\ 1 & 1 & 1 & ... & 0 \end{pmatrix}
\end{array}$$

Which is the same as the adjacency matrix of a complete graph K_k on k vertices. It is clear that rank of this matrix is k. For any positive integer n; the adjacency spectrum of K_n consists of $n-1$ and 1 with multiplicities 1 and $n-1$ respectively [4]. Hence we have the following theorem;

Theorem 7.3.10. *Let G be a connected graph with $mcc(G) = k \geq 2$ and $cdiam(G) = 1$, then*

- *Rank $CD(G) = mcc(G)$, number of maximal cyclic components in G.*
- *The cd-spectrum of G consists of $k-1$ with multiplicity 1 and 1 with multiplicity $k-1$.*

- *The determinant of $CD(G)$ equals $(-1)^{(k-1)}(k-1)$.*

In 1971, R.L Graham and H.O Pollak [23] Proved that if T is a tree of order n, then the determinant of the distance matrix $D(T)$ of T is $(-1^n)(n-1)2^{(n-2)}$. We will use this result to show that the determinant of the cyclic distance matrix of a graph, not containing any cyclic flower with more than two maximal cyclic components depends only on the number of maximal cyclic components of G and it is independent of the structure of G.

Lemma 7.3.11. *For any graph G, the cyclic distance matrix $CD(G)$ and distance matrix of its reduced graph $D(R(G))$ are the same.*

Proof. Let G be any graph and let $R(G)$ be the reduced graph of G. By Proposition 7.2.2, the number of maximal cyclic components in G and the order of $R(G)$ are the same. That is, $mcc(G) = |V(R(G))| = k$. Also, from the definition of distance between maximal cyclic components in a graph G, it is clear that for any two maximal cyclic components M_i, M_j with $1 \leq i, j \leq k$ in G, $dist(M_i, M_j) = dist(v_i, v_j)$ where v_i and v_j are the vertices in $R(G)$ representing to the maximal cyclic components M_i and M_j in G and $dist(v_i, v_j)$ is the distance between the vertices v_i and v_j in $R(G)$. Hence the proof. □

Theorem 7.3.12. *Let G be a graph with $mcc(G) = k$ which contains no cyclic flowers with more than two maximal cyclic components. Then the determinant of the cyclic distance matrix of G depends only on the number of maximal cyclic components in G and it is given by det $CD(G) = (-1^k)(k-1)2^{(k-2)}$*

Proof. By Lemma 7.3.11, cyclic distance matrix of a graph G and distance matrix of its reduced graph are the same. Also by Theorem 7.2.3, the reduced graph

$R(G)$ of a graph G is a tree if and only if G contains no cyclic flowers with more than two maximal cyclic components. Hence for any such graph G, $R(G)$ is a tree. Hence by using the classical result of Graham and Pollak [23], $det\ CD(G) = det\ D(R(G)) = (-1^k)(k-1)2^{(k-2)}$, (where $D(R(G))$ is the distance matrix of redced graph of G) which depends only on the the number of maximal cyclic components in G. □

Corollary 7.3.13. *Let G be a graph with $mcc(G) = k$, $k \geq 2$ which does not contain any cyclic flower with more than two maximal cyclic components. Then,*

1. *$CD(G)$ is nonsingular*
2. *The rank of $CD(G) = k$*
3. *$det\ CD(G)$ is independent of the structure of the graph G*
4. *$CD(G)$ is diagonalizable.*

Theorem 7.3.14. *Let G and H be two nonisomorphic graphs. Then the cd-spectrum of G and the cd-spectrum of H are the same if and only if $R(G) \cong R(H)$*

Proof. From the definition of cyclic distance matrix, it is clear that non isomorphic graphs may have the same cyclic distance matrix, since the entries in the matrix depend on the distance between maximal cyclic components. By Lemma 7.3.11, cyclic distance matrix of a graph G and distance matrix of its reduced graph are the same. Hence it follows that for any pair G and H of nonisomprphic graphs, the cd-spectrum of G and the cd-spectrum of H are the same if and only if $R(G) \cong R(H)$. □

Conclusion

The new vertex similarity measure called cyclic similar vertices which was introduced in chapter six is used here to combine cyclic similar vertices into a single vertex. By identifying cyclic similar vertices into a single vertex, a miniature graph called reduced graph of the original graph is obtained. This is a method of transforming the original graph into a smaller one or summarizing large graphs into small ones. This is similar to the technique used in map making and it is very useful in the study of large complicated graphs which usually appear in telecommunication networks, social networking systems etc,. Also a new graph matrix called cyclic distance matrix of a graph is introduced and some of its properties are studied. It is proved that the determinant of cyclic distance matrix of a graph whose reduced graph is free of cyclic flowers with more than two maximal cyclic components is independent of the structure of the graph but it depends only on the number of maximal cyclic components in the graph.

Epilogue

Some scope for future studies are suggested here.

1. Identify cyn^*-unique graph classes.
2. Characterize the polynomials over the set of non negative integers which may be the cycle neighbor polynomial of some simple finite graphs.
3. Characterize graphs G for which $\gamma(G) = \gamma_H(G)$.
4. Characterize graphs G for which $\gamma(G) = \gamma_{T_1}(G)$.
5. Characterize graphs G for which $\gamma(G) = \gamma_{T_0}(G)$.
6. Characterize graphs G for which $\gamma(G) = \gamma_{T_0}(G) = \gamma_{T_1}(G) = \gamma_H(G)$.
7. Characterize graphs G for which $\gamma(G) < \gamma_{T_0}(G) < \gamma_{T_1}(G) < \gamma_H(G)$.
8. **Conjecture:** There are no graphs G for which $\gamma_{T_0}(G) > \gamma(G) + 1$.
9. **Conjecture:** If G is not a cycle neighbor graph, then $CD(G)$ is non singular.

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Publications

Papers Published/Accepted

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2. Annie Sabitha Paul and Raji Pilakkat, "A note on Hausdorff domination" , Global Journal of Pure and Applied Mathematics, Vol.15, No.4, 2019, pp 349-364.

3. Annie Sabitha Paul and Raji Pilakkat, "On cycle neighbor equivalence and cycle neighbor roots of a graph" , Malaya Journal of Matematik, Vol. S, No. 1, Pages 27-31, 2020. DOI: 10.26637/MJM0S20/0006

4. Annie Sabitha Paul and Raji Pilakkat, "Modified cycle neighbor polynomial of graphs" , Advances in Mathematics Scientific Journal Vol.9, Issue 10, 2020, pp 8883-8889 DOI: https://doi.org/10.37418/amsj.9.10.111

5. Annie Sabitha Paul and Raji Pilakkat, "Cycle neighbor polynomial of some graph operations " , Malaya Journal of Mathematik, Vol.8, Issue 4, 2020,

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6. Annie Sabitha Paul and Raji Pilakkat, "Cycle neighbor polynomial of graphs" , Communications in Mathematics and Applications, Vol.11, Issue 4, 2020, pp 549-558 DOI: 10.26713/cma.v11i4.1441.

7. Annie Sabitha Paul and Raji Pilakkat, "Cyclic distance in graphs " , To appear in South East Asian Journal of Mathematics and Mathematical Sciences, Vol.17, Issue 1, 2021.

Paper Presentations

1. Presented a paper on "On cycle neighbor equivalence and cycle neighbor roots of a graph " in the International Conference on Advances in Applicable Mathematics - ICAAM 2020 organized by Department of Mathematics, Bharathiar University on 21 and 22 January 2020.

2. Presented a paper on "Cyclic Distance and Reduced graph" in the National Seminar on Mathematical Research in Education and Industries organized by Sir Syed College, Taliparamba on 16 and 17 January 2020.

3. Presented a paper on "T_0 Domination " in the National Conference on Discrete Mathematics and its Applications organized by Department of Mathematical Sciences, Kannur University on 28 and 29 January 2019.

4. Presented a paper on "Complementary circuit domnation in graph" in the National Seminar on Graph Theory and its Applications organized by Department of Mathematics, MPMM SN Trusts College, Shoranur on 28Th and 29th October 2015.

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