STUDY ON SOME GENERALIZATIONS OF WEIBULL DISTRIBUTION

Thesis submitted to

UNIVERSITY OF CALICUT

for the award of degree of

DOCTOR OF PHILOSOPHY

in

STATISTICS

under Faculty of Science

by

GIRISH BABU. M.

under the guidance of

Prof.(Dr.) K. JAYAKUMAR

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

KERALA - 673 635

INDIA

June 2019

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALICUT

Prof.(Dr.) K. Jayakumar
Professor & Head

Calicut University (P.O.)
Kerala, India 673 635.
Phone : 0494- 2407341 / 340
Mob : 09847533374
Email : jkumar19@rediffmail.com

Date : 03-01-2020.

CERTIFICATE

Certified that the corrections / suggestions from the adjudicators, of the Ph.D. thesis entitled **STUDY ON SOME GENERALIZATIONS OF WEIBULL DISTRIBUTION** submitted by **Sri. Girish Babu. M.**, Research Scholar of the Department under my supervision and guidance, have been incorporated in this copy of the thesis.

Prof.(Dr.) K. Jayakumar
Research Supervisor

DEPARTMENT OF STATISTICS

UNIVERSITY OF CALICUT

Prof.(Dr.) K. Jayakumar
Professor & Head

Calicut University (P.O.)
Kerala, India 673 635.
Phone : 0494- 2407341 / 340
Mob : 09847533374
Email : jkumar19@rediffmail.com

Date : 21-06-2019.

CERTIFICATE

I hereby certify that the work reported in this thesis entitled **STUDY ON SOME GENERALIZATIONS OF WEIBULL DISTRIBUTION** submitted by **Sri. Girish Babu. M.** for the award of Doctor of Philosophy in Statistics, to the University of Calicut, is based on bonafide research work carried out by him under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma of any other University or Institution. Also certify that the contents of the thesis have been checked using anti plagiarism data base and no unacceptable similarity was found through the software check.

Prof.(Dr.) K. Jayakumar
Research Supervisor

DECLARATION

I hereby declare that this thesis entitled **STUDY ON SOME GENERALIZATIONS OF WEIBULL DISTRIBUTION** submitted to University of Calicut for the award of the degree of **Doctor of Philosophy in Statistics** under the Faculty of Science is an independent work done by me under the guidance and supervision of **Dr. K. Jayakumar**, Professor & Head, Department of Statistics, University of Calicut.

I also declare that this thesis contains no materials which has been accepted for the award of any other degree or diploma of any University or Institution and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references made in the text of the thesis.

Calicut University Campus
21-06-2019

Girish Babu. M.

ACKNOWLEDGEMENTS

First of all, I wish to express my sincere gratitude to my teacher and esteemed research guide ***Dr. K.Jayakumar****, Professor and Head, Department of Statistics, University of Calicut, for giving me the opportunity to carry out research under his guidance. Without his advice, encouragement, criticism, support and motivation at various stages, I could not have been able to complete this research work.*

I am extremely thankful to my teachers ***Dr. N.Raju****,* ***Dr. M.Manoharan*** *and* ***Dr. C.Chandran****, Department of Statistics, University of Calicut for their constant advice and support on my research work.*

I remember with gratitude the ***Librarians and Office Staffs*** *of the Department of Statistics, University of Calicut for providing me necessary facilities and help.*

I am very much thankful to ***Sri. K.K.Sankaran****, Research Scholar(FDP), Department of Statistics, University of Calicut for his support and advice during my research work.*

I express my sincere thanks to all the the Research Scholars, M.Phil. and M.Sc. students of the Department of Statistics, University of Calicut for their help and support.

I acknowledge gratefully ***Dr. S.Jayasree****, Principal, Govt. Arts and Science College, Kozhikode for her moral support and encouragement. I also acknowledge the help provided by* ***Prof. B.K.Vijayan, Prof. P.M.Raghavan, Prof. M.Sreenivasan*** *and* ***Prof. P.A.Sivaramakrishnan*** *former Principals of Govt. Arts and Science College, Kozhikode.*

I am extremely thankful to ***Dr. Z.A.Ahamad Ashraf****, Head, Department of Statistics, Govt. Arts and Science College, Kozhikode for his constant support and helping hand. I am thankful to all my* ***Colleagues*** *of Govt. Arts and Science College, Kozhikode for their moral support and encouragement during the period of this research work.*

I extend my sincere gratitude to ***Prof. K.K.Chandrasekharan****, former Head, Department of Mathematics, Govt. Arts and Science College, Kozhikode for his valuable suggestions to made this thesis presentable in language and style.*

*I wish to thank all my loving **friends** at the Research Hostel, University of Calicut who had bestowed me their love, affection and encouragement in ups and downs of my life during the research period.*

The financial support received from the UGC, XII th Paln period through the Teacher Fellowship under FDP is gratefully acknowledged.

*Memories of love, affection and care of my father **Sri. Appu.M (Late)** and mother **Smt. Ammalu.M (Late)** have been a great inspiration and force which lead me to achieve higher levels in my life.*

*I am grateful to my wife **Manju.C**, son **Sarang.G**, daughter **Thrishara.G**, father-in-law **Sri. K.Chellappan Master**, mother-in-law **Smt. C.K.Sathiyabhama** and my family members for their moral support, love and encouragement.*

Once again, I express my heartfelt thanks to all who supported me in many ways for the completion of this research work.

Girish Babu. M.

STUDY ON SOME GENERALIZATIONS OF WEIBULL DISTRIBUTION

CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER

ONE

INTRODUCTION

1.1 Introduction

The modern statistical distribution theory has its stress on problem-solving, confronted by the practitioners and the applied researchers in various fields. It becomes necessary to introduce a variety of probability models that suits the problem for a better assessment and proper decision making in real life phenomenon.

Statistical distributions are widely applied in various fields for modelling of lifetime data, from reliability engineering for the study of machine life cycles, medical sciences for the modelling of survival times of patients after surgery or duration to recurrence of a kind of cancer after surgical removal, computer sciences for the modelling of the failure rates of a software system, the modelling of durations without claims of customer policies in the insurance sector, to the modelling of duration of marriage till divorce in social sciences. The well known fundamental continuous distributions such as exponential, gamma, Weibull and Rayleigh are very limited in their characteristics and are unable to show wide flexibility in lifetime data modelling. Generalized distributions have been widely studied in statistics literature and numerous authors have developed various classes of distributions.

The Weibull distribution is very popular in modelling lifetime data and modelling phenomenon with monotone failure rates. This distribution is named after Waloddi Weibull, who was the first to promote the uses of this distribution to model the breaking strength of materials (Weibull,1939). A similar model was proposed earlier by Rosin and Rammler (1933) in the context of modelling the variability in the diameter of powder particles which are being greater than a specific size. An earliest known publication dealing with the Weibull distribution is a work by Fisher and Tippett (1928), which is obtained as the limiting distribution of the smallest extremes in a sample. Gumbel (1958) refers to the Weibull distribution as the third asymptotic distribution of the smallest extremes. The Weibull and related models have been applied in many areas, and for solving a variety of problems from many disciplines. An extensive review of some modifications of the Weibull distribution is presented in Murthy et al. (2004). (See also, Pham and Lai (2007) and Lai et al. (2011)). Jayakumar and Babu (2015) studied some generalizations of Weibull distribution and the related time series models.

Here we give a brief description of some extended families of continuous and discrete Weibull distributions, which are mentioned in the various chapters of this thesis.

1.2 Weibull and Related Distributions

A continuous random variable X is said to follow two parameter Weibull distribution if its cumulative distribution function (cdf) is given by

$$F(x;\beta,\alpha) = 1 - e^{-(\beta x)^{\alpha}} \ ; \ x > 0, \tag{1.2.1}$$

where, $\beta > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. This distribution includes the exponential and Rayleigh distribution as special cases. Setting $\lambda = \beta^{\alpha}$, this distribution can be expressed as

$$F(x) = 1 - e^{-\lambda x^{\alpha}} \ ; \ x > 0. \tag{1.2.2}$$

The corresponding probability density function (pdf) and hazard rate function (hrf) are respectively

$$f(x) = \lambda \alpha x^{\alpha-1} e^{-\lambda x^{\alpha}} \ ; \ x > 0, \tag{1.2.3}$$

and

$$h(x) = \lambda \alpha x^{\alpha-1} \ ; \ x > 0, \tag{1.2.4}$$

where $\lambda, \alpha > 0$. The hrf can be increasing, decreasing or constant depending on $\alpha > 1$, $\alpha < 1$ or $\alpha = 1$. Weibull distribution does not exhibit any kind of non-monotonic hazard rate shape. For more details on Weibull and related distributions see, Rinne (2009).

The exponential distribution is one of the widely applied continuous distributions. A continuous random variable X is said to have an exponential distribution with scale parameter $\beta > 0$, if its cdf is given by

$$F(x;\beta) = 1 - e^{-\beta x} \ ; \ x > 0. \tag{1.2.5}$$

This distribution can be viewed as a continuous analogue of the geometric distribution.

A continuous random variable X is said to follow generalized exponential distribution if its cdf is given by

$$F(x;\beta,\theta) = (1 - e^{-\beta x})^{\theta} \ ; \ x > 0, \tag{1.2.6}$$

where, $\beta > 0$ is the scale parameter, $\theta > 0$ is the shape parameter. This distribution is introduced by Gupta and Kundu (1999).

If $Y = \frac{1}{X}$, where X follows the Weibull distribution, then Y has the inverse Weibull (IW) distribution. The cdf, pdf and hrf of an IW distribution are respectively given by

$$F(y) = e^{-\lambda y^{-\alpha}} \ ; \ y > 0, \tag{1.2.7}$$

$$f(y) = \lambda\alpha y^{-\alpha-1} e^{-\lambda y^{-\alpha}} \ ; \ y > 0, \tag{1.2.8}$$

and

$$h(y) = \lambda\alpha y^{-\alpha-1} \ ; \ y > 0, \tag{1.2.9}$$

where $\lambda, \alpha > 0$. The hrf of the IW distribution has unimodal shape. This distribution is also called as the type 2 extreme value or Fréchet distribution. Some generalizations of IW distribution are Kumaraswamy inverse Weibull distribution studied in Shahbaz et al. (2012) and beta inverse Weibull distribution studied in Khan (2010).

Let X be a Weibull random variable with parameters λ and α. Then $\frac{Y-a}{b} = \log(\lambda X^{\alpha})$ has the log-Weibull distribution with cdf, pdf and hrf are respectively given by

$$F(y) = 1 - \exp\{-e^{\frac{y-a}{b}}\} \; ; \; -\infty < y < \infty, \tag{1.2.10}$$

$$f(y) = \frac{1}{b} e^{\frac{y-a}{b}} \exp\{-e^{\frac{y-a}{b}}\}, \tag{1.2.11}$$

and

$$h(y) = \frac{1}{b} e^{\frac{y-a}{b}}, \tag{1.2.12}$$

where $-\infty < a < \infty$ and $b > 0$. The hrf is an increasing function of y. This distribution is also known as the type I extreme value distribution or Gumbel distribution.

Cohen (1973) introduced the reflected Weibull distribution by considering the transformation $Y = -X$, where X is a Weibull random variable. The cdf, pdf and hrf are respectively given by

$$F(y) = e^{-\lambda(-y)^{\alpha}} \; ; \; -\infty < y < 0, \tag{1.2.13}$$

$$f(y) = \lambda\alpha(-y)^{\alpha-1} e^{-\lambda(-y)^{\alpha}}, \tag{1.2.14}$$

and

$$h(y) = \frac{\lambda\alpha(-y)^{\alpha-1} e^{-\lambda(-y)^{\alpha}}}{1 - e^{-\lambda(-y)^{\alpha}}}, \tag{1.2.15}$$

where $\lambda, \alpha > 0$. The hrf is an increasing function of y.

Mudholkar and Srivastava (1993) introduced the exponentiated Weibull (EW) distribution by adding a new shape parameter θ to the Weibull distribution. The cdf, pdf and hrf are respectively given by

$$F(x) = (1 - e^{-\lambda x^\alpha})^\theta \; ; \; x > 0, \tag{1.2.16}$$

$$f(x) = \theta\lambda\alpha x^{\alpha-1} e^{-\lambda x^\alpha}(1 - e^{-\lambda x^\alpha})^{\theta-1}, \tag{1.2.17}$$

and

$$h(x) = \frac{\theta\lambda\alpha x^{\alpha-1} e^{-\lambda x^\alpha}(1 - e^{-\lambda x^\alpha})^{\theta-1}}{1 - (1 - e^{-\lambda x^\alpha})^\theta}, \tag{1.2.18}$$

where $\theta, \lambda, \alpha > 0$. This distribution accommodates unimodal, bathtub and a broad variety of monotone failure rates.

Xie and Lai (1995) introduced the additive Weibull (AW) distribution by combining the failure rates of two Weibull distributions of which one has a decreasing failure rate and the other has an increasing failure rate. The cdf, pdf and hrf of AW distribution are respectively given by

$$F(x) = 1 - e^{-(\beta x^\alpha + \gamma x^\delta)}, \tag{1.2.19}$$

$$f(x) = (\alpha\beta x^{\alpha-1} + \delta\gamma x^{\delta-1}) e^{-(\beta x^\alpha + \gamma x^\delta)}, \tag{1.2.20}$$

and

$$h(x) = (\alpha\beta x^{\alpha-1} + \delta\gamma x^{\delta-1}), \tag{1.2.21}$$

where $\beta > 0, \gamma > 0$ are scale parameters and $\alpha > \delta > 0$ or $\delta > \alpha > 0$ are shape parameters. The hrf increases when $\alpha > 1$ and $\delta > 1$, decreases when $\alpha < 1$ and $\delta < 1$, and bathtub shape when $\alpha > 1$ and $\delta < 1$ (or $\alpha < 1$ and $\delta > 1$).

Lai et al. (2003) introduced the modified Weibull distribution with cdf, pdf and hrf are respectively given by

$$F(x) = 1 - e^{-\lambda x^{\alpha} e^{\beta x}} \; ; \; x > 0, \tag{1.2.22}$$

$$f(x) = \lambda(\alpha + \beta x)x^{\alpha-1}e^{\beta x}e^{-\lambda x^{\alpha} e^{\beta x}}, \tag{1.2.23}$$

and

$$h(x) = \lambda(\alpha + \beta x)x^{\alpha-1}e^{\beta x}, \tag{1.2.24}$$

where $\lambda > 0, \alpha, \beta \geq 0$ and at most one of α or β is equal to zero. When $\beta = 0$, this becomes the Weibull distribution. The hrf can be increased or bathtub shaped.

The modified Weibull extension with bathtub shaped failure rate function was proposed by Xie et al. (2002). The cdf, pdf and hrf are respectively given by

$$F(x) = 1 - \exp\{-\lambda\alpha(1 - e^{(\frac{x}{\alpha})^{\beta}})\} \; ; \; x > 0, \tag{1.2.25}$$

$$f(x) = \lambda\beta(\frac{x}{\alpha})^{\beta-1}\exp\{(\frac{x}{\alpha})^{\beta} + \lambda\alpha(1 - e^{(\frac{x}{\alpha})^{\beta}})\}, \tag{1.2.26}$$

and

$$h(x) = \lambda\beta(\frac{x}{\alpha})^{\beta-1}\exp\{(\frac{x}{\alpha})^{\beta}\}, \tag{1.2.27}$$

where $\lambda, \alpha, \beta > 0$. The hrf is an increasing function when $\beta \geq 1$ and is bathtub shaped when $\beta < 1$.

Sarhan and Zaindin (2009) introduced a three parameter modified Weibull distribution with cdf, pdf and hrf are respectively given by

$$F(x) = 1 - e^{-(\beta x + \gamma x^{\delta})} \; ; \; x > 0, \tag{1.2.28}$$

$$f(x) = (\beta + \gamma \delta x^{\delta - 1}) e^{-(\beta x + \gamma x^{\delta})}, \tag{1.2.29}$$

and

$$h(x) = \beta + \gamma \delta x^{\delta - 1}, \tag{1.2.30}$$

where $\beta, \gamma, \delta > 0$. This distribution can be obtained as a particular case of additive Weibull distribution by setting one of the two shape parameters α or δ in Eqn.(1.2.19) to be equal to one. The hrf is monotonically increasing if $\delta > 1$ and is monotonically decreasing if $\delta < 1$.

The beta Weibull distribution was proposed by Famoye et al. (2005). The cdf and pdf are respectively given by

$$F(x) = I_{1 - e^{-\lambda x^{\alpha}}}(a, b) \; ; \; x > 0, \tag{1.2.31}$$

and

$$f(x) = \frac{1}{B(a,b)} \lambda \alpha x^{\alpha - 1} (1 - e^{-\lambda x^{\alpha}})^{a-1} e^{-abx^{\alpha}}, \tag{1.2.32}$$

where $a, b, \lambda, \alpha > 0$ and $I_{\{.\}}(a, b)$ denotes the incomplete gamma distribution. The hrf of this distribution increases if $\alpha \geq 1$ and $a\alpha > 1$, decreases if $\alpha < 1$ and $a\alpha < 1$, bathtub shaped if $\alpha > 1$ and $a\alpha < 1$, and unimodal shaped if $\alpha \leq 1$ and $a\alpha > 1$. When $a = b = 1$, this distribution becomes two parameter Weibull distribution.

The beta modified Weibull distribution with five parameters was introduced by Silva et al. (2010a). The cdf and pdf are respectively

$$F(x) = I_{1 - e^{-\lambda x^{\alpha} e^{\beta x}}}(a, b) \; ; \; x > 0, \tag{1.2.33}$$

and

$$f(x) = \frac{1}{B(a,b)} \lambda(\alpha + \beta x) x^{\alpha-1} e^{\beta x} (1 - e^{-\lambda x^{\alpha} e^{\beta x}})^{a-1} e^{-\lambda b x^{\alpha} e^{\beta x}}, \tag{1.2.34}$$

where $a, b, \alpha, \lambda > 0$ and $\beta \geq 0$. When $a = b = 1$ and $\beta = 0$, this distribution becomes the two parameter Weibull distribution. The hrf allows increasing, decreasing, bathtub and unimodal shapes.

Carrasco et al. (2008) introduced the generalized modified Weibull distribution. Its cdf, pdf and hrf are respectively given by

$$F(x) = (1 - e^{-\lambda x^{\alpha} e^{\beta x}})^{\theta} \; ; \; x > 0, \tag{1.2.35}$$

$$f(x) = \lambda \theta x^{\alpha-1} (\alpha + \beta x) e^{\beta x - \lambda x^{\alpha} e^{\beta x}} [1 - e^{-\lambda x^{\alpha} e^{\beta x}}]^{\theta-1}, \tag{1.2.36}$$

and

$$h(x) = \frac{\lambda \theta x^{\alpha-1} (\alpha + \beta x) e^{\beta x - \lambda x^{\alpha} e^{\beta x}} [1 - e^{-\lambda x^{\alpha} e^{\beta x}}]^{\theta-1}}{1 - (1 - e^{-\lambda x^{\alpha} e^{\beta x}})^{\theta}}, \tag{1.2.37}$$

where $\lambda, \theta > 0, \alpha, \beta \geq 0$ and at most one of the α or β is equal to zero. The hrf allows increasing, decreasing, bathtub and unimodal shapes.

Cordeiro et al. (2010) introduced the Kumaraswamy Weibull distribution. The cdf, pdf and hrf are respectively given by

$$F(x) = 1 - [1 - (1 - e^{-\lambda x^{\alpha}})^{a}]^{b} \; ; \; x > 0, \tag{1.2.38}$$

$$f(x) = ab\lambda\alpha x^{\alpha-1} e^{-\lambda x^{\alpha}} (1 - e^{-\lambda x^{\alpha}})^{a-1} [1 - (1 - e^{-\lambda x^{\alpha}})^{a}]^{b-1}, \tag{1.2.39}$$

and

$$h(x) = \frac{ab\lambda\alpha x^{\alpha-1} e^{-\lambda x^{\alpha}} (1 - e^{-\lambda x^{\alpha}})^{a-1}}{1 - (1 - e^{-\lambda x^{\alpha}})^{a}}, \tag{1.2.40}$$

where $\lambda, \alpha, a, b > 0$. The hrf can be constant, increasing, decreasing, bathtub and unimodal shapes. Cordeiro et al. (2014) studied the Kumaraswamy modified Weibull distribution. Eissa (2017) studied the exponentiated Kumaraswamy-Weibull distribution.

Now we discuss some discrete Weibull distributions which are used in our study for deriving new results.

1.2.1 Discrete Weibull distributions

Nakagawa and Osaki (1975) introduced the type I discrete Weibull distribution by considering lifetime as the integer part of the continuous Weibull distribution. The survival function, probability mass function (pmf) and hrf are given by

$$S(x) = 1 - q^{x^\alpha}, \tag{1.2.41}$$

$$p(x) = q^{x^\alpha} - q^{(x+1)^\alpha}, \tag{1.2.42}$$

and

$$h(x) = 1 - q^{(x+1)^\alpha - x^\alpha}, \tag{1.2.43}$$

where $x = 0, 1, 2, \ldots .; 0 < q < 1$ and $\alpha > 0$. The hrf increases when $\alpha > 1$, decreases when $\alpha < 1$ and constant when $\alpha = 1$.

Stein and Dattero (1984) proposed the type II discrete Weibull distribution. The hrf of this distribution is given by

$$h(x) = \begin{cases} \lambda\alpha x^{\alpha-1}, & \text{for } x = 1, 2, \ldots m, \\ 0, & \text{for } x = 0, \end{cases} \tag{1.2.44}$$

where $h(x) \leq 1$ and m is a positive integer defined as,

$$h(x) = \begin{cases} int\{\lambda^{-(\alpha-1)^{-1}}\}, & \text{if } \alpha > 1, \\ +\infty, & \text{if } \alpha \leq 1, \end{cases} \tag{1.2.45}$$

where $int\{.\}$ denotes the integer value.

Padgett and Spurrier (1985) proposed type III discrete Weibull distribution and its hrf is given by

$$h(x) = 1 - e^{-\lambda(x+1)^{\alpha}} \ ; \ x = 0, 1, 2, \ldots , \tag{1.2.46}$$

where $\lambda > 0$ and $-\infty < \alpha < \infty$.

Jazi et al. (2010) introduced the discrete inverse Weibull distribution. The cdf, pmf and hrf are respectively given by

$$F(x) = q^{x^{-\alpha}} \ ; \ x = 1, 2, \ldots , \tag{1.2.47}$$

$$p(x) = \begin{cases} q, & \text{if } x = 1, \\ q^{x^{-\alpha}} - q^{(x-1)^{-\alpha}}, & \text{if } x = 2, 3, \ldots , \end{cases} \tag{1.2.48}$$

and

$$h(x) = \frac{q^{x^{-\alpha}} - q^{(x-1)^{-\alpha}}}{1 - q^{x^{-\alpha}}} \ ; \ x = 1, 2, \ldots . \tag{1.2.49}$$

Noughabi et al. (2011) introduced the discrete modified Weibull distribution. The cdf, pmf and hrf are given by

$$F(x) = 1 - q^{x^{\alpha}c^{x}} \ ; \ x = 0, 1, 2, \ldots , \tag{1.2.50}$$

$$p(x) = q^{x^{\alpha}c^{x}} - q^{(x+1)^{\alpha}c^{x+1}}, \tag{1.2.51}$$

and

$$h(x) = 1 - q^{(x+1)^{\alpha}c^{x+1} - x^{\alpha}c^{x}}, \tag{1.2.52}$$

where $0 < q < 1, \alpha > 0$ and $c \geq 0$. The hrf shows increasing and bathtub shapes.

Bebbington et al. (2012) studied the discrete additive Weibull distribution. The cdf, pmf and hrf are respectively given by

$$F(x) = 1 - q_1^{x^\alpha} q_2^{x^\delta} \; ; \; x = 0, 1, 2, \dots , \tag{1.2.53}$$

$$p(x) = q_1^{x^\alpha} q_2^{x^\delta} - q_1^{(x+1)^\alpha} q_2^{(x+1)^\delta}, \tag{1.2.54}$$

and

$$h(x) = 1 - q_1^{(x+1)^\alpha - x^\alpha} q_2^{(x+1)^\delta - x^\delta}, \tag{1.2.55}$$

where $0 < q_1, q_2 < 1$ and $\alpha, \delta > 0$. The hrf shows increasing, decreasing and bathtub shapes.

1.3 Extended Classes of Distributions

Recently, several attempts have been made by various researchers to develop new families of distributions to extend well-known distributions. As a result, several classes of distributions were developed by adding one or more parameters. Some popular generators are, Marshall-Olkin family by Marshall and Olkin (1997), beta-G family by Eugene et al. (2002), gamma-G family by Zografos and Balakrishnan (2009), Kumaraswamy-G family by Cordeiro and de Castro (2011), Weibull-G family Bourguignon et al. (2014), and extended-G geometric family by Cordeiro et al. (2016). One of the main objectives for developing extended families is to explain how the lifetime phenomenon arises in various fields like public health, insurance, industry, engineering, life-testing and many others. In order to model both monotonic and non-monotonic failure rate shaped data, these families are very flexible for model fitting.

In the next section, we discuss some well known families which are used in this thesis to develop new classes of distributions.

1.3.1 Transmuted family of distributions

According to the Quadratic Rank Transmutation Map (QRTM), approach by Shaw and Buckley (2007), the cdf satisfies the relationship

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2 \; ; \; |\lambda| \leq 1, \tag{1.3.1}$$

where $G(x)$ is the cdf of the base distribution. When $\lambda = 0$, we get the cdf of the base random variable. Differentiating Eqn.(1.3.1) yields

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)] \; ; \; |\lambda| \leq 1, \tag{1.3.2}$$

where $f(x)$ and $g(x)$ are the pdfs corresponding to $F(x)$ and $G(x)$ respectively. The survival function of $F(x)$ in Eqn.(1.3.1) is given by

$$\bar{F}(x) = 1 - F(x) = 1 - G(x)[1 + \lambda\bar{G}(x)] \; ; \; |\lambda| \leq 1, \tag{1.3.3}$$

where $\bar{G}(x) = 1 - G(x)$.

Recently, various research papers have been appeared in the literature on transmuted generalization of distributions. Some of them are: transmuted extreme value distribution by Aryal and Tsokos (2009), transmuted Weibull distribution by Aryal and Tsokos (2011), transmuted modified Weibull distribution by Khan and King (2013a), transmuted generalized inverse Weibull distribution by Khan and King

(2013b), transmuted log-logistic distribution by Aryal (2013), transmuted additive Weibull distribution by Elbatal and Aryal (2013) and transmuted Weibull Lomax by Afify et al. (2015).

1.3.2 T-X family of distributions

Alzaatreh et al. (2013b) introduced the Transformed-Transformer (or T-X family) method to derive families of distributions by using any pdf as a generator. Let $r(t)$ be the pdf of a random variable $T \in [a, b]$, for $-\infty \leq a < b \leq \infty$. Let $W(F(x))$ be a function of the cdf $F(x)$ of any random variable X, so that, $W(F(x))$ satisfies the conditions

$$\left.\begin{array}{l} W(F(x))\epsilon[a,b], \\ W(F(x)) \quad \text{is absolutely continuous and monotonically non-decreasing,} \\ W(F(x)) \rightarrow a \quad \text{as} \quad x \rightarrow -\infty \quad \text{and} \quad W(F(x)) \rightarrow b \quad \text{as} \quad x \rightarrow \infty. \end{array}\right\} \tag{1.3.4}$$

Definition 1.3.1. *Let X be a random variable with pdf $f(x)$ and cdf $F(x)$ respectively. Let T be a continuous random variable with pdf $r(t)$ defined on $[a, b]$. Then, the cdf of T-X family of distributions is defined as*

$$G(x) = \int_a^{W(F(x))} r(t)dt = R\{W(F(x))\}, \tag{1.3.5}$$

where $R\{.\}$ is the cdf of the random variable T. The corresponding pdf is given by

$$g(x) = \frac{d}{dx}[W(F(x))]r(W(F(x))). \tag{1.3.6}$$

Different choices of $W(F(x))$ will give new family of distributions. The definition of $W(F(x))$ depends on the support of the random variable T. T-X family is a method for generating generalized distributions of X using T.

Aljarrah et al. (2014) introduced a wider class of $W(F(x))$ functions defined in Eqn.(1.3.4) as, $W : (0, 1) \to (a, b)$, where $-\infty \leq a < b \leq \infty$, is right continuous and non decreasing function, such that, $\lim_{x\to 0+} W(x) = a$ and $\lim_{x\to 1-} W(x) = b$. Then $F(x), -\infty < x < \infty$, is a distribution function.

Several research papers have appeared in the literature based on the T-X family. Some of them are: the Weibull-Pareto distribution by Alzaatreh et al. (2013a), gamma-half normal distribution by Alzaatreh and Knight (2013), Weibull-X family by Alzaatreh and Ghosh (2015), beta Marshall-Olkin family by Alizadeh et al. (2015), and so on.

1.3.3 Compounded-G classes of distributions

The compounded-G classes of distributions can be constructed as follows: Suppose a system has N components and are assumed to be independent and identically distributed (i.i.d.) at a given time. The lifetime of the i^{th} component is denoted by Y_i, and each component is made of α parallel units, so that the system will fail only if all the components fail. But for a series system, the failure of any component will destroy the entire system. Suppose that the random variable N follows a discrete distribution with pmf, $P(N = n), n = 1, 2, ...$. Let the failure times of the i^{th} components are $Z_{i1}, Z_{i2}, ..., Z_{i\alpha}$ and are i.i.d. with a suitable cdf depending on the parameter vector Θ.

Let $Y = \min(Y_1, Y_2, ..., Y_N)$, then the conditional cdf of Y given N is

$$F(y|N) = P(\min(Y_1, Y_2, ..., Y_N) < y|N) = 1 - [1 - G^{\alpha}(y; \Theta)]^N, \tag{1.3.7}$$

where $G(y;\Theta)$ is the baseline cdf. Also, define $Z = \max(Y_1, Y_2, ..., Y_N)$, then the conditional (or complementary) cdf of Z given N is

$$F(z|N) = P(\max(Y_1, Y_2, ..., Y_N) < y|N) = [G^{\alpha}(y;\Theta)]^N. \tag{1.3.8}$$

The unconditional cdf of Y follows from Eqn.(1.3.7) as

$$F(y) = \sum_{n=1}^{\infty} F(y|N = n)P(N = n) = 1 - \sum_{n=1}^{\infty}[1 - G^{\alpha}(y;\Theta)]^n P(N = n). \tag{1.3.9}$$

The unconditional cdf of Z follows from Eqn.(1.3.8) as

$$F(z) = \sum_{n=1}^{\infty} F(z|N = n)P(N = n) = \sum_{n=1}^{\infty}[G^{\alpha}(y;\Theta)]^n P(N = n). \tag{1.3.10}$$

By choosing a suitable discrete distribution for N, we can develop different compounding-G classes. Marshall and Olkin (1997) proposed a method of introducing a parameter in distributions. The Marshall-Olkin distribution is defined in terms of survival function as

$$\bar{F}(x;\theta) = \frac{\theta\bar{G}(x)}{1 - \bar{\theta}\bar{G}(x)} = \frac{\theta\bar{G}(x)}{G(x) + \theta\bar{G}(x)} \ ; \ -\infty < x < \infty, \theta > 0, \tag{1.3.11}$$

where, $\bar{\theta} = 1 - \theta$ and $\bar{F}(x) = 1 - F(x)$, which is the survival function of the random variable X. Many authors have studied the properties of various univariate distributions belonging to the family of Marshall-Olkin distributions, see Alice and Jose (2003,2005), Ghitany et al. (2005,2007) and Jayakumar and Thomas (2008).

Some well-known compound-G classes based on geometric distribution are, the exponential-geometric (EG) by Adamidis and Loukas (1998), the modified Weibull-geometric by Wang and Elbatal (2015), the generalized exponential-geometric by Silva et al. (2010b), the Weibull-geometric (WG) by Barreto-Souza et al. (2011), the beta exponential-geometric by Bidram (2012), the beta Weibull-geometric by Bidram et al. (2013), the exponentiated complementary exponential-geometric by Yamachi et al. (2013), the exponentiated Weibull-geometric by Chung and Kang (2014), the complementary Weibull-geometric (CWG) by Tojeiro et al. (2014), the exponentiated G-geometric (EGG) by Nadarajah et al. (2015) and the additive Weibull-geometric (AWG) by Elbatal et al. (2016).

The recent trends in developing compound classes of distributions can be classified as, compounding a G-class with discrete model, combining continuous model with compound power series class, combining compound G-class with the non-compound G-class and combining transmuted G-class with well-known compound distributions. Some notable works in this direction are Weibull-Power series by Morais and Barreto-Souza (2011), G-Poisson Lindley by Asgharzadeh et al. (2014), transmuted Weibull-geometric by Merovci and Elbatal (2014), transmuted complementary Weibull-geometric by Afify et al. (2014), transmuted exponentiated-Weibull by Saboor et al. (2015), exponentiated G-Poisson by Gomes et al. (2015), Poisson-X by Tahir et al. (2016), complementary generalized modified Weibull by Bagheri et al. (2016), Gompertz power series by Jafari and Tahmasebi (2016) and transmuted exponentiated Weibull-geometric by Saboor et al. (2016). For a recent survey on generalized compounded classes of distributions see Tahir and Cordeiro (2016).

1.4 Some Basic Statistical Concepts

1.4.1 Hazard rate function

If X is a lifetime random variable with cdf $F(x)$, pdf $f(x)$ and survival function $S(x)$, then the hrf is defined as

$$h(x) = \frac{f(x)}{S(x)},$$

and $h(x)\Delta x$ represents the approximate probability of failure in the interval $[x, x + \Delta x]$. The hrf plays an important role in lifetime data modelling. A lifetime distribution is said to have an increasing hrf if $h(x)$ is monotonically increasing over time and is a decreasing hrf if $h(x)$ is monotonically decreasing. If $h(x)$ initially decreases, followed by an approximate constant shape and then increasing, then the distribution is said to have a bathtub shape hrf. The distribution is said to have an upside-down bathtub (or unimodal) shape hrf, if its $h(x)$ has a unique mode. The different shapes of $h(x)$ can be investigated using the first order derivative $h'(x)$. If $h'(x) > 0$, for all x, then the shape of hrf is monotonically increasing; if $h'(x) < 0$, for all x, then the shape of hrf is monotonically decreasing; if $h'(x) = 0$, for all x, then hrf is constant; if $h'(x) < 0$, for all $x \in (0, x_0)$ and $h'(x) > 0$, for all $x > x_0$ and the value x_0 is unique and positive solution of $h'(x) = 0$, then the hrf shows a bathtub shape; and if $h'(x) > 0$, for all $x \in (0, x_0)$ and $h'(x) < 0$, for all $x > x_0$ and the value x_0 is unique and positive solution of $h'(x) = 0$, then the hrf shows upside down-bathtub (or unimodal) shape.

1.4.2 Order statistics

Order statistics deals with the properties and applications of ordered random variables and functions involving them. It has a great role in the statistical study of extremes such as, floods and droughts, problems of breaking strength and fatigue failure, etc. David and Nagaraja (2003) studied various properties and applications of order statistics. Let $X_1, X_2, ..., X_n$ are n independent random variables, each with cdf $F(x)$ and are arranged in the order of magnitude as $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, we call $X_{(r)}$ the r^{th} order statistic $(r = 1, 2, ..., n)$. The cdf of the largest order statistic $X_{(n)}$ is given by

$$F_{(n)}(x) = P(X_{(n)} \leq x) = F^n(x), \tag{1.4.1}$$

and the cdf of the smallest order statistics $X_{(1)}$ is given by

$$F_{(1)}(x) = P(X_{(1)} \leq x) = 1 - [1 - F(x)]^n. \tag{1.4.2}$$

The cdf of the r^{th} order statistics is

$$F_{(r)}(x) = P(X_{(r)} \leq x) = \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i}. \tag{1.4.3}$$

The pdf of the r^{th} order statistics is

$$f_{(r)}(x) = \frac{1}{B(r, n-r-1)} F^{r-1}(x)[1 - F(x)]^{n-r} f(x). \tag{1.4.4}$$

If X is a discrete random variable with pmf $p(x),\ x = 0, 1, ..., n$, then the pmf of the r^{th} order statistics is given by

$$p_{(r)}(x) = F_{(r)}(x) - F_{(r)}(x - 1). \tag{1.4.5}$$

1.4.3 Entropy

It is well known that the entropy is a measure of uncertainty of probability distribution. Or in other words, it is the expected value of the information about the distribution. The popular entropy measures are the Rényi and Shannon entropies (Rényi, 1961; Shannon, 1948). The Rényi entropy of a continuous random variable X with pdf $f(x)$ is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \int_0^\infty f^\gamma(x)dx \ ; \ \gamma > 0, \gamma \neq 1. \tag{1.4.6}$$

The Shannon entropy is defined by $E[-\log(f(X))]$ and is a particular case of Rényi entropy for $\gamma \uparrow 1$.

1.4.4 Stress-strength parameter

Stress is defined as the load which tends to produce a failure of a component or a material, whereas strength is the ability of a component or material to accomplish its required function satisfactorily without failure when applying the external load. In reliability study, the stress-strength parameter describes the life of a component which has a random strength Y and is subjected to a random stress Z. If the stress (Z) applied to a component exceeds the strength (Y), then it fails. Thus the stress-strength parameter $R = P(Y > Z)$ measures the component reliability. This idea was first introduced by Birnbaum (1956) and further developed in Birnbaum and Mc Carty (1958). Estimation of R when Y and Z are i.i.d. random variables have considered in the literature. For a review, see Kotz et al. (2003). In continuous case,

R is defined as

$$R = P(Y > Z) = \int_0^{\infty} F_Z(y) f_Y(y) dy, \tag{1.4.7}$$

and in discrete case

$$R = \sum_{y=0}^{\infty} F_Z(y) p_Y(y). \tag{1.4.8}$$

1.4.5 Minification processes

Models with minification structure have been introduced in the literature as an alternative to the additive time series models. The study on minification processes began with the work of Tavares (1980). One of the important nonlinear models used to generate $\{X_n\}$ of non negative random variables is defined by

$$X_n = \begin{cases} X_0, & n = 0, \\ k \min(X_{n-1}, \epsilon_n), & n \geq 1, \end{cases} \tag{1.4.9}$$

where $k > 1$, is a constant and $\{\epsilon_n\}$ is an innovative process of i.i.d. random variables, such that $\{X_n\}$ is a stationary Markov process. Sim (1986) developed a first order autoregressive Weibull process. He showed that $\{X_n\}$ are stationary Weibull random variables with survival function $e^{-\frac{\theta x^{\alpha}}{k^{\alpha}-1}}$ if and only if $\{\epsilon_n\}$ is a sequence of i.i.d. Weibull random variables with survival function $e^{-\theta x^{\alpha}}$. Lewis and McKenzie (1991) showed that the stationary, autoregressive, Markovian minification processes introduced by Tavares (1980) and Sim (1986) can be extended to give processes with marginals other than the exponential and Weibull distributions. Jose et al. (2010) developed different types of autoregressive processes with minification structure and max-min structure. Jose (2011) considered various Marshall-Olkin distributions and developed autoregressive minification processes with stationary marginals as exponential, Weibull, uniform, Pareto and Gumbel distributions.

1.5 Various Tests for Goodness-of-fit

Goodness-of-fit test is a method used to examine whether a random sample is taken from a specific distribution. In this section, four methods of goodness-of-fit tests applied in this thesis are discussed. They are, the likelihood ratio test, Kolmogorov-Smirnov (K-S) test, Cramér-von Mises test and the Anderson Darling test.

1.5.1 Likelihood ratio test

The likelihood ratio test (LRT) is used to examine how well a model fits the given data set. This test is used to compare two nested models. Suppose that a random variable X has a pdf given by $f(x;\theta)$ with unknown parameter θ. The main objective is to test the null and alternative hypotheses, $H_0 : \theta \in \boldsymbol{\theta_0}$ and $H_1 : \theta \in \boldsymbol{\theta_1}$, where $\boldsymbol{\theta_0}$ and $\boldsymbol{\theta_1}$ are the parameter space of the reduced and full model respectively. The test statistic is given by

$$\omega = -2\log\left[\frac{L_0(\hat{\theta})}{L_1(\hat{\theta})}\right], \tag{1.5.1}$$

where L_0 and L_1 are the likelihood functions of the reduced and the full model respectively. Under H_0, ω is asymptotically distributed as a chi-square random variable with degrees of freedom equal to the difference between the number of parameters of the two models. When the p value obtained is less than 0.05, we reject the null hypothesis at the 5% level of significance and it implies that the full model provides a better fit than the reduced model.

1.5.2 Kolmogorov-Smirnov (K-S) test

Let $X_1, X_2, ..., X_n$ be a random sample taken from a population. The K-S test is used to test whether this sample belongs to a population with a specific distribution. The K-S test statistic measures the difference between the empirical distribution function of the given sample and the estimated cdf of the candidate distribution. The null and alternative hypotheses for the test are H_0: *The sample follows the specific distribution* and H_1: *The sample does not follow the specific distribution.*

Let $F(x_i)$ is the values of the cdf of the candidate distribution at x_i and $\hat{F}(x_i)$ is the value of the empirical distribution at x_i. The value of the K-S test statistic is defined by

$$D = sup_{x_i}(|F(x_i) - \hat{F}(x_i)|) \ ; \ i = 1, 2, ..., n. \tag{1.5.2}$$

The computed value of the test statistic is then compared with tabulated K-S value at a given significance level. If there are more than one distributions to be compared, the distribution with smaller K-S value is the most appropriate to fit the given data.

1.5.3 Cramér-von Mises (W*) and Anderson Darling (A*) Criteria

Let $F(x;\Theta)$ be the cdf and the form of F is known but Θ is unknown. Then the statistics Cramér-von Mises (W*) and Anderson Darling (A*) are computed as follows:

(i). Compute $\xi_i = F(x_i;\hat{\Theta})$ where the x_i's are in ascending order;

(ii). Compute $x_i = \phi^{-1}(\xi_i)$, where $\phi(.)$ is the cdf of standard normal distribution and $\phi^{-1}(.)$ is its inverse;

(iii). Compute $y_i = \phi((x_i - \bar{x})/s_x)$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$;

(iv). Calculate,

$$W^2 = \sum_{i=1}^{n}\left(y_i - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}$$

and

$$A = -n - \frac{1}{n}\sum_{1=1}^{n}\Big[(2i-1)\log(y_i) + (2n+1-2i)\log(1-y_i)\Big];$$

(v). $W^* = W^2(1 + \frac{0.5}{n})$ and $A^* = A^2(1 + \frac{0.75}{n} + \frac{2.25}{n^2})$, see Chen and Balakrishnan (1995). While comparing the models, the one with smallest W* and A* is the best model.

1.5.4 Information criteria

The consequences of increasing the number of parameters, usually improves the fit of a given model and of course the likelihood also increases irrespective of whether the additional parameter is important or not. When the models to be compared are not nested, the likelihood ratio test is not the best option and therefore one has to employ other methods to compare the models. The information criteria enable us to do this comparison when the models are not nested. The most widely used information criteria are; the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC) and Bayesian Information Criterion (BIC). Here, AIC=$-2\log L + 2k$, CAIC=$-2\log L + (\frac{2kn}{n-k-1})$, HQIC=$-2\log L + 2k\log(\log(n))$ and BIC=$-2\log L + k\log n$, where, L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. The appropriate model is one with minimum AIC, CAIC, HQIC and BIC values.

1.6 Discretization of Continuous Distributions

In lifetime modelling, the observed measurements may be discrete in nature. We come across such type of situations where lifetime is measured on a discrete scale. For instance, the convalescing period of a particular disease measured in days and the survival time of cancer patients in months, see Krishna and Singh (2009). Here the continuous lifetime is not measured on a continuous scale, but counted as a discrete random variable.

Developing discrete version of continuous distributions have attracted the attention of researchers. In recent decades, a large number of research papers, dealing with discrete distribution, which are derived by discretizing continuous random variables have appeared in the statistics literature. Lisman and van Zuylen (1972) proposed and Kemp (1997) studied the discrete normal distribution. Salvia and Bollinger (1982) introduced the basic results about discrete reliability and illustrated them with discrete lifetime distributions with one parameter. Roy (2003) studied another version of discrete normal distribution. A discrete analogue of Weibull distribution was first proposed by Nakagawa and Osaki (1975). Stein and Dattero (1984) introduced a second type discrete Weibull distribution and a third one were proposed by Padgett and Spurrier (1985). Sato et al. (1999) proposed discrete exponential distribution and applied this distribution to model defect count in semiconductor deposition equipment and defect count distribution per chips. Inusah and Kozubowski (2006) and Kozubowski and Inusah (2006) introduced discrete analogues of Laplace and skew-Laplace distributions, respectively. Krishna and Pundir (2009) introduced the discrete Burr distribution which led to the discrete Pareto distribution. A discrete analogue of the generalized exponential distribution of Gupta and Kundu (1999) was proposed by Nekoukhou et al. (2012). Chakraborty and Chakravarty (2012) introduced the discrete gamma distribution. Jayakumar and Sankaran (2018) introduced a generalization of discrete Weibull distribution and studied its properties.

The continuous random variable may be characterized either by its pdf, hrf, moments, etc. Construction of a discrete analogue from a continuous distribution is based on the principle of preserving one or more characteristic property of the continuous one. Discretization of continuous distribution can be done using different methodologies. Some of them are: i) discretize the continuous cdf, ii) discretize the continuous pdf, iii) discretize the continuous hrf and iv) obtain discrete lifetime distributions from the alternative hazard rate. A detailed survey of the methods and constructions of discrete analogues of continuous distributions are discussed in Chakraborty (2015).

Let X, be a continuous random variable, then its discrete analogue Y, can be derived by using the survival function as follows:

$$P(Y = y) = P(X \geq y) - P(X \geq y+1) = S_X(y) - S_X(y+1) \ ; \ y = 0, 1, 2, \dots \ . \quad (1.6.1)$$

where, $Y = \lfloor X \rfloor$ = largest integer less than or equal to X and $S_X(.)$ is the survival function of the random variable X. For a given continuous distribution, it is possible to generate corresponding discrete distribution using the Eqn.(1.6.1). Suppose the underlying distribution is exponential with survival function, $S_X(x) = P(X \geq x) = e^{-\beta x}$, then the pmf of its discrete version is given by

$$P(Y = y) = e^{-\beta y} - e^{-\beta(y+1)} = q^y - q^{y+1} = (1-q)q^y \ ; \ y = 0, 1, 2, \dots \ , \qquad (1.6.2)$$

where, $q = e^{-\beta}$. This is the geometric distribution with parameter q. By the similar way, Nakagawa and Osaki (1975) proposed the discrete Weibull distribution.

1.7 Objectives of the Study

The main objectives of the present study are as follows:

- To study the recent developments in the construction of families of continuous and discrete distributions.

- To construct new families of Weibull and related distributions.
- To construct discrete analogues of well known continuous distributions related to Weibull distribution.
- To study the statistical properties of the newly constructed distributions.
- To obtain the estimates of the parameters of the new models.
- To analyze the flexibility of the new models for real life data modelling.
- To develop various autoregressive minification process of the new models.
- To extend the new models to bivariate case.

1.8 Organization of the work

This thesis contains seven chapters, including this one. The other chapters are organized as follows:

In Chapter 2, we introduce a new class of continuous distribution called "T-transmuted - X family". Some members of this family are also proposed and studied. Being a special case of the exponential transmuted Weibull (ETW) distribution, the exponential transmuted exponential (ETE) distribution is studied in detail. Shapes of density function and hazard rate function of this distribution are discussed. Characterizations of ETE distribution based on truncated moments and hazard rate function are also derived. Applications of the distribution to model real life data are presented.

A new generalization of Weibull distributions called "Weibull truncated negative binomial (WTNB) distribution" has been developed in the Chapter 3 and its various properties are derived. Characterizations of the WTNB distribution are studied based on truncated moments and hazard rate function. Minification process with WTNB distribution marginals are obtained. Bivariate extension of WTNB distribution is also developed. Real data applications of WTNB distribution are discussed with two data sets.

In Chapter 4, we introduced a new bivariate distribution with modified Weibull distribution as marginals. The marginal and conditional probability distributions, mathematical expectations and moment generating function of the new bivariate model are derived. The bivariate copula function of the new model is proposed and a real data application is presented.

Some new discrete analogues of Weibull geometric and additive Weibull geometric distributions are introduced in Chapter 5. Their properties and applications are also discussed in this Chapter.

Chapter 6 presents discrete analogue of the complementary Weibull geometric distribution. Its mathematical properties are studied and obtained the derivations for the quantile function, probability generating function and distributions of order statistics. The flexibility of this distribution for data modelling are illustrated with two real-life data sets.

Finally, Chapter 7 gives the concluding remarks of the thesis and presents the possible future works in this direction.

CHAPTER

TWO

T-TRANSMUTED X FAMILY OF DISTRIBUTIONS

2.1 Introduction

[1] Lifetime distributions are used to explain the life of a system, a device, and in general, time-to-event data. Modelling and analyzing lifetime data becomes a crucial problem in many applied fields such as medicine, engineering, insurance, finance and so on. The distributions such as exponential, gamma and Weibull have been frequently used in statistics literature to analyze lifetime data. The quality of the statistical analysis procedures heavily depend on the assumed probability distributions. Because of this, extensive efforts have been made by many researchers to develop new classes of distributions. Jayakumar and Babu (2015) introduced a

[1]Some results included in this chapter have appeared in the paper Jayakumar and Babu (2017).

class of distributions containing Marshall-Olkin extended Weibull distribution and studied the role of this distribution in the study of minification process.

In Section 2, we introduce a broad class of lifetime distributions, which are developed by combining the T-X family of Alzaatreh et al. (2013b) and the transmuted family of distributions by Shaw and Buckley (2007). Section 3 proposes some members of the T-transmuted X family, such as exponential-transmuted uniform distribution, exponential-transmuted Fréchet distribution and exponential-transmuted Raleigh distribution. In Section 4, we study the exponential-transmuted Weibull distribution and an extensive study of one of its special case, called exponential-transmuted exponential distribution. In this section, we study the shape properties of its pdf and hrf, expressions for quantile function, moments and moment generating function. Characterizations based on truncated moments and hrf are also study in this section. Maximum likelihood estimation of its parameters, simulation study, expressions for entropy and two real life data applications are also obtain in this section.

2.2 A new extended family of continuous distributions

In the composite function $W(F(x))$ given in the Eqn.(1.3.5), we assume that $F(x)$ follows the transmuted family given in the Eqn.(1.3.1) and obtain some more generalized family of distributions. As a special case, we take $W(F(x)) = -ln[1 - F(x)]$, the cumulative hazard function of $F(x)$, where $F(x)$ is a transmuted family. That is,

$$W(F(x)) = -ln\left[1 - G(x)[1 + \lambda\bar{G}(x)]\right], |\lambda| \leq 1, \tag{2.2.1}$$

where $G(x)$ is the base distribution and $\bar{G}(x) = 1 - G(x)$. Then, the cdf of the new family is

$$J(x) = \int_0^{-ln\left[1-G(x)[1+\lambda\bar{G}(x)]\right]} dR(t) = R\left\{-ln\left[1 - G(x)[1 + \lambda\bar{G}(x)]\right]\right\}, \tag{2.2.2}$$

where $R(t)$ is the cdf of the random variable T with pdf $r(t)$. We call $J(x)$ as the "T-transmuted X family" of distributions. The pdf of $J(x)$ is

$$j(x) = \frac{d}{dx}[J(x)] = \frac{g(x)[1 + \lambda - 2\lambda G(x)]}{1 - G(x)[1 + \lambda\bar{G}(x)]} \quad r\left\{-ln\left[1 - G(x)[1 + \lambda\bar{G}(x)]\right]\right\}. \tag{2.2.3}$$

The hrf is given by

$$h(x) = \frac{j(x)}{1 - J(x)} = \frac{g(x)[1 + \lambda - 2\lambda G(x)]}{1 - G(x)[1 + \lambda\bar{G}(x)]} \frac{r\left\{-ln[1 - G(x)(1 + \lambda\bar{G}(x))]\right\}}{1 - R\left\{-ln[1 - G(x)(1 + \lambda\bar{G}(x))]\right\}}. \tag{2.2.4}$$

The shapes of the pdf and hrf of T-transmuted X family can be described analytically. The critical points of the density function are the roots of the equation:

$$\begin{aligned} \frac{\partial\ ln[j(x)]}{\partial x} &= \frac{g'(x)}{g(x)} - \frac{2\lambda g(x)}{1 + \lambda - 2\lambda G(x)} - \frac{g(x)[1 + \lambda - 2\lambda G(x)]}{1 - G(x)[1 + \lambda\bar{G}(x)]} \\ &\quad \left[\frac{1}{r\left\{-ln[1 - G(x)(1 + \lambda\bar{G}(x))]\right\}} + 1\right] = 0. \end{aligned} \tag{2.2.5}$$

Here, the Eqn.(2.2.5) may have more than one root. If $x = x_0$ is a root, then it corresponds to a local maximum if $\frac{\partial^2 ln[j(x)]}{\partial x^2} < 0$, a local minimum if $\frac{\partial^2 ln[j(x)]}{\partial x^2} > 0$, and a point of inflection if $\frac{\partial^2 ln(j(x))}{\partial x^2} = 0$.

Similarly, the critical points of $h(x)$ are the roots of the equation:

$$\begin{aligned}\frac{\partial\ ln[h(x)]}{\partial x} =& \frac{g'(x)}{g(x)} - \frac{2\lambda g(x)}{1+\lambda-2\lambda G(x)} \\ & - \frac{g(x)[1+\lambda-2\lambda G(x)]}{1-G(x)[1+\lambda\bar{G}(x)]}\Bigg[\frac{1}{r\left\{-ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}} + \\ & \frac{r\left\{-ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}}{1-R\left\{-ln[1-G(x)(1+\lambda\bar{G}(x))]\right\}} + 1\Bigg] = 0.\end{aligned} \tag{2.2.6}$$

There may be more than one root to the Eqn.(2.2.6). If $x = x_0$ is a root, then it corresponds to a local maximum if $\frac{\partial^2 ln[h(x)]}{\partial x^2} < 0$, a local minimum if $\frac{\partial^2 ln[h(x)]}{\partial x^2} > 0$, and a point of inflection if $\frac{\partial^2 ln[h(x)]}{\partial x^2} = 0$.

Several families of distributions can be derived from T-transmuted X family for different choices of $r(t)$.

2.3 Some Members of T-transmuted X Family of Distributions and their Properties

In this section we discuss some members of the T-transmuted X family. Here we consider the case where T follows an exponential distribution with parameter $\theta > 0$ with cdf $R(t) = 1 - e^{-\theta t},\ t > 0$.

2.3.1 Exponential-transmuted uniform (ETU) distribution

We consider the base distribution as the uniform distribution with cdf and pdf are respectively given by, $G(x) = \frac{x}{\alpha}$ and $g(x) = \frac{1}{\alpha}$; $0 < x < \alpha$. Then the cdf and pdf of the ETU distribution are respectively given by

$$J(x) = 1 - \Big[1 - \frac{x}{\alpha}\big[1 + \lambda(1 - \frac{x}{\alpha})\big]\Big]^{\theta}, \tag{2.3.1}$$

and

$$j(x) = \frac{\theta}{\alpha}\Big[1 - \frac{x}{\alpha}\big[1 + \lambda(1 - \frac{x}{\alpha})\big]\Big]^{\theta-1}\Big(1 + \lambda\big(1 - \frac{2x}{\alpha}\big)\Big), \tag{2.3.2}$$

where, $\alpha > 0, \theta > 0, |\lambda| \leq 1$ and $0 < x < \alpha$.

2.3.2 Exponential-transmuted Fréchet (ETF) distribution

Here we consider the base distribution as a Fréchet distribution with cdf and pdf are respectively given by, $G(x) = e^{-(\frac{\beta}{x})^{\alpha}}$ and $g(x) = \alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}$; $x > 0, \alpha > 0, \beta > 0$. Then $W(F(x)) = -ln\big(1 - e^{-(\frac{\beta}{x})^{\alpha}}\big[1 + \lambda(1 - e^{-(\frac{\beta}{x})^{\alpha}})\big]\big)$. Now, the cdf and pdf of the ETF distribution are given by

$$\begin{aligned} J(x) &= R\Big[-ln\big(1 - e^{-(\frac{\beta}{x})^{\alpha}}\big[1 + \lambda(1 - e^{-(\frac{\beta}{x})^{\alpha}})\big]\big)\Big] \\ &= 1 - \Big[1 - e^{-(\frac{\beta}{x})^{\alpha}}\big[1 + \lambda(1 - e^{-(\frac{\beta}{x})^{\alpha}})\big]\Big]^{\theta}, \end{aligned} \tag{2.3.3}$$

and

$$j(x) = \theta\alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}\frac{(1 + \lambda - 2\lambda e^{-(\frac{\beta}{x})^{\alpha}})}{(1 - e^{-(\frac{\beta}{x})^{\alpha}}[1 + \lambda(1 - e^{-(\frac{\beta}{x})^{\alpha}})])^{1-\theta}}, \tag{2.3.4}$$

where, $\alpha > 0, \beta > 0, \theta > 0, |\lambda| \leq 1$ and $x > 0$. Some properties and applications of the ETF distribution are studied in Jayakumar and Babu (2018a).

2.3.3 Exponential-transmuted Rayleigh (ETR) distribution

We consider the base distribution as a Rayleigh distribution with cdf and pdf are respectively given by, $G(x) = 1 - e^{-\frac{x^2}{2\sigma^2}}$ and $g(x) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$. Then the cdf and pdf of the ETR distribution are respectively given by

$$J(x) = 1 - e^{-\frac{\theta x^2}{2\sigma^2}}\left[1 - \lambda + \lambda e^{-\frac{x^2}{2\sigma^2}}\right]^{\theta}, \tag{2.3.5}$$

and

$$j(x) = \frac{\theta x e^{-\frac{\theta x^2}{2\sigma^2}}}{\sigma^2}\frac{\left[1 - \lambda + 2\lambda e^{-\frac{x^2}{2\sigma^2}}\right]}{\left[1 - \lambda + \lambda e^{-\frac{x^2}{2\sigma^2}}\right]^{1-\theta}}, \tag{2.3.6}$$

where, $\sigma > 0, \theta > 0, |\lambda| \leq 1$ and $x > 0$.

2.4 Exponential-transmuted Weibull (ETW) distribution

Here we consider the base distribution as the Weibull distribution with cdf and pdf are respectively given by, $G(x) = 1 - e^{-(\beta x)^{\alpha}}$ and $g(x) = \alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}$. Then the cdf and pdf of the ETW distribution are respectively given by

$$J(x) = 1 - e^{-\theta(\beta x)^{\alpha}}\left[1 - \lambda + \lambda e^{-(\beta x)^{\alpha}}\right]^{\theta}, \tag{2.4.1}$$

and

$$j(x) = \theta\alpha\beta^{\alpha}x^{\alpha-1}e^{-\theta(\beta x)^{\alpha}}\frac{(1 - \lambda + 2\lambda e^{-(\beta x)^{\alpha}})}{(1 - \lambda + \lambda e^{-(\beta x)^{\alpha}})^{1-\theta}}, \tag{2.4.2}$$

where, $\alpha > 0, \beta > 0, \theta > 0, |\lambda| \leq 1$ and $x > 0$. When $\theta = 1$ and $\lambda = 0$, the ETW distribution becomes the well known two parameter Weibull distribution. Various shapes of the pdf of the ETW distribution are shown in Figure 2.1.

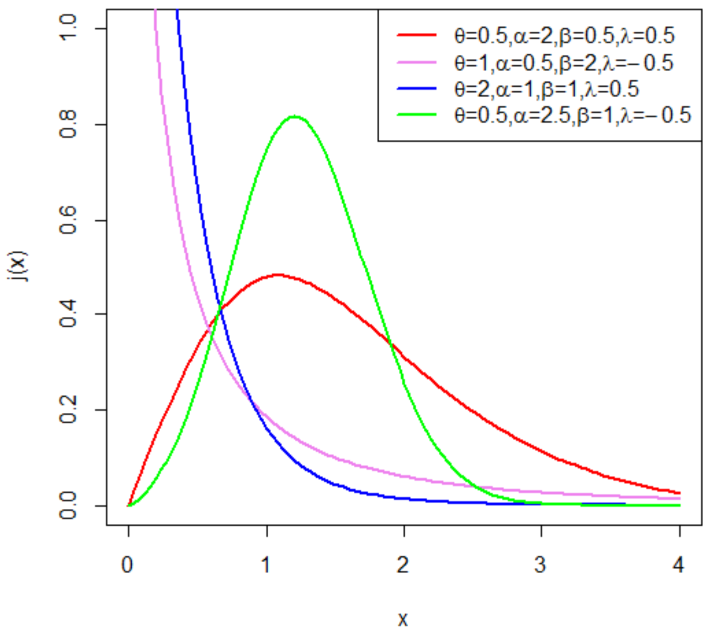

Figure 2.1: The pdf of ETW distribution for various choices of α, θ and λ.

2.4.1 Exponential-transmuted exponential (ETE) distribution

Let $W(F(x)) = -ln\big(1 - F(x)\big) = -ln\big[1 - G(x)(1 + \lambda\bar{G}(x))\big]$, where the base distribution is exponential with cdf, $G(x) = 1 - e^{-\beta x};\ x > 0, \beta > 0$. Then the cdf of the corresponding family is given by

$$J(x) = 1 - e^{-\theta\beta x}(1 - \lambda + \lambda e^{-\beta x})^{\theta};\ x > 0, \theta > 0, \beta > 0, |\lambda| \leq 1. \tag{2.4.3}$$

We call this new family of distributions as Exponential-transmuted Exponential (ETE) distribution with parameters θ, β and λ. The pdf of this distribution is

$$j(x) = \theta\beta e^{-\theta\beta x}\frac{\big(1 - \lambda + 2\lambda e^{-\beta x}\big)}{\big(1 - \lambda + \lambda e^{-\beta x}\big)^{1-\theta}};\ x > 0, \theta > 0, \beta > 0, |\lambda| \leq 1. \tag{2.4.4}$$

When $\lambda = 0$, the ETE distribution becomes exponential distribution. The ETE distribution is a special case of the ETW distribution when the shape parameter

$\alpha = 1$ in the Eqn.(2.4.1).

2.4.2 Properties of the ETE distribution

Using the binomial expansion, the cdf of the ETE distribution given in the Eqn.(2.4.3) can be expressed as

$$\begin{aligned} J(x) &= 1 - e^{-\theta\beta x}\big(1-\lambda(1-e^{-\beta x})\big)^{\theta} \\ &= 1 - e^{-\theta\beta x}\sum_{i=0}^{\infty}(-1)^i \begin{pmatrix} \theta \\ i \end{pmatrix} \lambda^i (1-e^{-\beta x})^i \\ &= 1 - e^{-\theta\beta x}\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}(-1)^{i+k}\lambda^i \begin{pmatrix} \theta \\ i \end{pmatrix}\begin{pmatrix} i \\ k \end{pmatrix} e^{-k\beta x} \\ &= 1 - \sum_{k=0}^{\infty} S_k(\theta,\lambda) e^{-(k+\theta)\beta x}, \end{aligned}$$

where

$$S_k(\theta,\lambda) = \sum_{i=k}^{\infty}(-1)^{i+k}\begin{pmatrix} \theta \\ i \end{pmatrix}\begin{pmatrix} i \\ k \end{pmatrix}\lambda^i. \tag{2.4.5}$$

That is

$$J(x) = 1 - \sum_{k=0}^{\infty} S_k(\theta,\lambda) e^{-(k+\theta)\beta x}. \tag{2.4.6}$$

Then the pdf can be expressed as

$$j(x) = \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta e^{-(k+\theta)\beta x}. \tag{2.4.7}$$

2.4.3 Shapes of the density function

The shapes of the density function can describe analytically. The critical points of the ETE density function are the roots of the equation, $\frac{\partial\ ln(j(x))}{\partial x} = 0$. That is,

$$\frac{\partial\ ln(j(x))}{\partial x} = -\theta\beta - \frac{\lambda\beta(\theta-1)e^{-\beta x}}{1-\lambda+\lambda e^{-\beta x}} - \frac{2\lambda\beta e^{-\beta x}}{1-\lambda+2\lambda e^{-\beta x}} = 0. \tag{2.4.8}$$

This implies

$$u^2[4\theta\lambda^2] + u[\lambda(\lambda-1)(4\theta+1)] + \theta(1-\lambda)^2 = 0, \tag{2.4.9}$$

where $u = e^{-\beta x}$. Here the Eqn.(2.4.9) is a quadratic equation in u and since, $0 < u < 1$, the possible root is

$$u = \frac{1-\lambda}{\lambda}\left[\frac{(8\theta+1)^{\frac{1}{2}} - (4\theta+1)}{8\theta}\right].$$

Therefore, the solution of the Eqn.(2.4.9) is $x_0 = \frac{-ln(u)}{\beta}$. Since, $\theta > 0$ and $0 < u < 1$, the root x_0 exists only if $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$. Thus the shape of the density function of the ETE distribution is unimodal for $x > 0, \theta > 0, \beta > 0$ and $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$. Also note that

$$\frac{\partial^2 ln(j(x))}{\partial x^2} = \lambda\beta^2(1-\lambda)e^{-\beta x}\left[\frac{\theta-1}{(1-\lambda+\lambda e^{-\beta x})^2} + \frac{2}{(1-\lambda+2\lambda e^{-\beta x})^2}\right]. \tag{2.4.10}$$

Since, $\lambda < 0, \theta > 0, \beta > 0$ and $0 < e^{-\beta x} < 1$, the Eqn.(2.4.10) is always negative. That is $\frac{\partial^2 ln(j(x))}{\partial x^2} < 0$. The third derivative $\frac{\partial^3 ln(j(x))}{\partial x^3}$ also exists. The mode of the ETE distribution is given by

$$x_0 = \frac{-1}{\beta}\ ln\left[\frac{1-\lambda}{\lambda}\left(\frac{(8\theta+1)^{\frac{1}{2}} - (4\theta+1)}{8\theta}\right)\right], \tag{2.4.11}$$

where $\theta > 0$ and $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$. Thus, the shape of the pdf of the ETE distribution is decreasing for $\frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < \lambda < 1$ and is unimodal for

$-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$. Various shapes of the pdf of the ETE distribution are shown in Figure 2.2.

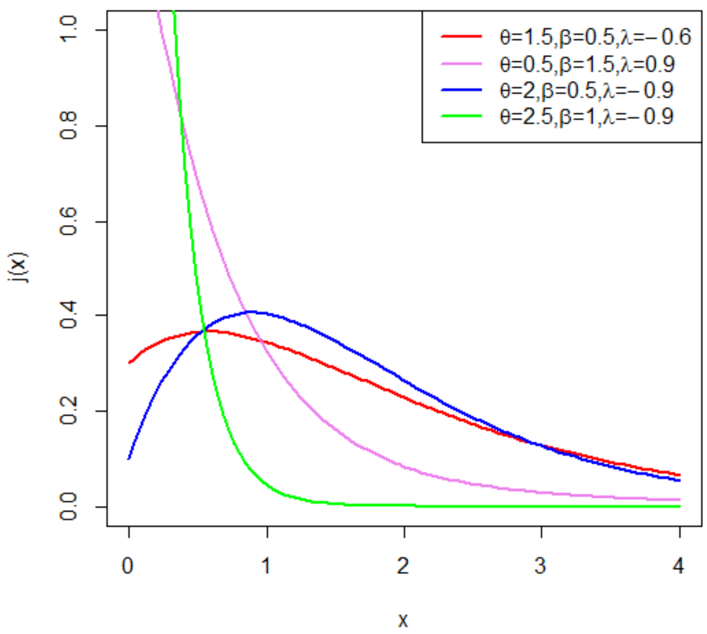

Figure 2.2: The pdf of ETE distribution for various choices of α, θ and λ.

2.4.4 Hazard rate function

The hrf of the ETE distribution is given by

$$h(x) = \frac{j(x)}{1 - J(x)} = \theta\beta \frac{1 - \lambda + 2\lambda e^{-\beta x}}{1 - \lambda + \lambda e^{-\beta x}}; \; x > 0, \theta > 0, \beta > 0, |\lambda| \leq 1. \qquad (2.4.12)$$

Here note that, $\lim_{x \to 0} h(x) = \theta\beta(\lambda + 1)$, and $\lim_{x \to \infty} h(x) = \theta\beta$. We have the following cases :

Case i. When $-1 \leq \lambda < 0$, $h(x)$ is an increasing function, increases from $(1 + \lambda)\theta\beta$ to $\theta\beta$.

Case ii. When $\lambda = 0$, $h(x) = \theta\beta$, a constant function.

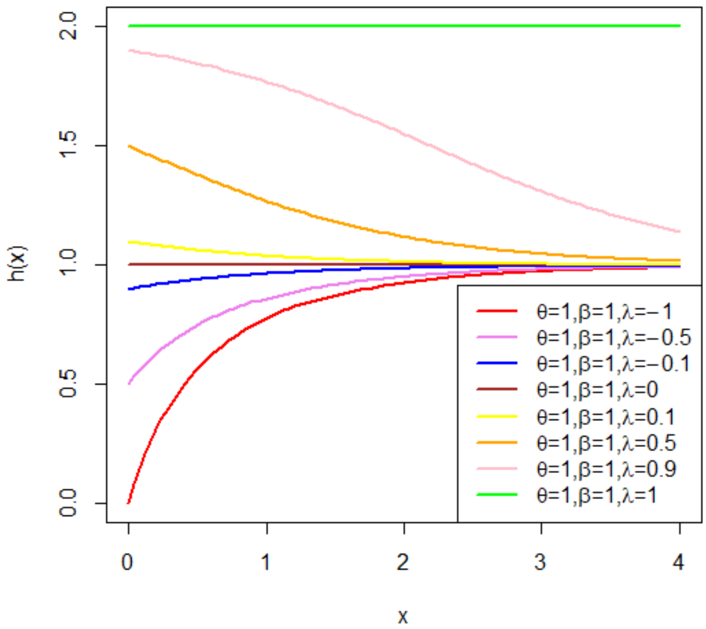

Figure 2.3: The hrf of ETE distribution for various parameter values.

Case iii. When $0 < \lambda < 1$, $h(x)$ is a decreasing function decreases from $(1+\lambda)\theta\beta$ to $\theta\beta$.

Case iv. When $\lambda = 1$, $h(x) = 2\theta\beta$, a constant function.

The shapes of the hazard rate function for various parameter values are presented in the Figure 2.3.

2.4.5 Quantile function

The p^{th}quantile x_p of the ETE distribution is the real solution of the equation $J(x_p) = p$. That is, $1 - e^{-\theta\beta x_p}(1 - \lambda + \lambda e^{-\beta x_p})^{\theta} = p$.

$$\Rightarrow e^{-\beta x_p}(1 - \lambda + \lambda e^{-\beta x_p}) = (1-p)^{\frac{1}{\theta}}. \tag{2.4.13}$$

Let $u = e^{-\beta x_p}$. Then $x_p = -\frac{ln(u)}{\beta}$. Hence from the Eqn.(2.4.13), $\lambda u^2 + (1-\lambda)u - (1-p)^{\frac{1}{\theta}} = 0 \Rightarrow u = \frac{-(1-\lambda)\pm\sqrt{(1-\lambda)^2+4\lambda(1-p)^{\frac{1}{\theta}}}}{2\lambda}$.

Since $0 < u < 1$, the possible root of u is

$$u = \frac{-(1-\lambda)+\sqrt{(1-\lambda)^2+4\lambda(1-p)^{\frac{1}{\theta}}}}{2\lambda}.$$

Therefore,

$$x_p = -\frac{1}{\beta}\, ln\left\{\left[\frac{1}{\lambda}(1-p)^{\frac{1}{\theta}} + \frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}} - \frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right\}. \tag{2.4.14}$$

In particular, the median is given by

$$\text{Median} = x_{0.5} = -\frac{1}{\beta}\, ln\left\{\left[\frac{1}{\lambda}\left(\frac{1}{2}\right)^{\frac{1}{\theta}} + \frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}} - \frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right\}. \tag{2.4.15}$$

2.4.6 Moments and moment generating function

Here we derive the expression for raw moments of the ETE distribution as

$$\begin{aligned}
\mu_r^{'} = E(X^r) &= \int_0^\infty x^r \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta e^{-(k+\theta)\beta x}dx \\
&= \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{\Gamma(r+1)}{[(k+\theta)\beta]^r}, \qquad (2.4.16)
\end{aligned}$$

where $S_k(\theta,\lambda)$ is as in Eqn.(2.4.5). The first four raw moments are respectively

$$\mu_1^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{1}{[(k+\theta)\beta]},$$

$$\mu_2^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{2}{[(k+\theta)\beta]^2},$$

$$\mu_3^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda)\frac{6}{[(k+\theta)\beta]^3},$$

$$\mu_4^{'} = \sum_{k=0}^{\infty} S_k(\theta,\lambda) \frac{24}{[(k+\theta)\beta]^4}.$$

Then, the skewness $= \frac{{\mu_3}^2}{{\mu_2}^3}$ and the kurtosis $= \frac{\mu_4}{{\mu_2}^2}$, where $\mu_2 = \mu_2^{'} - {\mu_1^{'}}^2$, $\mu_3 = \mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2{\mu_1^{'}}^3$ and $\mu_4 = \mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}{\mu_1^{'}}^2 - 3{\mu_1^{'}}^4$. Since the pdf of the ETE distribution is decreasing for $\frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < \lambda < 1$, it may be skewed to the right. Also for $-1 < \lambda < \frac{(8\theta+1)^{\frac{1}{2}}-(4\theta+1)}{(8\theta+1)^{\frac{1}{2}}+(4\theta-1)} < 0$, the pdf is unimodal and the mode value is always less than the mean value, it shows the right skewness. In Table 2.1, we present the raw moments, central moments, mode, skewness and kurtosis of the ETE distribution for different choices of parameter values. In all the cases, the distribution shows a positively skewed behavior. The moment generating function (mgf) of the ETE distribution is obtained as

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_0^{\infty} e^{tx} \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta e^{-(k+\theta)\beta x} dx \\ &= \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta \int_0^{\infty} e^{-[(k+\theta)\beta - t]x} dx \\ &= \sum_{k=0}^{\infty} S_k(\theta,\lambda)(k+\theta)\beta \frac{1}{(k+\theta)\beta - t}, \end{aligned} \tag{2.4.17}$$

where $S_k(\theta,\lambda)$ is as in Eqn.(2.4.5).

2.4.7 Characterization based on truncated moments

Characterization of a probability distribution plays an important role in statistics and mathematical sciences. In recent years, there has been a great interest in charac-

Table 2.1: Moments, Mode, Skewness and Kurtosis for various choices of parameters.

Parameter	Raw moments	Central moments	Mode	Skewness	Kurtosis
$\theta = 1.0$ $\beta = 5.0$ $\lambda = -0.5$	$\mu_1' = 0.25$ $\mu_2' = 0.11$ $\mu_3' = 0.07$ $\mu_4' = 0.06$	$\mu_2 = 0.048$ $\mu_3 = 0.019$ $\mu_4 = 0.020$	0.058	3.28	8.66
$\theta = 0.5$ $\beta = 0.5$ $\lambda = -0.5$	$\mu_1' = 4.61$ $\mu_2' = 38.44$ $\mu_3' = 467.34$ $\mu_4' = 7509.17$	$\mu_2 = 17.19$ $\mu_3 = 131.66$ $\mu_4 = 2438.05$	1.114	3.414	8.25
$\theta = 0.5$ $\beta = 10$ $\lambda = -0.5$	$\mu_1' = 0.231$ $\mu_2' = 0.096$ $\mu_3' = 0.058$ $\mu_4' = 0.047$	$\mu_2 = 0.043$ $\mu_3 = 0.016$ $\mu_4 = 0.015$	0.055	3.35	8.58
$\theta = 2.5$ $\beta = 0.5$ $\lambda = -0.5$	$\mu_1' = 1.14$ $\mu_2' = 2.25$ $\mu_3' = 6.19$ $\mu_4' = 21.74$	$\mu_2 = 0.95$ $\mu_3 = 1.46$ $\mu_4 = 5.99$	0.076	2.477	6.63
$\theta = 0.5$ $\beta = 1.0$ $\lambda = -0.5$	$\mu_1' = 2.31$ $\mu_2' = 9.61$ $\mu_3' = 58.42$ $\mu_4' = 469.32$	$\mu_2 = 4.27$ $\mu_3 = 16.48$ $\mu_4 = 151.78$	0.557	3.48	8.31
$\theta = 0.5$ $\beta = 1.0$ $\lambda = 0.5$	$\mu_1' = 1.62$ $\mu_2' = 5.95$ $\mu_3' = 34.54$ $\mu_4' = 273.16$	$\mu_2 = 3.33$ $\mu_3 = 14.13$ $\mu_4 = 122.37$	0	5.43	11.06
$\theta = 3.0$ $\beta = 2.0$ $\lambda = 0.5$	$\mu_1' = 0.116$ $\mu_2' = 0.028$ $\mu_3' = 0.011$ $\mu_4' = 0.006$	$\mu_2 = 0.015$ $\mu_3 = 0.004$ $\mu_4 = 0.003$	0	6.23	12.35

terizations of probability distributions by truncated moments. The development of the general theory of the characterizations of probability distributions by truncated moments began with the work of Galambos and Kotz (1978). Further development on the characterizations of probability distributions by truncated moments continued with the contributions of many authors and researchers, among them Kotz and Shanbhag (1980) and Glanzel (1987, 1990), are notable.

A probability distribution can be characterized through various methods, see, for example, Su and Huang (2000), Gupta and Ahsanullah (2006), Nair and Sudheesh (2010), Nanda (2010), Hamedani (2010), Huang and Su (2012), Ahsanullah et al. (2014), among others. Several characterizations of Weibull distribution are available in the literature (see, Janardan (1978), Janardan and Schaeffer (1978), Janardan and Taneja (1979a, 1979b), Shimizu and Davies (1981), Khan and Beg (1987)). Scholz (1990) characterizes a three parameter Weibull distribution using a quantile relationship. We present a characterization of the ETE distribution in terms of a simple relationship between truncated moments. This result is developed using the Theorem 1 of Glanzel (1987), which stated as follows:

Theorem 2.4.1. *Let $(\Omega, \mathcal{A}, \boldsymbol{P})$ be a given probability space, and let $H = [a, b]$ be an interval for some $a < b$ ($a = \infty$, $b = -\infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with distribution function $G(x)$ and let q_1 and q_2 be two real functions defined on H such that*

$$\mathbb{E}[q_1(X)|X \geq x] = \mathbb{E}[q_2(X)|X \geq x]\eta(x), \ x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$, and $G(x)$ is a twice continuously differentiable and strictly monotone function on the set H. Finally assume that the equation $q_2\eta = q_1$ has no real solution in the interior of H. Then G is uniquely determined by the functions q_1, q_2 and η. In particular,

$$G(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)q_2(u) - q_1(u)}\right| e^{-s(u)}du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_2}{\eta q_2 - q_1}$ and C is a constant chosen to make $\int_H dG = 1$.

The above theorem has the advantage that the cdf G is not required to have a closed form and is given in terms of an integral whose integrand depends on the solution of a first order differential equation, which can serve as a bridge between probability and differential equation. The following theorem gives the characterization of the ETE distribution in terms of a simple relationship between truncated moments.

Theorem 2.4.2. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable, and let $q_2(x) = (1-\lambda+\lambda e^{-\beta x})^{1-\theta}(1-\lambda+2\lambda e^{-\beta x})^{-1}$ and $q_1(x) = q_2(x)e^{-\theta\beta x}$ for $x > 0$. The pdf of X is Eqn.(2.4.4) if and only if the function η defined in Theorem 2.4.1 has the form*

$$\eta(x) = \frac{1}{2}e^{-\theta\beta x},\ x > 0. \tag{2.4.18}$$

Proof. Let X has the pdf given in Eqn.(2.4.4). Then

$$\begin{aligned}(1-J(x))\mathbb{E}[q_2(X)|X \geq x] &= e^{-\theta\beta x},\ x > 0,\\ (1-J(x))\mathbb{E}[q_1(X)|X \geq x] &= \frac{1}{2}e^{-2\theta\beta x},\ x > 0,\end{aligned}$$

and

$$\eta(x)q_2(x) - q_1(x) = -\frac{1}{2}\frac{e^{-\theta\beta x}(1-\lambda+\lambda e^{-\beta x})^{1-\theta}}{1-\lambda+2\lambda e^{-\beta x}} < 0, for\ x > 0. \tag{2.4.19}$$

Conversely, if η is given as Eqn.(2.4.18), then

$$s'(x) = \frac{\eta'(x)q_2(x)}{\eta(x)q_2(x) - q_1(x)} = \theta\beta,\ x > 0 \tag{2.4.20}$$

and hence $s(x) = \theta\beta x,\ x > 0$, or $e^{-s(x)} = e^{-\theta\beta x},\ x > 0$. Now, using Theorem 2.4.1, X has the pdf given in Eqn.(2.4.4). $\square$

Corollary 2.4.1. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable, and let $q_2(x) = (1-\lambda+\lambda e^{-\beta x})^{1-\theta}(1-\lambda+2\lambda e^{-(\beta x)})^{-1}$. The pdf of X is as Eqn.(2.4.4) if and only if there exist functions q_1 and η defined in Theorem 2.4.1 satisfying the differential equation*

$$\frac{\eta'(x)q_2(x)}{\eta(x)q_2(x)-q_1(x)} = \theta\beta, \; x>0. \tag{2.4.21}$$

Remark 2.4.1. *The general solution of the differential equation in Corollary 2.4.1 is obtained as follows:*

$$\eta'(x) - \theta\beta\eta(x) = [q_2(x)]^{-1}q_1(x)\theta\beta,$$

or

$$\frac{d}{dx}[\eta(x)e^{-\theta\beta x}] = [q_2(x)]^{-1}q_1(x)\theta\beta e^{-\theta\beta x}.$$

Thus we obtain

$$\eta(x) = e^{\theta\beta x}\left[-\int \theta\beta e^{-\theta\beta x}[q_2(x)]^{-1}q_1(x)dx + D\right], \tag{2.4.22}$$

where D is a constant.

2.4.8 Characterization based on hazard rate function

The hrf $h(x)$ of a twice differentiable distribution function $J(x)$ and pdf $j(x)$ satisfies the first order differential equation

$$\frac{j'(x)}{j(x)} = \frac{h'(x)}{h(x)} - h(x). \tag{2.4.23}$$

For many univariate continuous distributions, this is the only characterization available in terms of the hrf. Hamedani and Ahsanullah (2005) characterized certain well known distributions in terms of the hrf. The hrf, $h(x)$ of the ETE distribution has a twice differentiable cdf $J(x)$ and pdf $j(x)$ satisfies the first order differential equation Eqn.(2.4.23). The following characterization establishes a non-trivial characterization of the ETE distribution, when $\theta = 1$.

Theorem 2.4.3. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is given in Eqn.(2.4.4) if and only if its hrf $h(x)$ satisfies the differential equation*

$$\frac{1}{\beta}h'(x) = \frac{d}{dx}\left[\frac{1-\lambda+2\lambda e^{-\beta x}}{1-\lambda+\lambda e^{-\beta x}}\right]. \tag{2.4.24}$$

Proof. When $\theta = 1$, the pdf $j(x)$ and hrf $h(x)$ are respectively

$$j(x) = \beta e^{-\beta x}(1-\lambda+2\lambda e^{-\beta x}), \tag{2.4.25}$$

and

$$h(x) = \frac{\beta(1-\lambda+2\lambda e^{-\beta x})}{1-\lambda+\lambda e^{-\beta x}}. \tag{2.4.26}$$

Then we have

$$\frac{j'(x)}{j(x)} = \frac{-2\lambda\beta e^{-\beta x}}{1-\lambda+2\lambda e^{-\beta x}} - \beta. \tag{2.4.27}$$

Using the Eqn.(2.4.23) we get the first order differential equation

$$\frac{1}{\beta}h'(x) = -\frac{\lambda(1-\lambda)\beta e^{-\beta x}}{(1-\lambda+\lambda e^{-\beta x})^2}, \tag{2.4.28}$$

which implies

$$\frac{1}{\beta}h'(x) = \frac{d}{dx}\left[\frac{1-\lambda+2\lambda e^{-\beta x}}{1-\lambda+\lambda e^{-\beta x}}\right].$$

Now, if the differential equation Eqn.(2.4.24) holds, then

$$\frac{d}{dx}\left[\frac{1}{\beta}h(x)\right] = \frac{d}{dx}\left[\frac{1-\lambda+2\lambda e^{-\beta x}}{1-\lambda+\lambda e^{-\beta x}}\right], x > 0,$$

from which, we obtain

$$h(x) = \frac{\beta(1-\lambda+2\lambda e^{-\beta x})}{1-\lambda+\lambda e^{-\beta x}}, x > 0, \tag{2.4.29}$$

which is the hrf of the ETE distribution, when $\theta = 1$. □

2.4.9 Maximum likelihood estimation of the parameters of ETE distribution

The likelihood function of the ETE distribution is given by

$$L = (\theta\beta)^n e^{-\theta\beta\sum_{1=1}^{n} x_i} \prod_{i=1}^{n} \left(1-\lambda+2\lambda e^{-\beta x_i}\right) \prod_{i=1}^{n} \left(1-\lambda+\lambda e^{-\beta x_i}\right)^{\theta-1}. \quad (2.4.30)$$

The log likelihood function is

$$\begin{aligned}\log L &= n\ \log(\theta) + n\log(\beta) - \theta\beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(1-\lambda+2\lambda e^{-\beta x_i}) \\ &\quad +(\theta-1)\sum_{i=1}^{n}\log(1-\lambda+\lambda e^{-\beta x_i}). \end{aligned} \quad (2.4.31)$$

The Eqn.(2.4.31) can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating this with respect to θ, β and λ. The components of the score vector $V(\Theta) = \left(\frac{\partial \log L}{\partial\theta}, \frac{\partial \log L}{\partial\beta}, \frac{\partial \log L}{\partial\lambda}\right)$, are given by

$$\begin{aligned}\frac{\partial \log L}{\partial\theta} &= \frac{n}{\theta} - \beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(1-\lambda+\lambda e^{-\beta x_i}), \\ \frac{\partial \log L}{\partial\beta} &= \frac{n}{\beta} - \theta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\frac{2\lambda\beta e^{-\beta x_i}}{1-\lambda+2\lambda e^{-\beta x_i}} - \sum_{i=1}^{n}\frac{\lambda\beta(\theta-1)e^{-\beta x_i}}{1-\lambda+\lambda e^{-\beta x_i}}, \\ \frac{\partial \log L}{\partial\lambda} &= \sum_{i=1}^{n}\frac{2e^{-\beta x_i}}{1-\lambda+2\lambda e^{-\beta x_i}} + (\theta-1)\sum_{i=1}^{n}\frac{e^{-\beta x_i}-1}{1-\lambda+\lambda e^{-\beta x_i}}.\end{aligned}$$

That is, the normal equations takes the following form

$$\frac{n}{\theta} - \beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(1-\lambda+\lambda e^{-\beta x_i}) = 0, \quad (2.4.32)$$

$$\frac{n}{\beta} - \theta\sum_{i=1}^{n} x_i - 2\lambda\beta\sum_{i=1}^{n}\frac{e^{-\beta x_i}}{1-\lambda+2\lambda e^{-\beta x_i}} - \lambda\beta(\theta-1)\sum_{i=1}^{n}\frac{e^{-\beta x_i}}{1-\lambda+\lambda e^{-\beta x_i}} = 0, \quad (2.4.33)$$

$$\sum_{i=1}^{n} \frac{2e^{-\beta x_i}}{1-\lambda+2\lambda e^{-\beta x_i}} + (\theta-1)\sum_{i=1}^{n} \frac{e^{-\beta x_i}-1}{1-\lambda+\lambda e^{-\beta x_i}} = 0. \tag{2.4.34}$$

These equations do not have explicit solutions and they have to be obtained numerically. From the Eqn.(2.4.32), the MLE of the parameter θ can be obtained as

$$\hat{\theta} = \frac{n}{\beta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\log(1-\lambda+\lambda e^{-\beta x_i})}. \tag{2.4.35}$$

Substituting the Eqn.(2.4.35) in the Eqn.(2.4.32) and the Eqn.(2.4.33), we get the MLEs of β and λ. Statistical softwares like *nlm* or *optim* packages in **R** programming can be used to solve these equations numerically.

2.4.10 Simulation study

In order to check the performance of the maximum likelihood estimates, we conduct a simulation study. We take the sample sizes as $n = 50, 100, 200, 500, 800$ and 1000. The process is replicated 1000 times and the average estimates along with the standard errors are presented in Table 2.2. Here we can see that as the sample size increases the MLE estimates of the ETE distribution converge to the true value and the corresponding standard errors are decreasing.

Table 2.2: Average of MLEs of the ETE distribution with standard error for various choices of parameter values.

Parameters	n	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\beta}(\hat{SE}(\hat{\beta}))$	$\hat{\lambda}(\hat{SE}(\hat{\lambda}))$
$\theta = 1$, $\beta = 5$, $\lambda = -0.5$	50	0.990(1.108)	5.717(1.891)	-0.598(0.275)
	100	1.112(1.081)	5.228(1.840)	-0.595(0.299)
	200	1.049(0.717)	5.114(1.948)	-0.579(0.138)
	500	1.047(0.667)	5.007(1.901)	-0.554(0.101)
	800	1.023(0.651)	5.001(1.281)	-0.516(0.097)
	1000	1.006(0.560)	4.996(1.029)	-0.508(0.073)
$\theta = 5$, $\beta = 10$, $\lambda = 0.5$	50	4.416(1.231)	9.639(2.054)	0.622(0.342)
	100	4.695(0.843)	9.747(1.479)	0.581(0.189)
	200	4.831(1.207)	9.839(1.582)	0.513(0.221)
	500	4.919(0.943)	10.098(1.809)	0.507(0.226)
	800	5.099(0.886)	9.982(1.171)	0.483(0.118)
	1000	5.062(0.778)	9.997(1.066)	0.499(0.033)
$\theta = 0.5$, $\beta = 0.5$, $\lambda = -0.9$	50	0.418(0.299)	0.431(0.238)	-0.807(0.167)
	100	0.432(0.257)	0.452(0.370)	-0.814(0.121)
	200	0.456(0.248)	0.475(0.248)	-0.851(0.093)
	500	0.499(0.201)	0.507(0.164)	-0.883(0.071)
	800	5.133(0.177)	0.495(0.113)	-0.899(0.062)
	1000	5.065(0.132)	0.498(0.091)	-0.903(0.008)
$\theta = 10$, $\beta = 2.5$, $\lambda = 0.8$	50	9.120(1.155)	2.249(0.792)	0.869(0.149)
	100	9.435(0.827)	2.496(0.755)	0.837(0.153)
	200	9.971(0.729)	2.507(0.479)	0.825(0.071)
	500	10.094(0.226)	2.503(0.223)	0.818(0.039)
	800	9.991(0.217)	2.512(0.182)	0.811(0.026)
	1000	10.004(0.116)	2.499(0.177)	0.809(0.022)

2.4.11 Entropy

For the ETE distribution, the Rényi entropy is obtained as

$$\begin{aligned} I_R(\gamma) &= \frac{\gamma}{1-\gamma}\Big(\log(\theta)+\log(\beta)\Big) \\ &+\frac{\gamma}{1-\gamma}\log\bigg(\int_0^\infty e^{-\gamma\theta\beta x}\big(1-\lambda+\lambda e^{-\beta x}\big)^{\gamma(\theta-1)} \\ &\big(1-\lambda+2\lambda e^{-\beta x}\big)^{\gamma}dx\bigg). \end{aligned} \tag{2.4.36}$$

For given values of θ, β, λ and γ, the Rényi entropy can be numerically computed using **R** programming. Table 2.3 shows the values of entropy for given parameter values and γ. The Shannon entropy of T-transmuted X family of distributions is

Table 2.3: Rényi entropy for given values of θ, β, λ and γ.

θ	β	λ	$\gamma = 0.5$	$\gamma = 2.0$	$\gamma = 3.0$	$\gamma = 5.0$
1	1	-1	1.5963	1.0986	1.0075	0.9183
		-0.5	1.5216	0.9808	0.8836	0.7907
		0.5	1.1732	0.3449	0.1813	0.0189
		1	0.6931	0.0000	-0.1438	-0.2908
2	1	-1	1.0383	0.5935	0.5106	0.4284
		-0.5	0.9081	0.6768	0.2939	0.1999
		0.5	0.4029	-0.3769	-0.5306	-0.6852
		1	0.0001	-0.6931	-0.8369	-0.9835
2	2	-1	0.3452	-0.0996	1.4407	-0.1826
		-0.5	0.2149	-0.3028	-0.3993	-0.4933
		0.5	-0.2903	-1.0700	-1.2237	-1.3783
		1	-0.6932	-1.3863	-1.5301	-1.6771
0.5	0.5	-1	2.8918	2.3375	2.2357	2.1371
		-0.5	2.8510	2.2723	2.1683	2.0696
		0.5	1.9387	1.7819	1.6028	1.4261
		1	2.0794	1.3863	1.2427	1.0955

given by

$$\begin{aligned} E\Big(-\log\big(j(X)\big)\Big) &= -E\Big(\log\big(g(X)\big)\Big) - E\Big(\log L\big[1+\lambda-2\lambda G(X)\big]\Big) \\ &\quad +E\Big(\log\big[1-G(x)\big(1+\lambda\bar{G}(x)\big)\big]\Big) \\ &\quad +E\Big[\log\big[\gamma\big(-\log\big[1-G(X)\big(1+\lambda\bar{G}(X)\big)\big]\big)\big]\Big]. \end{aligned} \tag{2.4.37}$$

For the ETE distribution, the Shannon entropy obtained as

$$\begin{aligned} E\Big(-\log\big(j(X)\big)\Big) &= \theta\beta E(X) - E\Big[\log\Big(1-\lambda+\lambda e^{-\beta X}\Big)\Big] \\ &\quad -(\theta-1)E\Big[\log\Big(1-\lambda+\lambda e^{-\beta X}\Big)\Big]. \end{aligned} \tag{2.4.38}$$

2.4.12 Data applications of the ETE distribution

In this section, to show how the ETE distribution works in practice, we consider two real data sets. We compare the fit of the ETE distribution with the Kumaraswamy exponential (KuE) distribution, exponentiated Weibull (EW) distribution, Weibull (W) distribution and the exponential (E) distribution. The values of the $-\log L$, AIC, CAIC, BIC and K-S are calculated for the five distributions in order to verify which distribution fits better to these data.

The first data set is the life of fatigue of Kelvar 373 epoxy that are subject to constant pressure at the 90% stress level until all had failed. The data sets contains 76 observations, which are taken from Andrews and Herzberg (1985). The data are as follows:

0.0251, 0.6751, 1.0483, 1.4880, 1.8808, 2.2460, 3.4846, 0.0886, 0.6753, 1.0596, 1.5728, 1.8878, 2.2878, 3.7433, 0.0891, 0.7696, 1.0773, 1.5733, 1.8881, 2.3203, 3.7455, 0.2501, 0.8375, 1.1733 1.7083, 1.9316, 2.3470, 3.9143, 0.3113, 0.8391, 1.2570, 1.7263, 1.9558, 2.3513, 4.8073, 0.3451, 0.8425, 1.2766, 1.7460, 2.0048, 2.4951, 5.4005, 0.4763, 0.8645, 1.2985, 1.7630, 2.0408, 2.5260, 5.4435, 0.5650, 0.8851, 1.3211, 1.7746, 2.0903, 2.9941, 5.5295, 0.5671, 0.9113, 1.3503, 1.8275, 2.1093, 3.0256, 6.5541, 0.6566, 0.9120, 1.3551, 1.8375, 2.1330, 3.2678, 9.0960, 0.6748, 0.9836, 1.4595, 1.8503, 2.2100, 3.4045.

The second data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 taken from Lee and Wang (1992). The data are as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

The descriptive statistics of the two data sets are given in Table 2.4. The estimates of the parameters are obtained using method of maximum likelihood estimation. From the values in Table 2.5, even though the CAIC and BIC values of ETE distribution is observed as little higher than with Weibull distribution, all the other tests revealed that the ETE distribution gives a better fit for the first data set. Figure 2.4 shows the fitted density curves for the first data set. Also the values in Table 2.6 show that the BIC value of Weibull distribution is little lower than ETE distribution, but comparing with all other test values we can conclude that the ETE distribution is a better model for the second data set. Figure 2.5 shows the fitted density curves for the second data set.

Table 2.4: Descriptive statistics of the two data sets

Data	n	Min.	Max.	Mean	SD	Skewness	Kurtosis
First data set	76	0.0251	9.096	1.959	1.57	2.019	5.60
Second data set	121	0.30	154	46.33	35.28	1.056	0.471

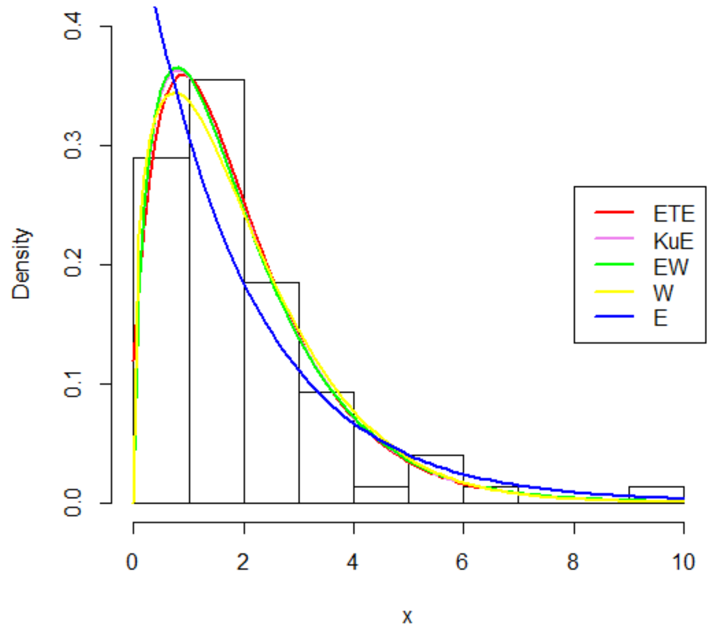

Figure 2.4: Fitted pdf plots of first data set

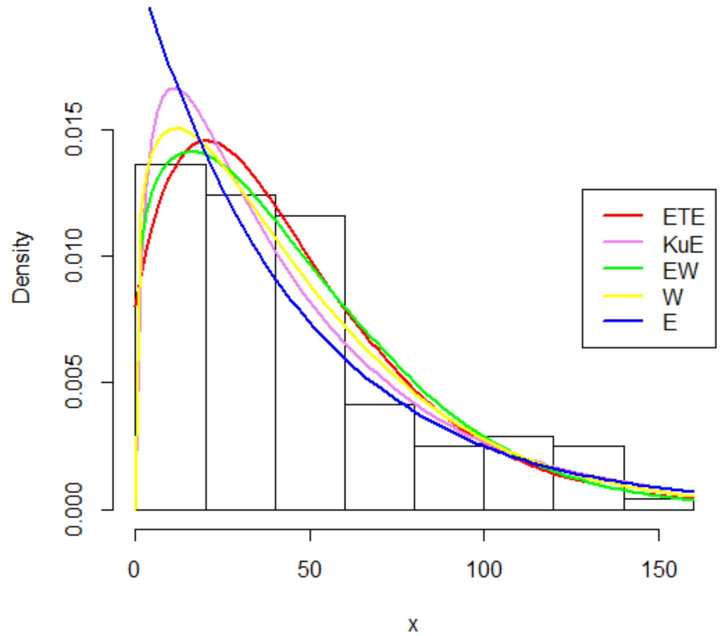

Figure 2.5: Fitted pdf plots of second data set

Table 2.5: Parameter estimates and goodness of fit statistics for various models fitted to the first data set.

Model	ML estimates	$-\log(L)$	AIC	CAIC	BIC	K-S	p value
ETE	$\hat{\theta} = 1.346$ $\hat{\beta} = 0.579$ $\hat{\lambda} = -0.848$	121.461	248.922	249.255	255.914	0.0984	0.4266
KuE	$\hat{\alpha} = 2.448$ $\hat{\beta} = 0.328$ $\hat{\theta} = 1.556$	122.094	250.188	250.521	257.180	0.0990	0.4191
EW	$\hat{\alpha} = 1.101$ $\hat{\beta} = 0.609$ $\hat{\theta} = 1.443$	122.166	250.332	250.665	257.324	0.0992	0.416
W	$\hat{\alpha} = 1.326$ $\hat{\beta} = 0.469$	122.526	249.052	249.216	253.714	0.1098	0.2968
E	$\hat{\beta} = 0.510$	127.114	256.228	256.282	258.559	0.512	0.0266

Table 2.6: Parameter estimates and goodness of fit statistics for various models fitted to the second data set.

Model	ML estimates	$-ln(L)$	AIC	CAIC	BIC	K-S	p value
ETE	$\hat{\theta} = 1.876$ $\hat{\beta} = 0.018$ $\hat{\lambda} = -0.765$	578.878	1163.76	1163.96	1172.14	0.0569	0.8284
KuE	$\hat{\beta} = 0.098$ $\hat{\alpha} = 0.231$ $\hat{\theta} = 1.651$	583.314	1172.63	1172.83	1181.02	0.1152	0.0803
EW	$\hat{\alpha} = 1.393$ $\hat{\beta} = 0.017$ $\hat{\theta} = 0.798$	579.879	1165.76	1165.96	1174.15	0.0664	0.6606
W	$\hat{\alpha} = 1.3056$ $\hat{\beta} = 0.0199$	580.024	1164.05	1164.15	1169.64	0.0588	0.7967
E	$\hat{\beta} = 0.022$	585.128	1172.26	1172.29	1175.05	0.1206	0.0594

2.5 Summary

In this chapter, we introduced a new family of continuous distributions called "T-transmuted X family". Many of the existing distributions are sub models of this family. The ETE, ETU, ETR, ETF and ETW models are derived from this family. Properties of ETE distribution are studied. This distribution is a generalization of exponential distribution. The hrf of the ETE distribution is increasing, decreasing or constant depending upon the various choices of parameter values. Expressions for moments, moment generating function, Rényi and Shannon entropies are derived. The method of maximum likelihood estimation is used to estimate the parameter values and a simulation study shows that the method is performed well. Two real data sets were analyzed to show the flexibility of this model for data modelling. The fit of the ETE distribution is compared with the Kumaraswamy exponential distribution, exponentiated Weibull distribution, Weibull distribution and the exponential distribution. The ETE distribution is found to be the best fitted model for these data sets compared with the other four models.

CHAPTER

THREE

A NEW GENERALIZATION OF WEIBULL DISTRIBUTION

3.1 Introduction

[1] The need for extended forms of the Weibull distribution arises in many applied areas. A number of extended families of Weibull distribution are available in the literature. Compounding Weibull distribution with well know discrete distributions is one among them. Rodrigues et al. (2011) introduced the Weibull binomial distribution using an idea due to Adamidis and Loukas (1998). Babu (2016) introduced the Weibull truncated negative binomial (WTNB) distribution using the method

[1]Some results included in this chapter have appeared in the papers Babu (2016) and Babu and Jayakumar (2018a).

suggested by Nadarajah et al. (2013). The WTNB distribution is a generalization of the Weibull distribution. This distribution is constructed by compounding Weibull distribution with truncated negative binomial distribution.

In a sequence of independent Bernoulli trials, let the random variable X denote the trial at which the r^{th} success occurs, where r is a fixed positive integer. Then X is said to follow a negative binomial distribution with parameters r and p, if its pmf is given by

$$P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \ ; \ x = r, r+1, \dots \ , \tag{3.1.1}$$

and is denoted by NB(r, p). This distribution is sometimes defined in terms of the number of failures before r^{th} success, say Y and $Y = X - r$. Then the alternative form of the distribution is

$$P(Y = y) = \binom{r+y-1}{y} p^r (1-p)^y \ ; \ y = 0, 1, \dots \ . \tag{3.1.2}$$

The geometric distribution with $P(X = x) = p(1-p)^{x-1} \ ; \ x = 1, 2, ...; 0 < p < 1$ is a special case of NB(r, p) when $r = 1$.

Let N be the total number of failures happened in a random experiment with the probability of success of an event in a single trial is δ. Let θ denotes the number of successes in the trial. Here the distribution of N follows a negative binomial (NB) distribution with pmf

$$P(N = n) = \binom{\theta+n-1}{\theta-1} \delta^\theta (1-\delta)^n \ ; \ n = 0, 1, 2, \ \dots \ , \tag{3.1.3}$$

and $P(0) = P(N = 0) = \delta^\theta$. To obtain the corresponding pmf for the truncated negative binomial (TNB) distribution, the pmf given in Eqn.(3.1.3) must be divided by $1 - P(0)$. Thus the pmf of the TNB random variables with parameters $0 < \delta < 1$ and $\theta > 0$ is given by

$$P(N = n) = \frac{\delta^\theta}{1 - \delta^\theta} \binom{\theta + n - 1}{\theta - 1} (1 - \delta)^n, \; n = 1, 2, ... \; . \tag{3.1.4}$$

Let $X_i, i = 1, 2, ...,$ be a sequence of i.i.d. random variables with survival function $\bar{F}(x)$ and N be a truncated negative binomial random variable independent of X_i's. Now, consider $U_N = \min(X_1, X_2, ..., X_N)$. Then

$$\begin{aligned} P(U_N > x) &= \bar{G}(x) \\ &= \frac{\delta^\theta}{1 - \delta^\theta} \sum_{n=1}^{\infty} \binom{\theta + n - 1}{\theta - 1} [(1 - \delta)\bar{F}(x)]^n \\ &= \frac{\delta^\theta}{1 - \delta^\theta} \left[\left(F(x) + \delta \bar{F}(x) \right)^{-\theta} - 1 \right]. \end{aligned} \tag{3.1.5}$$

Similarly, if $\delta > 1$ and N is a truncated negative binomial random variable with parameters δ^{-1} and $\theta > 0$, then $V_N = \max(X_1, X_2, ..., X_N)$ also has the survival function given in Eqn.(3.1.5). Here note that in the Eqn.(3.1.5), if $\delta \to 1$, then $\bar{G}(x) \to \bar{F}(x)$. If $\theta = 1$, then this family reduces to the family of Marshall-Olkin distributions. Thus the family of distributions described in the Eqn.(3.1.5) is a generalization of Marshall-Olkin distributions, see Nadarajah et al. (2013).

This family can be interpreted as follows: Suppose the failure times of a device are observed and at every time a failure occurs, the device is repaired to resume function. Suppose also that the device is deemed no usable when a failure occurs that exceeds a certain level of severity. Let $X_1, X_2, ...,$ denotes the failure times and N denote the number of failures. Then U_N will represent the time to the first failure of the device and V_N will represent the life time of the device. Thus, this family can be used to model both the time of the first failure and the life time. The pdf corresponding to Eqn.(3.1.5) is

$$g(x) = \frac{\theta\delta^{\theta}(1-\delta)f(x)}{(1-\delta^{\theta})[F(x)+\delta\bar{F}(x)]^{\theta+1}}. \tag{3.1.6}$$

The hrf is given by

$$h(x) = \frac{\theta(1-\delta)\bar{F}(x)h_F(x)}{[F(x)+\delta\bar{F}(x)]\big[1-(F(x)+\delta\bar{F}(x))^{\theta}\big]}, \tag{3.1.7}$$

where $h_F(x) = \frac{f(x)}{\bar{F}(x)}$ is the hrf corresponding to $F(x)$. Nadarajah et al. (2013) introduced and studied the exponential-truncated negative binomial (ETNB) distribution with parameters δ, θ and β by substituting $\bar{F}(x) = e^{-\beta x}; x > 0, \beta > 0$, in the survival function given in the Eqn.(3.1.5). That is,

$$\bar{G}(x) = \frac{\delta^{\theta}}{1-\delta^{\theta}}\left[(1-e^{-\beta x}+\delta e^{-\beta x})^{-\theta}-1\right] ; x > 0, \tag{3.1.8}$$

where $\delta > 0, \theta > 0$ and $\beta > 0$. By applying the above concept, Kamel et al. (2016) studied the uniform truncated negative binomial (UTNB) distribution, Jayakumar and Sankaran (2016) studied the generalized uniform distribution and Jayakumar and Sankaran (2017) studied the generalized exponential truncated negative binomial distribution.

In Section 2, we introduce the Weibull-truncated negative binomial distribution and study its sub models and shape properties of pdf and hrf. In this section, we derive the expressions for its moment, order statistics and entropy. Section 3 gives the characterizations based on truncated moments and hrf. Maximum likelihood estimation of the parameters of WTNB distribution and a simulation study are given in Section 4. Minification process with WTNB marginals are presents in Section 5. Bivariate extensions of WTNB distribution are proposes in Section 6 and two real-life data applications are illustrates in Section 7.

3.2 The Weibull-Truncated Negative Binomial Distribution

Babu (2016) studied a new family of distributions named as WTNB distribution with parameters $\delta > 0, \theta > 0, \alpha > 0$ and $\beta > 0$. Here $F(x)$ follows two parameters Weibull distribution with cdf, $F(x) = 1 - e^{-(\beta x)^{\alpha}}, \alpha > 0, \beta > 0$. Then the Eqn.(3.1.5) becomes

$$\bar{G}(x) = \frac{\delta^{\theta}}{1-\delta^{\theta}}\left[(1 - e^{-(\beta x)^{\alpha}} + \delta e^{-(\beta x)^{\alpha}})^{-\theta} - 1\right];\; x > 0, \delta > 0, \theta > 0. \tag{3.2.1}$$

The cdf is given by

$$G(x) = \frac{1 - \delta^{\theta}[1 - (1-\delta)e^{-(\beta x)^{\alpha}}]^{-\theta}}{1-\delta^{\theta}}. \tag{3.2.2}$$

The corresponding pdf is

$$g(x) = \frac{(1-\delta)\delta^{\theta}\theta\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{(1-\delta^{\theta})[1 - (1-\delta)e^{-(\beta x)^{\alpha}}]^{\theta+1}}, \tag{3.2.3}$$

where $x > 0, \delta > 0, \theta > 0, \alpha > 0$ and $\beta > 0$.

The following distributions are special cases of the WTNB distribution.

Case I: When $\theta = 1$,

$$g(x) = \frac{\delta\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{[1 - (1-\delta)e^{-(\beta x)^{\alpha}}]^{2}}. \tag{3.2.4}$$

This is the Marshall-Olkin extended Weibull (MOEW) distribution studied in Ghitany et al. (2005).

Case II: When $\theta = 1$ and $\alpha = 1$,

$$g(x) = \frac{\delta\beta e^{-\beta x}}{[1-(1-\delta)e^{-\beta x}]^2}. \tag{3.2.5}$$

This is the Marshall-Olkin extended exponential (MOEE) distribution studied in Singh et al. (2016).

Case III: When $\alpha = 1$,

$$g(x) = \frac{(1-\delta)\delta^\theta \beta e^{-\beta x}}{(1-\delta^\theta)[1-(1-\delta)e^{-\beta x}]^{\theta+1}}. \tag{3.2.6}$$

This is the ETNB distribution studied in Nadarajah et al. (2013).

Case IV: When $\theta = 1, \alpha = 1$ and $\delta = 2$,

$$g(x) = \frac{2\beta e^{-\beta x}}{[1+e^{-\beta x}]^2}, \tag{3.2.7}$$

which is the half-logistic distribution with scale parameter β.

Case V: When $\theta = 1$ and $\delta \to 1$,

$$g(x) = \alpha\beta^\alpha x^{\alpha-1} e^{-(\beta x)^\alpha}. \tag{3.2.8}$$

Here the WTNB distribution reduces to two parameter Weibull distribution. Also, the Eqn.(3.2.3) can be expressed as infinite mixtures of the pdf of some other distributions. Suppose $0 < \delta < 2$. Then

$$g(x) = \frac{\theta(1-\delta)\delta^\theta}{1-\delta^\theta} \sum_{k=0}^{\infty} \binom{\theta+k}{\theta} \frac{(1-\delta)^k}{k+1} f_{WE}(x;\alpha,\beta(k+1)^{\frac{1}{\alpha}}) \tag{3.2.9}$$

where $f_{WE}(x;\alpha,\beta(k+1)^{\frac{1}{\alpha}}) = \alpha(\beta(k+1)^{\frac{1}{\alpha}})^\alpha x^{\alpha-1} e^{-(\beta(k+1)^{\frac{1}{\alpha}} x)^\alpha}$ is the pdf of a Weibull random variable.

Random numbers from the WTNB distribution can be simulated using

$$X = \frac{1}{\beta}\left[-\ln\left(\frac{1-\delta[\delta^{\theta}+Y(1-\delta^{\theta})]^{-\frac{1}{\theta}}}{1-\delta}\right)\right]^{\frac{1}{\alpha}}, \text{ where } Y \sim U(0,1). \tag{3.2.10}$$

3.2.1 Shapes of the pdf of the WTNB distribution

In order to derive the shape properties of the pdf of WTNB distribution, we consider the first derivative of the logarithmic function of $g(x)$. That is

$$\begin{aligned}\frac{d}{dx}[\ln(g(x))] &= \frac{-\left[(1-\alpha)[1-(1-\delta)e^{-(\beta x)^{\alpha}}]+\alpha\beta^{\alpha}x^{\alpha}[1+\theta(1-\delta)e^{-(\beta x)^{\alpha}}]\right]}{x[1-(1-\delta)e^{-(\beta x)^{\alpha}}]}\\ &= \frac{-S(x)}{x[1-(1-\delta)e^{-(\beta x)^{\alpha}}]},\end{aligned} \tag{3.2.11}$$

where $S(x) = (1-\alpha)[1-(1-\delta)e^{-(\beta x)^{\alpha}}]+\alpha\beta^{\alpha}x^{\alpha}[1+\theta(1-\delta)e^{-(\beta x)^{\alpha}}]$. Here the function $S(x)$ is positive and this implies that $g(x)$ is a decreasing function with $\lim_{x\to 0^+} g(x) \to \infty$ and $\lim_{x\to\infty} g(x) \to 0$, when

(i). $0 < \delta \leq 1$, $\theta > 0$ and $0 < \alpha \leq 1$, and

(ii). $\delta > 1$, $0 < \theta < \frac{1}{\delta-1}$ and $0 < \alpha \leq 1$.

The pdf is unimodal, when

(i). $0 < \delta \leq 1$, $\theta > 0$ and $\alpha > 1$, and

(ii). $\delta > 1$, $0 < \theta < \frac{1}{\delta-1}$ and $\alpha > 1$.

The mode is obtained as the solution of the nonlinear equation $\frac{d}{dx}(\ln(g(x))) = 0$.

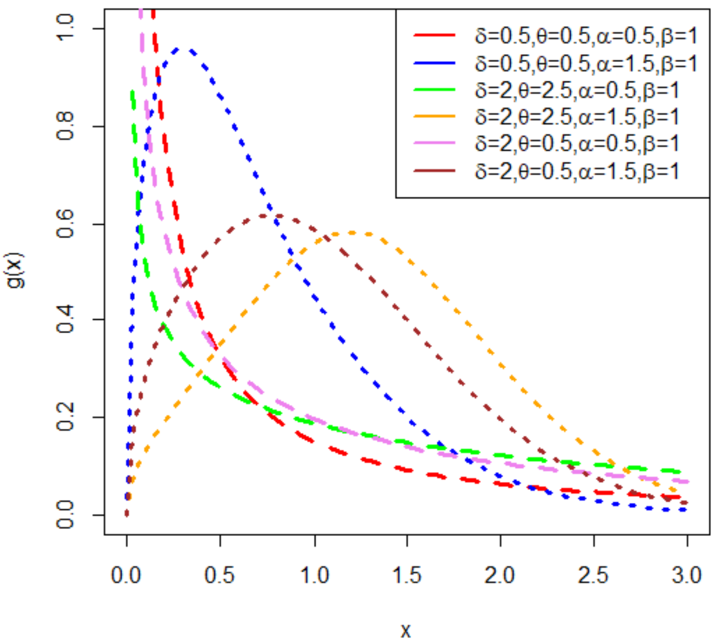

Figure 3.1: Shape of pdf of the WTNB for various choices of parameter values

Some possible shapes of the pdf of the WTNB distribution are presented in Figure 3.1.

3.2.2 Hazard rate function of the WTNB distribution

The hrf of the WTNB distribution is given by

$$h(x) = \frac{(1-\delta)\theta\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{[1-(1-\delta)e^{-(\beta x)^{\alpha}}][1-(1-(1-\delta)e^{-(\beta x)^{\alpha}})^{\theta}]}. \tag{3.2.12}$$

The hrf is decreasing from ∞ to 0, when,

(i). $0 < \delta \leq 1$, $\theta > 0$ and $0 < \alpha \leq 1$, and

(ii). $\delta > 1$, $0 < \theta < \frac{1}{\delta - 1}$ and $0 < \alpha \leq 1$,

and is increasing from 0 to ∞ when,

(i). $0 < \delta \leq 1$, $\theta > 0$ and $\alpha > 1$, and

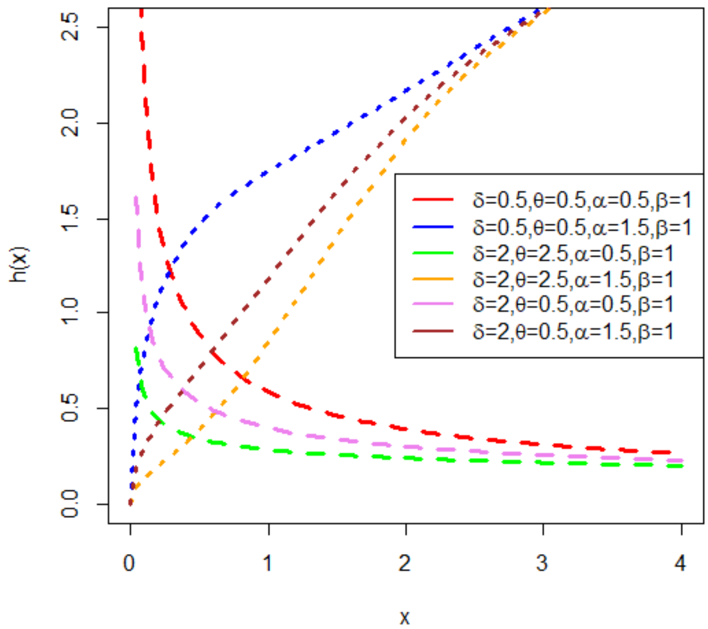

Figure 3.2: Shape of hrf of the WTNB distribution for various choices of parameter values

(ii). $\delta > 1$, $0 < \theta < \frac{1}{\delta-1}$ and $\alpha > 1$.

Some possible shapes of the hrf of the WTNB distribution are presented in Figure 3.2.

3.2.3 Moments of the WTNB distribution

The r^{th} raw moment can be written as

$$E(X^r) = \frac{(1-\delta)\delta^\theta \theta \alpha \beta^\alpha}{1-\delta^\theta} \int_0^\infty \frac{x^{r+\alpha-1} e^{-(\beta x)^\alpha}}{[1-(1-\delta)e^{-(\beta x)^\alpha}]^{\theta+1}} dx. \quad (3.2.13)$$

Taking $u = e^{-(\beta x)^\alpha}$, Eqn.(3.2.13) becomes

$$E(X^r) = \frac{(1-\delta)\delta^\theta \theta}{\beta^r(1-\delta^\theta)} \int_0^1 \frac{(-\log(u))^{\frac{r}{\alpha}}}{[1-(1-\delta)u]^{\theta+1}} du. \quad (3.2.14)$$

Case I. If $|1-\delta| < 1$, that is, $0 < \delta < 2$, then by the binomial series expansion, $(1-x)^{-n} = \sum_{k=0}^{\infty}\binom{n+k-1}{n-1}x^k$, the Eqn.(3.2.14) can be rewritten as sum of infinite series as

$$E(X^r) = \frac{(1-\delta)\delta^\theta\theta}{\beta^r(1-\delta^\theta)}\sum_{k=0}^{n}\binom{\theta+k}{\theta}(1-\delta)^k\int_0^1 u^k(-\log u)^{\frac{r}{\alpha}}du. \tag{3.2.15}$$

where $\int_0^1 u^k(-\log(u))^{\frac{r}{\alpha}}du = (k+1)^{-\frac{r}{\alpha}-1}\Gamma[\frac{r}{\alpha}+1, -(k+1)\log(u)]$.

Therefore,

$$E(X^r) = \frac{(1-\delta)\delta^\theta\theta}{\beta^r(1-\delta^\theta)}\sum_{k=0}^{n}\binom{\theta+k}{\theta}(1-\delta)^k(k+1)^{-\frac{r}{\alpha}-1}\Gamma[\frac{r}{\alpha}+1, -(k+1)\log(u)]. \tag{3.2.16}$$

Case II. If $|1-\delta| < \delta$, that is, $\delta > \frac{1}{2}$ and letting $u = 1-e^{-(\beta x)^\alpha}$, the Eqn.(3.2.13) can be written as

$$E(X^r) = \frac{(1-\delta)\theta}{\beta^r(1-\delta^\theta)\delta}\int_0^\infty \frac{(-\log(1-u))^{\frac{r}{\alpha}}}{[1+(\frac{1-\delta}{\delta})u]^{\theta+1}}du \tag{3.2.17}$$

Now using the series expansion by setting $u = 1-\nu$, the Eqn.(3.2.17) can be written as

$$\begin{aligned} E(X^r) &= \frac{(1-\delta)\theta}{\beta^r(1-\delta)\delta}\int_0^1 \frac{(-\log(\nu))^{\frac{r}{\alpha}}}{[1+(\frac{1-\delta}{\delta})(1-\nu)]^{\theta+1}}d\nu \\ &= \frac{\theta\beta^{-r}}{(1-\delta^\theta)}\sum_{k=0}^{\infty}\binom{\theta+k}{\theta}(-1)^k(\frac{1-\delta}{\delta})^{k+1}\int_0^1 \frac{(1-\nu)^k}{(-\log(\nu))^{\frac{-r}{\alpha}}}d\nu \end{aligned} \tag{3.2.18}$$

Now using the Eqn.(2.6.5.3) of Prudnikov et al. (1986),

$$\int_0^a \frac{x^{\alpha-1}}{(a^\mu - x^\mu)^{-m}}(\log(\frac{a}{x}))^\sigma dx = a^{\alpha+\mu m}\Gamma(\sigma+1)\sum_{j=0}^{m}\frac{(-1)^j(1+m-j)_j}{j!(\alpha+\mu k)^{\sigma+1}}, \tag{3.2.19}$$

we can express the Eqn.(3.2.18) as

$$E(X^r) = \frac{\theta\beta^{-r}}{(1-\delta^\theta)}\sum_{k=0}^{\infty}\binom{\theta+k}{\theta}(-1)^k(\frac{1-\delta}{\delta})^{k+1}\Gamma\left(\frac{r}{\alpha}+1\right)\sum_{j=0}^{k}\frac{(-1)^j(1+k-j)_j}{j!(1+j)^{\frac{r}{\alpha}+1}}. \tag{3.2.20}$$

3.2.4 Order statistics of the WTNB distribution

Let $X_1, X_2, ..., X_n$ are independent random variables following the WTNB distribution. Let $X_{i:n}$ denote the i^{th} order statistics. Then, the pdf of $X_{i:n}$ is

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(n-i)!(i-1)!}g(x)G^{i-1}(x)\bar{G}^{n-i}(x) \\ &= \frac{(-1)^{n-1}n!}{(n-i)!(i-1)!}\frac{(1-\delta)\theta\alpha\beta^\alpha\delta^{\theta(n+1-i)}x^{\alpha-1}e^{-(\beta x)^\alpha}}{(1-\delta^\theta)^n(1-(1-\delta)e^{-(\beta x)^\alpha})^{\theta+1}} \\ &\quad \left[1-\frac{\delta^\theta}{(1-(1-\delta)e^{-(\beta x)^\alpha})^\theta}\right]^{i-1} \\ &\quad \left[1-\frac{1}{(1-(1-\delta)e^{-(\beta x)^\alpha})^\theta}\right]^{n-i}. \end{aligned} \tag{3.2.21}$$

In particular the pdf of the largest order statistic, $X_{(n)}$ is

$$\begin{aligned} g_{X_{(n)}}(x) &= \frac{(-1)^{n-1}n(1-\delta)\theta\alpha\beta^\alpha\delta^\theta x^{\alpha-1}e^{-(\beta x)^\alpha}}{(1-\delta^\theta)^n(1-(1-\delta)e^{-(\beta x)^\alpha})^{\theta+1}} \\ &\quad \left[1-\frac{\delta^\theta}{(1-(1-\delta)e^{-(\beta x)^\alpha})^\theta}\right]^{n-1}. \end{aligned} \tag{3.2.22}$$

The pdf of the smallest order statistic, $X_{(1)}$ is

$$g_{X_{(1)}}(x) = \frac{(-1)^{n-1}n(1-\delta)\theta\alpha\beta^{\alpha}\delta^{n\theta}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{(1-\delta^{\theta})^{n}(1-(1-\delta)e^{-(\beta x)^{\alpha}})^{\theta+1}} \left[1-\frac{1}{(1-(1-\delta)e^{-(\beta x)^{\alpha}})^{\theta}}\right]^{n-1}. \qquad (3.2.23)$$

Using binomial series expansion, $g_{i:n}(x)$ can be expressed as

$$\begin{aligned} g_{i:n}(x) &= \frac{(-1)^{n-1}n!}{(n-i)!(i-1)!}\frac{(1-\delta)\theta\alpha\beta^{\alpha}\delta^{\theta(n+1-i)}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{(1-\delta^{\theta})^{n}(1-(1-\delta)e^{-(\beta x)^{\alpha}})^{\theta+1}} \\ &\quad \sum_{k=0}^{i-1}\sum_{l=0}^{n-i}(-1)^{k+l}\binom{i-1}{k}\binom{n-i}{l}\frac{\delta^{\theta k}}{[1-(1-\delta)e^{-(\beta x)^{\alpha}}]^{\theta(k+l)}} \\ &= \frac{(-1)^{n-1}n!\delta^{\theta(n-i)}}{(n-i)!(i-1)!(1-\delta^{\theta})^{n}}\sum_{k=0}^{i-1}\sum_{l=0}^{n-i}(-1)^{k+l}\binom{i-1}{k}\binom{n-i}{l} \\ &\quad \frac{1-\delta^{\theta(k+l+1)}}{\delta^{\theta l}(k+l+1)}\, g(x;\delta,\theta(k+l+1),\alpha,\beta), \end{aligned} \qquad (3.2.24)$$

where $g(x;\delta,\theta(k+l+1),\alpha,\beta) = \frac{(1-\delta)\delta^{\theta(k+l+1)}\theta(k+l+1)\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{(1-\delta^{\theta(k+l+1)})[1-(1-\delta)e^{-(\beta x)^{\alpha}}]^{\theta(k+l+1)+1}}$. Thus $X_{i:n}$ is a finite mixture of WTNB random variables.

3.2.5 Rényi and Shannon entropies of the WTNB distribution

The Rényi entropy of the WTNB distribution is given by

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left(\int_0^{\infty}\left[\frac{(1-\delta)\delta^{\theta}\theta\alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}}{(1-\delta^{\theta})[1-(1-\delta)e^{-(\beta x)^{\alpha}}]^{\theta+1}}\right]^{\gamma}dx\right). \qquad (3.2.25)$$

Letting $u = e^{-(\beta x)^{\alpha}}$, we get

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{(1-\delta)\delta^{\theta}\theta}{1-\delta^{\theta}}\right)^{\gamma}\frac{(\alpha\beta^{\gamma})^{\gamma-1}}{\beta^{(\gamma-1)(\alpha-1)}}\int_0^1\frac{(-\log u)^{(\gamma-1)(1-\frac{1}{\alpha})}u^{\gamma-1}}{[1-(1-\delta)u]^{\gamma(\theta+1)}}du\right]. \tag{3.2.26}$$

The Shannon entropy is given by

$$\begin{aligned} E[-\log(g(X))] &= \log\left(\frac{1-\delta^{\theta}}{(1-\delta)\delta^{\theta}\theta\alpha\beta^{\alpha}}\right) - (\alpha-1)E(\log(X)) + \beta^{\alpha}E(X^{\alpha}) \\ &\quad +(\theta+1)E\big(\log(1-(1-\delta)e^{-(\beta X)^{\alpha}})\big). \end{aligned} \tag{3.2.27}$$

3.3 Characterization Based on Truncated Moments

We present a characterization of the WTNB distribution in terms of a simple relationship between truncated moments. This results are developed using the Theorem 1 of Glanzel (1987).

Theorem 3.3.1. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable, and let $q_2(x) = [1-(1-\delta)e^{-(\beta x)^{\alpha}}]^{\theta+1}$ and $q_1(x) = q_2(x)e^{-(\beta x)^{\alpha}}$ for $x > 0$. The pdf of X is Eqn.(3.2.3) if and only if the function η defined in Theorem 2.4.1 has the form*

$$\eta(x) = \frac{1}{2}e^{-(\beta x)^{\alpha}},\ x > 0. \tag{3.3.1}$$

Proof. Let X have pdf given in Eqn.(3.2.3). Then

$$\begin{aligned} (1-G(x))\mathbb{E}[q_2(X)|X \geq x] &= \frac{(1-\delta)\delta^{\theta}\theta}{1-\delta^{\theta}}e^{-(\beta x)^{\alpha}},\ x > 0, \\ (1-G(x))\mathbb{E}[q_1(X)|X \geq x] &= \frac{(1-\delta)\delta^{\theta}\theta}{2(1-\delta^{\theta})}e^{-2(\beta x)^{\alpha}},\ x > 0, \end{aligned}$$

and hence

$$\eta(x)q_2(x) - q_1(x) = -\frac{1}{2}e^{-(\beta x)^\alpha}[1-(1-\delta)e^{-(\beta x)^\alpha}]^{\theta+1} < 0, for\ x > 0. \quad (3.3.2)$$

Conversely, if η is given as Eqn.(3.3.1), then

$$s'(x) = \frac{\eta'(x)q_2(x)}{\eta(x)q_2(x) - q_1(x)} = \alpha\beta^\alpha x^{\alpha-1},\ x > 0, \quad (3.3.3)$$

and hence $s(x) = (\beta x)^\alpha,\ x > 0$, or $e^{-s(x)} = e^{-(\beta x)^\alpha},\ x > 0$. Now, using Theorem 2.4.1, X has the pdf given in Eqn.(3.2.3). □

Corollary 3.3.1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable, and let $q_2(x) = [1-(1-\delta)e^{-(\beta x)^\alpha}]^{\theta+1}$. The pdf of X is Eqn.(3.2.3) if and only if there exist functions q_1 and η defined in Theorem 2.4.1 satisfying the differential equation*

$$\frac{\eta'(x)q_2(x)}{\eta(x)q_2(x) - q_1(x)} = \alpha\beta^\alpha x^{\alpha-1},\ x > 0. \quad (3.3.4)$$

Remark 3.3.1. *The general solution of the differential equation in Corollary 3.3.1 is obtained as follows:*

$$\eta'(x) - \alpha\beta^\alpha x^{\alpha-1}\eta(x) = [q_2(x)]^{-1}\alpha\beta^\alpha x^{\alpha-1}q_1(x),$$

or

$$\frac{d}{dx}[\eta(x)e^{-(\beta x)^\alpha}] = [q_2(x)]^{-1}\alpha\beta^\alpha x^{\alpha-1}e^{-(\beta x)^\alpha}q_1(x).$$

Thus we obtain

$$\eta(x) = e^{(\beta x)^\alpha}\left[-\int [q_2(x)]^{-1}\alpha\beta^\alpha x^{\alpha-1}e^{-(\beta x)^\alpha}q_1(x)dx + D\right], \quad (3.3.5)$$

where D is a constant.

3.3.1 Characterization based on hazard rate function

The following Theorem is a non-trivial characterization of the WTNB distribution, when $\theta = 1$.

Theorem 3.3.2. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The pdf of X is given in Eqn.(3.2.3) if and only if its hrf, $h(x)$ satisfies the differential equation*

$$x^{-(\alpha-1)}h'(x) - (\alpha-1)x^{-\alpha}h(x) = \frac{d}{dx}\left[\frac{\alpha\beta^\alpha}{1-(1-\delta)e^{-(\beta x)^\alpha}}\right]. \tag{3.3.6}$$

Proof. When $\theta = 1$, the pdf $g(x)$ and hrf $h(x)$ of X are respectively

$$g(x) = \frac{\delta\alpha\beta^\alpha x^{\alpha-1}e^{-(\beta x)^\alpha}}{[1-(1-\delta)e^{-(\beta x)^\alpha}]^2}, \tag{3.3.7}$$

and

$$h(x) = \frac{\alpha\beta^\alpha x^{\alpha-1}}{1-(1-\delta)e^{-(\beta x)^\alpha}}. \tag{3.3.8}$$

Then we have

$$\frac{g'(x)}{g(x)} = \frac{\alpha-1}{x} - \alpha\beta^\alpha x^{\alpha-1} - \frac{2(1-\delta)\alpha\beta^\alpha x^{\alpha-1}e^{-(\beta x)^\alpha}}{1-(1-\delta)e^{-(\beta x)^\alpha}}. \tag{3.3.9}$$

We get the first order differential equation as

$$h'(x) - \frac{\alpha-1}{x}h(x) = -\frac{\alpha^2\beta^{2\alpha}(1-\delta)x^{2(\alpha-1)e^{-(\beta x)^\alpha}}}{[1-(1-\delta)e^{-(\beta x)^\alpha}]^2}, \tag{3.3.10}$$

which implies,

$$x^{-(\alpha-1)}h'(x) - (\alpha-1)x^{-\alpha}h(x) = \frac{d}{dx}\left[\frac{\alpha\beta^\alpha}{1-(1-\delta)e^{-(\beta x)^\alpha}}\right].$$

Now, the differential equation Eqn.(3.3.6) holds, then

$$\frac{d}{dx}[x^{-(\alpha-1)}h(x)] = \frac{d}{dx}\left[\frac{\alpha\beta^\alpha}{1-(1-\delta)e^{-(\beta x)^\alpha}}\right], x > 0,$$

from which, we obtain

$$h(x) = \frac{\alpha\beta^\alpha x^{\alpha-1}}{1-(1-\delta)e^{-(\beta x)^\alpha}}, x > 0, \tag{3.3.11}$$

which is the hrf of WTNB distribution when $\theta = 1$. □

3.4 Estimation of Parameters of the WTNB Distribution

The moments of the WTNB distribution is not in a closed form. Therefore, we consider the method of maximum likelihood for the estimation of unknown parameters. For a given sample $(x_1, x_2, ..., x_n)$, the log-likelihood function is given by

$$\begin{aligned}\log(L) &= n\log(1-\delta) + n\theta\log(\delta) + n\log(\theta) + n\log(\alpha) + n\alpha\log(\beta) \\ &\quad -n\log(1-\delta^{\theta}) + (\alpha-1)\sum_{i=1}^{n}(\log(x_i)) - \beta^{\alpha}\sum_{i=1}^{n}(x_i^{\alpha}) \\ &\quad -(\theta+1)\sum_{i=1}^{n}\log[1-(1-\delta)e^{-(\beta x_i)^{\alpha}}]. \qquad (3.4.1)\end{aligned}$$

The partial derivatives of the log-likelihood function with respect to the parameters are

$$\frac{\partial \log L}{\partial \delta} = -\frac{n}{1-\delta} + \frac{n\theta}{\delta} + \frac{n\theta\delta^{\theta-1}}{1-\delta^{\theta}} - (\theta+1)\sum_{i=1}^{n}\frac{e^{-(\beta x)^{\alpha}}}{1-(1-\delta)e^{-(\beta x)^{\alpha}}}, \qquad (3.4.2)$$

$$\frac{\partial \log L}{\partial \theta} = n\log(\delta) + \frac{n}{\theta} - \frac{n\delta^{\theta}\log(\delta)}{1-\delta^{\theta}} - \sum_{i=1}^{n}\log[1-(1-\delta)e^{-(\beta x_i)^{\alpha}}], \qquad (3.4.3)$$

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} &= \frac{n}{\alpha} + n\log(\beta) + \sum_{i=1}^{n}\log(x_i) - \beta^{\alpha}\left[\sum_{i=1}^{n}[\log(\beta)+\log(x_i)]x_i^{\alpha}\right] \\ &\quad -(\theta+1)\sum_{i=1}^{n}\frac{(1-\delta)\beta^{\alpha}x_i^{\alpha}\log(\beta x_i)e^{-(\beta x_i)^{\alpha}}}{[1-(1-\delta)e^{-(\beta x_i)^{\alpha}}]}, \qquad (3.4.4)\end{aligned}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n\alpha}{\beta} - \alpha\beta^{\alpha-1}\sum_{i=1}^{n}x_i^{\alpha} - (\theta+1)\sum_{i=1}^{n}\frac{(1-\delta)\beta^{\alpha-1}\alpha x_i^{\alpha}e^{-(\beta x_i)^{\alpha}}}{[1-(1-\delta)e^{-(\beta x_i)^{\alpha}}]}. \qquad (3.4.5)$$

The MLEs of $\psi = (\delta, \theta, \alpha, \beta)^T$ say $\hat{\psi} = (\hat{\delta}, \hat{\theta}, \hat{\alpha}, \hat{\beta})^T$ are the solutions of the simultaneous equations $\frac{\partial \log L}{\partial \delta} = 0, \frac{\partial \log L}{\partial \theta} = 0, \frac{\partial \log L}{\partial \alpha} = 0$ and $\frac{\partial \log L}{\partial \beta} = 0$. The solution of these nonlinear equations does not have closed forms and have to solve numerically by the Newton Raphson method. *optim* or *nlm* packages in **R** software can be used for the solution of these equations. For interval estimation and hypothesis testing on the parameters of the WTNB distribution, we require the information matrix. The information matrix is as follows:

$$I_n(\hat{\psi}) = \begin{bmatrix} \frac{\partial^2 \log L}{\partial \delta^2} & \frac{\partial^2 \log L}{\partial \delta \partial \theta} & \frac{\partial^2 \log L}{\partial \delta \partial \alpha} & \frac{\partial^2 \log L}{\partial \delta \partial \beta} \\ \frac{\partial^2 \log L}{\partial \theta \partial \delta} & \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \delta} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \log L}{\partial \beta \partial \delta} & \frac{\partial^2 \log L}{\partial \beta \partial \theta} & \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} \end{bmatrix} \tag{3.4.6}$$

The elements of the Eqn.(3.4.6) are obtained by partial differentiation of the first order partial derivatives of the likelihood function with respect to the parameters δ, θ, α and β. Here the WTNB distribution satisfies the regularity conditions, which are fulfilled for the parameters in the interior parameter space but not on the boundary. Hence, $\hat{\psi}$ is consistent and asymptotically normal. That is, $\sqrt{(I_n(\hat{\psi}))}(\hat{\psi} - \psi)$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix.

We can use the normal distribution of $\hat{\psi}$ to construct approximate confidence region for a particular parameter. The asymptotic $100(1-\eta)\%$ confidence interval for the parameters δ, θ, α and β can be determined as $\hat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{(V(\hat{\delta}))}$, $\hat{\theta} \pm Z_{\frac{\eta}{2}} \sqrt{(V(\hat{\theta}))}$, $\hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{(V(\hat{\alpha}))}$ and $\hat{\beta} \pm Z_{\frac{\eta}{2}} \sqrt{(V(\hat{\beta}))}$ respectively, where $V(\hat{\delta})$, $V(\hat{\theta})$, $V(\hat{\alpha})$ and $V(\hat{\beta})$

are the variances of $\hat{\delta}, \hat{\theta}, \hat{\alpha}$ and $\hat{\beta}$, obtained by the diagonal elements of $I_n^{-1}(\hat{\psi})$, and $Z_{\frac{\eta}{2}}$ is the $(1-\frac{\eta}{2})^{th}$ quantile of the standard normal distribution.

3.4.1 Simulation study of the WTNB distribution

We conduct a Monte Carlo simulation study to assess the performance of the MLEs of the unknown parameters for the WTNB distribution. The performance of the estimates is evaluated in terms of their average values and mean squared errors (MSEs). The **R** programming software is used to generate 1000 samples of the WTNB distribution for different sample sizes, where n = (50,100,200,500), and for different parameters combinations, where $\delta = (0.5, 2.5), \theta = (0.5, 1.5), \alpha = (0.5, 1.0)$ and $\beta = (1.0, 1.5)$. The average values of estimates, average biases, MSEs and coverage probabilities (CP) are provided in Table 3.1. It is observed from Table 3.1, that the MSE decreases as the sample size increases. Thus, the MLE method works very well to estimate the model parameters of the WTNB distribution.

3.5 Minification Process of the WTNB Distribution

Here we develop an autoregressive (AR(1)) minification process of order one with WTNB distribution as marginals. Consider an AR(1) minification process,

$$X_n = \begin{cases} \epsilon_n & w.p.\ \ \rho \\ min(X_{n-1}, \epsilon_n) & w.p.\ \ 1-\rho \end{cases} \tag{3.5.1}$$

Table 3.1: The parameter estimates, average biases, MSEs and CPs of WTNB distribution.

Parameter	Samples(n)	Average Values	Average Biases	MSEs	CP
$\delta = 0.5$	50	0.4492	-0.1362	0.0192	0.925
	100	0.4736	-0.0110	0.0135	0.936
	200	0.4843	-0.1037	0.0092	0.939
	500	0.5114	0.0093	0.0038	0.941
$\theta = 0.5$	50	0.3182	-0.1634	0.0826	0.881
	100	0.3601	-0.1521	0.0631	0.903
	200	0.4027	-0.0933	0.0447	0.917
	500	0.4223	-0.0892	0.0401	0.919
$\alpha = 0.5$	50	0.4043	-0.1122	0.0193	0.895
	100	0.5624	0.0644	0.0038	0.906
	200	0.5559	0.0426	0.0027	0.918
	500	0.5354	0.0115	0.0019	0.928
$\beta = 1.0$	50	0.7120	-0.2982	0.0252	0.879
	100	0.8214	-0.1732	0.0216	0.893
	200	0.8435	-0.1513	0.0189	0.907
	500	0.9117	-0.0922	0.0147	0.916
$\delta = 2.5$	50	1.9864	-0.3130	0.0165	0.904
	100	2.1308	-0.2869	0.0134	0.918
	200	2.2234	-0.2411	0.0021	0.920
	500	2.4138	-0.1036	0.0019	0.936
$\theta = 1.5$	50	1.4112	-0.1271	0.0631	0.911
	100	1.5276	0.1133	0.0447	0.917
	200	1.5117	0.0926	0.0312	0.926
	500	1.4968	-0.0014	0.0019	0.935
$\alpha = 1.0$	50	0.7836	-0.2113	0.0291	0.921
	100	0.8109	-0.1906	0.0182	0.935
	200	1.1937	0.0028	0.0133	0.939
	500	1.0821	0.0016	0.0068	0.944
$\beta = 1.5$	50	0.9824	-0.3227	0.0917	0.885
	100	0.9932	-0.3190	0.0893	0.896
	200	1.1384	-0.2081	0.0722	0.901
	500	1.3709	-0.1271	0.0315	0.919

where $0 < \rho < 1, n \geq 1$ and $\{\epsilon_n\}$ is a sequence of i.i.d random variables.

Theorem 3.5.1. *The minification process given in Eqn.(3.5.1), defines a stationary AR(1) minification process with WTNB$(\delta, \theta, \alpha, \beta)$ as marginal distribution if and only if ϵ_n's are i.i.d. WTNB$(\rho, \delta, \theta, \alpha, \beta)$ distribution with $X_0 \underline{\underline{d}}\, WTNB(\delta, \theta, \alpha, \beta)$.*

Proof. Let $X \sim WTNB(\delta, \theta, \alpha, \beta)$. The survival function given in Eqn.(3.2.1) can be expressed as

$$\bar{G}_X(x) = \frac{1}{1 + \left[\frac{[1-(1-\delta)e^{-(\beta x)^{\alpha}}]^{\theta} - \delta^{\theta}}{\delta^{\theta}\left(1-[1-(1-\delta)e^{-(\beta x)^{\alpha}}]^{\theta}\right)}\right]}. \tag{3.5.2}$$

The model given in Eqn.(3.5.1) in terms of survival function is,

$$P(X_n > x) = P(\epsilon_n > x)[\rho + (1-\rho)P(X_{n-1} > x)]. \tag{3.5.3}$$

That is,

$$\bar{G}_{X_n}(x) = \bar{G}_{\epsilon_n}(x)[\rho + (1-\rho)\bar{G}_{X_{n-1}}(x)]. \tag{3.5.4}$$

If $\{X_n\}$ is stationary with WTNB$(\delta, \theta, \alpha, \beta)$ distribution marginal, then

$$\begin{aligned} \bar{G}_{\epsilon_n}(x) &= \frac{\bar{G}_X(x)}{\rho + (1-\rho)\bar{G}_X(x)} \\ &= \frac{1}{1+\rho\left[\frac{[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta - \delta^\theta}{\delta^\theta\left(1-[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta\right)}\right]}. \end{aligned} \tag{3.5.5}$$

That is, ϵ_n's are WTNB$(\rho, \delta, \theta, \alpha, \beta)$ distribution.

Conversely, if ϵ_n's are WTNB$(\rho, \delta, \theta, \alpha, \beta)$ distribution with $X_0 \underline{\underline{d}}\, WTNB(\delta, \theta, \alpha, \beta)$ distribution, then $\{X_n\}$ defines a stationary process, with $WTNB(\delta, \theta, \alpha, \beta)$ distribution as the stationary marginal. From the Eqn.(3.5.4),

$$\begin{aligned} \bar{G}_{X_1}(x) &= \bar{G}_{\epsilon_1}(x)[\rho + (1-\rho)\bar{G}_{X_0}(x)] \\ &= \frac{1}{1+\rho\left[\frac{[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta - \delta^\theta}{\delta^\theta\left(1-[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta\right)}\right]} \\ &\quad \left[\rho + (1-\rho)\frac{1}{1+\left[\frac{[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta - \delta^\theta}{\delta^\theta\left(1-[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta\right)}\right]}\right] \\ &= \frac{1}{1+\left[\frac{[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta - \delta^\theta}{\delta^\theta\left(1-[1-(1-\delta)e^{-(\beta x)^\alpha}]^\theta\right)}\right]} \end{aligned} \tag{3.5.6}$$

That is, $X_1 \underline{\underline{d}}\, WTNB(\delta, \theta, \alpha, \beta)$ distribution.

If we assume that $X_{n-1} \underline{\underline{d}}\, WTNB(\delta, \theta, \alpha, \beta)$ distribution, then by induction we get $X_n \underline{\underline{d}}\, WTNB(\delta, \theta, \alpha, \beta)$ distribution. Hence the process $\{X_n\}$ is stationary with $WTNB(\delta, \theta, \alpha, \beta)$ distribution marginals. □

3.6 Bivariate WTNB Distribution

Let $\bar{G}(x, y)$ be the survival function of a bivariate random vector (X, Y). Now by applying the bivariate set up suggested in Marshall and Olkin (1997), we can develop bivariate truncated negative binomial distribution with survival function

$$\bar{F}(x, y) = \frac{\delta^{\theta}}{1-\delta^{\theta}} \left[\left[1 - (1-\delta)\bar{G}(x, y)\right]^{-\theta} - 1 \right]; 0 < x, y < \infty, \delta > 0, \theta > 0. \quad (3.6.1)$$

Clearly when $\theta = 1$, we get the bivariate Marshall and Olkin distribution studied in Marshall and Olkin (1997). When $\theta = 1$ and $\delta \to 1$, we get the original survival function.

Definition 3.6.1. *A bivariate random vector (X, Y) has bivariate WTNB distribution with parameters $\delta > 0, \theta > 0, \lambda_1 > 0, \lambda_2 > 0, \lambda_{12} > 0, \alpha_1 > 0$ and $\alpha_2 > 0$ if its survival function is of the form,*

$$\bar{F}(x, y) = \frac{\delta^{\theta}}{1-\delta^{\theta}} \left[\left[1 - (1-\delta)e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} max(x^{\alpha_1}, y^{\alpha_2})}\right]^{-\theta} - 1 \right]. \quad (3.6.2)$$

Remark 3.6.1. *If $\alpha_1 = \alpha_2 = 1$, then bivariate WTNB becomes a bivariate ETNB.*

Remark 3.6.2. *If $\delta = 1$ and $\theta = 1$, then bivariate WTNB becomes a type I Marshall-Olkin bivariate distribution studied in Jose (2011).*

Remark 3.6.3. *If $\delta = 1$, $\theta = 1$ and $\alpha_1 = \alpha_2 = 1$ then bivariate WTNB becomes a Marshall-Olkin bivariate exponential distribution (see Marshall and Olkin (1967)).*

Remark 3.6.4. *If $\delta = 1$, $\theta = 1$ and $\alpha_1 = \alpha_2 = \alpha$ then bivariate WTNB becomes a Marshall-Olkin bivariate Weibull distribution studied in Hanagal (1996).*

3.7 Data Applications of the WTNB Distribution

In order to check how the WTNB distribution works in practice, we use two real data sets. The first data set is taken from Lee and Krutchkoff (1980), which represents the actual mercury concentrations found in 115 swordfish and the data are follows:

0.05, 0.07, 0.07, 0.13, 0.13, 0.19, 0.24, 0.25, 0.28, 0.32, 0.39, 0.45, 0.46, 0.53, 0.54, 0.56, 0.60, 0.60, 0.61, 0.62, 0.65, 0.71, 0.72, 0.75, 0.76, 0.79, 0.81, 0.81, 0.82, 0.82, 0.82, 0.83, 0.83, 0.83, 0.84, 0.85, 0.89, 0.90, 0.91, 0.92, 0.92, 0.93, 0.95, 0.95, 0.97, 0.97, 0.98, 1.00, 1.00, 1.01, 1.02, 1.04, 1.05, 1.05, 1.08, 1.10, 1.12, 1.12, 1.14, 1.14, 1.15, 1.16, 1.20, 1.20, 1.20, 1.20, 1.20, 1.21, 1.22, 1.25, 1.25, 1.26, 1.27, 1.27, 1.29, 1.29, 1.29, 1.29, 1.30, 1.31, 1.32, 1.32, 1.37, 1.37, 1.39, 1.39, 1.40, 1.40, 1.41, 1.42, 1.43, 1.44, 1.45, 1.54, 1.54, 1.58, 1.58, 1.60, 1.60, 1.62, 1.62, 1.66, 1.66, 1.68, 1.69, 1.72, 1.74, 1.85, 1.89, 1.96, 2.06, 2.10, 2.23, 2.25, 2.72.

The second data set is taken from Murthy et al. (2004, p.245) and the data are as follows:

1, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 10, 11, 12, 14.

The descriptive statistics of the two data sets are presented in Table 3.2. The parameters are estimated by using the MLE method. We compare the fit of the data sets with the sub models of the WTNB distribution. The parameter estimates of

Table 3.2: Descriptive statistics of the two data sets

Data	Sample size (n)	Mean	SD	Min.	Max.	Skewness	Kurtosis
Data set I	115	1.102	0.499	0.05	2.72	0.173	0.441
Data set II	16	6.188	3.674	1.00	14.00	0.908	-0.076

the first data set are presented in Table 3.3. The goodness of fit statistics are pre-

Table 3.3: The parameter estimates of the first data set

Model	ML estimates	-log L
WTNB	$\hat{\delta} = 1.9383, \hat{\theta} = 1.8831, \hat{\alpha} = 1.8388, \hat{\beta} = 1.0703$	82.327
MOEW	$\hat{\delta} = 1.3213, \hat{\alpha} = 1.9064, \hat{\beta} = 0.8941$	85.624
W	$\hat{\alpha} = 1.9328, \hat{\beta} = 0.8362$	87.610
ETNB	$\hat{\delta} = 1.9252, \hat{\theta} = 1.5481, \hat{\beta} = 1.5009$	108.578
MOEE	$\hat{\delta} = 1.9877, \hat{\beta} = 1.2847$	111.638

sented in Table 3.4. Results from Table 3.3 and Table 3.4 shows that the -logL, AIC,

Table 3.4: Goodness of fit statistics for the first data set

Model	AIC	CAIC	BIC	HQIC	A*	W*	K-S	p - value
WTNB	172.655	173.018	183.634	177.111	0.8688	0.1344	0.0814	0.4306
MOEW	177.248	177.464	185.483	180.591	1.4848	0.2364	0.1257	0.0529
W	179.220	179.328	184.710	181.449	1.7766	0.2855	0.1495	0.0117
ETNB	223.157	223.373	231.392	226.499	2.1952	0.3539	0.2829	<0.001
MOEE	227.275	227.382	232.765	229.503	2.3913	0.3875	0.2234	<0.001

CAIC, BIC, HQIC, A^*, W^* and K-S distance are lowest and p value is highest for the WTNB distribution. This is a clear evidence that the WTNB distribution is a better model to fit the given data set compared to the other four sub models used for comparison. From Table 3.5, we have rejected the null hypothesis based on the LR test in all the four hypotheses and it shows the relevance of the four parameter WTNB distribution for data modelling. The plots of the fitted cdfs with the empirical distribution of the first data set is shown in Figure 3.3. The parameter estimates of the second data set are presented in Table 3.6. The goodness of fit statistics for the second data set are given in Table 3.7. Results from Table 3.6 and Table 3.7,

Table 3.5: Likelihood ratio test results for the first data set

Distribution	Hypotheses	LR statistic	p - value
WTNB vs ETNB	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	52.502	<0.001
WTNB vs MOEW	$H_0 : \theta = 1$ vs $H_1 : H_0$ is false	6.594	0.0102
WTNB vs MOEE	$H_0 : \theta = 1, \alpha = 1$ vs $H_1 : H_0$ is false	58.622	<0.001
WTNB vs W	$H_0 : \delta = 1$ vs $H_1 : H_0$ is false	10.566	0.0051

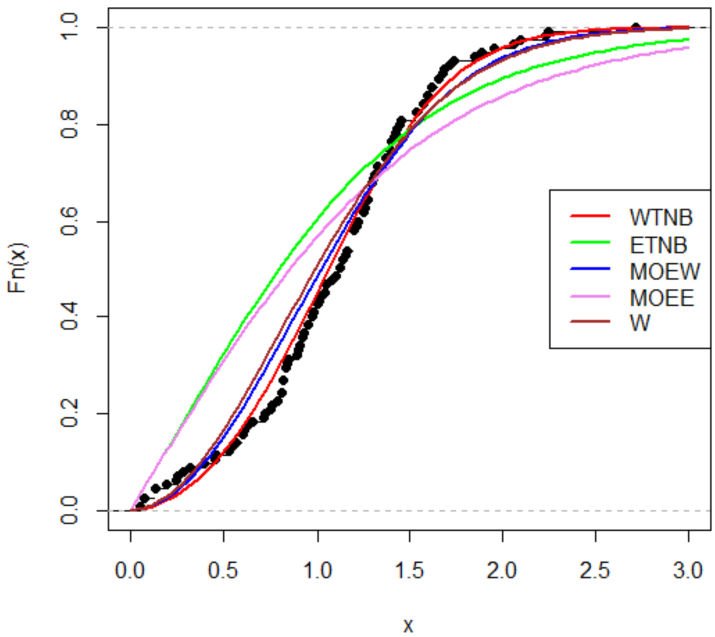

Figure 3.3: Fitted cdfs for the first data set

Table 3.6: The parameter estimates of the second data set

Model	ML estimates	-log L
WTNB	$\hat{\delta} = 1.5187, \hat{\theta} = 0.6924, \hat{\alpha} = 1.7379, \hat{\beta} = 0.1709$	41.686
ETNB	$\hat{\delta} = 1.9903, \hat{\theta} = 1.8713, \hat{\beta} = 0.2508$	43.966
MOEW	$\hat{\delta} = 1.4826, \hat{\alpha} = 1.6352, \hat{\beta} = 0.1623$	43.652
MOEE	$\hat{\delta} = 1.9245, \hat{\beta} = 0.2224$	45.198
W	$\hat{\alpha} = 1.4523, \hat{\beta} = 0.1789$	45.600

Table 3.7: Goodness of fit statistics for the second data set

Model	AIC	CAIC	BIC	HQIC	A*	W*	K-S	p - value
WTNB	91.372	95.008	94.462	91.530	0.556	0.101	0.143	0.8988
MOEW	93.304	95.304	95.621	93.422	0.557	0.116	0.167	0.7584
ETNB	93.932	95.932	96.249	94.051	0.578	0.114	0.231	0.3604
MOEE	94.396	95.319	95.941	94.475	0.561	0.119	0.268	0.2016
W	95.200	96.123	96.745	95.279	0.575	0.183	0.272	0.1875

shows that the -logL, AIC, CAIC, BIC, HQIC, A^*, W^* and K-S distance are lowest and p value is highest for the WTNB distribution. Thus, the WTNB distribution is a better model to fit the second data set compared to the other four sub models. From Table 3.8, we have rejected the null hypothesis based on the LR test in all the

Table 3.8: Likelihood Ratio test results for the second data set

Distribution	Hypotheses	LR statistic	p - value
WTNB vs ETNB	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	4.560	0.0327
WTNB vs MOEW	$H_0 : \theta = 1$ vs $H_1 : H_0$ is false	3.932	0.0474
WTNB vs MOEE	$H_0 : \theta = 1, \alpha = 1$ vs $H_1 : H_0$ is false	7.024	0.0298
WTNB vs W	$H_0 : \delta = 1$ vs $H_1 : H_0$ is false	7.828	0.0199

four hypotheses. The plots of the fitted cdfs with the empirical distribution of the second data set is presented in Figure 3.4 .

3.8 Summary

In this chapter, the Weibull truncated negative binomial distribution is introduced. The sub models of this distribution are identified as ETNB, MOEW, MOEE, half-logistic and Weibull distributions. The shape properties of pdf and hrf are studied. Some statistical properties, such as quantiles, moments, Shannon and Rényi entropies, distribution of order statistics are derived. Estimation of the unknown parameters are done using MLE method and a simulation study is conducted to study

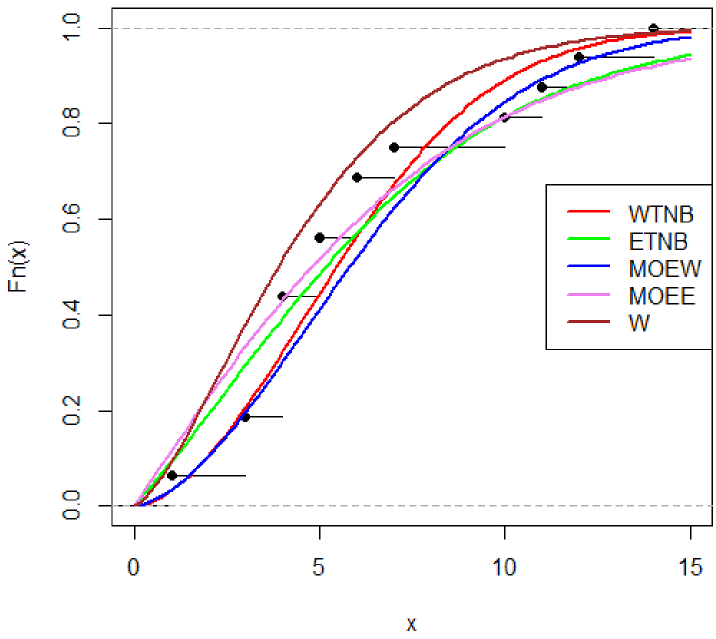

Figure 3.4: Fitted cdfs for the second data set

the performance of the estimates. Also, minification process with WTNB marginals are obtained. Finally, real data applications of the WTNB distribution is discussed with two data sets.

CHAPTER

FOUR

A NEW BIVARIATE DISTRIBUTION WITH MODIFIED WEIBULL DISTRIBUTION AS MARGINALS

4.1 Introduction

[1] Construction of bivariate and multivariate distribution functions attracted the attention of researchers over a long period of time. These distributions are important in modelling dependent random variables in many areas such as reliability, survival analysis, queuing models, insurance risk analysis and so on. The problem of constructing bivariate distributions with specified marginals have been discussed by many authors (see Plackett (1965), Mardia (1967) and Farlie (1960)). Marshall and Olkin (1967) introduced the Marshall-Olkin bivariate exponential (MOBE) dis-

[1]Some results included in this chapter have appeared in the paper Babu and Jayakumar (2018b).

tribution, in which both the marginals have exponential distribution. If the bivariate data shows non-constant hazard rate function then MOBE distribution may not be an appropriate choice. Because of that, Marshall and Olkin (1967) suggested the Marshall-Olkin bivariate Weibull (MOBW) distribution, where the marginals are Weibull distributions. Sarhan and Balakrishnan (2007) studied a new class of bivariate distribution using a latent random variable with exponential distribution. For more details on bivariate distributions, see Balakrishnan and Lai (2009).

Kundu and Dey (2009) have considered the maximum likelihood estimation of the model parameters of the MOBW distribution via expectation maximization (EM) algorithm. Using the maximum instead of the minimum in the Marshall and Olkin scheme, Kundu and Gupta (2009, 2010) introduced the bivariate generalized exponential and singular bivariate generalized exponential distribution, respectively. Some developments on the construction of bivariate distributions with fixed marginals are discussed in Lin et al. (2014). Sarhan et al. (2011) introduced the bivariate generalized linear failure rate distribution. Kundu and Gupta (2014) introduced a five parameter bivariate Weibull-geometric distribution. Muhammed (2016) introduced bivariate inverse Weibull distribution with marginals as inverse Weibull distribution.

The objective of this Chapter is to propose a new bivariate distribution with modified Weibull distribution as marginals. Since this distribution has a singular component, it is suitable for modelling the situation where ties present in the data set. This model can be applied in shock modelling situation. Consider three independent shock sources, say, S_1, S_2 and S_3. These sources are affecting a system with two components, say, C_1 and C_2. Here assume that, if the shock from S_1 hits the system, it destroys C_1 and if the shock is from S_2 it destroys C_2, while the shock is from S_3 it destroys both the components suddenly. Let U_i denote the inter-interval times between the shocks $S_i, i = 1, 2, 3$. Assume that U_1 and U_2 follow Weibull distribution and U_3 follows exponential distribution. Define the random variables X_1 and X_2 as

$$X_i = \min(U_i, U_3),\ \ i = 1, 2. \tag{4.1.1}$$

Here the random variables X_1 and X_2 are dependent because of the common (latent) random variable U_3 and the distribution of (X_1, X_2) follows the bivariate modified Weibull distribution. Similar applications may occur in modelling competing risk, stress of the components of a system and the maintenance time of the component of a system, see Muhammed (2016).

In Section 2, we introduce a new bivariate distribution with modified Weibull distribution as marginals. We develop the marginal and conditional distributions of the new bivariate distribution in Section 3. The expressions for its mathematical expectation are derives in Section 4 and the bivariate reliability function, hrf, mean waiting time and reverse hazard rate function are discuss in Section 5. Maximum likelihood estimation of the parameters of the new bivariate distribution and a simulation study are discuss in Section 6. An application of data modelling with the new bivariate distribution is illustrates in Section 7. Bivariate copula function for the new bivariate distribution is propose in Section 8.

4.2 The New Bivariate Distribution

Here we introduce a new bivariate modified Weibull (NBMW) distribution. We consider the situation where the overall lifetime of the system follows exponential distribution and the components follow Weibull distribution. Such cases may occur in reliability and survival analysis. An example of such situation is the modelling of lifetime distribution of human beings or any other species with pairs of organs like kidneys. The cdf of modified Weibull distribution of Sarhan and Zaindin (2009) represents the lifetime of a series system consists of two independent units, where the lifetime of one unit follows exponential distribution and that of the other follows Weibull distribution. This distribution has several desirable properties and the distributions such as, exponential, Rayleigh, linear failure rate and Weibull distributions are special cases of this distribution.

We consider three mutually independent random variables U_1, U_2 and U_3 with the following distributions: $U_i \sim W(\alpha_i, \beta_i), i = 1, 2$ and $U_3 \sim E(\beta_0)$, where $W(\alpha_i, \beta_i)$ denotes the two parameter Weibull distribution with parameters (α_i, β_i), and $E(\beta_0)$ denotes the exponential distribution with parameter β_0. That is, the random variable $U_i (i = 1, 2)$ has Weibull distribution with distribution function $G_i(t) = 1 - e^{-\alpha_i t^{\beta_i}}, t > 0, \alpha_i > 0, \beta_i > 0 (i = 1, 2)$, and the variable U_3 has exponential distribution with a constant failure rate $\beta_0 > 0$ with distribution function $G_3(t) = 1 - e^{-\beta_0 t}, t > 0, \beta_0 > 0$. The survival functions of $U_i (i = 1, 2, 3)$, are given by

$$\bar{G}_i(t) = e^{-\alpha_i t^{\beta_i}}, t > 0, \alpha_i > 0, \beta_i > 0 \ (i = 1, 2), \tag{4.2.1}$$

and

$$\bar{G}_3(t) = e^{-\beta_0 t}, t > 0, \beta_0 > 0. \tag{4.2.2}$$

Next we study the joint distribution of the random variables X_1 and X_2. The following lemma gives the joint survival function of X_1 and X_2 , which is the survival function of the NBMW distribution.

Lemma 4.2.1. *The joint survival function of the dependent random variables X_1 and X_2 is*

$$\bar{F}_{X_1,X_2}(x_1, x_2) = e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 z)}, \tag{4.2.3}$$

where $z = \max(x_1, x_2)$.

Proof. We have

$$\begin{aligned}
\bar{F}_{X_1,X_2}(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2) \\
&= P\left(\min(U_1, U_3) > x_1, \min(U_2, U_3) > x_2\right) \\
&= P\left(U_1 > x_1, U_2 > x_2, U_3 > \max(x_1, x_2)\right).
\end{aligned}$$

Since $U_i (i = 1, 2, 3)$ are mutually independent, we have

$$\begin{aligned}
\bar{F}_{X_1,X_2}(x_1, x_2) &= P(U_1 > x_1)P(U_2 > x_2)P(U_3 > \max(x_1, x_2)) \\
&= \bar{G}_1(x_1)\bar{G}_2(x_2)\bar{G}_3(z) \\
&= e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 z)}.
\end{aligned}$$

where $z = max(x_1, x_2)$. □

Now we have the following theorem which gives the joint pdf of the NBMW distribution.

Theorem 4.2.1. *If the joint survival function of* (X_1, X_2) *is*

$$\bar{F}_{X_1,X_2}(x_1, x_2) = e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 z)},$$

where $z = \max(x_1, x_2)$, *then the joint probability density function of* (X_1, X_2) *is given by*

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & if \quad x_1 > x_2 > 0, \\ f_2(x_1, x_2) & if \quad x_2 > x_1 > 0, \\ f_0(x_1, x_1) & if \quad x_1 = x_2 > 0, \end{cases} \tag{4.2.4}$$

where

$$f_1(x_1, x_2) = \alpha_2 \beta_2 x_2^{\beta_2 - 1} (\alpha_1 \beta_1 x_1^{\beta_1 - 1} + \beta_0) e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 x_1)},$$
$$f_2(x_1, x_2) = \alpha_1 \beta_1 x_1^{\beta_1 - 1} (\alpha_2 \beta_2 x_2^{\beta_2 - 1} + \beta_0) e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 x_2)},$$

and

$$f_0(x_1, x_1) = \beta_0 e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)}.$$

Proof. First we assume that $x_1 > x_2$. Then from Eqn.(4.2.3) we have

$$\bar{F}_{X_1,X_2}(x_1, x_2) = e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 x_1)}.$$

Then

$$f_1(x_1, x_2) = \frac{\partial^2 \bar{F}_{X_1,X_2}(x_1, x_2)}{\partial x_1 \partial x_2} = \alpha_2 \beta_2 x_2^{\beta_2 - 1} (\alpha_1 \beta_1 x_1^{\beta_1 - 1} + \beta_0) e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 x_1)}$$

Similarly for $x_2 > x_1$,

$$f_2(x_1, x_2) = \alpha_1 \beta_1 x_1^{\beta_1 - 1} (\alpha_2 \beta_2 x_2^{\beta_2 - 1} + \beta_0) e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 x_2)}.$$

But, $f_0(x_1, x_1)$ cannot be obtained in a similar way. For this reason, we use the identity

$$\int_0^\infty f_0(x_1, x_1) dx_1 + \int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 + \int_0^\infty \int_0^{x_2} f_2(x_1, x_2) dx_1 dx_2 = 1. \tag{4.2.5}$$

where,

$$\int_0^\infty \int_0^{x_1} f_1(x_1,x_2)dx_2dx_1 = 1 - \int_0^\infty (\alpha_1\beta_1 x_1^{\beta_1-1} + \beta_0)e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)}dx_1,$$

and

$$\int_0^\infty \int_0^{x_2} f_2(x_1,x_2)dx_1dx_2 = 1 - \int_0^\infty (\alpha_2\beta_2 x_2^{\beta_2-1} + \beta_0)e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)}dx_1.$$

Then from Eqn.(4.2.5)

$$\begin{aligned}
\int_0^\infty f_0(x_1,x_1)dx_1 &= \int_0^\infty (\alpha_1\beta_1 x_1^{\beta_1-1} + \alpha_2\beta_2 x_1^{\beta_2-1} + \beta_0)e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)}dx_1 \\
&+ \int_0^\infty \beta_0 e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)}dx_1 - 1 \\
&= \int_0^\infty \beta_0 e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)}dx_1
\end{aligned}$$

This implies that

$$f_0(x_1,x_1) = \beta_0 e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)}.$$

This completes the proof of the theorem. □

Now setting $\beta_1 = \beta_2 = 1$, we get the joint probability density function of the bivariate exponential distribution as

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \alpha_2(\alpha_1+\beta_0)e^{-(\alpha_1 x_1 + \alpha_2 x_2 + \beta_0 x_1)} & if \quad x_1 > x_2 > 0, \\ \alpha_1(\alpha_2+\beta_0)e^{-(\alpha_1 x_1 + \alpha_2 x_2 + \beta_0 x_2)} & if \quad x_2 > x_1 > 0, \\ \beta_0 e^{-(\alpha_1+\alpha_2+\beta_0)x_1} & if \quad x_1 = x_2 > 0, \end{cases} \tag{4.2.6}$$

The shape of the joint probability density function of the NBMW distribution for various choices of parameters are shown in Figure 4.1.

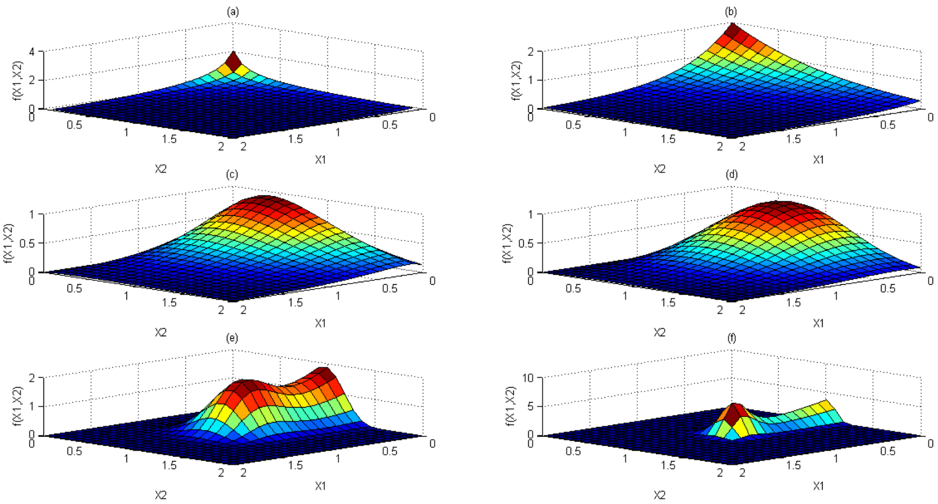

Figure 4.1: Scatter plots of the absolute continuous part of the joint pdf of the NBMW distribution for different parameter values of $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_0)$: (a). (1,1,0.5,0.5,1); (b). (1,1,1,1,1); (c). (1,1,1.5,1.5,1); (d). (1,1,2,2,1); (e). (1,1,5,5,1); (f). (1,1,10,10,1).

4.3 Marginal and Conditional Probability Density Functions

Now we derive the marginal density function of X_i and the conditional density functions of $X_i/X_j, i \neq j = 1, 2$.

Theorem 4.3.1. *The marginal probability density function of $X_i (i = 1, 2)$ is given by*

$$f_{X_i}(x_i) = (\alpha_i \beta_i x_i^{\beta_i - 1} + \beta_0) e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)}, \quad x_i > 0, \ i = 1, 2. \tag{4.3.1}$$

Proof. First we shall derive $f_{X_1}(x_1)$. We have

$$f_{X_1}(x_1) = \int_0^\infty f_{X_1, X_2}(x_1, x_2) dx_2.$$

We can express

$$f_{X_1}(x_1) = \lambda(x_1) + \mu(x_1) + f_0(x_1, x_1), \tag{4.3.2}$$

where

$$\lambda(x_1) = \int_0^{x_1} f_1(x_1, x_2) dx_2$$

and

$$\mu(x_1) = \int_{x_1}^{\infty} f_2(x_1, x_2) dx_2. \tag{4.3.3}$$

Then using the Eqn.(4.2.4) we have

$$\begin{aligned} \lambda(x_1) &= \int_0^{x_1} \alpha_2 \beta_2 x_2^{\beta 2-1} (\alpha_1 \beta_1 x_1^{\beta_1 - 1} + \beta_0) e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 x_1)} dx_2 \\ &= (\alpha_1 \beta_1 x_1^{\beta_1 - 1} + \beta_0)(1 - e^{-\alpha_2 x_1^{\beta_2}}) e^{-(\alpha_1 x_1^{\beta_1} + \beta_0 x_1)} \end{aligned} \tag{4.3.4}$$

and

$$\begin{aligned} \mu(x_1) &= \int_{x_1}^{\infty} \alpha_1 \beta_1 x_1^{\beta 1-1} (\alpha_2 \beta_2 x_2^{\beta_2 - 1} + \beta_0) e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \beta_0 x_2)} dx_2 \\ &= \alpha_1 \beta_1 x_1^{\beta_1 - 1} e^{-(\alpha_1 x_1^{\beta_1} + \alpha_2 x_1^{\beta_2} + \beta_0 x_1)} \end{aligned} \tag{4.3.5}$$

Substituting the Eqn.(4.3.4) and the Eqn.(4.3.5) in the Eqn.(4.3.2) and using the expression of $f_0(x_1, x_1)$ from Theorem 4.2.1, we obtain

$$f_{X_1}(x_1) = (\alpha_1 \beta_1 x_1^{\beta_1 - 1} + \beta_0) e^{-(\alpha_1 x_1^{\beta_1} + \beta_0 x_1)}.$$

Proceeding similarly we can derive $f_{X_2}(x_2)$ as

$$f_{X_2}(x_2) = (\alpha_2 \beta_2 x_2^{\beta_2 - 1} + \beta_0) e^{-(\alpha_2 x_2^{\beta_2} + \beta_0 x_2)}.$$

This completes the proof of the theorem. □

Here note that the marginal distribution function of the NBMW distribution follows the modified Weibull distribution, see Sarhan and Zaindin (2009). The cdf is given by

$$F_{X_i}(x_i) = 1 - e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)}, \quad i = 1, 2. \tag{4.3.6}$$

The marginal pdf of X_i in the case of the bivariate exponential distribution with joint pdf as given in the Eqn.(4.2.6), is

$$f_{X_i}(x_i) = (\alpha_i + \beta_0) e^{-(\alpha_i + \beta_0) x_i}; x_i > 0, i = 1, 2. \tag{4.3.7}$$

Now we derive the conditional probability density function as presented in the following theorem.

Theorem 4.3.2. *The conditional pdf of X_i, given $X_j = x_j$, denoted by $f_{i/j}(x_i/x_j)(i \neq j = 1, 2)$, is given by*

$$f_{i/j}(x_i/x_j) = \begin{cases} f_{i/j}^{(1)}(x_i/x_j) & if \quad x_i > x_j > 0, \\ f_{i/j}^{(2)}(x_i/x_j) & if \quad x_i < x_j > 0, \\ f_{i/j}^{(0)}(x_i/x_j) & if \quad x_i = x_j > 0, \end{cases} \tag{4.3.8}$$

where

$$\begin{aligned} f_{i/j}^{(1)}(x_i/x_j) &= \frac{\alpha_j \beta_j x_j^{\beta_j - 1} (\alpha_i \beta_i x_i^{\beta_i - 1} + \beta_0) e^{-(\alpha_i x_i^{\beta_i} + \beta_0 (x_i - x_j))}}{\alpha_j \beta_j x_j^{\beta_j - 1} + \beta_0}, \\ f_{i/j}^{(2)}(x_i/x_j) &= \alpha_i \beta_i x_i^{\beta_i - 1} e^{-\alpha_i x_i^{\beta_i}}, \end{aligned}$$

and

$$f_{i/j}^{(0)}(x_i/x_j) = \frac{\beta_0 e^{-\alpha_j x_i^{\beta_j}}}{\alpha_i \beta_i x_i^{\beta_i - 1} + \beta_0}.$$

Proof. The theorem follows readily upon substituting for the joint pdf of (X_1, X_2) given in the Eqn.(4.2.4) and the marginal pdf of $X_i (i = 1, 2)$ given in the Eqn.(4.3.1), which completes the proof of the theorem. □

For the case of the bivariate exponential distribution, we obtain upon setting $\beta_1 = \beta_2 = 1$ given in Eqn.(4.3.8)

$$f_{i/j}(x_i/x_j) = \begin{cases} \frac{\alpha_j(\alpha_i+\beta_0)e^{-(\alpha_i+\beta_0)x_i+\beta_0 x_j}}{\alpha_j+\beta_0} & if \quad x_i > x_j > 0, \\ \alpha_i e^{-\alpha_i x_i} & if \quad x_i < x_j > 0, \\ \frac{\beta_0 e^{-\alpha_j x_i}}{\alpha_i+\beta_0} & if \quad x_i = x_j > 0. \end{cases} \qquad (4.3.9)$$

4.4 Mathematical Expectations

In this section, we derive the mathematical expectation, second order moments and the marginal moment generating function of $X_i (i = 1, 2)$.

Theorem 4.4.1. *The mathematical expectation of* $X_i (i = 1, 2)$ *is given by*

$$E(X_i) = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k \beta_i}{k!} \left[\frac{\alpha_i \beta_i (k+1) \Gamma(\beta_i(k+1))}{\beta_0^{\beta_i(k+1)+1}} + \frac{k(k\beta_i+1)\Gamma(k\beta_i)}{\beta_0^{k\beta_i+1}} \right] \qquad (4.4.1)$$

Proof. We have $E(X_i) = \int_0^\infty x_i f_{X_i}(x_i) dx_i$ and substituting for $f_{X_i}(x_i)$ from the Eqn.(4.3.1), we get

$$\begin{aligned} E(X_i) &= \int_0^\infty x_i(\alpha_i \beta_i x_i^{\beta_i - 1} + \beta_0) e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)} dx_i \\ &= \alpha_i \beta_i \int_0^\infty x_i^{\beta_i} e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)} dx_i + \beta_0 \int_0^\infty x_i e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)} dx_i \end{aligned}$$

Now, using

$$e^{-\alpha_i x_i^{\beta_i}} = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k x_i^{k\beta_i}}{k!}, \ i = 1, 2;$$

we have

$$\begin{aligned}
E(X_i) &= \int_0^\infty \alpha_i\beta_i \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k x_i^{(k+1)\beta_i}}{k!} e^{-\beta_0 x_i} dx_i \\
&\quad + \beta_0 \int_0^\infty \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k x_i^{k\beta_i+1}}{k!} e^{-\beta_0 x_i} dx_i \\
&= \alpha_i\beta_i \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k}{k!} \frac{\Gamma(\beta_i(k+1)+1)}{\beta_0^{\beta_i(k+1)+1}} + \beta_0 \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k}{k!} \frac{\Gamma(k\beta_i+2)}{\beta_0^{k\beta_i+2}} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k \beta_i}{k!} \left[\frac{\alpha_i\beta_i(k+1)\Gamma(\beta_i(k+1))}{\beta_0^{\beta_i(k+1)+1}} + \frac{k(k\beta_i+1)\Gamma(k\beta_i)}{\beta_0^{k\beta_i+1}} \right]
\end{aligned}$$

This completes the proof. □

Theorem 4.4.2. *The second order moment of $X_i (i = 1, 2)$ is given by*

$$\begin{aligned}
E(X_i^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k \beta_i}{k!} \left[\frac{\alpha_i(\beta_i(k+1)+1)(k+1)\Gamma(\beta_i(k+1))}{\beta_0^{\beta_i(k+1)+2}} \right. \\
&\quad \left. + \frac{k(k\beta_i+1)(k\beta_i+2)\Gamma(k\beta_i)}{\beta_0^{k\beta_i+2}} \right]
\end{aligned}$$

Proof. We have $E(X_i^2) = \int_0^\infty x_i^2 f_{X_i}(x_i) dx_i$ and substituting for $f_{X_i}(x_i)$ from the Eqn.(4.3.1), we get

$$\begin{aligned}
E(X_i^2) &= \int_0^\infty x_i^2 (\alpha_i\beta_i x_i^{\beta_i-1} + \beta_0) e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)} dx_i \\
&= \alpha_i\beta_i \int_0^\infty x_i^{\beta_i+1} e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)} dx_i + \beta_0 \int_0^\infty x_i^2 e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)} dx_i \\
&= \int_0^\infty \alpha_i\beta_i \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k x_i^{(k+1)\beta_i+1}}{k!} e^{-\beta_0 x_i} dx_i \\
&\quad + \beta_0 \int_0^\infty \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k x_i^{k\beta_i+2}}{k!} e^{-\beta_0 x_i} dx_i
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k \beta_i}{k!} \left[\frac{\alpha_i(\beta_i(k+1)+1)(k+1)\Gamma(\beta_i(k+1))}{\beta_0^{\beta_i(k+1)+2}} + \frac{k(k\beta_i+1)(k\beta_i+2)\Gamma(k\beta_i)}{\beta_0^{k\beta_i+2}} \right]$$

This completes the proof. □

Theorem 4.4.3. *The moment generating function of $X_i (i = 1, 2)$ is given by*

$$M_{X_i}(t_i) = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k}{k!} \left[\frac{\alpha_i \beta_i \Gamma(\beta_i(k+1))}{(\beta_0 + t_i)^{\beta_i(k+1)}} + \frac{\beta_0 \Gamma(k\beta_i + 1)}{(\beta_0 + t_i)^{k\beta_i + 1}} \right] \tag{4.4.2}$$

Proof. We have $M_{X_i}(t_i) = E\left[e^{-t_i X_i}\right] = \int_0^\infty e^{-t_i x_i} f_{X_i}(x_i) dx_i$ and substituting for $f_{X_i}(x_i)$ from the Eqn.(4.3.1), we get

$$\begin{aligned} M_{X_i}(t_i) &= \int_0^\infty e^{-t_i X_i} (\alpha_i \beta_i x_i^{\beta_i - 1} + \beta_0) e^{-(\alpha_i x_i^{\beta_i} + \beta_0 x_i)} dx_i \\ &= \alpha_i \beta_i \int_0^\infty x_i^{\beta_i - 1} e^{-\alpha_i x_i^{\beta_i}} e^{-(\beta_0 + t_i) x_i} dx_i + \beta_0 \int_0^\infty e^{-\alpha_i x_i^{\beta_i}} e^{-(\beta_0 + t_i) x_i} dx_i \\ &= \alpha_i \beta_i \int_0^\infty \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k x_i^{(k+1)\beta_i - 1}}{k!} e^{-(\beta_0 + t_i) x_i} dx_i \\ &\quad + \beta_0 \int_0^\infty \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k x_i^{k\beta_i}}{k!} e^{-(\beta_0 + t_i) x_i} dx_i \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_i^k}{k!} \left[\frac{\alpha_i \beta_i \Gamma(\beta_i(k+1))}{(\beta_0 + t_i)^{\beta_i(k+1)}} + \frac{\beta_0 \Gamma(k\beta_i + 1)}{(\beta_0 + t_i)^{k\beta_i + 1}} \right] \end{aligned}$$

This completes the proof. □

Theorem 4.4.4. *The joint moment generating function of* (X_1, X_2) *is given by*

$$\begin{aligned} M(t_1,t_2) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k\alpha_2^{k+1}t_2^l\alpha_1^m}{k!l!m!}\frac{\Gamma[l+\beta_2(k+1)+\beta_1(m+1)]}{(\beta_0+t_1)^{l+\beta_2(k+1)+\beta_1(m+1)}} \\ &+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+l+m}\alpha_1^{k+1}t_1^l\alpha_2^m}{k!l!m!}\frac{\Gamma[l+\beta_1(k+1)+\beta_2(m+1)]}{(\beta_0+t_2)^{l+\beta_1(k+1)+\beta_2(m+1)}} \\ &+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+j}\alpha_1^k\alpha_2^l}{k!l!}\frac{\Gamma[k\beta_1+l\beta_2-1]}{(\beta_0+t_1+t_2)^{k\beta_1+l\beta_2-1}} \end{aligned} \tag{4.4.3}$$

Proof. The joint moment generating function of (X_1, X_2) is given by

$$\begin{aligned} M(t_1,t_2)=E[e^{-t_1X_1-t_2X_2}] &= \int_0^{\infty}\int_0^{\infty}e^{-t_1x_1-t_2x_2}f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\ &= I_1+I_2+I_3 \end{aligned} \tag{4.4.4}$$

where

$$\begin{aligned} I_1 &= \int_0^{\infty}\int_0^{x_1}e^{-t_1x_1-t_2x_2}f_1(x_1,x_2)dx_2dx_1 \\ &= \int_0^{\infty}\int_0^{x_1}e^{-t_1x_1-t_2x_2}\alpha_2\beta_2x_2^{\beta_2-1}(\alpha_1\beta_1x_1^{\beta_1-1}+\beta_0)e^{-(\alpha_1x_1^{\beta_1}+\alpha_2x_2^{\beta_2}+\beta_0x_1)}dx_2dx_1 \\ &= \int_0^{\infty}e^{-t_1x_1}(\alpha_1\beta_1x_1^{\beta_1-1}+\beta_0)e^{-(\alpha_1x_1^{\beta_1}+\beta_0x_1)}\int_0^{x_1}\alpha_2\beta_2x_2^{\beta_2-1}e^{-t_2x_2}e^{-\alpha_2x_2^{\beta_2}}dx_2dx_1 \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k\alpha_2^{k+1}t_2^l\alpha_1^m}{k!l!m!}\frac{\Gamma[l+\beta_2(k+1)+\beta_1(m+1)]}{(\beta_0+t_1)^{l+\beta_2(k+1)+\beta_1(m+1)}} \end{aligned} \tag{4.4.5}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^\infty \int_0^{x_2} e^{-t_1x_1-t_2x_2} f_2(x_1,x_2)dx_1dx_2 \\ &= \int_0^\infty \int_0^{x_2} e^{-t_1x_1-t_2x_2}\alpha_1\beta_1 x_1^{\beta_1-1}(\alpha_2\beta_2 x_2^{\beta_2-1}+\beta_0)e^{-(\alpha_1x_1^{\beta_1}+\alpha_2x_2^{\beta_2}+\beta_0x_2)}dx_1dx_2 \\ &= \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty \frac{(-1)^{k+l+m}\alpha_1^{k+1}t_1^l\alpha_2^m}{k!l!m!}\frac{\Gamma[l+\beta_1(k+1)+\beta_2(m+1)]}{(\beta_0+t_2)^{l+\beta_1(k+1)+\beta_2(m+1)}} \qquad (4.4.6) \end{aligned}$$

Also

$$\begin{aligned} I_3 &= \int_0^\infty e^{-(t_1+t_2)x_1} f_0(x_1,x_1)dx_1 \\ &= \int_0^\infty e^{-(t_1+t_2)x_1}\beta_0 e^{-(\alpha_1x_1^{\beta_1}+\alpha_2x_1^{\beta_2}+\beta_0x_1)}dx_1 \\ &= \sum_{k=0}^\infty\sum_{l=0}^\infty \frac{(-1)^{k+j}\alpha_1^k\alpha_2^l}{k!l!}\frac{\Gamma[k\beta_1+l\beta_2-1]}{(\beta_0+t_1+t_2)^{k\beta_1+l\beta_2-1}} \qquad (4.4.7) \end{aligned}$$

Using I_1, I_2 and I_3 in the Eqn.(4.4.4) we get

$$\begin{aligned} M(t_1,t_2) &= \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty \frac{(-1)^{k}\alpha_2^{k+1}t_2^l\alpha_1^m}{k!l!m!}\frac{\Gamma[l+\beta_2(k+1)+\beta_1(m+1)]}{(\beta_0+t_1)^{l+\beta_2(k+1)+\beta_1(m+1)}} \\ &\quad+\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{m=0}^\infty \frac{(-1)^{k+l+m}\alpha_1^{k+1}t_1^l\alpha_2^m}{k!l!m!}\frac{\Gamma[l+\beta_1(k+1)+\beta_2(m+1)]}{(\beta_0+t_2)^{l+\beta_1(k+1)+\beta_2(m+1)}} \\ &\quad+\sum_{k=0}^\infty\sum_{l=0}^\infty \frac{(-1)^{k+j}\alpha_1^k\alpha_2^l}{k!l!}\frac{\Gamma[k\beta_1+l\beta_2-1]}{(\beta_0+t_1+t_2)^{k\beta_1+l\beta_2-1}} \end{aligned}$$

□

4.5 Bivariate Reliability Function

In this section we introduce the joint reliability function, joint hazard rate function, joint mean waiting time and the joint reverse hazard rate function of the NBMW distribution. The joint reliability function of the random variables X_1 and X_2 is defined by

$$\begin{aligned} R_{X_1,X_2}(x_1,x_2) &= 1-\Big[F_{X_1}(x_1)+F_{X_2}(x_2)-F_{X_1,X_2}(x_1,x_2)\Big] \\ &= \bar{F}_{X_1}(x_1)+\bar{F}_{X_2}(x_2)-\bar{F}_{X_1,X_2}(x_1,x_2). \end{aligned} \tag{4.5.1}$$

Theorem 4.5.1. *The joint reliability function of the random variables X_1 and X_2 from the NBMW distribution is given by*

$$R_{X_1,X_2}(x_1,x_2)=\begin{cases} R_1(x_1,x_2) & if \quad x_1>x_2, \\ R_2(x_1,x_2) & if \quad x_1<x_2, \\ R_3(x_1,x_1) & if \quad x_1=x_2, \end{cases} \tag{4.5.2}$$

where

$$\begin{aligned} R_1(x_1,x_2) &= e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_2^{\beta_2}+\beta_0 x_1)}\Big[e^{\alpha_2 x_2^{\beta_2}}+e^{\alpha_1 x_1^{\beta_1}+\beta_0(x_1-x_2)}-1\Big], \\ R_2(x_1,x_2) &= e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_2^{\beta_2}+\beta_0 x_2)}\Big[e^{\alpha_1 x_1^{\beta_1}}+e^{\alpha_2 x_2^{\beta_2}+\beta_0(x_2-x_1)}-1\Big], \end{aligned}$$

and

$$R_3(x_1,x_1)=e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_1^{\beta_2}+\beta_0 x_1)}\Big[e^{\alpha_2 x_1^{\beta_2}}+e^{\alpha_1 x_1^{\beta_1}}-1\Big].$$

Proof. Applying the Eqn.(4.2.3) and the Eqn.(4.3.6) in the Eqn.(4.5.1), we get the joint reliability function of NBMW distribution as

$$R_{X_1,X_2}(x_1,x_2)=e^{-(\alpha_1 x_1^{\beta_1}+\beta_0 x_1)}+e^{-(\alpha_2 x_2^{\beta_2}+\beta_0 x_2)}-e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_2^{\beta_2}\beta_0 z)},$$

where $z = \max(x_1, x_2), \alpha_i > 0, i = 1, 2$ and $\beta_0 > 0$. This completes the proof. □

4.5.1 Hazard rate function

The bivariate hazard rate function of NBMW distribution, $h(x_1, x_2)$ (see Basu, 1971) is defined as

$$h(x_1, x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{\bar{F}_{X_1,X_2}(x_1, x_2)} = \begin{cases} h_1(x_1, x_2) & if \quad x_1 > x_2, \\ h_2(x_1, x_2) & if \quad x_1 < x_2, \\ h_0(x_1, x_1) & if \quad x_1 = x_2, \end{cases} \tag{4.5.3}$$

where

$$\begin{aligned} h_1(x_1, x_2) &= \frac{f_1(x_1, x_2)}{\bar{F}_1(x_1, x_2)} = \frac{\alpha_2\beta_2 x_2^{\beta_2 - 1}\left(\alpha_1\beta_1 x_1^{\beta_1 - 1} + \beta_0\right)}{\left[e^{\alpha_2 x_2^{\beta_2}} + e^{\alpha_1 x_1^{\beta_1} + \beta_0(x_1 - x_2)} - 1\right]}, \\ h_2(x_1, x_2) &= \frac{f_2(x_1, x_2)}{\bar{F}_2(x_1, x_2)} = \frac{\alpha_1\beta_1 x_1^{\beta_1 - 1}\left(\alpha_2\beta_2 x_2^{\beta_2 - 1} + \beta_0\right)}{\left[e^{\alpha_1 x_1^{\beta_1}} + e^{\alpha_2 x_2^{\beta_2} + \beta_0(x_2 - x_1)} - 1\right]}, \end{aligned}$$

and

$$h_0(x_1, x_1) = \frac{f_3(x_1, x_2)}{\bar{F}_3(x_1, x_2)} = \frac{\beta_0}{\left[e^{\alpha_2 x_1^{\beta_2}} + e^{\alpha_1 x_1^{\beta_1}} - 1\right]}.$$

The shape of the bivariate hazard rate function of the NBMW distribution for various choices of parameters are shown in Figure 4.2.

4.5.2 Mean waiting time

The mean waiting time function $\mu_w(t)$ for the marginal distributions of X_i's, $i = 1, 2$ can be defined by

$$\mu_{w_i}(t) = \frac{1}{F_{X_i}(t)} \int_0^t F_{X_i}(x_i) dx_i. \tag{4.5.4}$$

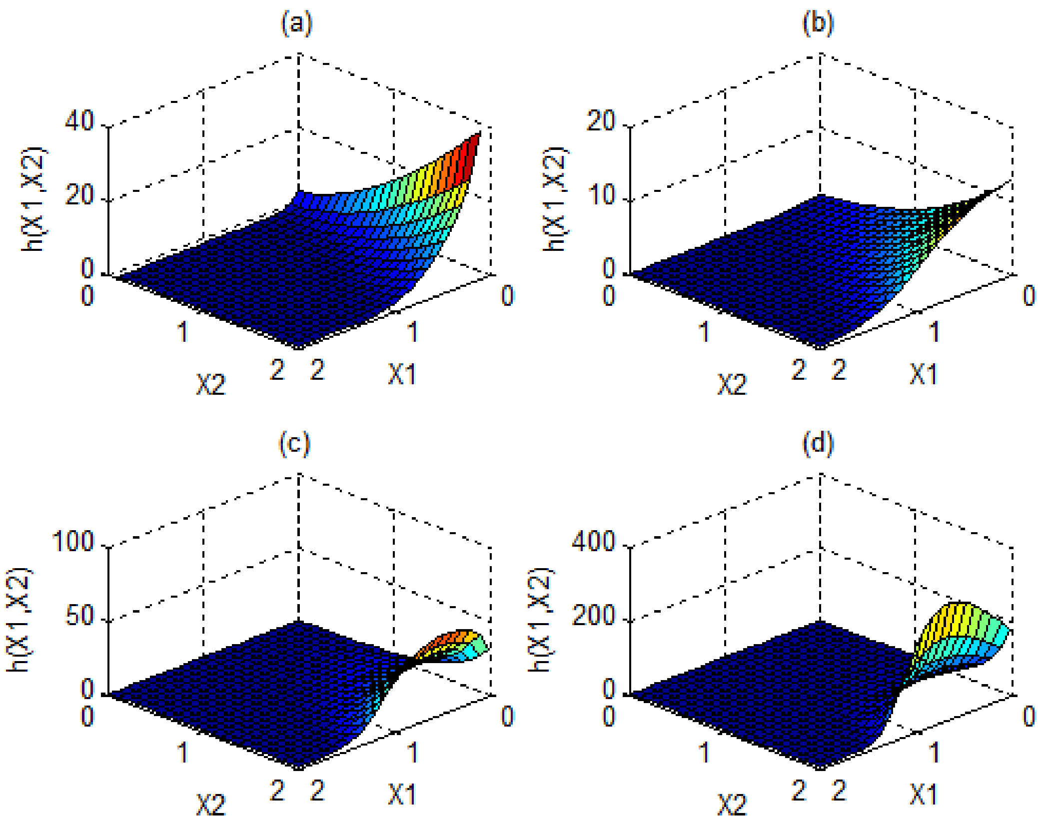

Figure 4.2: Scatter plots of the absolute continuous part of the joint hazard rate function of the NBMW distribution for different parameter values of $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_0)$: (a). (2,2,0.5,0.5,1); (b). (1,1,1,1,1); (c). (1,1,1.5,1.5,1); (d). (1,1,2,2,1).

Then the mean waiting time of (X_1, X_2) is defined as

$$\mu_w(t_1, t_2) = \frac{1}{F(t_1, t_2)} \int_0^{t_1} \int_0^{t_2} F_{X_1,X_2}(x_1, x_2) dx_1 dx_2. \tag{4.5.5}$$

The following Lemma gives the mean waiting time of (X_1, X_2).

Lemma 4.5.1. *The joint mean waiting time $\mu_w(t_1, t_2)$ to the random variables X_1 and X_2 is*

$$\mu_w(t_1, t_2) = \begin{cases} \mu_{w_1}(t_1, t_2) & if \quad t_1 > t_2, \\ \mu_{w_2}(t_1, t_2) & if \quad t_1 < t_2, \\ \mu_{w_0}(t_1, t_1) & if \quad t_1 = t_2, \end{cases} \tag{4.5.6}$$

where

$$\mu_{w_1}(t_1,t_2) = \frac{1}{F(t_1,t_2)}\Bigg[t_2t_1 - \sum_{j=0}^{\infty}\frac{(-1)^j\alpha_1^j t_2\gamma(j\beta_1+1,\beta_0t_1)}{j!\beta_0^{j\beta_1+1}} - \sum_{j=0}^{\infty}\frac{(-1)^k\alpha_2^k t_1\gamma(k\beta_2+1,\beta_0t_2)}{k!\beta_0^{k\beta_2+1}} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\alpha_1^j\alpha_2^k t_2^{k\beta_2+1}\gamma(j\beta_1+1,\beta_0t_1)}{j!k!(k\beta_2+1)\beta_0^{j\beta_1+1}}\Bigg],$$

$$\mu_{w_2}(t_1,t_2) = \frac{1}{F(t_1,t_2)}\Bigg[t_1t_2 - \sum_{j=0}^{\infty}\frac{(-1)^j\alpha_1^j t_2\gamma(j\beta_1+1,\beta_0t_1)}{j!\beta_0^{j\beta_1+1}} - \sum_{j=0}^{\infty}\frac{(-1)^k\alpha_2^k t_1\gamma(k\beta_2+1,\beta_0t_2)}{k!\beta_0^{k\beta_2+1}} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\alpha_1^j\alpha_2^k t_1^{k\beta_1+1}\gamma(j\beta_2+1,\beta_0t_2)}{j!k!(k\beta_1+1)\beta_0^{j\beta_2+1}}\Bigg],$$

and

$$\mu_{w_0}(t_1,t_1) = \frac{1}{F(t_1,t_1)}\Bigg[t_1^2 - \sum_{j=0}^{\infty}\frac{(-1)^j\alpha_1^j t_1\gamma(j\beta_1+1,\beta_0t_1)}{j!\beta_0^{j\beta_1+1}} - \sum_{j=0}^{\infty}\frac{(-1)^k\alpha_2^k t_1\gamma(k\beta_2+1,\beta_0t_1)}{k!\beta_0^{k\beta_2+1}} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\alpha_1^j\alpha_2^k t_1\gamma(j\beta_1+k\beta_2+1,\beta_0t_1)}{j!k!\beta_0^{j\beta_1+k\beta_2+1}}\Bigg].$$

Here $\gamma(.)$ is the incomplete gamma function.

Proof. From the Eqn.(4.5.1) we have

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) &= 1-\bar{F}_{X_1}(x_1)-\bar{F}_{X_2}(x_2)+\bar{F}_{X_1,X_2}(x_1,x_2) \\ &= 1-e^{-(\alpha_1 x_1^{\beta_1}+\beta_0 x_1)}-e^{-(\alpha_2 x_2^{\beta_2}+\beta_0 x_2)}+e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_2^{\beta_2}+\beta_0 z)}, \end{aligned}$$

where $z = max(x_1, x_2)$.

First we consider the case when $x_1 > x_2$. From the Eqn.(4.2.4), Eqn.(4.3.6) and the Eqn.(4.5.5) we get

$$\begin{aligned} \mu_{w_1}(t_1,t_2) = \frac{1}{F(t_1,t_2)}\int_0^{t_1}\int_0^{t_2}\Big[1 &- e^{-(\alpha_1 x_1^{\beta_1}+\beta_0 x_1)}-e^{-(\alpha_2 x_2^{\beta_2}+\beta_0 x_2)} \\ &+ e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_2^{\beta_2}+\beta_0 x_1)}\Big]dx_2 dx_1. \end{aligned}$$

Now, using

$$e^{-\alpha_i x_i^{\beta_i}} = \sum_{k=0}^{\infty}\frac{(-1)^k\alpha_i^k x_i^{k\beta_i}}{k!},\ i=1,2,$$

we have

$$\begin{aligned} \mu_{w_1}(t_1,t_2) = \frac{1}{F(t_1,t_2)}\Bigg[t_2 t_1 &- \sum_{j=0}^{\infty}\frac{(-1)^j\alpha_1^j t_2\gamma(j\beta_1+1,\beta_0 t_1)}{j!\beta_0^{j\beta_1+1}} \\ &- \sum_{j=0}^{\infty}\frac{(-1)^k\alpha_2^k t_1\gamma(k\beta_2+1,\beta_0 t_2)}{k!\beta_0^{k\beta_2+1}} \\ &+ \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\alpha_1^j\alpha_2^k t_2^{k\beta_2+1}\gamma(j\beta_1+1,\beta_0 t_1)}{j!k!(k\beta_2+1)\beta_0^{j\beta_1+1}}\Bigg]. \end{aligned}$$

Proceeding in the similar way, for $x_1 < x_2$, we get

$$\mu_{w_2}(t_1,t_2) = \frac{1}{F(t_1,t_2)}\Bigg[t_1 t_2 - \sum_{j=0}^{\infty}\frac{(-1)^j\alpha_1^j t_2\gamma(j\beta_1+1,\beta_0 t_1)}{j!\beta_0^{j\beta_1+1}} - \sum_{j=0}^{\infty}\frac{(-1)^k\alpha_2^k t_1\gamma(k\beta_2+1,\beta_0 t_2)}{k!\beta_0^{k\beta_2+1}} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\alpha_1^j\alpha_2^k t_1^{k\beta_1+1}\gamma(j\beta_2+1,\beta_0 t_2)}{j!k!(k\beta_1+1)\beta_0^{j\beta_2+1}}\Bigg],$$

and for $x_1 = x_2$

$$\mu_{w_0}(t_1,t_1) = \frac{1}{F(t_1,t_1)}\Bigg[t_1^2 - \sum_{j=0}^{\infty}\frac{(-1)^j\alpha_1^j t_1\gamma(j\beta_1+1,\beta_0 t_1)}{j!\beta_0^{j\beta_1+1}} - \sum_{j=0}^{\infty}\frac{(-1)^k\alpha_2^k t_1\gamma(k\beta_2+1,\beta_0 t_1)}{k!\beta_0^{k\beta_2+1}} + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\alpha_1^j\alpha_2^k t_1\gamma(j\beta_1+k\beta_2+1,\beta_0 t_1)}{j!k!\beta_0^{j\beta_1+k\beta_2+1}}\Bigg].$$

This completes the proof. □

4.5.3 Reverse hazard rate function

Reversed hazard rates are important in the study of systems. Hazard rates have an affinity to series systems where as reversed hazard rates are more appropriate for studying parallel systems, see Block et al. (1998). The bivariate reverse hazard rate function is defined by

$$r_{X_1,X_2}(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{F_{X_1,X_2}(x_1,x_2)} = \begin{cases} r_1(x_1,x_2) & if \;\; x_1 > x_2, \\ r_2(x_1,x_2) & if \;\; x_1 < x_2, \\ r_0(x_1,x_1) & if \;\; x_1 = x_2, \end{cases} \tag{4.5.7}$$

where

$$r_1(x_1, x_2) = \frac{\alpha_2\beta_2 x_2^{\beta_2-1}(\alpha_1\beta_1 x_1^{\beta_1-1} + \beta_0)e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_2^{\beta_2}+\beta_0 x_1)}}{1 - e^{-(\alpha_2 x_2^{\beta_2}+\beta_0 x_2)} - e^{-(\alpha_1 x_1^{\beta_1}+\beta_0 x_1)}(1 - e^{-\alpha_2 x_2^{\beta_2}})}, \quad (4.5.8)$$

$$r_2(x_1, x_2) = \frac{\alpha_1\beta_1 x_1^{\beta_1-1}(\alpha_2\beta_2 x_2^{\beta_2-1} + \beta_0)e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_2^{\beta_2}+\beta_0 x_2)}}{1 - e^{-(\alpha_1 x_1^{\beta_1}+\beta_0 x_1)} - e^{-(\alpha_2 x_2^{\beta_2}+\beta_0 x_2)}(1 - e^{-\alpha_1 x_1^{\beta_1}})}, \quad (4.5.9)$$

$$r_0(x_1, x_1) = \frac{\beta_0 e^{-(\alpha_1 x_1^{\beta_1}+\alpha_2 x_1^{\beta_2}+\beta_0 x_1)}}{1 - e^{-(\alpha_2 x_1^{\beta_2}+\beta_0 x_1)} - e^{-(\alpha_1 x_1^{\beta_1}+\beta_0 x_1)}(1 - e^{-\alpha_2 x_1^{\beta_2}})}. \quad (4.5.10)$$

4.6 Maximum Likelihood Estimation of Parameters

Here we discuss the method of computing maximum likelihood estimates of the unknown parameters of the NBMW distribution. Suppose $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ be a random sample drawn from the NBMW$(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_0)$ distribution. We use the following notations:

$$I_1 = \{i;\ x_i < y_i\},\ I_2 = \{i;\ x_i > y_i\},\ I_3 = \{i;\ x_i = y_i\},\ I = I_1 \cup I_2 \cup I_3,$$

$$|I_1| = n_1, |I_2| = n_2, |I_3| = n_3,\ \text{and}\ n_1 + n_2 + n_3 = n.$$

The likelihood function is

$$L = \left(\prod_{i=1}^{n_1} f_1(x_i, y_i)\right)\left(\prod_{i=1}^{n_2} f_2(x_i, y_i)\right)\left(\prod_{i=1}^{n_3} f_3(x_i, x_i)\right), \quad (4.6.1)$$

where

$$\begin{aligned}
f_1(x_i, y_i) &= \alpha_2\beta_2 y_i^{\beta_2-1}\big(\alpha_1\beta_1 x_i^{\beta_1-1}+\beta_0\big)e^{-(\alpha_1 x_i^{\beta_1}+\alpha_2 y_i^{\beta_2}+\beta_0 x_i)}, \quad \text{for } 0<y_i<x_i,\\
f_2(x_i, y_i) &= \alpha_1\beta_1 x_i^{\beta_1-1}\big(\alpha_2\beta_2 y_i^{\beta_2-1}+\beta_0\big)e^{-(\alpha_1 x_i^{\beta_1}+\alpha_2 y_i^{\beta_2}+\beta_0 y_i)}, \quad \text{for } 0<x_i<y_i,\\
f_3(x_i, x_i) &= \beta_0 e^{-(\alpha_1 x_i^{\beta_1}+\alpha_2 x_i^{\beta_2}+\beta_0 x_i)}, \quad \text{for } 0<x_i=y_i.
\end{aligned}$$

The log-likelihood function is

$$\begin{aligned}
\ln(L) &= n_1\ln(\alpha_2)+n_2\ln(\alpha_1)+n_1\ln(\beta_2)+n_2\ln(\beta_1)+n_3\ln(\beta_0)\\
&+\sum_{i=1}^{n_1}\ln(\alpha_1\beta_1 x_i^{\beta_1-1}+\beta_0)+\sum_{i=1}^{n_2}\ln(\alpha_2\beta_2 y_i^{\beta_2-1}+\beta_0)\\
&-\sum_{i=1}^{n_1}(\alpha_1 x_i^{\beta_1}+\alpha_2 y_i^{\beta_2}+\beta_0 x_i)-\sum_{i=1}^{n_2}(\alpha_1 x_i^{\beta_1}+\alpha_2 y_i^{\beta_2}+\beta_0 y_i)\\
&-\sum_{i=1}^{n_3}(\alpha_1 x_i^{\beta_1}+\alpha_2 x_i^{\beta_2}+\beta_0 x_i)\\
&+(\beta_2-1)\sum_{i=1}^{n_1}\ln(y_i)+(\beta_1-1)\sum_{i=1}^{n_2}\ln(x_i). \qquad (4.6.2)
\end{aligned}$$

On differentiating the Eqn.(4.6.2) with respect to $\alpha_1, \alpha_2, \beta_1, \beta_2$, and β_0 and equating to zero, we get the following likelihood equations:

$$\begin{aligned}
\frac{\partial\ln(L)}{\partial\alpha_1}=0 \quad \Rightarrow \quad & \frac{n_2}{\alpha_1}+\sum_{i=1}^{n_1}\frac{\beta_1 x_i^{\beta_1-1}}{\alpha_1\beta_1 x_i^{\beta_1-1}+\beta_0}-\\
&\sum_{i=1}^{n_1}x_i^{\beta_1}-\sum_{i=1}^{n_2}x_i^{\beta_1}-\sum_{i=1}^{n_3}x_i^{\beta_1}=0. \qquad (4.6.3)
\end{aligned}$$

$$\frac{\partial \ln(L)}{\partial \alpha_2} = 0 \;\Rightarrow\; \frac{n_1}{\alpha_2} + \sum_{i=1}^{n_2} \frac{\beta_2 y_i^{\beta_2 - 1}}{\alpha_2 \beta_2 y_i^{\beta_2 - 1} + \beta_0} - \sum_{i=1}^{n_1} y_i^{\beta_2} - \sum_{i=1}^{n_2} y_i^{\beta_2} - \sum_{i=1}^{n_3} x_i^{\beta_2} = 0. \qquad (4.6.4)$$

$$\frac{\partial \ln(L)}{\partial \beta_1} = 0 \;\Rightarrow\; \frac{n_2}{\beta_1} + \sum_{i=1}^{n_1} \frac{\alpha_1 x_i^{\beta_1 - 1}(\beta_1 \ln(x_i) + 1)}{\alpha_1 \beta_1 x_i^{\beta_1 - 1} + \beta_0} - \sum_{i=1}^{n_1} \alpha_1 x_i^{\beta_1} \ln(x_i) - \sum_{i=1}^{n_2} \alpha_1 x_i^{\beta_1} \ln(x_i) - \sum_{i=1}^{n_3} \alpha_1 x_i^{\beta_1} \ln(x_i) + \sum_{i=1}^{n_2} \ln(x_i) = 0. \qquad (4.6.5)$$

$$\frac{\partial \ln(L)}{\partial \beta_2} = 0 \;\Rightarrow\; \frac{n_1}{\beta_2} + \sum_{i=1}^{n_2} \frac{\alpha_2 y_i^{\beta_2 - 1}(\beta_2 \ln(y_i) + 1)}{\alpha_2 \beta_2 y_i^{\beta_2 - 1} + \beta_0} - \sum_{i=1}^{n_1} \alpha_2 y_i^{\beta_2} \ln(y_i) - \sum_{i=1}^{n_2} \alpha_2 y_i^{\beta_2} \ln(y_i) - \sum_{i=1}^{n_3} \alpha_2 x_i^{\beta_2} \ln(y_i) + \sum_{i=1}^{n_1} \ln(y_i) = 0. \qquad (4.6.6)$$

$$\frac{\partial \ln(L)}{\partial \beta_0} = 0 \;\Rightarrow\; \frac{n_3}{\beta_0} + \sum_{i=1}^{n_1} \frac{1}{\alpha_1 \beta_1 x_i^{\beta_1 - 1} + \beta_0} + \sum_{i=1}^{n_2} \frac{1}{\alpha_2 \beta_2 y_i^{\beta_2 - 1} + \beta_0} - \sum_{i=1}^{n_1} x_i - \sum_{i=1}^{n_2} y_i - \sum_{i=1}^{n_3} x_i = 0. \qquad (4.6.7)$$

The Eqns.(4.6.3) to (4.6.7) have no explicit form and their solutions $(\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_0)$ are numerically obtained using the Newton-Raphson method. The asymptotic variance covariance matrix of $(\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_0)$ is obtained by inverting the following

Fisher information matrix

$$I=-\begin{bmatrix}\frac{\partial^2\ln(L)}{\partial^2\alpha_1} & \frac{\partial^2\ln(L)}{\partial\alpha_1\partial\alpha_2} & \frac{\partial^2\ln(L)}{\partial\alpha_1\partial\beta_1} & \frac{\partial^2\ln(L)}{\partial\alpha_1\partial\beta_2} & \frac{\partial^2\ln(L)}{\partial\alpha_1\partial\beta_0}\\ \frac{\partial^2\ln(L)}{\partial\alpha_2\partial\alpha_1} & \frac{\partial^2\ln(L)}{\partial^2\alpha_2} & \frac{\partial^2\ln(L)}{\partial\alpha_2\partial\beta_1} & \frac{\partial^2\ln(L)}{\partial\alpha_2\partial\beta_2} & \frac{\partial^2\ln(L)}{\partial\alpha_2\partial\beta_0}\\ \frac{\partial^2\ln(L)}{\partial\beta_1\partial\alpha_1} & \frac{\partial^2\ln(L)}{\partial\beta_1\partial\alpha_2} & \frac{\partial^2\ln(L)}{\partial^2\beta_1} & \frac{\partial^2\ln(L)}{\partial\beta_1\partial\beta_2} & \frac{\partial^2\ln(L)}{\partial\beta_1\partial\beta_0}\\ \frac{\partial^2\ln(L)}{\partial\beta_2\partial\alpha_1} & \frac{\partial^2\ln(L)}{\partial\beta_1\partial\alpha_2} & \frac{\partial^2\ln(L)}{\partial\beta_2\partial\beta_1} & \frac{\partial^2\ln(L)}{\partial^2\beta_2} & \frac{\partial^2\ln(L)}{\partial\beta_2\partial\beta_0}\\ \frac{\partial^2\ln(L)}{\partial\beta_0\partial\alpha_1} & \frac{\partial^2\ln(L)}{\partial\beta_0\partial\alpha_2} & \frac{\partial^2\ln(L)}{\partial\beta_0\partial\beta_1} & \frac{\partial^2\ln(L)}{\partial\beta_0\partial\beta_2} & \frac{\partial^2\ln(L)}{\partial^2\beta_0}\end{bmatrix} \tag{4.6.8}$$

The second order partial derivatives of the log-likelihood function are given by:

$$\begin{aligned}
\frac{\partial^2\ln(L)}{\partial\alpha_1^2} &= -\frac{n_2}{\alpha_1^2}-\sum_{i=1}^{n_1}\frac{(\beta_1 x_i^{\beta_1-1})^2}{(\alpha_1\beta_1 x_i^{\beta_1-1}+\beta_0)^2},\\
\frac{\partial^2\ln(L)}{\partial\alpha_2^2} &= -\frac{n_1}{\alpha_2^2}-\sum_{i=1}^{n_2}\frac{(\beta_2 y_i^{\beta_2-1})^2}{(\alpha_2\beta_2 y_i^{\beta_2-1}+\beta_0)^2},\\
\frac{\partial^2\ln(L)}{\partial\beta_1^2} &= -\frac{n_2}{\beta_1^2}+\sum_{i=1}^{n_1}\frac{\alpha_1 x_i^{\beta_1-1}[\alpha_1 x_i^{\beta_1-1}+\beta_0\ln(x_i)(\beta_1\ln(x_i)+2)]}{(\alpha_1\beta_1 x_i^{\beta_1-1}+\beta_0)^2}\\
&\quad -\sum_{i=1}^{n_1}\alpha_1 x_i^{\beta_1}(\ln(x_i))^2-\sum_{i=1}^{n_2}\alpha_1 x_i^{\beta_1}(\ln(x_i))^2-\sum_{i=1}^{n_3}\alpha_1 x_i^{\beta_1}(\ln(x_i))^2,\\
\frac{\partial^2\ln(L)}{\partial\beta_2^2} &= -\frac{n_1}{\beta_2^2}+\sum_{i=1}^{n_2}\frac{\alpha_2 y_i^{\beta_2-1}[\alpha_2 y_i^{\beta_2-1}+\beta_0\ln(y_i)(\beta_2\ln(y_i)+2)]}{(\alpha_2\beta_2 y_i^{\beta_2-1}+\beta_0)^2}\\
&\quad -\sum_{i=1}^{n_1}\alpha_2 y_i^{\beta_2}(\ln(y_i))^2-\sum_{i=1}^{n_2}\alpha_2 y_i^{\beta_2}(\ln(y_i))^2-\sum_{i=1}^{n_3}\alpha_2 x_i^{\beta_2}(\ln(y_i))^2,\\
\frac{\partial^2\ln(L)}{\partial\beta_0^2} &= -\frac{n_3}{\beta_0^2}-\sum_{i=1}^{n_1}\frac{1}{(\alpha_1\beta_1 x_i^{\beta_1-1}+\beta_0)^2}-\sum_{i=1}^{n_2}\frac{1}{(\alpha_2\beta_2 y_i^{\beta_2-1}+\beta_0)^2},\\
\frac{\partial^2\ln(L)}{\partial\alpha_1\partial\alpha_2} &= 0,\\
\frac{\partial^2\ln(L)}{\partial\alpha_1\partial\beta_1} &= \sum_{i=1}^{n_1}\frac{\beta_0 x_i^{\beta_1-1}(\beta_1\ln(x_i)+1)}{(\alpha_1\beta_1 x_i^{\beta_1-1}+\beta_0)^2}-\sum_{i=1}^{n_1}x_i^{\beta_1}\ln(x_i)\\
&\quad -\sum_{i=1}^{n_2}x_i^{\beta_1}\ln(x_i)-\sum_{i=1}^{n_3}x_i^{\beta_1}\ln(x_i),
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \beta_2} &= 0, \\
\frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \beta_0} &= -\sum_{i=1}^{n_1} \frac{\beta_1 x_i^{\beta_1 - 1}}{(\alpha_1 \beta_1 x_i^{\beta_1 - 1} + \beta_0)^2}, \\
\frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \beta_1} &= 0, \\
\frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \beta_2} &= \sum_{i=1}^{n_2} \frac{\beta_0 y_i^{\beta_2 - 1}(\beta_2 \ln(y_i) + 1)}{(\alpha_2 \beta_2 y_i^{\beta_2 - 1} + \beta_0)^2} - \sum_{i=1}^{n_1} y_i^{\beta_2} \ln(y_i) \\
&\quad - \sum_{i=1}^{n_2} y_i^{\beta_2} \ln(y_i) - \sum_{i=1}^{n_3} y_i^{\beta_2} \ln(y_i), \\
\frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \beta_0} &= -\sum_{i=1}^{n_2} \frac{\beta_2 y_i^{\beta_2 - 1}}{(\alpha_2 \beta_2 y_i^{\beta_2 - 1} + \beta_0)^2}, \\
\frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \beta_2} &= 0, \\
\frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \beta_0} &= -\sum_{i=1}^{n_1} \frac{\alpha_1 x_i^{\beta_1 - 1}}{(\alpha_1 \beta_1 x_i^{\beta_1 - 1} + \beta_0)^2}, \\
\frac{\partial^2 \ln(L)}{\partial \beta_2 \partial \beta_0} &= -\sum_{i=1}^{n_2} \frac{\alpha_2 y_i^{\beta_2 - 1}}{(\alpha_2 \beta_2 y_i^{\beta_2 - 1} + \beta_0)^2}.
\end{aligned}
$$

where $\hat{\theta} = (\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_0)$ is the maximum likelihood estimator of the parameter $\theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_0)$.

4.6.1 Simulation study

The sample observations from the NBMW distribution can be generated based on the following algorithm:

Step 1. Generate U_1, U_2 and U_3 from $U(0, 1)$.

Step 2. Input $\alpha_1, \alpha_2, \beta_1, \beta_2$ and β_0

Step 3. Compute

$$Z_1 = \left[-\frac{\ln(1-U_1)}{\alpha_1}\right]^{\frac{1}{\beta_1}}, Z_2 = \left[-\frac{\ln(1-U_2)}{\alpha_2}\right]^{\frac{1}{\beta_2}} \text{ and } Z_3 = \left[-\frac{\ln(1-U_3)}{\beta_0}\right].$$

Step 4. Obtain $X_1 = \max(Z_1, Z_3)$ and $X_2 = \max(Z_2, Z_3)$.

Step 5. For a given sample size (n), compute the sizes n_1, n_2, and n_3.

Step 8. Maximize the Eqn.(4.6.2) to obtain the estimates $(\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_0)$.

We have performed a simulation to study the behavior of the MLEs. We have considered two different sets of model parameters; (1). $\alpha_1 = 0.5, \alpha_2 = 0.5, \beta_1 = 0.5, \beta_2 = 0.5$ and $\beta_0 = 0.5$, and (2). $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 1$ and $\beta_0 = 1$. We have used the sample sizes as n=25, 50, 100 and 500. The process is repeated 1000 times and the average estimates and the square root of the mean squared errors are computed and presented in Table 4.1. The results show that as the sample size increases, the bias and the mean square errors decrease, which gives the consistency property of MLEs.

4.7 Data Application

In this section we present an analysis of a bivariate real data set to illustrate that the NBMW distribution can be used as a good lifetime model. We have taken the American football league data reported in Csorgo and Welsh (1989). We compare the fit of the NBMW distribution with the following distributions:

(i). Bivariate generalized Gompertz (BGG) distribution of El-Sherpieny et al. (2013)

Table 4.1: Parameter estimates (the mean square errors) for different sample sizes.

Parameters	n=25	n=50	n=100	n=500
$\alpha_1 = 0.5$	0.195 (0.0630)	0.266 (0.0030)	0.330 (0.0019)	0.445 (0.0002)
$\alpha_2 = 0.5$	0.107 (0.0031)	0.199 (0.0015)	0.205 (0.0007)	0.413 (0.0001)
$\beta_1 = 0.5$	1.199 (0.0790)	1.052 (0.0356)	0.824 (0.0218)	0.667 (0.0042)
$\beta_2 = 0.5$	0.683 (0.0768)	0.589 (0.0478)	0.542 (0.0245)	0.533 (0.0045)
$\beta_0 = 0.5$	0.130 (0.0024)	0.247 (0.0013)	0.392 (0.0002)	0.494 (0.0001)
$\alpha_1 = 1$	0.692 (0.0064)	0.847 (0.0037)	0.899 (0.0017)	0.942 (0.0004)
$\alpha_2 = 1$	0.812 (0.0066)	0.903 (0.0064)	0.932 (0.0015)	0.945 (0.0002)
$\beta_1 = 1$	1.649 (0.1285)	1.608 (0.0467)	1.520 (0.0239)	1.318 (0.0046)
$\beta_2 = 1$	0.672 (0.0464)	0.846 (0.0255)	0.950 (0.0129)	0.962 (0.0029)
$\beta_0 = 1$	0.558 (0.0037)	0.635 (0.0008)	0.722 (0.0007)	0.803 (0.0003)

with cdf

$$F_{X_1,X_2}(x_1,x_2) = \left[1 - e^{\frac{-\lambda}{\alpha}(e^{\alpha x_1}-1)}\right]^{\beta_1}\left[1 - e^{\frac{-\lambda}{\alpha}(e^{\alpha x_2}-1)}\right]^{\beta_2}\left[1 - e^{\frac{-\lambda}{\alpha}(e^{\alpha z}-1)}\right]^{\beta_3},$$

where $z = \min(x_1, x_2)$, $\lambda > 0, \alpha > 0$ and $\beta_i > 0, i = 1, 2, 3$.

(ii). Bivariate exponentiated generalized Weibull-Gompertz (BEGWG) distribution of El-Damcese et al. (2015) with cdf

$$F_{X_1,X_2}(x_1,x_2) = \left[1 - e^{-x_1(e^{x_1}-1)}\right]^{\alpha_1}\left[1 - e^{-x_2(e^{x_2}-1)}\right]^{\alpha_2}\left[1 - e^{-z(e^{z}-1)}\right]^{\alpha_3},$$

where $z = \min(x_1, x_2)$ and $\alpha_i > 0, i = 1, 2, 3$.

(iii). Bivariate exponentiated modified Weibull extension (BEMWE) distribution of El-Gohary et al. (2016) with cdf

$$F_{X_1,X_2}(x_1,x_2) = \left[1 - e^{-\lambda\alpha(e^{(\frac{x_1}{\alpha})^{\gamma}}-1)}\right]^{\beta_1}\left[1 - e^{-\lambda\alpha(e^{(\frac{x_2}{\alpha})^{\gamma}}-1)}\right]^{\beta_2}\left[1 - e^{-\lambda\alpha(e^{(\frac{z}{\alpha})^{\gamma}}-1)}\right]^{\beta_3},$$

where $z = \min(x_1, x_2)$, $\lambda > 0, \alpha > 0, \gamma > 0$ and $\beta_i > 0, i = 1, 2, 3$.

(iv). Bivariate exponentiated Pareto (BEP) distribution of Ashwag et al. (2017)

with survival function

$$\begin{aligned}\bar{F}_{X_1,X_2}(x_1,x_2) &= 1-\theta_1\lambda_1\big[1-(1+x_1)^{-\lambda_1}\big]^{\theta_1-1}\big[1+x_1\big]^{-(\lambda_1+1)} \\ &\quad -\theta_2\lambda_2\big[1-(1+x_2)^{-\lambda_2}\big]^{\theta_2-1}\big[1+x_2\big]^{-(\lambda_2+1)}+C_G(u_1,u_2;\rho)\end{aligned}$$

where $C_G(u_1,u_2;\rho)$ is the Gaussian copula function, $\theta_i > 0, \lambda_i > 0, i = 1,2$ and $\rho \in (-1,1)$.

We use the -log L, AIC, CAIC, BIC for the comparison of the models. Here the variables X_1 and X_2 are respectively the game time to the first points scored by kicking the ball between goal posts and by moving the ball into the end zone. The data represents the the time in minutes and are given by:

$(X_1,X_2):$ $(2.05, 3.98), (9.05, 9.05), (0.85, 0.85), (3.43, 3.43), (7.78, 7.78), (10.57, 14.28),$ $(7.05, 7.05), (2.58, 2.58), (7.23, 9.68), (6.85, 34.58), (32.45, 42.35), (8.53, 14.57),$ $(31.13, 49.88), (14.58, 20.57), (5.78, 25.98), (13.80, 49.75), (7.25, 7.25), (4.25, 4.25),$ $(1.65, 1.65), (6.42, 15.08), (4.22, 9.48), (15.53, 15.53), (2.90, 2.90), (7.02, 7.02),$ $(6.42, 6.42), (8.98, 8.98), (10.15, 10.15), (8.87, 8.87), (10.40, 10.25), (2.98, 2.98),$ $(3.88, 6.43), (0.75, 0.75), (11.63, 17.37), (1.38, 1.38), (10.53, 10.53), (12.13, 12.13),$ $(14.58, 14.58), (11.82, 11.82), (5.52, 11.27), (19.65, 10.70), (17.83, 17.83), (10.85, 38.07).$

Table 4.2 shows the descriptive statistics of the variables X_1 and X_2. Here $X_1 < X_2$

Table 4.2: Descriptive statistics of the variables X_1 and X_2.

Variables	Size(n)	Min.	Max.	Mean	Median	Skewness
X_1	42	0.75	32.45	9.079	7.515	1.726
X_2	42	0.75	49.88	13.349	9.915	1.756

means the first score is a field goal, $X_1 > X_2$ means the first score is an unconverted touch down and $X_1 = X_2$ means the first score is converted touch down. Table 4.3 presents the MLEs of unknown parameters and the values of -log(L), AIC, CAIC and BIC. From Table 4.3 it is evident that the NBMW model fits the data better than the other four models.

Table 4.3: The MLEs of parameters and the goodness of fit test statistics.

Model	ML estimates	-log L	AIC	CAIC	BIC
BGG	$\hat{\beta}_1 = 0.024, \hat{\beta}_2 = 0.150, \hat{\beta}_3 = 0.310,$ $\hat{\lambda} = 0.004, \hat{\alpha} = 0.100$	260.50	531.00	532.67	539.69
BEMWE	$\hat{\beta}_1 = 0.212, \hat{\beta}_2 = 1.315, \hat{\beta}_3 = 2.645,$ $\hat{\lambda} = 0.096, \hat{\alpha} = 0.100, \hat{\gamma} = 0.420$	239.86	491.72	494.12	502.15
BEGWG	$\hat{\alpha_1} = 0.032, \hat{\alpha_2} = 0.186, \hat{\alpha_3} = 0.406$	354.03	714.06	714.69	719.27
BEP	$\hat{\theta}_1 = 9.948, \hat{\theta}_2 = 8.013, \hat{\lambda}_1 = 1.375,$ $\hat{\lambda}_2 = 1.142, \hat{\rho} = 0.927$	252.27	514.56	516.22	523.25
NBMW	$\hat{\alpha_1} = 0.022, \hat{\alpha_2} = 0.001, \hat{\beta}_1 = 0.545,$ $\hat{\beta}_2 = 1.947, \hat{\beta}_0 = 0.010$	208.37	426.74	428.41	435.43

4.8 Copula Function

Copula is a function that joins two or more marginal distribution functions to construct bivariate or multivariate distributions, see Sklar (1959). This function can link any type of marginal distribution and construct unknown bivariate distributions from known marginals.

A bivariate copula is defined as follows: Let X_1 and X_2 be continuous random variables with distribution functions, $F_{X_1}(x_1) = P(X_1 \leq x_1)$ and $F_{X_2}(x_2) = P(X_2 \leq x_2)$. If $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ are continuous and differentiable and C is unique, then the joint density can be written as

$$f_X(x_1, x_2) = \prod_{i=1}^{2} f_{X_i}(x_i) C^{'}(F_{X_1}(x_1), F_{X_2}(x_2)), \quad (4.8.1)$$

where $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are the probability density function corresponding to $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$. Also $C^{'}(F_{X_1}(x_1), F_{X_2}(x_2)) = \frac{\partial^2 C(F_{X_1}(x_1), F_{X_2}(x_2))}{\partial F_{X_1}(x_1).\partial F_{X_2}(x_2)}$, is the copula density. Many copula functions have been introduced in the literature and some of them are by Trivedi and Zimmer (2007), Nelson (2007) and Balakrishnan and Lai (2009).

For the NBMW distribution, the marginal survival functions are $\bar{F}_{X_1}(x_1) = e^{-\alpha_1 x_1^{\beta_1} + \beta_0 x_1}$ and $\bar{F}_{X_2}(x_2) = e^{-\alpha_2 x_2^{\beta_2} + \beta_0 x_2}$. In order to find the survival copula $\hat{C}(u, v)$ for this distribution, we may express $\bar{H}(x_1, x_2) = \hat{C}(\bar{F}_{X_1}(x_1), \bar{F}_{X_2}(x_2))$. That is,

$$\begin{aligned} \bar{H}(x_1, x_2) &= e^{-(\alpha_1 x_1^{\beta_1} + \beta_0 x_1)}.e^{-(\alpha_2 x_2^{\beta_2} + \beta_0 x_2)}.\min(e^{\beta_0 x_1}, e^{\beta_0 x_2}) \\ &= \bar{F}_{X_1}(x_1)\bar{F}_{X_2}(x_2).\min(e^{\beta_0 x_1}, e^{\beta_0 x_2}). \quad (4.8.2) \end{aligned}$$

The Gaussian copula with correlation parameter ρ of the bivariate modified Weibull distribution takes the form

$$\begin{aligned} C_G(u, v; \rho) &= \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v); \rho) \\ &= \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\Phi^{-1}(u)} \frac{\exp\{\frac{-1}{2(1-\rho^2)}(y_1^2 - 2\rho y_1 y_2 + y_2^2)\}}{2\pi\sqrt{1-\rho^2}} dy_1 dy_2, \quad (4.8.3) \end{aligned}$$

where Φ_ρ denotes the bivariate standard normal distribution function with correlation parameter $\rho \in (-1, 1)$ and Φ^{-1} denotes the inverse of the univariate standard normal distribution function. The density of the bivariate Gaussian copula is given by

$$C_G^{'}(u,v;\rho) = \frac{e^{\frac{-1}{2(1-\rho^2)}(y_1^2-2\rho y_1 y_2+y_2^2)}}{2\pi\sqrt{1-\rho^2}}, \tag{4.8.4}$$

where $y_1 = \Phi^{-1}(u)$, $y_2 = \Phi^{-1}(v)$, $u = F_{X_1}(x_1)$ and $v = F_{X_2}(x_2)$ are the marginal distributions for the random variables X_1 and X_2, respectively.

4.9 Summary

This chapter discussed the construction of a new bivariate distribution with modified Weibull distribution as marginals. The marginal and conditional probability distributions, mathematical expectations and moment generating function of the new bivariate distribution are derived. Expressions for bivariate reliability function, joint hazard rate function, mean waiting time and the reverse hazard rate function are obtained. The maximum likelihood estimators of the unknown parameters are derived. A simulation study is carried out to show the performance of the MLEs. Bivariate copula function for the new model is proposed and a real data application is illustrated.

CHAPTER

FIVE

DISCRETE ANALOGUES OF WEIBULL DISTRIBUTION AND ITS PROPERTIES

5.1 Introduction

[1] Developing the discrete analogues of continuous distributions have drawn much attention among the researchers and number of papers in this area are appeared in various journals. In reliability analysis and lifetime modelling, there are situations where the data shows discrete behavior like: the number of rounds fired a weapon till the first failure, the number of cancer deaths reported from a place over a given time period and the number of cycles successfully completed prior to the first failure

[1] Some results included in this chapter have appeared in the papers Jayakumar and Babu (2018) and Jayakumar and Babu (2019a).

when a particular device work in cycles. In some situations, if the data is huge or the individual observations are unknown then the data are grouped as discrete.

It is well known that the geometric distribution and the negative binomial distribution are the discrete analogues of the exponential distribution and gamma distribution respectively. A discrete version of the normal distribution was introduced in Lisman and van Zuylen (1972) and studied in Kemp (1997). Another version of discrete normal distribution was studied in Roy (2003). Nakagawa and Osaki (1975), Stein and Dattero (1984) and Padgett and Spurrier (1985) were proposed three different analogues of the discrete Weibull distribution and are further studied in Khan et al. (1989) and Kulasekera (1994).

Recently, several forms of discrete lifetime distributions derived from continuous distributions are proposed by many authors. Some of them are: discrete half-normal distribution in Kemp (2008); discrete Burr and Pareto distributions in Krishna and Pundir (2009); discrete modified Weibull distribution in Noughabi et al. (2011); discrete generalized exponential distribution in Gómez-Déniz (2010) and Nekoukhou et al. (2012); discrete gamma distribution in Chakraborty and Chakravarty (2012); discrete additive Weibull distribution in Bebbington et al. (2012); discrete inverse Weibull distribution in Jazi et al. (2010); discrete reduced modified Weibull distribution in Almalki and Nadarajah (2014); discrete Lindley distribution in Bakouch et al. (2014); discrete logistic distribution in Chakraborty and Chakravarty (2016), discrete Weibull geometric distribution in Jayakumar and Babu (2018), discrete additive Weibull geometric distribution in Jayakumar and Babu (2019a) and discrete type I half logistic Weibull distribution in Jayakumar and Babu (2019b).

In Section 2, we introduce the discrete Weibull geometric distribution and study its various properties such as, shape of pmf and hrf, quantile function and random number generation, moments, maximum likelihood estimation of parameters, simulation study and stress-strength parameter. Also, two real-life data applications are illustrates in this section to show the flexibility of this distribution in data modelling. We introduce the discrete additive Weibull geometric distribution in Section 3 and presents its various sub models. Its various properties are study in this section and illustrates a real-life data modelling.

5.2 Discrete Weibull Geometric Distribution

Suppose that $X_1, X_2, ..., X_n$ are i.i.d. random variables having Weibull distribution $W(\beta, \alpha)$, with scale parameter $\beta > 0$, shape parameter $\alpha > 0$ and pdf

$$g(x; \beta, \alpha) = \alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}, \quad x > 0. \tag{5.2.1}$$

Also let N be a discrete random variable having geometric distribution with pmf, $P(n; p) = (1-p)p^{n-1}$ for $n \ \epsilon \ \mathbf{N}$ and $p \ \epsilon \ (0, 1)$.

Let, $X_{(1)} = \min\{X_i\}_{i=1}^{N}$. The conditional cumulative distribution of $X_{(1)} \mid N = n$, is given by

$$G_{\{X_{(1)}|N=n\}} = 1 - [1 - F(x)]^n = 1 - e^{-n(\beta x)^{\alpha}}. \tag{5.2.2}$$

The cdf of $X_{(1)}$ is given by

$$\begin{aligned} F(x;p,\beta,\alpha) &= (1-p)\sum_{n=1}^{\infty} p^{n-1}[1-e^{-n(\beta x)^{\alpha}}] \\ &= \frac{1-e^{-(\beta x)^{\alpha}}}{1-pe^{-(\beta x)^{\alpha}}}; \quad x>0,\ 0<p<1,\ \beta>0,\ \alpha>0. \end{aligned} \tag{5.2.3}$$

The marginal pdf of $X_{(1)}$ is

$$f(x;p,\beta,\alpha) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}(1-pe^{-(\beta x)^{\alpha}})^{-2}; \quad x>0. \tag{5.2.4}$$

The distribution of $X_{(1)}$ is called Weibull geometric and is denoted by $WG(p,\beta,\alpha)$. This distribution is studied by Barreto-Souza et al. (2011). The survival and hrf are

$$S(x;p,\beta,\alpha) = \frac{(1-p)e^{-(\beta x)^{\alpha}}}{1-pe^{-(\beta x)^{\alpha}}}; \quad x>0, \tag{5.2.5}$$

and

$$h(x;p,\beta,\alpha) = \frac{\alpha\beta^{\alpha}x^{\alpha-1}}{1-pe^{-(\beta x)^{\alpha}}}; \quad x>0, \tag{5.2.6}$$

respectively.

Using the discretization method of difference in survival functions as shown in Eqn.(1.6.1) and after the re-parametrization $\rho = e^{-\beta^{\alpha}}$, the pmf of the discrete version, say Y of the Weibull geometric distribution is derived as

$$p_Y(y;p,\rho,\alpha) = P(Y=y) = \frac{(1-p)(\rho^{y^{\alpha}} - \rho^{(y+1)^{\alpha}})}{(1-p\rho^{y^{\alpha}})(1-p\rho^{(y+1)^{\alpha}})}, \tag{5.2.7}$$

where $y = 0,1,2,...;\ \alpha>0,\ 0<p<1$ and $0<\rho<1$. We call this distribution as discrete Weibull geometric distribution with parameters $p,\ \rho$ and α, and is denoted as DWG(p,ρ,α).

Here note that, $\sum_{y=0}^{\infty} P(Y=y) = S_X(0) - S_X(1) + S_X(1) - S_X(2) + ... = S_X(0) = 1.$

In particular, when $\alpha = 1$, the pmf becomes, $p_Y(y; p, \rho) = \frac{(1-p)(\rho^y - \rho^{(y+1)})}{(1-p\rho^y)(1-p\rho^{(y+1)})}$, which is the pmf of the discrete exponential geometric distribution.

When $p \to 0$, we get, $p_Y(y; \rho, \alpha) = \rho^{y^\alpha} - \rho^{(y+1)^\alpha}$, which is the discrete Weibull distribution of Nakagawa and Osaki (1975) with parameters ρ and α.

When $p \to 0$ and $\alpha \to 2$, then, $p_Y(y; \rho) = \rho^{y^2} - \rho^{(y+1)^2}$, which is the discrete Rayleigh distribution of Roy (2004).

When $p \to 0$ and $\alpha \to 1$, then, $p_Y(y; \rho) = \rho^y - \rho^{(y+1)}$, which is geometric distribution with parameter ρ.

5.2.1 Structural properties of the DWG distribution

The pmf plots of DWG(p, ρ, α) for various choices of the values of the parameters have been presented in Figure 5.1. The probabilities can be calculated recursively using the following relation :

$$p_Y(y+1) = \frac{(1 - p\rho^{y^\alpha})(\rho^{(y+1)^\alpha} - \rho^{(y+2)^\alpha})}{(1 - p\rho^{(y+2)^\alpha})(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}\, p_Y(y). \tag{5.2.8}$$

Gupta et al. (1997) proposed analogous statements for discrete distributions with unbounded support as :

a). The distribution is log-concave if and only if $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is decreasing. Then the hazard rate is increasing (IFR).

b). The distribution is log-convex if and only if $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is increasing. Then the hazard rate is decreasing (DFR).

c). If the sequence $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is constant, the hazard rate is constant and the

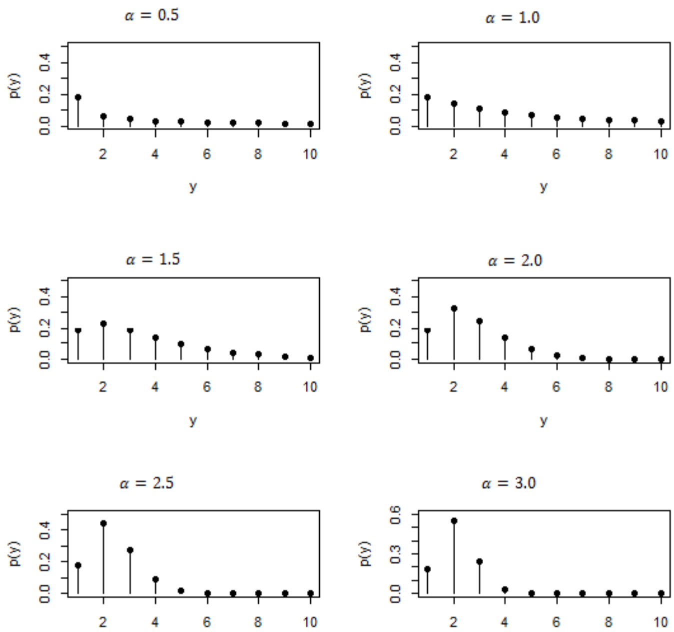

Figure 5.1: Plots of the pmf of DWG distribution for $p = 0.5$, $\rho = 0.9$ and $\alpha = (0.5, 1.0, 1.5, 2.0, 2.5, 3.0)$.

distribution is geometric.

For the DWG(p, ρ, α)

$$\frac{p(y+1)}{p(y)} = \frac{(1 - p\rho^{y^\alpha})(\rho^{(y+1)^\alpha} - \rho^{(y+2)^\alpha})}{(1 - p\rho^{(y+2)^\alpha})(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}.$$

Let, $\delta(y) = 1 - \frac{p(y+1)}{p(y)}$ and $\Delta\delta(y) = \delta(y+1) - \delta(y)$. Then

a). If $\Delta\delta(y) > 0$ (log concavity), then the hrf $h(y)$ is increasing.

b). If $\Delta\delta(y) < 0$ (log convexity), then $h(y)$ is decreasing.

c). If $\Delta\delta(y) = 0$, $h(y)$ is constant hazard rate.

All the above three cases are justified to the DWG distribution based on the values of the parameters as shown by the plots of pmf.

The cdf of DWG(p, ρ, α) distribution is obtained as

$$F(y) = P(Y \leq y) = 1 - S_X(y) + P(Y = y) = \frac{1 - \rho^{(y+1)^\alpha}}{1 - p\rho^{(y+1)^\alpha}}, \tag{5.2.9}$$

where $y = 0, 1, 2, ...;\ 0 < p < 1,\ 0 < \rho < 1$ and $\alpha > 0$.

Here note that, $F(0) = \frac{1-\rho}{1-p\rho}$. The proportion of positive values, $1 - F(0) = \frac{\rho(1-p)}{1-p\rho}$.

Also,

$$P(a < Y \leq b) = \frac{1 - \rho^{(b+1)^\alpha}}{1 - p\rho^{(b+1)^\alpha}} - \frac{1 - \rho^{(a+1)^\alpha}}{1 - p\rho^{(a+1)^\alpha}}.$$

The survival function of DWG(p, ρ, α) distribution is given by

$$S(y) = P(Y > y) = 1 - P(Y \leq y) = \frac{(1-p)\rho^{(y+1)^\alpha}}{1 - p\rho^{(y+1)^\alpha}}. \tag{5.2.10}$$

Discrete hazard rates may applicable in several common situations in reliability theory and survival analysis where clock time is not the best scale on which to describe lifetime. For example, in weapons like tanks, the number of rounds fired until failure is more important than lifetime in failure. In other situations, a device is monitored only once per time period and the observation then is the number of time periods successfully completed prior to the failure of the device. Similarly, in survival analysis, one may be interested in the length of stay (usually measured as number of days) in an observation ward or survival time (measured in number of weeks) of leukemia patients. In all these cases, the lifetimes are not measured on continuous scale but are simply counted and hence are discrete random variables. For the application of the hazard rate functions to characterizations of aging properties of discrete lifetimes distributions, one can see Shaked et al. (1995).

The hrf of DWG(p, ρ, α) is given by

$$h(y) = P(Y = y/Y \geq y) = \frac{P(Y = y)}{P(Y \geq y)} = \frac{1 - \rho^{(y+1)^\alpha - y^\alpha}}{1 - p\rho^{(y+1)^\alpha}} \tag{5.2.11}$$

provided, $P(Y \geq y) > 0$. It indicates the conditional probability of the system at time y, given that it did not fail before time y.

When $y \to 0$, from the Eqn.(5.2.11), $h(y) \to \frac{1-\rho}{1-p\rho} = p_Y(0)$.

For $\alpha = 1$, $h(y) = \frac{1-\rho}{1-p\rho^{(y+1)}}$. Also note that, as $y \to \infty$, $h(y) \to 1 - \rho$.

We have

$$h(0) = \frac{1-\rho}{1-p\rho},\ h(1) = \frac{1-\rho}{1-p\rho^2},\ h(2) = \frac{1-\rho}{1-p\rho^3},\ \dots .$$

That is, $h(0) > h(1) > h(2) > \dots$. Therefore, $h(y)$ is decreasing from $\frac{1-\rho}{1-p\rho}$ to $1 - \rho$.

Also let, $h(y) = \frac{1-\rho}{1-p\rho^{y+1}} = m$, where, m is a constant, such that, $1 - \rho < m < \frac{1-\rho}{1-p\rho}$.

Then the value of y corresponds to m is obtained as

$$y = \frac{\ln(m + \rho - 1) - \ln(p) - \ln(m)}{\ln(\rho)} - 1.$$

Since, y is discrete, we may take the floor value of y.

Now, for $0 < \alpha < 1$, as $y \to \infty$, $h(y) \to 0$. In this case $h(y)$ is decreasing from $\frac{1-\rho}{1-p\rho}$ to 0.

For $\alpha > 1$, $h(y)$ is an increasing failure rate function (IFR).

The accumulated hazard function, $H(y)$ is given by

$$H(y) = \sum_{t=0}^{y} h(t) = \sum_{t=0}^{y} \frac{1 - \rho^{(t+1)^\alpha - t^\alpha}}{1 - p\rho^{(t+1)^\alpha}}. \tag{5.2.12}$$

The mean residual life function (MRL) is given by

$$L(y) = E[(Y-y)|Y \geq y] = \frac{\sum_{j\geq y} jp(j)}{\sum_{j\geq y} p(j)} - y = \frac{\sum_{j>y} S(j)}{S(y)} = \sum_{j\geq y}\prod_{i=y}^{j}(1-h(i))$$

$$= \sum_{j\geq y}\prod_{i=y}^{j}\frac{\rho^{(i+1)^\alpha}(1-p\rho^{i^\alpha})}{\rho^{i^\alpha}(1-p\rho^{(i+1)^\alpha})}; \quad y \geq 0.$$

Also, from Roy and Gupta (1999)

$$\mu(y) = E[(Y-y)|Y > y] = L(y+1)+1 = \sum_{j\geq y+1}\prod_{i=y+1}^{j}\frac{\rho^{(i+1)^\alpha}(1-p\rho^{i^\alpha})}{\rho^{i^\alpha}(1-p\rho^{(i+1)^\alpha})} + 1; \quad y > 0.$$

Assume that, the MRL function at time $y = 0$, is equal to the mean of the lifetime distribution, that is, $L(0) = \mu$.

Then,

$$\mu(0) = \frac{\mu}{1-p(0)} = \frac{\mu(1-p\rho)}{\rho(1-p)}.$$

Figure 5.2, shows the shape of hazard rate function for different choice of parameter values.

The reverse hazard rate function is given by

$$h^*(y) = P(Y = y/Y \leq y) = \frac{P(Y = y)}{P(Y \leq y)} = \frac{(1-p)(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}{(1-p\rho^{y^\alpha})(1-\rho^{(y+1)^\alpha})}. \tag{5.2.13}$$

The second rate of failure is given by

$$h^{**}(y) = \log\left\{\frac{S(y)}{S(y+1)}\right\} = \log\left\{\frac{(\frac{1}{\rho})^{(y+2)^\alpha} - p}{(\frac{1}{\rho})^{(y+1)^\alpha} - p}\right\}. \tag{5.2.14}$$

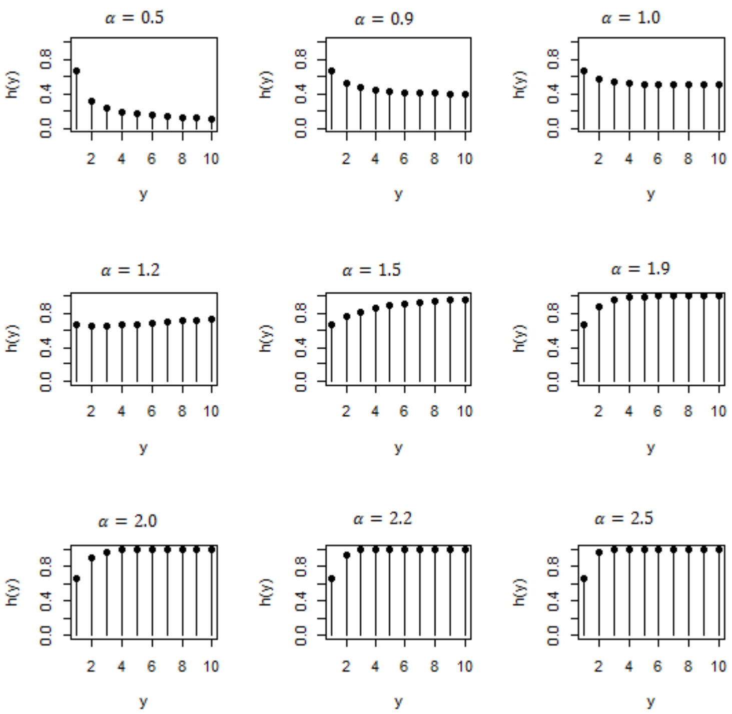

Figure 5.2: Shapes of hrf for $p = 0.5, \rho = 0.5$ and various values of α.

5.2.2 Quantile function and random number generation

From Rohatgi and Saleh (2001), the point y_u is known as the u^{th} quantile of a discrete random variable Y, if it satisfies, $P(Y \leq y_u) \geq u$ and $P(Y \geq y_u) \geq 1 - u$. Using this result, we have the following theorem.

Theorem 5.2.1. *The u^{th} quantile $\phi(u)$ of $DWG(p, \rho, \alpha)$ is given by*

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left(\ln\left(\frac{u-1}{up-1} \right) / \ln(\rho) \right)^{\frac{1}{\alpha}} - 1 \right\rceil, \tag{5.2.15}$$

where, $\lceil y_u \rceil$ denotes the smallest integer greater than or equal to y_u.

Proof. First suppose that, $P(Y \leq y_u) \geq u$. Then

$$\frac{1-\rho^{(y_u+1)^\alpha}}{1-p\rho^{(y_u+1)^\alpha}} \geq u \Rightarrow y_u \geq \left[\ln\left(\frac{1-u}{1-up}\right)/\ln(\rho)\right]^{\frac{1}{\alpha}} - 1, \text{ since } \ln(\rho) < 0. \quad (5.2.16)$$

Similarly, $P(Y \geq y_u) \geq 1-u$ gives,

$$y_u \leq \left[\ln\left(\frac{1-u}{1-up}\right)/\ln(\rho)\right]^{\frac{1}{\alpha}}. \quad (5.2.17)$$

Combining the Eqns.(5.2.16) and (5.2.17) we get

$$\left[\ln\left(\frac{1-u}{1-up}\right)/\ln(\rho)\right]^{\frac{1}{\alpha}} - 1 < y_u \leq \left[\ln\left(\frac{1-u}{1-up}\right)/\ln(\rho)\right]^{\frac{1}{\alpha}}.$$

Hence, $\phi(u)$ is an integer given by

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left(\ln\left(\frac{1-u}{1-up}\right)/\ln(\rho)\right)^{\frac{1}{\alpha}} - 1 \right\rceil.$$

This completes the proof. □

Using the usual inverse transformation method, a random number (integer) can be sampled from the proposed model. Let, U be a random number drawn from a uniform distribution on (0, 1), then a random number Y following DWG(p,ρ,α) distribution is obtained by the Eqn.(5.2.15).

In particular, the median is given by

$$\phi(0.5) = \lceil y_{0.5} \rceil = \left\lceil \left(\ln\left(\frac{1}{2-p}\right)/\ln(\rho)\right)^{\frac{1}{\alpha}} - 1 \right\rceil. \quad (5.2.18)$$

5.2.3 Simulation study

Table 5.1, presents the MLEs of DWG(p,ρ,α) distribution and their standard errors for different values of n, of various simulated samples. Standard errors are attained by means of the asymptotic covariance matrix of the MLEs of parameters when the Newton-Raphson procedure converges.

Table 5.1: MLEs of DWG(p, ρ, α) for various samples(n).

$Parameters$	n	$\hat{p}(\hat{SE}(\hat{p}))$	$\hat{\rho}(\hat{SE}(\hat{\rho}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$
$p = 0.5$, $\rho = 0.5$, $\alpha = 0.5$	50	0.539(2.781)	0.553(1.498)	0.557(0.915)
	100	0.581(0.901)	0.554(0.696)	0.485(0.391)
	500	0.599(0.794)	0.534(0.415)	0.501(0.272)
	1000	0.491(0.824)	0.535(0.404)	0.533(0.234)
$p = 0.75$, $\rho = 0.5$, $\alpha = 1.0$	50	0.735(0.933)	0.458(0.869)	1.017(1.072)
	100	0.734(0.779)	0.503(0.730)	1.269(1.062)
	500	0.805(0.363)	0.553(0.459)	1.104(0.553)
	1000	0.751(0.294)	0.531(0.349)	1.026(0.392)
$p = 0.6$, $\rho = 0.9$, $\alpha = 1.5$	50	0.543(1.709)	0.887(0.411)	1.359(1.065)
	100	0.709(0.531)	0.913(0.156)	1.556(0.564)
	500	0.638(0.343)	0.907(0.087)	1.549(0.294)
	1000	0.623(0.259)	0.908(0.063)	1.527(0.211)
$p = 0.9$, $\rho = 0.5$, $\alpha = 2.0$	50	0.906(0.383)	0.481(1.010)	1.455(1.746)
	100	0.936(0.249)	0.634(0.899)	2.197(2.230)
	500	0.938(0.103)	0.614(0.393)	2.094(0.925)
	1000	0.902(0.069)	0.637(0.276)	2.012(0.691)
$p = 0.8$, $\rho = 0.6$, $\alpha = 2.5$	50	0.862(0.895)	0.673(1.434)	2.413(3.940)
	100	0.858(0.687)	0.701(1.012)	2.580(3.038)
	500	0.753(0.565)	0.541(0.568)	2.607(1.621)
	1000	0.808(0.145)	0.633(0.278)	2.562(0.965)

5.2.4 Moments

The r^{th} moment about origin is given by

$$\mu_r^{'} = E(Y^r) = \sum_{y=0}^{\infty} y^r \frac{(1-p)(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}{(1-p\rho^{y^\alpha})(1-p\rho^{(y+1)^\alpha})}. \tag{5.2.19}$$

For given values of p, ρ and α, the moments can be numerically computed using **R** programming. Table 5.2 shows the moments, skewness and kurtosis for DWG distribution for given values of p, ρ and α.

Table 5.2: Moments, skewness and kurtosis for $p = 0.9$, $\rho = 0.9$ and various values of α.

Parameter	Raw moments	Central moments	Skewness	Kurtosis
$\alpha = 0.5$	$\mu_1^{'} = 0.96$ $\mu_2^{'} = 5.22$ $\mu_3^{'} = 35.98$ $\mu_4^{'} = 279.68$	$\mu_2 = 4.29$ $\mu_3 = 22.71$ $\mu_4 = 167.61$	6.53	9.11
$\alpha = 1.0$	$\mu_1^{'} = 1.20$ $\mu_2^{'} = 5.60$ $\mu_3^{'} = 34.97$ $\mu_4^{'} = 257.59$	$\mu_2 = 4.16$ $\mu_3 = 18.27$ $\mu_4 = 257.59$	4.63	7.62
$\alpha = 1.5$	$\mu_1^{'} = 0.97$ $\mu_2^{'} = 3.23$ $\mu_3^{'} = 15.84$ $\mu_4^{'} = 98.35$	$\mu_2 = 2.29$ $\mu_3 = 8.27$ $\mu_4 = 52.47$	5.69	10.01
$\alpha = 2.0$	$\mu_1^{'} = 0.73$ $\mu_2^{'} = 1.51$ $\mu_3^{'} = 4.32$ $\mu_4^{'} = 15.90$	$\mu_2 = 0.98$ $\mu_3 = 1.71$ $\mu_4 = 15.90$	3.06	7.80
$\alpha = 5$	$\mu_1^{'} = 0.487$ $\mu_2^{'} = 0.484$ $\mu_3^{'} = 0.498$ $\mu_4^{'} = 0.527$	$\mu_2 = 0.26$ $\mu_3 = 0.02$ $\mu_4 = 0.08$	0.03	1.25

5.2.5 Maximum likelihood estimation of the parameters of DWG distribution

Consider a random sample $(y_1, y_2, ..., y_n)$ of size n, from the DWG(p, ρ, α). Then, the log-likelihood function is given by

$$\log(L) = n\log(1-p)+\sum_{i=1}^{n}\log(\rho^{y_i^\alpha}-\rho^{(y_i+1)^\alpha})-\sum_{i=1}^{n}\log(1-p\rho^{y_i^\alpha})-\sum_{i=1}^{n}\log(1-p\rho^{(y_i+1)^\alpha}). \tag{5.2.20}$$

The likelihood equations are

$$\frac{\partial\log(L)}{\partial p} = \frac{-n}{1-p}+\sum_{i=1}^{n}\frac{\rho^{y_i^\alpha}}{1-p\rho^{y_i^\alpha}}+\sum_{i=1}^{n}\frac{\rho^{(y_i+1)^\alpha}}{1-p\rho^{(y_i+1)^\alpha}} = 0, \tag{5.2.21}$$

$$\begin{aligned}\frac{\partial \log(L)}{\partial \rho} &= \sum_{i=1}^{n} \frac{y_i^{\alpha}\rho^{y_i^{\alpha}-1} - (y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}-1}}{\rho^{y_i^{\alpha}} - \rho^{(y_i+1)^{\alpha}}} \\ &\quad + p\sum_{i=1}^{n} \frac{y_i^{\alpha}\rho^{y_i^{\alpha}-1}}{1-p\rho^{y_i^{\alpha}}} + p\sum_{i=1}^{n} \frac{(y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}-1}}{1-p\rho^{(y_i+1)^{\alpha}}} = 0, \qquad (5.2.22)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log(L)}{\partial \alpha} &= \log(\rho)\sum_{i=1}^{n} \frac{y_i^{\alpha}\rho^{y_i^{\alpha}}\log(y_i) - (y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\log(y_i+1)}{\rho^{y_i^{\alpha}} - \rho^{(y_i+1)^{\alpha}}} \\ &\quad + p\log(\rho)\sum_{i=1}^{n} \frac{y_i^{\alpha}\rho^{y_i^{\alpha}}\log(y_i)}{1-p\rho^{y_i^{\alpha}}} \\ &\quad + p\log(\rho)\sum_{i=1}^{n} \frac{(y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\log(y_i+1)}{1-p\rho^{(y_i+1)^{\alpha}}} = 0. \qquad (5.2.23)\end{aligned}$$

These equations do not have explicit solutions and they have to be obtained numerically by using statistical softwares like *nlm* or *optim* packages in **R** programming. Let the estimators be, $\hat{\theta} = (\hat{p}, \hat{\rho}, \hat{\alpha})^T$. The Fisher's information matrix is given by

$$I_y(\theta) = \begin{bmatrix} -E(\frac{\partial^2 L}{\partial p^2}) & -E(\frac{\partial^2 L}{\partial p \partial \rho}) & -E(\frac{\partial^2 L}{\partial p \partial \alpha}) \\ -E(\frac{\partial^2 L}{\partial \rho \partial p}) & -E(\frac{\partial^2 L}{\partial \rho^2}) & -E(\frac{\partial^2 L}{\partial \rho \partial \alpha}) \\ -E(\frac{\partial^2 L}{\partial \alpha \partial p}) & -E(\frac{\partial^2 L}{\partial \alpha \partial \rho}) & -E(\frac{\partial^2 L}{\partial \alpha^2}) \end{bmatrix}.$$

Here, the DWG distribution satisfies the regularity conditions which are full filled for the parameters in the interior of the parameter space, but not on the boundary (see Ferguson, 1996). Hence, the vector $\hat{\theta}$ is consistent and asymptotically normal. That is $\sqrt{I_y(\theta)}[\hat{\theta} - \theta]$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix. The Fisher's information matrix can be computed using the approximation

$$I_y(\hat{\theta}) \approx \begin{bmatrix} -\frac{\partial^2 L}{\partial p^2}|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial p \partial \rho}|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial p \partial \alpha}|_{(\hat{p},\hat{\rho},\hat{\alpha})} \\ -\frac{\partial^2 L}{\partial \rho \partial p}|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \rho^2}|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \rho \partial \alpha}|_{(\hat{p},\hat{\rho},\hat{\alpha})} \\ -\frac{\partial^2 L}{\partial \alpha \partial p}|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \alpha \partial \rho}|_{(\hat{p},\hat{\rho},\hat{\alpha})} & -\frac{\partial^2 L}{\partial \alpha^2}|_{(\hat{p},\hat{\rho},\hat{\alpha})} \end{bmatrix},$$

where $\hat{p}$, $\hat{\rho}$ and $\hat{\alpha}$ are the MLEs of p, ρ and α, respectively.

5.2.6 Stress-strength parameter

Let, $Y \sim DWG(\theta_1)$ and $Z \sim DWG(\theta_2)$, where, $\theta_1 = (p_1, \rho_1, \alpha_1)^T$ and $\theta_2 = (p_2, \rho_2, \alpha_2)^T$. Then, from Eqns.(5.2.7) and (5.2.9), we have

$$R = \sum_{y=0}^{\infty} \frac{(1-p_1)(\rho_1^{y^{\alpha_1}} - \rho_1^{(y+1)^{\alpha_1}})(1-\rho_2^{(y+1)^{\alpha_2}})}{(1-p_1\rho_1^{y^{\alpha_1}})(1-p_1\rho_1^{(y+1)^{\alpha_1}})(1-p_2\rho_2^{(y+1)^{\alpha_2}})}. \tag{5.2.24}$$

Assume that, $(y_1, y_2, ..., y_n)$ and $(z_1, z_2, ..., z_m)$ are independent observations drawn from DWG(θ_1) and DWG(θ_2), respectively. The total likelihood function is given by, $L_R(\theta^*) = L_n(\theta_1)\ L_m(\theta_2)$, where, $\theta^* = (\theta_1,\ \theta_2)$. The score vector is given by

$$U_R(\theta^*) = \left(\frac{\partial L_R}{\partial p_1}, \frac{\partial L_R}{\partial \rho_1}, \frac{\partial L_R}{\partial \alpha_1}, \frac{\partial L_R}{\partial p_2}, \frac{\partial L_R}{\partial \rho_2}, \frac{\partial L_R}{\partial \alpha_2}\right).$$

The MLE, $\hat{\theta^*}$ may be obtained from the solution of the nonlinear equation, $U_R(\hat{\theta^*}) = 0$. Applying $\hat{\theta^*}$, in the Eqn.(5.2.24), we obtain the stress-strength parameter R.

5.2.7 Data applications of the DWG distribution

In this section, to show how the DWG(p, ρ, α) distribution works in practice, we use two real data sets, of which the first data set is discrete version of a continuous data and the second data set is a count data. The parameters are estimated by using the method of maximum likelihood. We compare the fit of the DWG distribution with the geometric (G) distribution, discrete Weibull (DW) distribution, discrete logistic (DLOG) distribution and exponentiated discrete Weibull (EDW) distribution. The values of the -log L, K-S, AIC, CAIC and BIC are calculated for the five distributions in order to verify which distribution fits better to these data.

The first data set represents remission times(in months) of 128 bladder cancer patients taken from Lee and Wang (2003). The data are :

0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.340, 10.660, 10.750, 11.250, 11.640, 11.790, 11.980, 12.020, 12.030, 12.070, 12.630, 13.110, 13.290, 13.800, 14.240, 14.760, 14.770, 14.830, 15.960, 16.620, 17.120, 17.140, 17.360, 18.100, 19.130, 20.280, 21.730, 22.690, 23.630, 25.740, 25.820, 26.310, 32.150, 34.260, 36.660, 43.010, 46.120, 79.050.

Since the data set is continuous, here first we discretize the data by considering the floor value (y). The values in Table 5.3, indicates that the DWG distribution leads

Table 5.3: Parameter estimates and goodness of fit for various models fitted for the first data set.

Model	ML estimates	-log L	AIC	CAIC	BIC	K-S	p value
G	$\hat{p} = 0.8991$	414.836	831.672	831.704	831.779	0.1000	0.1549
DW	$\hat{q} = 0.9114$ $\hat{\beta} = 1.0511$	414.556	833.112	837.304	833.326	0.1131	0.0758
DLOG	$\hat{p} = 0.8000$ $\hat{\mu} = 7.6149$	456.825	917.650	917.746	917.864	0.1860	0.0003
EDW	$\hat{p} = 0.4689$ $\hat{\alpha} = 0.5397$ $\hat{\gamma} = 4.9697$	409.766	825.532	825.726	825.854	0.1237	0.0399
DWG	$\hat{p} = 0.9529$ $\hat{\rho} = 0.9982$ $\hat{\alpha} = 1.7025$	409.277	824.554	824.748	824.876	0.0905	0.2458

to a better fit compared to the other four models. Figure 5.3, shows the structure of the cdf of the five models with the empirical distribution of the given data. Here the dotted line indicates the empirical cdf of the data.

The second data set is the number of shocks before failure reported in Murthy et al. (2004, p.245). The data are: 1, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 10, 11, 12, 14.

The values in Table 5.4, indicates that the DWG distribution leads to a better fit for the second data set compared to the other four models. Figure 5.4, shows the structure of the cdf of the five models in comparison with the empirical distribution function of the given data. The dotted line indicates the empirical cdf of the second data set.

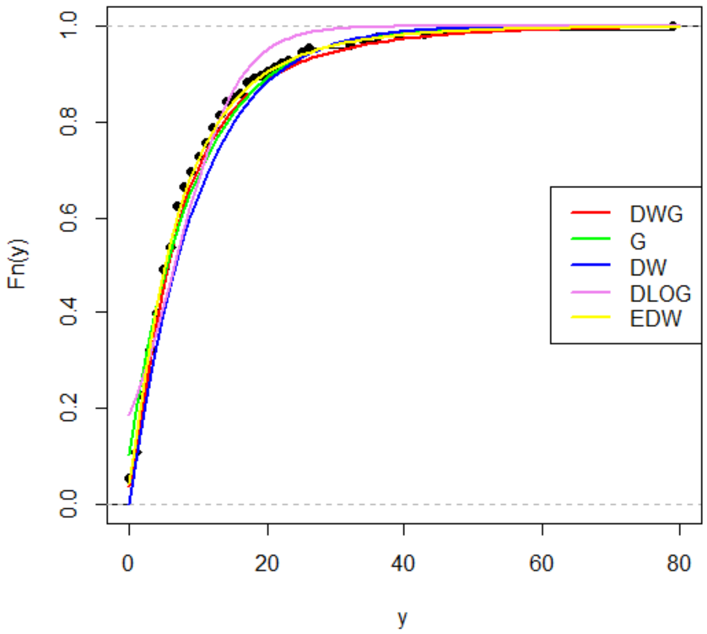

Figure 5.3: Empirical and fitted cdfs for the first data set.

5.3 Discrete Additive Weibull Geometric Distribution

Xie and Lai (1995) proposed the additive Weibull (AW) distribution by combining the failure rates of two Weibull distributions of which one has a decreasing failure rate and the other has an increasing failure rate. The cdf of AW distribution is given by

$$F(x; \alpha, \beta, \gamma, \delta) = 1 - e^{-(\alpha x^{\beta} + \gamma x^{\delta})} \tag{5.3.1}$$

where $\alpha > 0, \gamma > 0$ and $\beta > \delta > 0$ or $(\delta > \beta > 0)$, which gives identifiability to the model. The corresponding pdf is given by

$$f(x; \alpha, \beta, \gamma, \delta) = (\alpha\beta x^{\beta-1} + \gamma\delta x^{\delta-1})e^{-(\alpha x^{\beta} + \gamma x^{\delta})}. \tag{5.3.2}$$

Table 5.4: Parameter estimates and goodness of fit for various models fitted for the second data set.

Model	ML estimates	-log L	AIC	CAIC	BIC	K-S	p value
G	$\hat{p} = 0.8609$	46.389	94.778	95.064	95.551	0.2995	0.1133
DW	$\hat{q} = 0.9831$ $\hat{\beta} = 2.0111$	41.637	87.274	88.197	88.819	0.2227	0.4057
DLOG	$\hat{p} = 0.6079$ $\hat{\mu} = 6.2330$	43.170	90.340	91.263	91.885	0.1851	0.6435
EDW	$\hat{p} = 0.7183$ $\hat{\alpha} = 1.0020$ $\hat{\gamma} = 4.6559$	41.224	88.448	90.448	90.766	0.1868	0.6317
DWG	$\hat{p} = 0.8637$ $\hat{\rho} = 0.9993$ $\hat{\alpha} = 2.8921$	41.050	88.100	90.100	90.418	0.1715	0.7341

Here α and γ are scale parameters, and β and δ are shape parameters. Lemonte et al. (2014) examined some structural properties of AW distribution.

Suppose that $X_1, X_2, ..., X_N$ are N i.i.d. random variables from AW distribution with cdf given in Eqn.(5.3.1). Let N be a discrete random variable following a geometric distribution (truncated at zero) with the pmf given by

$$P(N = n) = (1 - p)p^n, \text{ for } n \,\epsilon\, N \text{ and } p \,\epsilon\, (0, 1). \tag{5.3.3}$$

Let $X_{(1)} = Min\{X_i\}_{i=1}^N$, then the cdf of $X_{(1)} \mid N = n$, is given by

$$G_{\{X_{(1)}|N=n\}} = 1 - [1 - F(x)]^n = 1 - e^{-n(\alpha x^\beta + \gamma x^\delta)}, \tag{5.3.4}$$

the cdf of $X_{(1)}$ is given by

$$\begin{aligned} F(x; \alpha, \beta, \gamma, \delta, p) &= (1-p)\sum_{n=1}^{\infty} p^{n-1}[1 - e^{-n(\alpha x^\beta + \gamma x^\delta)}] \\ &= \frac{1 - e^{-(\alpha x^\beta + \gamma x^\delta)}}{1 - pe^{-(\alpha x^\beta + \gamma x^\delta)}}, \end{aligned} \tag{5.3.5}$$

where $x > 0,\ 0 < p < 1,\ \alpha > 0,\ \beta > 0,\ \gamma > 0$ and $\delta > 0$. The corresponding pdf of

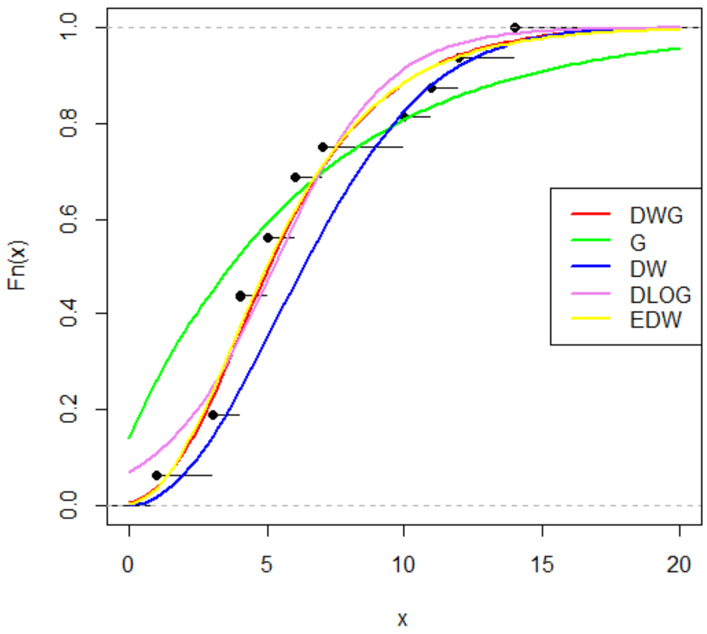

Figure 5.4: Empirical and fitted cdfs for the second data set.

$X_{(1)}$ is given by

$$f(x;\alpha,\beta,\gamma,\delta,p)=\frac{(1-p)(\alpha\beta x^{\beta-1}+\gamma\delta x^{\delta-1})e^{-(\alpha x^{\beta}+\gamma x^{\delta})}}{(1-pe^{-(\alpha x^{\beta}+\gamma x^{\delta})})^{2}},\quad x\geq 0. \tag{5.3.6}$$

The distribution of $X_{(1)}$ is called additive Weibull geometric and its is survival function is given by

$$S(x;\alpha,\beta,\gamma,\delta,p)=\frac{(1-p)e^{-(\alpha x^{\beta}+\gamma x^{\delta})}}{1-pe^{-(\alpha x^{\beta}+\gamma x^{\delta})}}. \tag{5.3.7}$$

This distribution is studied by Elbatal et al. (2016). Marshall and Olkin (1997) introduced a method of adding a parameter into a family of distributions. According to them if $\bar{F}(x)$ denote the survival function of a continuous random variable X, then the usual device of adding a new parameter results in another survival function

$\bar{G}(x)$ defined by

$$\bar{G}(x) = \frac{\theta \bar{F}(x)}{1 - \bar{\theta}\bar{F}(x)}\ , -\infty < x < \infty,\ \theta > 0, \tag{5.3.8}$$

where $\bar{\theta} = 1 - \theta$. In particular when $\theta = 1$, $\bar{G}(x) = \bar{F}(x)$.

Let Y be the discrete analogue of the continuous random variable X with survival function defined in the Eqn.(5.3.8). Gómez-Déniz (2010) obtained the discrete analogue of Marshall-Olkin scheme by applying the Eqn.(5.3.8) in the Eqn.(1.6.1). The corresponding random variable Y has the pmf

$$p_Y(y) = P(Y = y) = \frac{\theta[\bar{F}(y) - \bar{F}(y+1)]}{[1 - \bar{\theta}\bar{F}(y)][1 - \bar{\theta}\bar{F}(y+1)]} \tag{5.3.9}$$

Now, we apply the additive Weibull geometric distribution with survival function defined in the Eqn.(5.3.7) into the Eqn.(5.3.9) and after re-parametrization as $\rho = e^{-\alpha}$ and $\eta = e^{-\gamma}$, the pmf becomes

$$p_Y(y) = \frac{\theta(1-p)[\rho^{y^\beta}\eta^{y^\delta} - \rho^{(y+1)^\beta}\eta^{(y+1)^\delta}]}{[1 - (1 - \theta(1-p))\rho^{y^\beta}\eta^{y^\delta}][1 - (1 - \theta(1-p))\rho^{(y+1)^\beta}\eta^{(y+1)^\delta}]},\quad y = 0, 1, 2, \dots\ , \tag{5.3.10}$$

where $\theta > 0, 0 < p < 1, 0 < \rho < 1, 0 < \eta < 1, \beta > \delta > 0$ (or $\delta > \beta > 0$). We call this distribution as the generalized discrete additive Weibull geometric distribution.

When $\theta = 1$, Eqn.(5.3.10) becomes

$$p_Y(y; p, \rho, \eta, \beta, \delta) = \frac{(1-p)(\rho^{y^\beta}\eta^{y^\delta} - \rho^{(y+1)^\beta}\eta^{(y+1)^\delta})}{(1 - p\rho^{y^\beta}\eta^{y^\delta})(1 - p\rho^{(y+1)^\beta}\eta^{(y+1)^\delta})},\quad y = 0, 1, 2, \dots\ , \tag{5.3.11}$$

where $0 < p < 1,\ 0 < \rho < 1,\ 0 < \eta < 1,\ \beta > \delta > 0$ (or $\delta > \beta > 0$). We call this distribution as discrete additive Weibull geometric (DAWG) distribution with parameters $p,\ \rho,\ \eta,\ \beta$ and δ and is denoted as DAWG$(p, \rho, \eta, \beta, \delta)$, see Jayakumar and Babu (2019a).

We have the following cases

1. When $\theta = 1$ and $\rho \uparrow 1$ or $\eta \uparrow 1$, then the equation Eqn.(5.3.10) reduces to the discrete Weibull geometric distribution introduced in Jayakumar and Babu (2018).

2. When $\eta = \rho$, $\delta = \beta$, then also it becomes the discrete Weibull geometric distribution with parameters ρ^2 and β.

3. When $\beta = 1$ and $\eta = 1$, the it becomes the discrete exponential geometric distribution.

4. When $p \downarrow 0$ and $\beta = 1$, it becomes the discrete modified Weibull distribution.

5. When $p \downarrow 0$ and $\eta = 1$, then it becomes the discrete Weibull distribution with parameters ρ and β.

6. When $\beta = 2$ and $\eta = 1$, it becomes discrete Rayleigh geometric distribution.

7. When $p \downarrow 0$, $\beta = 2$ and $\eta = 1$, then it becomes discrete Rayleigh distribution.

8. When $p \downarrow 0$, $\beta = 1$ and $\eta = 1$, then it becomes geometric distribution with parameter ρ.

5.3.1 Structural properties of the DAWG distribution

Figure 5.5, provides the pmf plots of DAWG$(p, \rho, \eta, \beta, \delta)$ for various choices of values of the parameters. The probabilities can be calculate recursively using the following

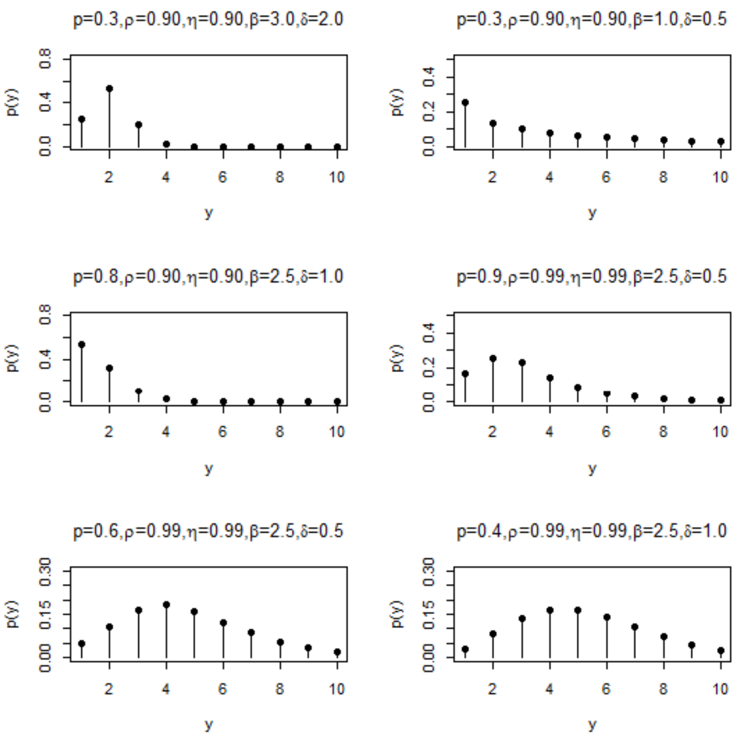

Figure 5.5: Shape of the pmf of DAWG$(p, \rho, \eta, \beta, \delta)$ distribution.

relation

$$p_Y(y+1) = \frac{(1 - p\rho^{y^\beta}\eta^{y^\delta})(\rho^{(y+1)^\beta}\eta^{(y+1)^\delta} - \rho^{(y+2)^\beta}\eta^{(y+2)^\delta})}{(1 - p\rho^{(y+2)^\beta}\eta^{(y+2)^\delta})(\rho^{y^\beta}\eta^{y^\delta} - \rho^{(y+1)^\beta}\eta^{(y+1)^\delta})}\, p_Y(y. \qquad (5.3.12)$$

We have the analogous statements for the DAWG distributions as:

i). The distribution is log-concave if and only if $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is decreasing.

ii). The distribution is log-convex if and only if $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is increasing.

iii). If the sequence $\{\frac{p_Y(y+1)}{p_Y(y)}\}_{y\geq 0}$ is constant, the hazard rate is constant and the distribution is geometric.

The cdf of DAWG$(p,\rho,\eta,\beta,\delta)$ is

$$F(y)=P(Y\leq y)=1-S_X(y)+P(Y=y)=\frac{1-\rho^{(y+1)^\beta}\eta^{(y+1)^\delta}}{1-p\rho^{(y+1)^\beta}\eta^{(y+1)^\delta}}. \qquad (5.3.13)$$

where $y=0,1,2,...;\ \beta>\delta>0$ (or $\delta>\beta>0$), $0<p<1, 0<\rho<1$ and $0<\eta<1$. Here note that, $F(0)=\frac{1-\rho\eta}{1-p\rho\eta}$. The proportion of positive values, $1-F(0)=\frac{\rho\eta(1-p)}{1-p\rho\eta}$. The survival function of DAWG$(p,\rho,\eta,\beta,\delta)$ is given by

$$S(y)=P(Y>y)=1-P(Y\leq y)=\frac{(1-p)\rho^{(y+1)^\beta}\eta^{(y+1)^\delta}}{1-p\rho^{(y+1)^\beta}\eta^{(y+1)^\delta}}. \qquad (5.3.14)$$

The hrf of DAWG$(p,\rho,\eta,\beta,\delta)$ is given by

$$h(y)=P(Y=y/Y\geq y)=\frac{P(Y=y)}{P(Y\geq y)}=\frac{1-\rho^{(y+1)^\beta-y^\beta}\eta^{(y+1)^\delta-y^\delta}}{1-p\rho^{(y+1)^\beta}\eta^{(y+1)^\delta}}, \qquad (5.3.15)$$

provided, $P(Y\geq y)>0$. Figure 5.6, provides the plots of hrf of DAWG$(p,\rho,\eta,\beta,\delta)$ for various choices of values of the parameters. When $y\rightarrow 0$, we have from the Eqn.(5.3.15), $h(y)\rightarrow\frac{1-\rho\eta}{1-p\rho\eta}=p_Y(0)$. Now to study the limit of $h(y)$ as $y\rightarrow\infty$ we consider the following five cases based on the values of the shape parameters β and δ.

Case (i). When $\beta>1$ and $\delta>1$ (provided $\beta>\delta$ or $\beta<\delta$). Here note that, $\lim_{y\rightarrow\infty}h(y)=1$. Now in this case, $h(0)=\frac{1-\rho\eta}{1-p\rho\eta}$, $h(1)=\frac{1-\rho^{2^\beta-1}\eta^{2^\delta-1}}{1-p\rho^{2^\beta}\eta^{2^\delta}}$, $h(2)=\frac{1-\rho^{3^\beta-2^\beta}\eta^{3^\delta-2^\delta}}{1-p\rho^{3^\beta}\eta^{3^\delta}}$,... . That is, $h(0)<h(1)<h(2)<...<1$. Therefore, $h(y)$ is an increasing function increases from $\frac{1-\rho\eta}{1-p\rho\eta}$ to 1.

Case (ii). When $\beta>1$ and $\delta=1$. Here, $\lim_{y\rightarrow\infty}h(y)=1$. Also seen that $h(0)<h(1)<h(2)<...<1$. Therefore, $h(y)$ is an increasing function increases from $\frac{1-\rho\eta}{1-p\rho\eta}$ to 1.

Case (iii). When $0<\beta<1$ and $\delta>1$ or $0<\delta<1$ and $\beta>1$. Here also

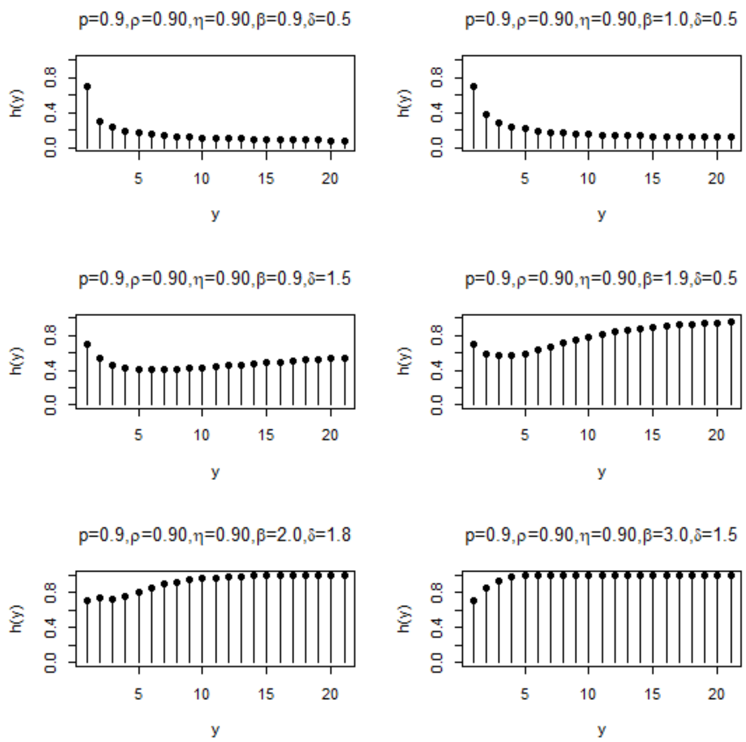

Figure 5.6: Shape of the hrf of DAWG$(p, \rho, \eta, \beta, \delta)$ distribution.

$\lim_{y\to\infty} h(y) = 1$. But, $h(y)$ is initially decreases from $h(0)$ to the minimum point $h(m)$ and then increases to 1. The minimum point m can be numerically identify by solving the conditions, $h(m) - h(m-1) \leq 0$ and $h(m+1) - h(m) \geq 0$.

Case (iv). When $\beta < 1$ and $\delta = 1$. In this case $\lim_{y\to\infty} h(y) = 1 - \eta$. Also, $h(0) > h(1) > h(2) > ... > 1 - \eta$. That is, $h(y)$ is a decreasing function.

Case (v). When $\beta < 1$ and $\delta < 1$ (provided $\beta > \delta$ or $\beta < \delta$). Here, $\lim_{y\to\infty} h(y) = 0$. Also seen that $h(0) > h(1) > h(2) > ... > 0$. That is, in this case also, $h(y)$ is decreasing.

The reverse hazard rate function is given by

$$h^*(y) = P(Y = y/Y \leq y) = \frac{(1-p)(\rho^{y^\beta}\eta^\delta - \rho^{(y+1)^\beta}\eta^{(y+1)^\delta})}{(1-p\rho^{y^\beta}\eta^\delta)(1-\rho^{(y+1)^\beta}\eta^{(y+1)^\delta})}. \tag{5.3.16}$$

The second rate of failure is given by

$$h^{**}(y) = \log\left\{\frac{S(y)}{S(y+1)}\right\} = \log\left\{\frac{(\frac{1}{\rho})^{(y+2)^{\beta}}(\frac{1}{\eta})^{(y+2)^{\delta}} - p}{(\frac{1}{\rho})^{(y+1)^{\beta}}(\frac{1}{\eta})^{(y+1)^{\delta}} - p}\right\}. \quad (5.3.17)$$

The accumulated hazard function, $H(y)$ is given by

$$H(y) = \sum_{t=0}^{y} h(t) = \sum_{t=0}^{y} \frac{1 - \rho^{(t+1)^{\beta} - t^{\beta}}\eta^{(t+1)^{\delta} - t^{\delta}}}{1 - p\rho^{(t+1)^{\beta}}\eta^{(t+1)^{\delta}}}. \quad (5.3.18)$$

The mean residual life function (MRL) is given by

$$\begin{aligned} L(y) &= E[(Y-y)|Y \geq y] = \frac{\sum_{j>y} S(j)}{S(y)} = \sum_{j\geq y}\prod_{i=y}^{j}(1-h(i)) \\ &= \sum_{j\geq y}\prod_{i=y}^{j}\frac{\rho^{(i+1)^{\beta}}\eta^{(i+1)^{\delta}}(1-p\rho^{i^{\beta}}\eta^{i^{\delta}})}{\rho^{i^{\beta}}\eta^{i^{\delta}}(1-p\rho^{(i+1)^{\beta}}\eta^{(i+1)^{\delta}})}; \quad y = 0, 1, \ldots . \end{aligned} \quad (5.3.19)$$

5.3.2 Quantile function

Since the cdf of the DAWG distribution is not invertible, we use the method discussed in Lemonte et al. (2014) to obtain the quantile function. We take

$$F(y) = \frac{1 - \rho^{(y+1)^{\beta}}\eta^{(y+1)^{\delta}}}{1 - p\rho^{(y+1)^{\beta}}\eta^{(y+1)^{\delta}}} = u,$$

where $u \epsilon (0,1)$. This implies

$$(y+1)^{\beta} ln(\rho) + (y+1)^{\delta} ln(\eta) = ln\left(\frac{1-u}{1-up}\right). \quad (5.3.20)$$

We obtain the nonlinear equation, $at^{\beta} + ct^{\delta} = x$, where $a = ln(\rho), c = ln(\eta), x = ln(\frac{1-u}{1-up})$ and $t = y+1$. We can expand t^{β} in Taylor series as $t^{\beta} = \sum_{k=0}^{\infty}(\beta)_k(t-1)^k/k! = \sum_{k=0}^{\infty} f_j t^j$, where $f_j = \sum_{k=j}^{\infty}(-1)^{k-j}\binom{k}{j}(\beta)^{[k]}/k!$, $(\beta)_k = \beta(\beta-1)\ \ldots\ (\beta -$

$k+1)$ is the falling factorial and $(\beta)^{[k]} = \beta(\beta+1)\ ...\ (\beta+k-1)$ is the ascending factorial. Analogously, we can expand t^{δ} as $t^{\delta} = \sum_{k=0}^{\infty} g_j t^j$, where $g_j = \sum_{k=j}^{\infty} (-1)^{k-j} \binom{k}{j} (\delta)^{[k]}/k!$. Now

$$x = H(t) = \sum_{j=0}^{\infty} (af_j + cg_j)t^j = \sum_{j=0}^{\infty} h_j t^j, \tag{5.3.21}$$

where $h_j = af_j + cg_j$. To obtain an expansion for the quantile function of the DAWG distribution we use the Lagrange theorem. Now suppose that the power series expansion holds $x = H(t) = h_0 + \sum_{j=1}^{\infty} h_j t^j$, $h_1 = H'(t)|_{t=0} \neq 0$, where $H(t)$ is analytic at a zero point. Then, the inverse power series $t = H^{-1}(x)$ exists, it is a single-valued in the neighborhood of the point $x = 0$ and it is given by $t = H^{-1}(x) = \sum_{j=1}^{\infty} \upsilon_j x^j$, where the coefficients υ_j can be given by

$$\upsilon_j = \frac{1}{j!}\left(\frac{d^{j-1}}{dt^{j-1}}\left[\phi(t)\right]^j\right)\Bigg|_{t=0}, \quad \phi(t) = \frac{t}{H(t) - h_0}.$$

Hence, the quantile function can be expressed as

$$Q(u) = \sum_{j=1}^{\infty} \upsilon_j \left(ln\left(\frac{1-u}{1-up}\right)\right)^j - 1. \tag{5.3.22}$$

5.3.3 Moments

The r^{th} moment about origin is given by

$$\mu_r^{'} = E(Y^r) = \sum_{y=0}^{\infty} y^r \frac{(1-p)(\rho^{y^\beta}\eta^{y^\delta} - \rho^{(y+1)^\beta}\eta^{(y+1)^\delta})}{(1-p\rho^{y^\beta}\eta^{y^\delta})(1-p\rho^{(y+1)^\beta}\eta^{(y+1)^\delta})}. \tag{5.3.23}$$

Since the function is not in a tractable form, for given values of p, ρ, η, β and δ, the moments can be numerically computed using **R** programming. The following

Table 5.5 shows the moments, skewness and kurtosis for DAWG distribution for given values of parameters.

Table 5.5: The Moments, skewness and kurtosis for $p = 0.9,\ \rho = 0.8,\ \eta = 0.9$ and various choices of β and δ.

Parameter	Raw moments	Central moments	Skewness	Kurtosis
$\beta = 1.5$ $\delta = 2$	$\mu_1^{'} = 0.27$ $\mu_2^{'} = 0.45$ $\mu_3^{'} = 0.98$ $\mu_4^{'} = 2.70$	$\mu_2 = 0.38$ $\mu_3 = 0.65$ $\mu_4 = 1.82$	2.79	12.59
$\beta = 1.5$ $\delta = 1$	$\mu_1^{'} = 0.32$ $\mu_2^{'} = 0.73$ $\mu_3^{'} = 2.37$ $\mu_4^{'} = 10.24$	$\mu_2 = 0.63$ $\mu_3 = 1.73$ $\mu_4 = 7.59$	3.50	19.44
$\beta = 0.5$ $\delta = 1.5$	$\mu_1^{'} = 0.46$ $\mu_2^{'} = 1.76$ $\mu_3^{'} = 10.13$ $\mu_4^{'} = 76.97$	$\mu_2 = 1.55$ $\mu_3 = 7.88$ $\mu_4 = 60.28$	4.09	25.18
$\beta = 0.5$ $\delta = 1$	$\mu_1^{'} = 0.82$ $\mu_2^{'} = 8.69$ $\mu_3^{'} = 175.90$ $\mu_4^{'} = 5284.45$	$\mu_2 = 8.01$ $\mu_3 = 155.61$ $\mu_4 = 4740.30$	6.86	73.86
$\beta = 0.2$ $\delta = 0.9$	$\mu_1^{'} = 1.38$ $\mu_2^{'} = 27.89$ $\mu_3^{'} = 1092.96$ $\mu_4^{'} = 63894.82$	$\mu_2 = 26.00$ $\mu_3 = 983.02$ $\mu_4 = 58185.39$	7.41	86.06

5.3.4 Stress-strength parameter

Let, $Y \sim ADWG(\theta_1)$ and $Z \sim ADWG(\theta_2)$, where, $\theta_1 = (p_1, \rho_1, \eta_1, \beta_1, \delta_1)^T$ and $\theta_2 = (p_2, \rho_2, \eta_2, \beta_2, \delta_2)^T$. Then, from Eqns.(5.3.10) and (5.3.13), the stress-strength parameter is

$$R = \sum_{y=0}^{\infty} \frac{(1-p_1)(\rho_1^{y^{\beta_1}}\eta_1^{y^{\delta_1}} - \rho_1^{(y+1)^{\beta_1}}\eta_1^{(y+1)^{\delta_1}})(1-\rho_2^{(y+1)^{\beta_2}}\eta_2^{(y+1)^{\delta_2}})}{(1-p_1\rho_1^{y^{\beta_1}}\eta_1^{y^{\delta_1}})(1-p_1\rho_1^{(y+1)^{\beta_1}}\eta_1^{(y+1)^{\delta_1}})(1-p_2\rho_2^{(y+1)^{\beta_2}}\eta_2^{(y+1)^{\delta_2}})}. \qquad (5.3.24)$$

Assume that, $(y_1, y_2, ..., y_n)$ and $(z_1, z_2, ..., z_m)$ are independent observations drawn from ADWG(θ_1) and ADWG(θ_2), respectively. The total likelihood function is given by, $L_R(\theta^*) = L_n(\theta_1)\ L_m(\theta_2)$, where, $\theta^* = (\theta_1,\ \theta_2)$. The score vector is given by

$$U_R(\theta^*) = \left(\frac{\partial L_R}{\partial p_1}, \frac{\partial L_R}{\partial \rho_1}, \frac{\partial L_R}{\partial \eta_1}, \frac{\partial L_R}{\partial \beta_1}, \frac{\partial L_R}{\partial \delta_1}, \frac{\partial L_R}{\partial p_2}, \frac{\partial L_R}{\partial \rho_2}, \frac{\partial L_R}{\partial \eta_2}, \frac{\partial L_R}{\partial \beta_2}, \frac{\partial L_R}{\partial \delta_2}\right). \quad (5.3.25)$$

The MLE, $\hat{\theta^*}$ may be obtained from the solution of the nonlinear equation, $U_R(\hat{\theta^*}) = 0$. Applying $\hat{\theta^*}$, in the Eqn.(5.3.24), the stress-strength parameter R can be obtained. The stress strength reliability value for different choices of $p_1, \rho_1, \eta_1, \beta_1, \delta_1$ and $p_2, \rho_2, \eta_2, \beta_2, \delta_2$ are computed and presented in Table 5.6. We can see that the values of R are decreasing when β_1 and δ_1 increase and increasing when β_2 and δ_2 increases.

5.3.5 Maximum likelihood estimation of parameters of DAWG Distribution

Consider a random sample $(y_1, y_2, ..., y_n)$ of size n, from the DAWG$(p, \rho, \eta, \beta, \delta)$. Then, the likelihood function is given by

$$L = \frac{(1-p)^n \prod_{i=1}^n (\rho^{y_i^\beta} \eta^{y_i^\delta} - \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta})}{\prod_{i=1}^n (1 - p\rho^{y_i^\beta} \eta^{y_i^\delta}) \prod_{i=1}^n (1 - p\rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta})}. \quad (5.3.26)$$

The log-likelihood function is

$$\begin{aligned}\log(L) &= n\log(1-p) + \sum_{i=1}^n \log(\rho^{y_i^\beta} \eta^{y_i^\delta} - \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}) \\ &\quad - \sum_{i=1}^n \log(1 - p\rho^{y_i^\beta} \eta^{y_i^\delta}) - \sum_{i=1}^n \log(1 - p\rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}). \quad (5.3.27)\end{aligned}$$

Table 5.6: Values of stress-strength parameter(R) for various choices of parameter values.

		$p_1 = 0.8, p_2 = 0.8$		
		$\rho_1 = 0.5, \rho_2 = 0.5$	$\eta_1 = 0.5, \eta_2 = 0.5$	
$(\beta_1, \delta_1) \rightarrow$ $(\beta_2, \delta_2) \downarrow$	(0.5,1)	(1,1.5)	(1.5,2)	(2,2.5)
$(0.5, 1)$	0.9404	0.9402	0.9402	0.9401
$(1, 1.5)$	0.9411	0.9410	0.9409	0.9409
$(1.5, 2)$	0.9413	0.9413	0.9412	0.9412
(2,2.5)	0.9413	0.9413	0.9413	0.9413
		$\rho_1 = 0.2, \rho_2 = 0.6$	$\eta_1 = 0.2, \eta_2 = 0.6$	
$(\beta_1, \delta_1) \rightarrow$ $(\beta_2, \delta_2) \downarrow$	(0.5,1)	(1,1.5)	(1.5,2)	(2,2.5)
$(0.5, 1)$	0.8994	0.8993	0.8993	0.8993
$(1, 1.5)$	0.8996	0.8996	0.8995	0.8995
$(1.5, 2)$	0.8997	0.8997	0.8997	0.8996
(2,2.5)	0.8977	0.8997	0.8997	0.8997
		$p_1 = 0.5, p_2 = 0.8$		
		$\rho_1 = 0.5, \rho_2 = 0.5$	$\eta_1 = 0.5, \eta_2 = 0.5$	
$(\beta_1, \delta_1) \rightarrow$ $(\beta_2, \delta_2) \downarrow$	(0.5,1)	(1,1.5)	(1.5,2)	(2,2.5)
$(0.5, 1)$	0.9443	0.9438	0.9436	0.9435
$(1, 1.5)$	0.9457	0.9455	0.9454	0.9453
$(1.5, 2)$	0.9463	0.9462	0.9462	0.9461
(2,2.5)	0.9464	0.9464	0.9464	0.9463
		$\rho_1 = 0.2, \rho_2 = 0.6$	$\eta_1 = 0.2, \eta_2 = 0.6$	
$(\beta_1, \delta_1) \rightarrow$ $(\beta_2, \delta_2) \downarrow$	(0.5,1)	(1,1.5)	(1.5,2)	(2,2.5)
$(0.5, 1)$	0.9002	0.9001	0.9001	0.9001
$(1, 1.5)$	0.9006	0.9006	0.9005	0.9005
$(1.5, 2)$	0.9008	0.9008	0.9008	0.9008
(2,2.5)	0.9009	0.9009	0.9009	0.9009
		$p_1 = 0.8, p_2 = 0.5$		
		$\rho_1 = 0.5, \rho_2 = 0.5$	$\eta_1 = 0.5, \eta_2 = 0.5$	
$(\beta_1, \delta_1) \rightarrow$ $(\beta_2, \delta_2) \downarrow$	(0.5,1)	(1,1.5)	(1.5,2)	(2,2.5)
$(0.5, 1)$	0.8637	0.8632	0.8630	0.8630
$(1, 1.5)$	0.8653	0.8651	0.8650	0.8649
$(1.5, 2)$	0.8659	0.8658	0.8658	0.8657
(2,2.5)	0.8660	0.8660	0.8660	0.8660
		$\rho_1 = 0.2, \rho_2 = 0.6$	$\eta_1 = 0.2, \eta_2 = 0.6$	
$(\beta_1, \delta_1) \rightarrow$ $(\beta_2, \delta_2) \downarrow$	(0.5,1)	(1,1.5)	(1.5,2)	(2,2.5)
$(0.5, 1)$	0.7816	0.7815	0.7815	0.7815
$(1, 1.5)$	0.7820	0.7819	0.7819	0.7819
$(1.5, 2)$	0.7822	0.7821	0.7821	0.7821
(2,2.5)	0.7823	0.7823	0.7823	0.7823

The likelihood equations are the following

$$\frac{\partial \log(L)}{\partial p} = \frac{-n}{1-p} + \sum_{i=1}^{n} \frac{\rho^{y_i^\beta} \eta^{y_i^\delta}}{1 - p\rho^{y_i^\beta} \eta^{y_i^\delta}} + \sum_{i=1}^{n} \frac{\rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}}{1 - p\rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}} = 0, \quad (5.3.28)$$

$$\begin{aligned}\frac{\partial \log(L)}{\partial \rho} &= \sum_{i=1}^{n} \frac{y_i^\beta \rho^{y_i^\beta - 1} \eta^{y_i^\delta} - (y_i+1)^\beta \rho^{(y_i+1)^\beta - 1} \eta^{(y_i+1)^\delta}}{\rho^{y_i^\beta} \eta^{y_i^\delta} - \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}} \\ &+ p \sum_{i=1}^{n} \frac{y_i^\beta \rho^{y_i^\beta - 1} \eta^{y_i^\delta}}{1 - p\rho^{y_i^\beta} \eta^{y_i^\delta}} + p \sum_{i=1}^{n} \frac{(y_i+1)^\beta \rho^{(y_i+1)^\beta - 1} \eta^{(y_i+1)^\delta}}{1 - p\rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}} = 0, \quad (5.3.29)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log(L)}{\partial \eta} &= \sum_{i=1}^{n} \frac{y_i^\delta \rho^{y_i^\beta} \eta^{y_i^\delta - 1} - (y_i+1)^\delta \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta - 1}}{\rho^{y_i^\beta} \eta^{y_i^\delta} - \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}} \\ &+ p \sum_{i=1}^{n} \frac{y_i^\delta \rho^{y_i^\beta} \eta^{y_i^\delta - 1}}{1 - p\rho^{y_i^\beta} \eta^{y_i^\delta}} + p \sum_{i=1}^{n} \frac{(y_i+1)^\delta \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta - 1}}{1 - p\rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}} = 0, \quad (5.3.30)\end{aligned}$$

$$\begin{aligned}\frac{\partial \log(L)}{\partial \beta} &= \log(\rho) \sum_{i=1}^{n} \frac{y_i^\beta \rho^{y_i^\beta} \eta^{y_i^\delta} \log(y_i) - (y_i+1)^\beta \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta} \log(y_i+1)}{\rho^{y_i^\beta} \eta^{y_i^\delta} - \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}} \\ &+ p \log(\rho) \sum_{i=1}^{n} \frac{y_i^\beta \rho^{y_i^\beta} \eta^{y_i^\delta} \log(y_i)}{1 - p\rho^{y_i^\beta} \eta^{y_i^\delta}} \\ &+ p \log(\rho) \sum_{i=1}^{n} \frac{(y_i+1)^\beta \rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta} \log(y_i+1)}{1 - p\rho^{(y_i+1)^\beta} \eta^{(y_i+1)^\delta}} = 0. \quad (5.3.31)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \log(L)}{\partial \delta} &= \log(\eta)\sum_{i=1}^{n}\frac{y_i^{\delta}\rho^{y_i^{\beta}}\eta^{y_i^{\delta}}\log(y_i)-(y_i+1)^{\delta}\rho^{(y_i+1)^{\beta}}\eta^{(y_i+1)^{\delta}}\log(y_i+1)}{\rho^{y_i^{\beta}}\eta^{y_i^{\delta}}-\rho^{(y_i+1)^{\beta}}\eta^{(y_i+1)^{\delta}}}\\ &+ p\log(\eta)\sum_{i=1}^{n}\frac{y_i^{\beta}\rho^{y_i^{\beta}}\eta^{y_i^{\delta}}\log(y_i)}{1-p\rho^{y_i^{\beta}}\eta^{y_i^{\delta}}}\\ &+ p\log(\eta)\sum_{i=1}^{n}\frac{(y_i+1)^{\beta}\rho^{(y_i+1)^{\beta}}\eta^{(y_i+1)^{\delta}}\log(y_i+1)}{1-p\rho^{(y_i+1)^{\beta}}\eta^{(y_i+1)^{\delta}}}=0. \qquad (5.3.32)\end{aligned}$$

These equations do not have explicit solutions and they have to be obtained numerically by using statistical softwares like *nlm* package in **R** programming. Let the estimators be, $\hat{\theta}=(\hat{p},\hat{\rho},\hat{\eta},\hat{\beta},\hat{\delta})^T$. The Fisher's information matrix is given by

$$I_y(\theta)=\begin{bmatrix} -E(\frac{\partial^2\log(L)}{\partial p^2}) & -E(\frac{\partial^2\log(L)}{\partial p\partial\rho}) & -E(\frac{\partial^2\log(L)}{\partial p\partial\eta}) & -E(\frac{\partial^2\log(L)}{\partial p\partial\beta}) & -E(\frac{\partial^2\log(L)}{\partial p\partial\delta})\\ -E(\frac{\partial^2\log(L)}{\partial\rho\partial p}) & -E(\frac{\partial^2\log(L)}{\partial\rho^2}) & -E(\frac{\partial^2\log(L)}{\partial\rho\partial\eta}) & -E(\frac{\partial^2\log(L)}{\partial\rho\partial\beta}) & -E(\frac{\partial^2\log(L)}{\partial\rho\partial\eta})\\ -E(\frac{\partial^2\log(L)}{\partial\eta\partial p}) & -E(\frac{\partial^2\log(L)}{\partial\eta\partial\rho}) & -E(\frac{\partial^2\log(L)}{\partial\eta^2}) & -E(\frac{\partial^2\log(L)}{\partial\eta\partial\beta}) & -E(\frac{\partial^2\log(L)}{\partial\eta\partial\delta})\\ -E(\frac{\partial^2\log(L)}{\partial\beta\partial p}) & -E(\frac{\partial^2\log(L)}{\partial\beta\partial\rho}) & -E(\frac{\partial^2\log(L)}{\partial\beta\partial\eta}) & -E(\frac{\partial^2\log(L)}{\partial\beta^2}) & -E(\frac{\partial^2\log(L)}{\partial\beta\partial\delta})\\ -E(\frac{\partial^2\log(L)}{\partial\delta\partial p}) & -E(\frac{\partial^2\log(L)}{\partial\delta\partial\rho}) & -E(\frac{\partial^2\log(L)}{\partial\delta\partial\eta}) & -E(\frac{\partial^2\log(L)}{\partial\delta\partial\beta}) & -E(\frac{\partial^2\log(L)}{\partial\delta^2}) \end{bmatrix}.$$

Here, the DAWG distribution satisfies the regularity conditions which are full filled for the parameters in the interior of the parameter space, but not on the boundary. Hence, the vector $\hat{\theta}$ is consistent and asymptotically normal. That is, $\sqrt{I_y(\theta)}[\hat{\theta}-\theta]$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix. The Fisher's information matrix can be computed using the approximation

$$I_y(\hat{\theta}) \approx \begin{bmatrix} -\frac{\partial^2 \log(L)}{\partial p^2}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial p \partial \rho}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial p \partial \eta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial p \partial \beta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial p \partial \delta}\big|_{\hat{\theta}} \\ -\frac{\partial^2 \log(L)}{\partial \rho \partial p}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \rho^2}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \rho \partial \eta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \rho \partial \beta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \rho \partial \delta}\big|_{\hat{\theta}} \\ -\frac{\partial^2 \log(L)}{\partial \eta \partial p}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \eta \partial \rho}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \eta^2}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \eta \partial \beta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \eta \partial \delta}\big|_{\hat{\theta}} \\ -\frac{\partial^2 \log(L)}{\partial \beta \partial p}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \beta \partial \rho}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \beta \partial \eta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \beta^2}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \beta \partial \delta}\big|_{\hat{\theta}} \\ -\frac{\partial^2 \log(L)}{\partial \delta \partial p}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \delta \partial \rho}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \delta \partial \eta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \delta \partial \beta}\big|_{\hat{\theta}} & -\frac{\partial^2 \log(L)}{\partial \delta^2}\big|_{\hat{\theta}} \end{bmatrix},$$

where $\hat{\theta}$ is the MLE of $\theta = (p,\ \rho,\ \eta,\ \beta,\ \delta)^T$.

We compute the maximized unrestricted and restricted log-likelihood ratio (LR) test statistic for testing on some DAWG sub models. Here, $H_0 : \eta = 1$ versus $H_1 : \eta \neq 1$ is equivalent to compare the DAWG distribution and the DWG distribution. The LR test statistic reduces to $\omega = 2(l(\hat{p}, \hat{\rho}, \hat{\eta}, \hat{\beta}, \hat{\delta}) - l(\hat{p}', \hat{\rho}', 1, \hat{\beta}', \hat{\delta}'))$, where $(\hat{p}, \hat{\rho}, \hat{\eta}, \hat{\beta}, \hat{\delta})$ and $(\hat{p}', \hat{\rho}', \hat{\beta}', \hat{\delta}')$ are the MLEs under H_1 and H_0, respectively. The test statistic ω is asymptotically (as $n \to \infty$) distributed as $\chi^2_{(k)}$, where k is the length of the parameter vector of interest. The LR test rejects H_0 if $\omega > \chi^2_{(k,\alpha)}$ where $\chi^2_{(k,\alpha)}$ denotes the upper $100(1-\alpha)\%$ quantile of the $\chi^2_{(k)}$ distribution.

5.3.6 Simulation study

This section explains the performance of the MLEs of the model parameters of DAWG distribution using Monte Carlo simulation for various sample sizes and for selected parameter values. The algorithm for the simulation study are given below.

Step 1. Input the value of replications (N);

Step 2. Specify the sample size n and the values of the parameters p, ρ, η, β and δ;

Step 3. Generate $u_i \sim Uniform(0,1),\ i = 1, 2, ..., n.$;

Step 4. Obtain the random observations from the DAWG distribution by solving for the real roots of the Eqn.(5.3.22) and take the floor value;

Step 5. Compute the MLEs of the five parameters;

Step 6. Repeat steps 3 to 5, N times;

Step 7. Compute the average bias, mean square error (MSE) and coverage probability (CP) for each parameter.

Here the expected value of the estimator is $E(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i$, average bias $= \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)$, MSE$= \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2$ and the coverage probability = Probability of $\theta_i \in \left(\hat{\theta}_i \pm 1.96\sqrt{-\frac{\partial^2 \log(L)}{\partial\theta_i^2}}\right)$. We take the parameter values as $p = 0.8, \rho = 0.5, \eta = 0.5, \beta = 0.5$ and $\delta = 1.5$ arbitrarily and generated random samples of size n=20, 40, 60, 80 and 100 respectively. The MLEs of p, ρ, η, β and δ are determined by maximizing the log-likelihood function in the equation (5.3.22) using the *nlm* package of **R** software, based on each generated samples. This simulation is repeated 500 times and the average estimates of bias, MSE and coverage probability are computed and presented in Table 5.7. From Table 5.7 it can be seen that, as sample size increases the estimates of bias and MSE decreases. Also note that the CP values are quite close to the 95% nominal level.

Table 5.7: Values of the average bias, MSE and CP for given parameter values.

Sample size	Actual value	Estimates	Average bias	MSE	CP
20	$p = 0.8$	0.921	0.115	0.074	0.873
	$\rho = 0.5$	0.346	-0.164	0.086	0.926
	$\eta = 0.5$	0.723	0.213	0.017	0.932
	$\beta = 0.5$	0.661	0.165	0.038	0.896
	$\delta = 1.5$	1.833	0.301	0.099	0.882
40	$p = 0.8$	0.893	0.099	0.032	0.901
	$\rho = 0.5$	0.453	-0.057	0.034	0.930
	$\eta = 0.5$	0.632	0.101	0.013	0.938
	$\beta = 0.5$	0.643	0.110	0.015	0.905
	$\delta = 1.5$	1.608	0.096	0.082	0.899
60	$p = 0.8$	0.866	0.071	0.016	0.926
	$\rho = 0.5$	0.486	-0.013	0.018	0.936
	$\eta = 0.5$	0.610	0.102	0.008	0.943
	$\beta = 0.5$	0.612	0.110	0.012	0.912
	$\delta = 1.5$	1.598	0.096	0.073	0.917
80	$p = 0.8$	0.841	0.046	0.012	0.931
	$\rho = 0.5$	0.489	-0.009	0.015	0.940
	$\eta = 0.5$	0.596	0.092	0.006	0.947
	$\beta = 0.5$	0.593	0.089	0.009	0.923
	$\delta = 1.5$	1.573	0.069	0.062	0.927
100	$p = 0.8$	0.833	0.028	0.009	0.938
	$\rho = 0.5$	0.491	-0.003	0.007	0.942
	$\eta = 0.5$	0.552	0.057	0.005	0.949
	$\beta = 0.5$	0.587	0.083	0.006	0.929
	$\delta = 1.5$	1.554	0.052	0.011	0.934

5.3.7 Data applications of the DAWG distribution

In this section, to show how the DAWG$(p, \rho, \eta, \beta, \delta)$ distribution works in practice, we use the data set representing remission times (in months) of 128 bladder cancer patients (Lee and Wang (2003)). The data are given in Section 5.2.7. (page 144). Since the data set is continuous, here first we discretize the data by considering the floor value (y). The parameters are estimated by using the method of maximum likelihood. We compare the fit of the DAWG distribution with the geometric (G) distribution, the discrete Weibull (DW) distribution, the discrete Logistic (DLOG) distribution, the exponentiated discrete Weibull (EDW) distribution and the discrete Weibull geometric (DWG) distribution.

Table 5.8: The parameter estimates and goodness of fit for various models fitted for the first data set.

Model	ML estimates	-log L	AIC	CAIC	BIC	K-S	p value
G	$\hat{p} = 0.8991$	414.836	831.672	831.704	831.779	0.1000	0.1549
DW	$\hat{q} = 0.9114$ $\hat{\beta} = 1.0511$	414.556	833.112	837.304	833.326	0.1131	0.0758
DLOG	$\hat{p} = 0.8000$ $\hat{\mu} = 7.6149$	456.825	917.650	917.746	917.864	0.1860	0.0003
EDW	$\hat{p} = 0.4689$ $\hat{\alpha} = 0.5397$ $\hat{\gamma} = 4.9697$	409.766	825.532	825.726	825.854	0.1237	0.0399
DWG	$\hat{p} = 0.9529$ $\hat{\rho} = 0.9982$ $\hat{\alpha} = 1.7025$	409.277	824.554	824.748	824.876	0.0905	0.2458
DAWG	$\hat{p} = 0.9589$ $\hat{\rho} = 0.9989$ $\hat{\eta} = 0.9995$ $\hat{\beta} = 1.7018$ $\hat{\delta} = 1.7016$	405.230	820.460	820.9518	820.996	0.0882	0.2727

The values of -log L, K-S, AIC, CAIC and BIC are calculated for the six distributions in order to verify which distribution fits better to these data. The values in Table 5.8, indicates that the DAWG distribution leads to a better fit compared to the other five models. The following Figure 5.7, shows the structure of the cdf's of the six models with the empirical distribution of the given data. Here the dotted line indicates the empirical cdf of the data.

The LR test statistic to test the hypothesis $H_0 : \eta = 1$ versus $H_1 : \eta \neq 1$ is $\omega = 8.094 > 5.991$ with p value 0.0175. So we reject the null hypothesis.

5.4 Summary

In this Chapter, we introduced some discrete versions of Weibull distribution by discretizing the Weibull geometric and additive Weibull geometric distributions. This

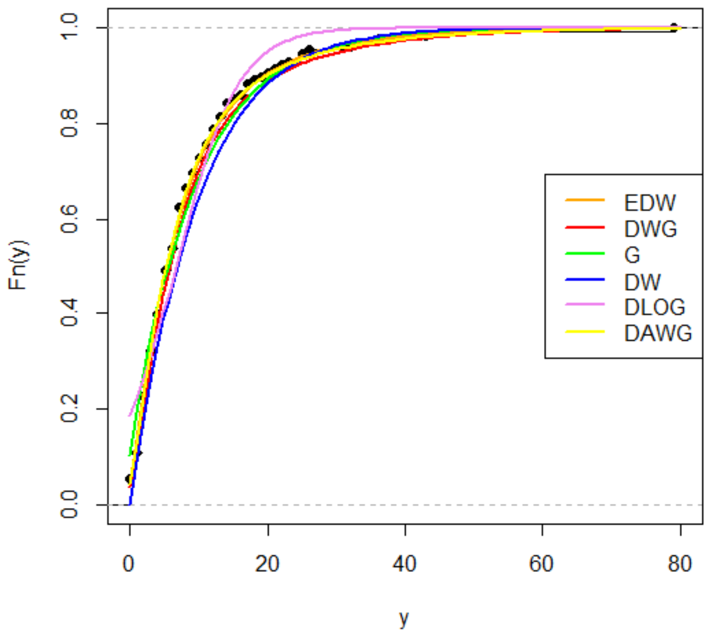

Figure 5.7: Fitted cdfs of the data with empirical distribution.

new discrete distributions are found to be better for modelling real life data, when the data exhibits over or under dispersion. The discrete Weibull, discrete Rayleigh and the geometric distributions are the sub models of the DWG distribution. The hrf of the DWG distribution showed increasing, decreasing and bathtub shapes. The parameters of the DWG distribution are estimated by the method of maximum likelihood. A simulation study is also carried out to check the performance of the method. Two data sets are used to check the flexibility of the DWG distribution for lifetime data modelling.

The DAWG distribution is a more generalized family than the DWG distribution. The sub models of this distribution are DWG, discrete exponential geometric, discrete modified Weibull, discrete Weibull, discrete Rayleigh geometric, discrete Rayleigh and the geometric distribution. Structural properties of the pmf and hrf of the DAWG distribution are studied in detail. The hrf of DAWG distribution is decreasing, increasing and bathtub shaped. The estimates of the five parameters of the DAWG distribution are obtained by using MLE method. A simulation study is also conducted to check the performances of the method. In order to check the flexibility of the data modelling, we applied this model for fitting the remission times of 128 bladder cancer patients data.

CHAPTER

SIX

DISCRETE COMPLEMENTARY WEIBULL GEOMETRIC DISTRIBUTION: PROPERTIES AND APPLICATIONS

6.1 Introduction

In competing risk analysis, the lifetime associated with a specific risk is not observable, but the minimum lifetime of all the risks are available. Whereas in complementary risk analysis, we observe only the maximum lifetime among all the risks. In most of the reliability analysis, the real cause of failure of the system can be hidden. To address such problems statistically, we need more generalized family of distributions.

Let N be a discrete random variable denoting the number of complementary risks related to the occurrence of an event of interest and follows geometric distribution with pmf,

$$P(N = n) = p(1 - p)^{n-1}, \tag{6.1.1}$$

where $0 < p < 1$ and $n = 1, 2, ...$. Also, let X_i denotes the time-to-event due to the i^{th} complementary risk, which are independent of N, $i = 1, 2, ...$. Then for given $N = n$, the random variable $X_i, i = 1, 2, ...n$ are assumed to be i.i.d. and following the Weibull distribution $W(\beta, \alpha)$ with scale parameter $\beta > 0$, shape parameter $\alpha > 0$ and the pdf

$$g(x; \beta, \alpha) = \alpha\beta^{\alpha}x^{\alpha-1}e^{-(\beta x)^{\alpha}}, x > 0. \tag{6.1.2}$$

In the latent complementary risks scenario, the number of causes N and the lifetime X_i are associated with a particular cause which are not observable (latent variables), but only the maximum lifetime Z among all the causes is usually observed. So the complementary lifetime is defined as

$$Z = \max(X_1, X_2, ..., X_N). \tag{6.1.3}$$

From Tojeiro et al. (2014), we can see that, if the random variable Z is defined as in Eqn.(6.1.3), then considering the Eqn.(6.1.1) and the Eqn.(6.1.2), Z is distributed according to complementary Weibull geometric (CWG) distribution with pdf

$$f(z; p, \beta, \alpha) = \frac{p\alpha\beta^{\alpha}z^{\alpha-1}e^{-(\beta z)^{\alpha}}}{[p + (1 - p)e^{-(\beta z)^{\alpha}}]^2}, \ p > 0, \beta > 0, \alpha > 0, z > 0. \tag{6.1.4}$$

The cdf, survival function and hrf of CWG distribution are respectively given by

$$F(z) = \frac{p(1 - e^{-(\beta z)^{\alpha}})}{p + (1 - p)e^{-(\beta z)^{\alpha}}}, \tag{6.1.5}$$

$$S(z) = \frac{e^{-(\beta z)^{\alpha}}}{p + (1 - p)e^{-(\beta z)^{\alpha}}}, \tag{6.1.6}$$

and

$$h(z) = \frac{p\alpha\beta^{\alpha} z^{\alpha-1}}{p + (1-p)e^{-(\beta z)^{\alpha}}}. \tag{6.1.7}$$

The properties of the CWG distribution are studied in Tojeiro et al. (2014). This distribution is complementary to the Weibull geometric (WG) distribution proposed in Barreto-Souza et al. (2011).

In Section 2, we introduce the discrete complementary Weibull geometric distribution and identify its sub models. We study the various properties of this distribution such as, shapes of pmf and hrf, quantile function and probability generating function in Section 3. Maximum likelihood estimation of its parameters and their existence and uniqueness are discuss in Section 4. The performance of the MLEs of the model parameters are discuss by conducting a simulation study and the results are presents in this section. Two real-life data applications of this distribution are illustrates in Section 5.

6.2 The Discrete Complementary Weibull Geometric Distribution

Now, by applying the method of discretization given in Eqn.(1.6.1) and after a reparametrization, $\rho = e^{-\beta^{\alpha}}$, the pmf of the discrete analogues say, Y, of the CWG distribution is obtained as

$$\begin{aligned} P_Y(y;p,\rho,\alpha) = P(Y=y) &= S_Z(y) - S_Z(y+1) \\ &= \frac{p(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})}{[p+(1-p)\rho^{y^\alpha}][p+(1-p)\rho^{(y+1)^\alpha}]}, \end{aligned} \quad (6.2.1)$$

where $y = 0, 1, 2, ..., p > 0, \alpha > 0$ and $0 < \rho < 1$. Here the parameter p and ρ can be interpreted as concentration parameters, while α is a shape parameter. We call this distribution as discrete complementary Weibull geometric (DCWG) distribution and is denoted as DCWG(p, ρ, α).

In particular, when $\alpha = 1$, the pmf becomes

$$P_Y(y;p,\rho) = \frac{p(\rho^y - \rho^{(y+1)})}{[p+(1-p)\rho^y][p+(1-p)\rho^{(y+1)}]}, \quad (6.2.2)$$

which is called the discrete complementary exponential geometric distribution.

When $p \to 1$, $P_Y(y;\rho,\alpha) = \rho^{y^\alpha} - \rho^{(y+1)^\alpha}$ which is the discrete Weibull distribution of Nakagawa and Osaki (1975).

When $p \to 1$ and $\alpha \to 2$, then $P_Y(y;\rho) = \rho^{y^2} - \rho^{(y+1)^2}$ which is the discrete Rayleigh distribution of Roy (2004).

When $p \to 1$ and $\alpha \to 1$, then $P_Y(y;\rho) = \rho^y - \rho^{(y+1)}$ which is the geometric distribution with parameter ρ.

When $p \to 0+$, the DCWG tends to a distribution degenerate at zero.

6.3 Structural Properties of the DCWG Distribution

The shape of the pmf of the DCWG(p, ρ, α) distribution for various choices of parameter values are shown in Figure 6.1. It can be seen that the distribution is unimodal and highly positively skewed. For $\alpha > 0$, $p > 0$ and $0 < \rho < 1$ we have $p_Y(0) > p_Y(1) > p_Y(2) > ....$, and hence $p_Y(y)$ is strictly decreasing function. Also note that when $\alpha = 1$, $p_Y(y)$ is geometric and when $\alpha > 1$, $p_Y(y)$ is initially increasing to the maximum point and then decreasing. The recurrence relation for

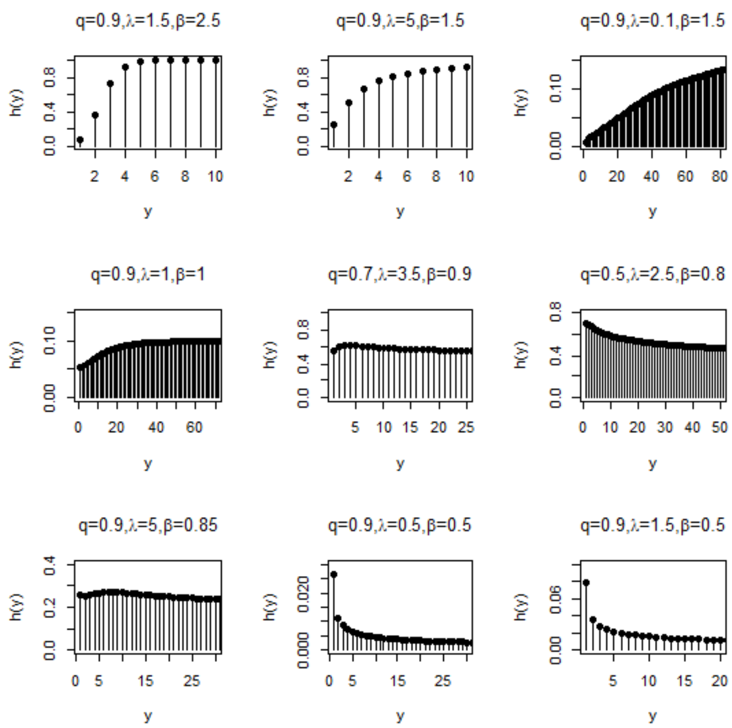

Figure 6.1: Shape of pmf for various parameter values.

probabilities of the DCWG(p, ρ, α) distribution is as follows:

$$P_Y(y+1) = \frac{(\rho^{(y+1)^\alpha} - \rho^{(y+2)^\alpha})(p + (1-p)\rho^{y^\alpha})}{(\rho^{y^\alpha} - \rho^{(y+1)^\alpha})(p + (1-p)\rho^{(y+2)^\alpha})} P_Y(y) \tag{6.3.1}$$

The distribution is

(a) log-concave if and only if $\{\frac{P_Y(y+1)}{P_Y(y)}\}_{y \geq 0}$ is decreasing,

(b) log-convex if and only if $\{\frac{P_Y(y+1)}{P_Y(y)}\}_{y \geq 0}$ is increasing and

(c) geometric if $\{\frac{P_Y(y+1)}{P_Y(y)}\}_{y \geq 0}$ is constant. The cdf of the DCWG(p, ρ, α) distribution is obtained as

$$F(y) = P(Y \leq y) = 1 - S_Z(y+1) = \frac{p(1 - \rho^{(y+1)^\alpha})}{p + (1-p)\rho^{(y+1)\alpha}}, \tag{6.3.2}$$

where $y = 0, 1, 2, ...; p > 0, 0 < \rho < 1$ and $\alpha > 0$.

Remark 6.3.1. *The cdf of the DCWG(p, ρ, α) distribution can be expressed as*

$$F(y) = \frac{1 - \rho^{(y+1)^\alpha}}{1 - (\frac{p-1}{p})\rho^{(y+1)\alpha}}. \tag{6.3.3}$$

This becomes the discrete Weibull-geometric distribution of Jayakumar and Babu (2018) if $0 < \frac{p-1}{p} < 1$ and is satisfied only if $p \in (1, \infty)$.

The survival function of the DCWG(p, ρ, α) distribution is given by

$$S(y) = P(Y > y) = 1 - P(Y \leq y) = \frac{\rho^{(y+1)^\alpha}}{p + (1-p)\rho^{(y+1)\alpha}}. \tag{6.3.4}$$

The hrf is given by

$$h(y) = \frac{Y = y}{P(Y \geq y)} = \frac{1 - \rho^{(y+1)^\alpha - y^\alpha}}{1 - \frac{p-1}{p}\rho^{(y+1)^\alpha}}, \tag{6.3.5}$$

provided $P(Y \geq y) > 0$. Here note that as $y \to 0$, $h(y) \to \frac{p(1-\rho)}{p+(1-p)\rho}$.

For $\alpha = 1$, we have $\lim_{y\to\infty} h(y) \to 1 - \rho$, for $0 < \alpha < 1$, $\lim_{y\to\infty} h(y) \to 0$ and for

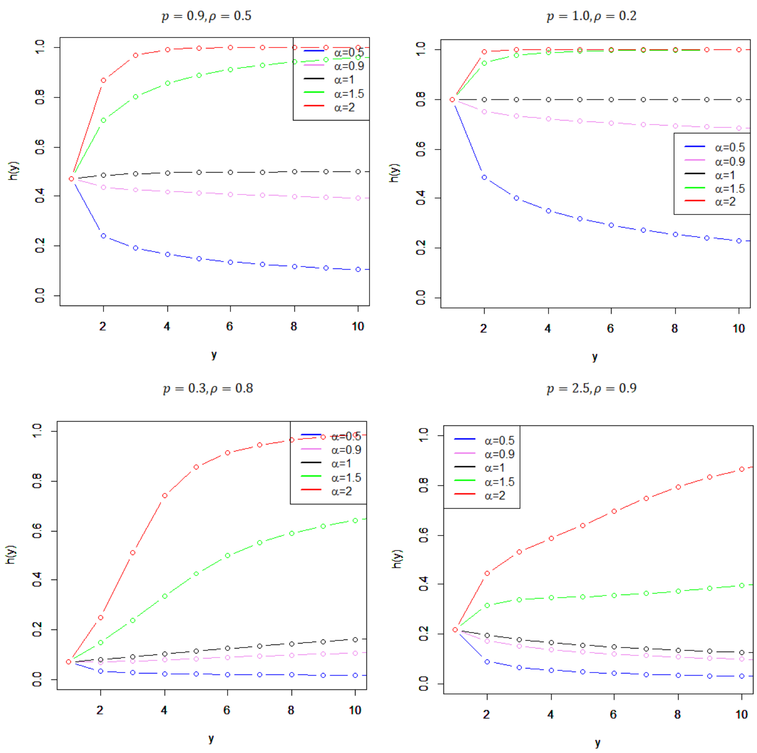

Figure 6.2: Shape of hrf for various parameter values.

$\alpha > 1$, $\lim_{y\to\infty} h(y) \to 1$. The shape of hrf for various choices of parameter values are shown in Figure 6.2. The reverse hazard rate function is given by

$$h^*(y) = \frac{P(Y=y)}{P(Y \leq y)} = \frac{\rho^{y^\alpha} - \rho^{(y+1)^\alpha}}{(1-\rho^{(y+1)^\alpha})(p + (1-p)\rho^{y^\alpha})}. \tag{6.3.6}$$

The second rate of failure is given by

$$h^{**}(y) = \log\left[\frac{S(y)}{S(y+1)}\right] = \log\left[\frac{1 - \frac{1}{1-\frac{1}{p}}\left(\frac{1}{\rho}\right)^{(y+2)^\alpha}}{1 - \frac{1}{1-\frac{1}{p}}\left(\frac{1}{\rho}\right)^{(y+1)^\alpha}}\right]. \tag{6.3.7}$$

6.3.1 Quantile function

The u^{th} quantile $\phi(u)$ of the DCWG(p, ρ, α) distribution is obtained as

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left(\frac{\ln(p(1-u)) - \ln(p + (1-p)u)}{\ln(\rho)}\right)^{\frac{1}{\alpha}} - 1 \right\rceil, \tag{6.3.8}$$

where $\lceil y_u \rceil$ denotes the smallest integer greater than or equal to y_u.

The median is

$$\phi(0.5) = \lceil y_{0.5} \rceil = \left\lceil \left(\frac{\ln(p) - \ln(p+1)}{\ln(\rho)} \right)^{\frac{1}{\alpha}} - 1 \right\rceil. \tag{6.3.9}$$

Let u follows $Uniform(0,1)$ distribution, then using the expression given in the Eqn.(6.3.8), we can generate random samples from the DCWG(p, ρ, α) distribution.

6.3.2 Probability generating function

The pgf of the DCWG(p, ρ, α) distribution is

$$P_Y(s) = 1 + (s-1) \sum_{y=1}^{\infty} \frac{s^{y-1} \rho^{(y+1)^\alpha}}{p + (1-p)\rho^{(y+1)^\alpha}}. \tag{6.3.10}$$

The expressions for mean and variance are

$$E(Y) = \sum_{y=1}^{\infty} \frac{\rho^{(y+1)^\alpha}}{p + (1-p)\rho^{(y+1)^\alpha}}, \tag{6.3.11}$$

and

$$V(Y) = \sum_{y=1}^{\infty} \frac{(2y-1)\rho^{(y+1)^\alpha}}{p + (1-p)\rho^{(y+1)^\alpha}} - \left[\sum_{y=1}^{\infty} \frac{\rho^{(y+1)^\alpha}}{p + (1-p)\rho^{(y+1)^\alpha}} \right]^2. \tag{6.3.12}$$

The mean, variance, skewness and kurtosis of the DCWG(p, ρ, α) distribution for various choices of parameter values are numerically computed and presented in Table6.1. The results shows that this distribution is suitable for modelling both over and under dispersed data.

Table 6.1: The mean, variance, skewness and kurtosis of the DCWG distribution for various parameter values.

Parameters	$\alpha \rightarrow$	0.5	1.0	1.5	2.0
$p = 0.5$	Mean	6.419	1.529	0.975	0.788
$\rho = 0.5$	Variance	149.154	2.814	0.785	0.418
	Skewness	5.144	1.589	0.760	0.323
	Kurtosis	54.088	6.696	3.435	2.695
$p = 1.0$	Mean	3.788	1.00	0.672	0.564
$\rho = 0.5$	Variance	85.699	2.00	0.638	0.379
	Skewness	6.695	2.121	1.151	0.660
	Kurtosis	89.298	9.500	4.275	2.746
$p = 1.5$	Mean	28.983	3.162	1.609	1.159
$\rho = 0.8$	Variance	5728.556	16.094	2.463	0.959
	Skewness	7.787	2.350	1.283	0.751
	Kurtosis	120.295	11.255	5.059	3.435
$p = 2.0$	Mean	0.274	0.137	0.117	0.112
$\rho = 0.2$	Variance	1.441	0.181	0.114	0.101
	Skewness	9.973	3.846	2.832	2.537
	Kurtosis	188.106	22.100	10.624	7.741

6.4 Maximum Likelihood Estimation of Parameters of the DCWG Distribution

Consider a random sample $(y_1, y_2, ..., y_n)$ of size n from the DCWG(p, ρ, α). Then the likelihood function is given by

$$L = \prod_{i=1}^{n} \frac{p[\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}]}{[p + (1-p)\rho^{y_i^\alpha}][p + (1-p)\rho^{(y_i+1)^\alpha}]} \tag{6.4.1}$$

The log-likelihood function is

$$\begin{aligned}\log L &= n\log(p) + \sum_{i=1}^{n} \log[\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}] \\ &\quad - \sum_{i=1}^{n} \log[p + (1-p)\rho^{y_i^\alpha}] - \sum_{i=1}^{n} \log[p + (1-p)\rho^{(y_i+1)^\alpha}] \end{aligned} \tag{6.4.2}$$

Hence, the likelihood equations are respectively

$$\frac{\partial \log L}{\partial p} = \frac{n}{p} - \sum_{i=1}^{n} \frac{1-\rho^{y_i^\alpha}}{p+(1-p)\rho^{y_i^\alpha}} - \sum_{i=1}^{n} \frac{1-\rho^{(y_i+1)^\alpha}}{p+(1-p)\rho^{(y_i+1)^\alpha}} = 0, \tag{6.4.3}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \rho} &= \sum_{i=1}^{n} \frac{y_i^\alpha \rho^{y_i^\alpha - 1} - (y_i+1)^\alpha \rho^{(y_i+1)^\alpha - 1}}{\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}} - (1-p)\sum_{i=1}^{n} \frac{y_i^\alpha \rho^{y_i^\alpha - 1}}{p+(1-p)\rho^{y_i^\alpha}} \\ & -(1-p)\sum_{i=1}^{n} \frac{(y_i+1)^\alpha \rho^{(y_i+1)^\alpha - 1}}{p+(1-p)\rho^{(y_i+1)^\alpha}} = 0, \end{aligned} \tag{6.4.4}$$

and

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} &= \sum_{i=1}^{n} \frac{\log(\rho)[y_i^\alpha \rho^{y_i^\alpha} \ln(y_i) - (y_i+1)^\alpha \rho^{(y_i+1)^\alpha} \log(y_i+1)]}{\rho^{y_i^\alpha} - \rho^{(y_i+1)^\alpha}} \\ & -(1-p)\log(\rho)\sum_{i=1}^{n} \frac{y_i^\alpha \rho^{y_i^\alpha} \log(y_i)}{p+(1-p)\rho^{y_i^\alpha}} \\ & -(1-p)\log(\rho)\sum_{i=1}^{n} \frac{(y_i+1)^\alpha \rho^{(y_i+1)^\alpha} \log(y_i+1)}{p+(1-p)\rho^{(y_i+1)^\alpha}} = 0. \end{aligned} \tag{6.4.5}$$

These equations do not have explicit solutions and they have to be obtained numerically by using statistical softwares like *nlm* or *optim* packages in **R** programming. Let the estimators be, $\hat{\theta} = (\hat{p}, \hat{\rho}, \hat{\alpha})^T$. The Fisher's information matrix is given by

$$I_Y(\theta) = \begin{bmatrix} -E(\frac{\partial^2 \log L}{\partial p^2}) & -E(\frac{\partial^2 \log L}{\partial p \partial \rho}) & -E(\frac{\partial^2 \log L}{\partial p \partial \alpha}) \\ -E(\frac{\partial^2 \log L}{\partial \rho \partial p}) & -E(\frac{\partial^2 \log L}{\partial \rho^2}) & -E(\frac{\partial^2 \log L}{\partial \rho \partial \alpha}) \\ -E(\frac{\partial^2 \log L}{\partial \alpha \partial p}) & -E(\frac{\partial^2 \log L}{\partial \alpha \partial \rho}) & -E(\frac{\partial^2 \log L}{\partial \alpha^2}) \end{bmatrix}. \tag{6.4.6}$$

The DCWG distribution satisfies the regularity conditions which are full filled for the parameters in the interior of the parameter space, but not on the boundary. Hence, the vector $\hat{\theta}$ is consistent and asymptotically normal. That is, $\sqrt{I_Y(\theta)}[\hat{\theta}-\theta]$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix. The Fisher's information matrix can be computed using the approximation

$$I_Y(\hat{\theta}) \approx \begin{bmatrix} -\frac{\partial^2 \log L}{\partial p^2}\big|_{\hat{\theta}} & -\frac{\partial^2 \log L}{\partial p \partial \rho}\big|_{\hat{\theta}} & -\frac{\partial^2 \log L}{\partial p \partial \alpha}\big|_{\hat{\theta}} \\ -\frac{\partial^2 \log L}{\partial \rho \partial p}\big|_{\hat{\theta}} & -\frac{\partial^2 \log L}{\partial \rho^2}\big|_{\hat{\theta}} & -\frac{\partial^2 \log L}{\partial \rho \partial \alpha}\big|_{\hat{\theta}} \\ -\frac{\partial^2 \log L}{\partial \alpha \partial p}\big|_{\hat{\theta}} & -\frac{\partial^2 \log L}{\partial \alpha \partial \rho}\big|_{\hat{\theta}} & -\frac{\partial^2 \log L}{\partial \alpha^2}\big|_{\hat{\theta}} \end{bmatrix}, \tag{6.4.7}$$

where $\hat{\theta}$ is the MLE of θ. The existence and uniqueness of MLEs of the parameters of the DCWG(p, ρ, α) distribution are established when the other parameters are known as suggested in Popović et al. (2016) and are explained in the following theorems.

Theorem 6.4.1. *From the Eqn.(6.4.3), let* $f_1(p; \rho, \alpha, y) = \frac{\partial \ln(L)}{\partial p}$*, where* ρ *and* α *are the true values of the parameters. Then there exist a unique solution for* $f_1(p; \rho, \alpha, y) = 0$*, for* $\hat{p} \in (0, \infty)$*.*

Proof. We have

$$f_1(p; \rho, \alpha, y) = \frac{n}{p} - \sum_{i=1}^{n} \frac{1-\rho^{y_i^\alpha}}{p+(1-p)\rho^{y_i^\alpha}} - \sum_{i=1}^{n} \frac{1-\rho^{(y_i+1)^\alpha}}{p+(1-p)\rho^{(y_i+1)^\alpha}}. \tag{6.4.8}$$

The limiting values of $f_1(p; \rho, \alpha, y)$ as $p \to 0$ and $p \to \infty$ are obtained as follows:

$$\lim_{p \to 0} f_1(p; \rho, \alpha, y) = \infty + 2n - \left[\sum_{i=1}^{n} \frac{1}{\rho^{y_i^\alpha}} + \sum_{i=1}^{n} \frac{1}{\rho^{(y_i+1)^\alpha}}\right] = \infty, \tag{6.4.9}$$

and

$$\lim_{p \to \infty} f_1(p; \rho, \alpha, y) = 0, \tag{6.4.10}$$

since $\lim_{p\to\infty}\sum_{i=1}^{n}\frac{1-\rho^{y_i^\alpha}}{p+(1-p)\rho^{y_i^\alpha}}=0$ and $\lim_{p\to\infty}\sum_{i=1}^{n}\frac{1-\rho^{(y_i+1)^\alpha}}{p+(1-p)\rho^{(y_i+1)^\alpha}}=0.$

Thus, there exist at least on root, say $\hat{p}\in(0,\infty)$, such that $f_1(p;\rho,\alpha,y)=0$.

Now to show the uniqueness, we have to show that $\frac{\partial f_1(p;\rho,\alpha,y)}{\partial p}<0$, that is,

$$-\frac{n}{p^2}+\sum_{i=1}^{n}\frac{(1-\rho^{y_i^\alpha})^2}{(p+(1-p)\rho^{y_i^\alpha})}+\sum_{i=1}^{n}\frac{(1-\rho^{(y_i+1)^\alpha})^2}{(p+(1-p)\rho^{(y_i+1)^\alpha})}<0, \tag{6.4.11}$$

and this is possible when

$$\sum_{i=1}^{n}\frac{(1-\rho^{y_i^\alpha})^2}{(p+(1-p)\rho^{y_i^\alpha})}+\sum_{i=1}^{n}\frac{(1-\rho^{(y_i+1)^\alpha})^2}{(p+(1-p)\rho^{(y_i+1)^\alpha})}<\frac{n}{p^2}. \tag{6.4.12}$$

This completes the proof. □

Theorem 6.4.2. *From the Eqn.(6.4.4), let* $f_2(\rho;p,\alpha,y)=\frac{\partial\ln(L)}{\partial\rho}$*, where* p *and* α *are the true values of the parameters. Then there exist a unique solution for* $f_2(\rho;p,\alpha,y)=0$*, for* $\hat{\rho}\in(0,1)$*.*

Proof. We have

$$\begin{aligned}f_2(\rho;p,\alpha,y) &= \sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1}-(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{\rho^{y_i^\alpha}-\rho^{(y_i+1)^\alpha}}-(1-p)\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1}}{p+(1-p)\rho^{y_i^\alpha}}\\ &\quad -(1-p)\sum_{i=1}^{n}\frac{(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{p+(1-p)\rho^{(y_i+1)^\alpha}}.\end{aligned} \tag{6.4.13}$$

Now we can see that

$$\lim_{\rho\to\infty}f_2(\rho;p,\alpha,y)=\infty, \tag{6.4.14}$$

since $\lim_{p\to 0^+}\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1}-(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{\rho^{y_i^\alpha}-\rho^{(y_i+1)^\alpha}}=\infty$, $\lim_{p\to 0}\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1}}{p+(1-p)\rho^{y_i^\alpha}}=0$ and $\lim_{p\to 0}\sum_{i=1}^{n}\frac{(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{p+(1-p)\rho^{(y_i+1)^\alpha}}=0.$

Also

$$\lim_{\rho\to\infty}f_2(\rho;p,\alpha,y)=-\infty-(1-p)\big[\sum_{i=1}^{n}y_i^\alpha+\sum_{i=1}^{n}(y_i+1)^\alpha\big]=-\infty, \tag{6.4.15}$$

since $\lim_{p\to 1^-}\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1}-(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{\rho^{y_i^\alpha}-\rho^{(y_i+1)^\alpha}}=-\infty$, $\lim_{p\to 1}\sum_{i=1}^{n}\frac{y_i^\alpha\rho^{y_i^\alpha-1}}{p+(1-p)\rho^{y_i^\alpha}}=\sum_{i=1}^{n}y_i^\alpha$ and $\lim_{p\to 0}\sum_{i=1}^{n}\frac{(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}}{p+(1-p)\rho^{(y_i+1)^\alpha}}=\sum_{i=1}^{n}(y_i+1)^\alpha$. Hence there exist a root for $\rho\in(0,1)$. The first derivative of $f_2(\rho;p,\alpha,y)$ is given by

$$\begin{aligned}\frac{\partial f_2(\rho;p,\alpha,y)}{\partial\rho} = & \sum_{i=1}^{n}\left[\frac{y_i^\alpha\rho^{y_i^\alpha}(y_i^\alpha-1)-(y_i+1)^\alpha\rho^{(y_i+1)^\alpha}[(y_i+1)^\alpha-1]}{\rho^{y_i^\alpha}-\rho^(y_i+1)^\alpha}-\right.\\ & \left.\frac{[y_i^\alpha\rho^{y_i^\alpha-1}-(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}]^2}{[\rho^{y_i^\alpha}-\rho^(y_i+1)^\alpha]^2}\right]\\ & -(1-p)\sum_{i=1}^{n}\left[\frac{y_i^\alpha\rho^{y_i^\alpha-1}(y_i^\alpha-1)}{\rho[p+(1-p)\rho^{y_i\alpha}]}-\frac{(1-p)y_i^{2\alpha}\rho^{2(y_i^\alpha-1)}}{[p+(1-p)\rho^{y_i^\alpha}]^2}\right]\\ & -(1-p)\sum_{i=1}^{n}\left[\frac{(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}[(y_i+1)^\alpha-1]}{\rho[p+(1-p)\rho^{(y_i+1)^\alpha}]}-\right.\\ & \left.\frac{(1-p)(y_i+1)^{2\alpha}\rho^{2[(y_i+1)^\alpha-1]}}{[p+(1-p)\rho^{(y_i+1)^\alpha}]^2}\right].\end{aligned} \qquad (6.4.16)$$

The roots are unique when

$$\begin{aligned}\sum_{i=1}^{n} & \left[\frac{y_i^\alpha\rho^{y_i^\alpha}(y_i^\alpha-1)-(y_i+1)^\alpha\rho^{(y_i+1)^\alpha}[(y_i+1)^\alpha-1]}{\rho^{y_i^\alpha}-\rho^(y_i+1)^\alpha}-\right.\\ & \left.\frac{[y_i^\alpha\rho^{y_i^\alpha-1}-(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}]^2}{[\rho^{y_i^\alpha}-\rho^(y_i+1)^\alpha]^2}\right]\\ < & \ (1-p)\sum_{i=1}^{n}\left[\frac{y_i^\alpha\rho^{y_i^\alpha-1}(y_i^\alpha-1)}{\rho[p+(1-p)\rho^{y_i\alpha}]}-\frac{(1-p)y_i^{2\alpha}\rho^{2(y_i^\alpha-1)}}{[p+(1-p)\rho^{y_i^\alpha}]^2}\right]\\ + & \ (1-p)\sum_{i=1}^{n}\left[\frac{(y_i+1)^\alpha\rho^{(y_i+1)^\alpha-1}[(y_i+1)^\alpha-1]}{\rho[p+(1-p)\rho^{(y_i+1)^\alpha}]}-\right.\\ & \left.\frac{(1-p)(y_i+1)^{2\alpha}\rho^{2[(y_i+1)^\alpha-1]}}{[p+(1-p)\rho^{(y_i+1)^\alpha}]^2}\right].\end{aligned} \qquad (6.4.17)$$

This completes the proof. □

Theorem 6.4.3. *From the Eqn.(6.4.5), let* $f_3(\alpha;p,\rho,y)=\frac{\partial\ln(L)}{\partial\alpha}$, *where* p *and*

ρ are the true values of the parameters. Then there exist a unique solution for $f_3(\alpha; p, \rho, y) = 0$, for $\hat{\alpha} \in (0, \infty)$.

Proof. We have

$$\begin{aligned} f_3(\alpha; p, \rho, y) &= \sum_{i=1}^{n} \frac{\ln(\rho)[y_i^{\alpha}\rho^{y_i^{\alpha}}\ln(y_i) - (y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\ln(y_i+1)]}{\rho^{y_i^{\alpha}} - \rho^{(y_i+1)^{\alpha}}} \\ &\quad -(1-p)\ln(\rho)\sum_{i=1}^{n}\frac{y_i^{\alpha}\rho^{y_i^{\alpha}}\ln(y_i)}{p+(1-p)\rho^{y_i^{\alpha}}} \\ &\quad -(1-p)\ln(\rho)\sum_{i=1}^{n}\frac{(y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\ln(y_i+1)}{p+(1-p)\rho^{(y_i+1)^{\alpha}}}. \end{aligned} \tag{6.4.18}$$

Then for $y_i > 0$, we have

$$\lim_{\alpha\to 0} f_3(\alpha; p, \rho, y) = \infty, \tag{6.4.19}$$

since $\lim_{\alpha\to 0^+}\sum_{i=1}^{n}\frac{\ln(\rho)[y_i^{\alpha}\rho^{y_i^{\alpha}}\ln(y_i)-(y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\ln(y_i+1)]}{\rho^{y_i^{\alpha}}-\rho^{(y_i+1)^{\alpha}}} = \infty$,
$\lim_{\alpha\to 0}(1-p)\ln(\rho)\sum_{i=1}^{n}\frac{y_i^{\alpha}\rho^{y_i^{\alpha}}\ln(y_i)}{p+(1-p)\rho^{y_i^{\alpha}}} = \frac{(1-p)\ln(\rho)}{p+(1-p)\rho}\sum_{i=1}^{n}\ln(y_i)$
and $\lim_{\alpha\to 0}(1-p)\ln(\rho)\sum_{i=1}^{n}\frac{(y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\ln(y_i+1)}{p+(1-p)\rho^{(y_i+1)^{\alpha}}} = \frac{(1-p)\ln(\rho)}{p+(1-p)\rho}\sum_{i=1}^{n}\ln(y_i+1)$.

Also

$$\lim_{\alpha\to\infty} f_3(\rho; p, \alpha, y) = 0, \tag{6.4.20}$$

since $\lim_{\alpha\to\infty}\sum_{i=1}^{n}\frac{\ln(\rho)[y_i^{\alpha}\rho^{y_i^{\alpha}}\ln(y_i)-(y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\ln(y_i+1)]}{\rho^{y_i^{\alpha}}-\rho^{(y_i+1)^{\alpha}}} = 0$,
$\lim_{\alpha\to\infty}(1-p)\ln(\rho)\sum_{i=1}^{n}\frac{y_i^{\alpha}\rho^{y_i^{\alpha}}\ln(y_i)}{p+(1-p)\rho^{y_i^{\alpha}}} = 0$ and
$\lim_{\alpha\to\infty}(1-p)\ln(\rho)\sum_{i=1}^{n}\frac{(y_i+1)^{\alpha}\rho^{(y_i+1)^{\alpha}}\ln(y_i+1)}{p+(1-p)\rho^{(y_i+1)^{\alpha}}} = 0$. Hence there exist a root for $\alpha \in (0, \infty)$. The first derivative of $f_3(\alpha; p, \rho, y)$ with respect to α is given by

$$\frac{\partial f_3(\alpha; p, \rho, y)}{\partial\alpha} = \ln(\rho)D_1 - (1-p)\ln(\rho)(D_2 + D_3), \tag{6.4.21}$$

where

$$\begin{aligned} D_1 &= \sum_{i=1}^{n} \left[\frac{y_i^{\alpha} \rho^{y_i^{\alpha}} [\ln(y_i)]^2 [1 + y_i^{\alpha} \ln(\rho)] - (y_i+1)^{\alpha} \rho^{(y_i+1)^{\alpha}} [\ln(y_i+1)]^2 [1 + (y_i+1)^{\alpha} \ln(\rho)]}{\rho^{y_i^{\alpha}} - \rho^{(y_i+1)^{\alpha}}} \right. \\ &- \left. \frac{[y_i^{\alpha} \rho^{y_i^{\alpha}} \ln(y_i) - (y_i+1)^{\alpha} \rho^{(y_i+1)^{\alpha}} \ln(y_i+1)]^2}{[\rho^{y_i^{\alpha}} - \rho^{(y_i+1)^{\alpha}}]^2} \right], \\ D_2 &= \sum_{i=1}^{n} \left[\frac{y_i^{\alpha} \ln(y_i) \rho^{y_i^{\alpha}} [1 + y_i^{\alpha} \ln(y_i) \ln(\rho)]}{p + (1-p)\rho^{y_i^{\alpha}}} - \frac{(1-p) \ln(\rho) y_i^{2\alpha} \rho^{2y_i^{\alpha}} [\ln(y_i)]^2}{[p + (1-p)\rho^{y_i^{\alpha}}]^2} \right], \end{aligned}$$

and

$$\begin{aligned} D_3 &= \sum_{i=1}^{n} \left[\frac{(y_i+1)^{\alpha} \ln(y_i+1) \rho^{(y_i+1)^{\alpha}} [1 + (y_i+1)^{\alpha} \ln(y_i+1) \ln(\rho)]}{p + (1-p)\rho^{(y_i+1)^{\alpha}}} \right. \\ &- \left. \frac{(1-p) \ln(\rho) (y_i+1)^{2\alpha} \rho^{2(y_i+1)^{\alpha}} [\ln(y_i+1)]^2}{[p + (1-p)\rho^{(y_i+1)^{\alpha}}]^2} \right]. \end{aligned}$$

The roots are unique when

$$D_1 < (1-p)(D_2 + D_3). \tag{6.4.22}$$

This completes the proof. □

6.4.1 Simulation study

This section explains the performance of the MLEs of the model parameters of the DCWG distribution using Monte Carlo simulation for various sample sizes and for selected parameter values. The algorithm for the simulation study is given below:

Step 1. Input the value of replication (N);

Step 2. Specify the sample size n and the values of the parameters p, ρ and α;

Step 3. Generate $u_i \sim Uniform(0,1),\ i = 1, 2, ..., n.$;

Step 4. Obtain the random observations from the DCWG distribution using the Eqn.(6.3.8);

Step 5. Compute the MLEs of the three parameters;

Step 6. Repeat steps 3 to 5, N times;

Step 7. Compute the parameter estimate, standard error of estimate, average bias, MSE and CP for each parameter.

We take random samples of size n=50,100,200 and 500 respectively. The MLEs of the parameter vector $\theta = (p, \rho, \alpha)^T$ are determined by maximizing the log-likelihood function given in Eqn.(6.4.2) by using the *optim* package of **R** software based on each generated samples. This simulation is repeated 1000 times and the average estimate and its standard error, average bias, MSE and CP are computed and presented in Table 6.2. From Table 6.2, it can be seen that, as sample size increases the estimates of bias and MSE decreases. Also note that the CP values are quite closer to the 95% nominal level.

6.5 Data Applications of the DCWG Distribution

In this section, we analyze two real data sets to prove empirically the flexibility of the DCWG distribution. The goodness-of-fit statistics for this model are compared with other competitive models and the MLEs of the model parameters are determined numerically. The first data set is taken from University of Bosphoros, Kandilli Observatory and Earthquake Research Institute-National Earthquake Mon-

Table 6.2: The parameter estimate, standard error, average bias, MSE and CP for given parameters.

Parameter(θ)	Samples(n)	$E(\hat{\theta})(E(SE(\hat{\theta})))$	Average bias	MSE	CP
$p = 0.5$	50	0.3481(0.1202)	-0.151	0.049	86.4
	100	0.3992(0.0911)	-0.101	0.011	88.3
	200	0.4235(0.0531)	-0.077	0.006	90.6
	500	0.4807(0.0327)	-0.018	0.003	92.5
$\rho = 0.9$	50	0.5861(0.1925)	-0.324	0.099	83.8
	100	0.6032(0.1322)	-0.288	0.089	85.1
	200	0.7864(0.1103)	-0.104	0.013	90.6
	500	0.8429(0.1051)	-0.061	0.001	93.2
$\alpha = 0.5$	50	0.5886(0.1317)	0.091	0.008	90.3
	100	0.5432(0.0633)	0.034	0.003	92.6
	200	0.5277(0.0432)	0.028	0.002	93.8
	500	0.5013(0.021)	0.011	0.001	94.5
$p = 0.8$	50	0.4138(0.1316)	-0.391	0.147	88.2
	100	0.5883(0.1172)	-0.201	0.045	89.3
	200	0.7114(0.1082)	-0.089	0.008	91.6
	500	0.7938(0.0922)	-0.016	0.001	94.1
$\rho = 0.5$	50	0.4832(0.1132)	-0.017	0.019	90.7
	100	0.4891(0.1071)	-0.011	0.012	92.6
	200	0.4986(0.0192)	-0.009	0.002	93.1
	500	0.4993(0.0112)	-0.007	0.001	94.6
$\alpha = 1.5$	50	2.1336(0.2218)	0.534	0.402	87.9
	100	1.9817(0.1677)	0.451	0.238	89.8
	200	1.7926(0.1281)	0.283	0.088	91.3
	500	1.6013(0.0927)	0.101	0.010	93.9

itoring Research Center and is studied in Kus (2007). The data represents the time interval of the successive earthquakes and are as follows:

1163, 3258, 323, 159, 501, 616, 398, 67, 2039, 217, 9, 633, 4863, 143, 182, 2117, 756, 409, 896, 8592, 461, 1821, 3709, 979.

The second data set represents the lifetime of eighteen electronic devices studied in Wang (2000). The data are as follows:

5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224, 245, 293, 321, 330, 350, 420.

The fit of the new discrete distribution is compared with discrete Weibull (DW) and geometric (G) distributions. To compare the goodness of fit of distributions, we consider the criteria like, Kolmogorov-Smirnov (K-S) statistic, Cramér - von Mises criterion (W*), and Anderson - Darling criterion (A*). The best distribution corresponds to lower -logL, K-S distance, A* and W* statistics values and high p value. The parameter estimates and goodness of fit statistics for the two data sets are presented in Table 6.3 and Table 6.4 respectively. The values in Table 6.3 show

Table 6.3: The parameter estimates and goodness of fit for the first data set.

Model	ML estimates	-log L	K-S	A*	W*	p value
G	$\hat{\rho} = 0.9993$	198.372	0.1836	1.0642	0.2033	0.3499
DW	$\hat{\rho} = 0.0.9963$, $\hat{\alpha} = 0.7868$	197.001	0.1002	0.2519	0.0435	0.9500
DCWG	$\hat{p} = 0.0006$, $\hat{\rho} = 0.0841$, $\hat{\alpha} = 0.1677$	196.912	0.0776	0.1639	0.0238	0.9963

that the DCWG distribution leads to a better fit to the first data set. Figure 6.3, shows the fitted cdfs with the empirical distribution of the first data set. The values

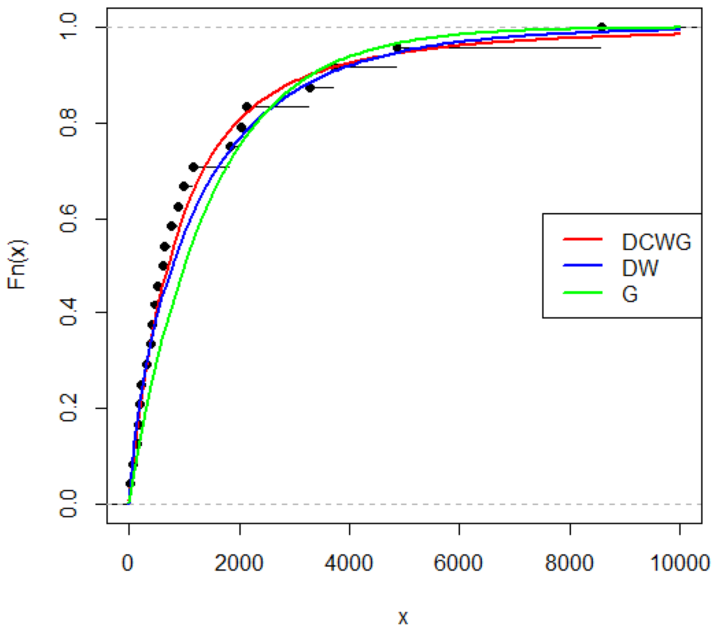

Figure 6.3: Fitted cdf plots for the first data set.

Table 6.4: The parameter estimates and goodness of fit for the second data set.

Model	ML estimates	-log L	K-S	A*	W*	p value
G	$\hat{\rho} = 0.9942$	110.719	0.1253	0.4279	0.0739	0.9071
DW	$\hat{\rho} = 0.0.9975$, $\hat{\alpha} = 1.1560$	110.466	0.1175	0.4634	0.0672	0.9403
DCWG	$\hat{p} = 0.3448$, $\hat{\rho} = 0.9875$, $\hat{\alpha} = 0.9415$	110.186	0.1056	0.3469	0.0441	0.9751

in Table 6.4 show that the DCWG distribution leads to a better fit to the second data set. Figure 6.4, shows the fitted cdfs with the empirical distribution of the second data set.

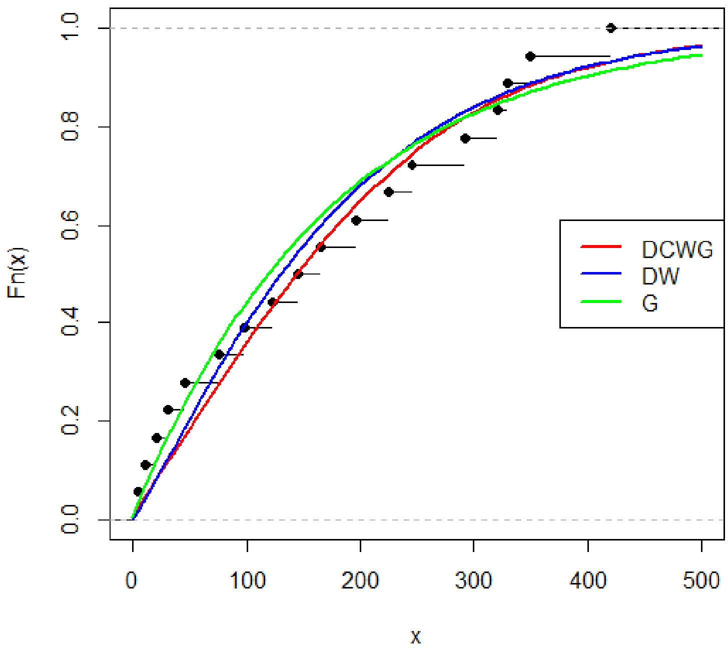

Figure 6.4: Fitted cdf plots for the second data set.

6.6 Summary

In this Chapter, we have introduced a new three-parameter discrete complementary Weibull-geometric (DCWG) distribution. We have studied some of its mathematical and statistical properties. The expressions for the quantile function, probability generating function and order statistics are derived. The model parameters are estimated using maximum likelihood estimation method and present a simulation study to illustrate the performance of the estimates. The new distribution is applied to two real data sets to show its flexibility for data modelling.

CHAPTER

SEVEN

CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

7.1 Conclusions

In this thesis, in Chapter 2 we have introduced a class of distributions called "T-transmuted X family" by combining the T-X family and the transmuted family. Since this class of distributions is a combination of T-X family and transmuted family of distributions, it shows more flexibility in data modelling. We have generated different combinations of distributions from this family. The study is mainly focussed on a special case of exponential-transmuted Weibull distribution namely, exponential-transmuted exponential (ETE) distribution. Statistical properties of ETE distribution are studied in this Chapter. The hazard rate function (hrf) of ETE distribution have shown the various shapes like, increasing, decreasing and constant. The flexibility of this model for data fitting was illustrated with two real-life data sets, one on the fatigue life of 76 Kelvar 373 epoxy and the other on the survival times of 121 patients with breast cancer. While comparing the goodness of fit statistics such as, -log(L), AIC, CAIC, K-S and p value, we have obtained that ETE distribution is a better model for these data sets compared with the exponential, Weibull, exponentiated Weibull and Kumaraswamy exponential distribution. We believe that this family can generate more classes of distributions which are suitable for modelling different types of real life data.

In Chapter 3, we discussed the construction and statistical properties of the Weibull truncated negative binomial (WTNB) distribution. Being a generalization of Weibull distribution, WTNB distribution is identified as a good model for real life data modelling. We have studied its characterizations by truncated moments and hrf. This distribution is identified as a better model for fitting the mercury concentrations of swordfish in marine sciences than the Marshall-Olkin extended Weibull, Marshall-Olkin extended exponential, exponential-truncated negative binomial and Weibull distribution. We have also developed a first order autoregressive minification process with WTNB distribution as marginal and is a competitive model to the time series data sets modelled with minification process having marginal distributions, such as exponential, Weibull, etc.

We introduced a new bivariate distribution with modified Weibull distribution as marginals in the Chapter 4. Here we considered the shock modelling situation with three independent shock sources, say, S_1, S_2 and S_3 are affecting a system with two components, say, C_1 and C_2. Also assume that, if the shock from S_1 hits the system, it destroys C_1 and if the shock is from S_2, it destroys C_2, while the shock from S_3 destroys both the components suddenly. Let U_i denote the inter interval times between the shocks $S_i, i = 1, 2, 3$ and assume that U_1 and U_2 follow Weibull distribution and U_3 follows exponential distribution. The random variables X_1 and X_2 are defined by

$$X_i = \min(U_i, U_3),\ \ i = 1, 2.$$

Here the random variables X_1 and X_2 are dependent because of the common (latent) random variable U_3 and the distribution of (X_1, X_2) follows the bivariate modified Weibull distribution. We derived the marginal and conditional probability distributions, bivariate reliability function, joint hazard rate function, mean waiting time and the reverse hazard rate function of the new bivariate distribution. The flexibility of data modelling using the proposed bivariate distribution was illustrated with the American football data (see page 123). The goodness of fit statistics showed that the new bivariate modified Weibull distribution is a better model than the bivariate generalized Gompertz, bivariate exponentiated generalized Weibull-Gompertz, bivariate exponentiated Weibull extension and the bivariate exponentiated Pareto distribution.

Discretization of continuous distributions are discussed in Chapter 5 and Chapter 6. We have developed the discrete versions of Weibull geometric, additive Weibull geometric and the complementary Weibull geometric distributions. Shape properties of pmf and hrf of these distributions showed that they are good choice for modelling over and under dispersed data. While modelling the data sets of the number of shocks before failure of a component, we have obtained that the discrete Weibull geometric (DWG) distribution is a good choice than the exponentiated discrete Weibull distribution, discrete logistic distribution, discrete Weibull distribution and the geometric distribution. Where as, the discrete additive Weibull geometric distribution is obtained as a good model more fitting the data sets of remission times of 128 bladder cancer patients than the DWG distribution. For the modelling of the data sets like, time intervals of the successive earthquakes and the lifetime of electronic devices, the discrete complementary Weibull geometric distribution was identified as a good choice.

To conclude, we have identified and studied some distributions that are capable of modelling some real life data sets, compared with existing distributions in the literature.

7.2 Future Work

Based on the major findings of this research work, we propose some future works as follows:

- Since the T-transmuted X family is a combination of T-X family and transmuted family, we expect to conduct more studies to explore its model identifiability and further developments by using the quantile function. Construction of bivariate and multivariate cases of this family are also expected to be a future work.

- Bivariate and Multivariate extensions of WTNB distribution.
- Bivariate copula functions of the newly introduced models.
- We expected to propose a more generalized family of Weibull distribution by considering X_i's, $i = 1, 2, ...,$ specified in Eqn.(6.1.3) are i.i.d. having additive Weibull distribution proposed in Xie and Lai (1995), where $\alpha > 0, \gamma > 0$ and $\beta > \delta > 0$, or $(\delta > \beta > 0)$ with pdf
$$f(x; \alpha, \beta, \gamma, \delta) = (\alpha\beta x^{\beta-1} + \gamma\delta x^{\delta-1})e^{-(\alpha x^{\beta}+\gamma x^{\alpha})}. \tag{7.2.1}$$
Then the distribution of $Z = \max(X_1, X_2, ..., X_N)$ is distributed as the complementary additive Weibull geometric (CAWG) distribution with cdf
$$F(z) = \frac{p(1 - e^{-(\alpha z^{\beta}+\gamma z^{\delta})})}{p + (1 - p)e^{-(\alpha z^{\beta}+\gamma z^{\delta})}}. \tag{7.2.2}$$
The pdf, survival function and hrf of the CAWG distribution are respectively,
$$f(z) = \frac{p(\alpha\beta z^{\beta-1} + \gamma\delta z^{\delta-1})e^{-(\alpha z^{\beta}+\gamma z^{\delta})}}{[p + (1 - p)e^{-(\alpha z^{\beta}+\gamma z^{\delta})}]^2}, \tag{7.2.3}$$
$$S(z) = \frac{e^{-(\alpha z^{\beta}+\gamma z^{\delta})}}{p + (1 - p)e^{-(\alpha z^{\beta}+\gamma z^{\delta})}}, \tag{7.2.4}$$
and
$$h(z) = \frac{p(\alpha\beta z^{\beta-1} + \gamma\delta z^{\delta-1})}{p + (1 - p)e^{-(\alpha z^{\beta}+\gamma z^{\delta})}}. \tag{7.2.5}$$
When $\gamma \to 0$, this distribution becomes the complementary Weibull geometric distribution. Using the method of difference in survival function as mentioned

in Eqn.(1.6.1), the discrete version of the CAWG distribution is obtained as

$$P_Y(y) = \frac{p(q_1^{z^\beta} q_2^{z^\delta} - q_1^{(z+1)^\beta} q_2^{(z+1)^\delta})}{[p + (1-p)q_1^{z^\beta} q_2^{z^\delta}][p + (1-p)q_1^{(z+1)^\beta} q_2^{(z+1)^\delta}]}, \tag{7.2.6}$$

where $q_1 = e^{-\alpha}$ and $q_2 = e^{-\delta}$. When $q_2 \to 1$, this distribution reduces to the DCWG distribution. The properties and applications of this distribution are to be explored in the future work.

- We have followed the method of differences in survival functions to discretize continuous distributions. As a future work, the other well known methods are to be used to discretize these distributions and to compare there model adequacy. The problems of information loss occurred while discretizing continuous data are also to be addressed.

- In all the distributions discussed in this thesis we used only MLE method for estimating the parameter values. There are different methods are available in the literature, including Bayesian method and as a future work we will use these estimation procedures and compare the results obtained.

REFERENCES

1. **Adamidis, K. and Loukas, S.** (1998). A lifetime distribution with decreasing failure rate, *Statistics and Probability Letters*, **39**, 35-42.

2. **Afify, A.Z., Nofal, Z.M. and Butt, N.S.** (2014). Transmuted complementary Weibull geometric distribution, *Pakistan Journal of Statistics and Operation Research*, **10**, 435-454.

3. **Afify, A.Z., Nofal, Z.M., Yousof, H.M., El Gebaly, Y.M. and Butt, N.S.** (2015). The transmuted Weibull Lomax distribution: properties and application, *Pakistan Journal of Statistics and Operation Research*, **11**, 135-152.

4. **Ahsanullah, M., Kibria, B.M.G. and Shakil, M.** (2014). Normal and Students t distributions and their applications, *Atlantis Press, France.*

5. **Alice, T. and Jose, K.K.** (2003). Marshall-Olkin Pareto process, *Far East Journal of Theoretical Statistics*, **9**, 117-132.

6. **Alice, T. and Jose, K.K.** (2005). Marshall-Olkin semi-Weibull minification processes, *Recent Advances in Statistical Theory and Applications*, 6-17.

7. **Alizadeh, M., Cordeiro, G.M., de Brito, E. and Demetrio, C.G.B.** (2015). The beta Marshall-Olkin family of distributions, *Journal of Statistical Distributions and Applications*, **2**, 2-18.

8. **Aljarrah, M.A., Lee, C. and Famoye, F.** (2014). On generating T-X family of distributions using quantile functions, *Journal of Statistical Distributions and Applications*, **1**, 1-17.

9. **Almalki, S.J. and Nadarajah, S.** (2014). A new discrete modified Weibull distribution, *IEEE Transactions on Reliability*, **63**, 68-80.

10. **Alzaatreh, A., Famoye, F. and Lee, C.** (2013a). Weibull-Pareto distribution and its applications, *Communications in Statistics - Theory and Methods*, **42**, 1673-1691.

11. **Alzaatreh, A. and Ghosh, I.** (2015). On the Weibull-X family of distributions, *Journal of Statistical Theory and Applications*, **14**, 169-183.

12. **Alzaatreh, A., Lee, C. and Famoye, F.** (2013b). A new method for generating families of continuous distributions, *Metron*, **71**, 63-79.

13. **Alzaatreh, A. and Knight, K.** (2013). On the gamma-half normal distribution and its applications, *Journal of Modern Applied Statistical Methods*, **12**, 103-119.

14. **Andrews, D.F. and Herzberg, A.M.** (1985). Data: a collection of problems from many fields for the student and research worker, *Springer-Verlag, New York*.

15. **Aryal, G.R.** (2013). Transmuted log-logistic distribution, *Journal of Statistics Applications and Probability*, **2**, 11-20.

16. **Aryal, G.R. and Tsokos, C.P.** (2009). On the transmuted extreme value distribution with applications, *Nonlinear Analysis: Theory, Methods and Applications*, **71**, 1401-1407.

17. **Aryal, G.R. and Tsokos, C.P.** (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution, *European Journal of Pure and Applied Mathematics*, **4**, 89-102.

18. **Asgharzadeh, A., Bakouch, H.S., Nadarajah, S. and Esmaeili, L.** (2014). A new family of compound lifetime distributions, *Kybernetika*, **50**, 142-169.

19. **Ashwag, S., Al-Urwi. and Baharith, L.A.** (2017). A bivariate exponentiated Pareto distribution derived from Gaussian copula, *Internationa Journal of Advanced and Applied Sciences*, **4**, 66-73.

20. **Babu, M.G.** (2016). On a generalization of Weibull distribution and its applications, *International Journal of Statistics and Applications*, **6**, 168-176.

21. **Babu, M.G. and Jayakumar, K.** (2018a). Characterizations of Weibull truncated negative binomial distribution, *Proceedings of the National Conference on Advances in Statistical Methods, Kannur University*, 105-118.

22. **Babu, M.G. and Jayakumar, K.** (2018b). A new bivariate distribution with modified Weibull distribution as marginals, *Journal of the Indian Society for Probability and Statistics*, **19**, 271-297.

23. **Bagheri, S.F., Samani, E.B. and Ganjali, M.** (2016). The generalized modified Weibull power series distribution: theory and applications, *Computational Statistics and Data Analysis*, **94**, 136-160.

24. **Bakouch, H.S., Jazi, M.A. and Nadarajah, S.** (2014). A new discrete distribution, *Statistics*, **48**, 200-240.

25. **Balakrishnan, N. and Lai, C.D.** (2009). Continuous bivariate distributions (2nd edn.), *Springer, New York*.

26. **Barreto-Souza, W., de Morais, A.L. and Cordeiro, G.M.** (2011). The Weibull-geometric distribution, *Journal of Statistical Computation and Simulation*, **81**, 645-657.

27. **Basu, A.P.** (1971). Bivariate failure rate, *Journal of the American Statistical Association*, **66**, 103-104.

28. **Bebbington, M., Lai, C.D., Wellington, M. and Zitikis, R.** (2012). The discrete additive Weibull distribution: A bathtub-shaped hazard for discontinuous failure data, *Reliability Engineering and System Safety*, **106**, 37-44.

29. **Bidram, H.** (2012). The beta exponential-geometric distribution, *Communications in Statistics-Simulation and Computation.*, **41**, 1606-1622.

30. **Bidram, H., Behboodian, J. and Towhidi, M.** (2013). The beta Weibull-geometric distribution, *Journal of Statistical Computation and Simulation*, **83**, 52-67.

31. **Birnbaum, Z.W.** (1956). On a use of the Mann-Whitney statistic, *Third Berkeley Symposium on Mathematical Statistics and Probability*, **1**, 13-17.

32. **Birnbaum, Z.W. and Mc Carty, B.C.** (1958). A distribution-free upper confidence bounds for $P(Y < X)$ based on independent samples of X and Y, *The Annals of Mathematical Statistics*, **29**, 558-562.

33. **Block, H., Savits, T. and Singh, H.** (1998). The reversed hazard rate function, *Probability in the Engineering and Informational Sciences*, **12**, 69-90.

34. **Bourguignon, M., Silva, R.B. and Cordeiro, G.M.** (2014). The Weibull-G family of probability distributions, *Journal of Data Science*, **12**, 53-68.

35. **Carrasco, J.M., Ortega, E.M.M. and Cordeiro, G.M.** (2008). A generalized modified Weibull distribution for lifetime modeling, *Computational Statistics and Data Analysis*, **53**, 450-462.

36. **Chakraborty, S.** (2015). Generating discrete analogues of continuous probability distributions - A survey of methods and constructions, *Journal of Statistical Distributions and Applications*, **2**, 1-30.

37. **Chakraborty, S. and Chakravarty, D.** (2012). Discrete gamma distribution: properties and parameter estimation, *Communications in Statistics - Theory and Methods*, **41**, 3301-3324.

38. **Chakraborty, S. and Chakravarty, D.** (2016). A new discrete probability distribution with integer support on $(-\infty, \infty)$, *Communication in Statistics -*

Theory and Methods, **45**, 492-505.

39. **Chen, G. and Balakrishnan, N.** (1995). A general purpose approximate goodness-of-fit test, *Journal of Quality Technology*, **27**, 154-161.

40. **Chung, Y. and Kang, Y.** (2014). The exponentiated Weibull-geometric distribution: properties and estimations, *Communications for Statistical Applications and Methods*, **21**, 147-160.

41. **Cohen, A.C.** (1973). The reflected Weibull distribution, *Technometrics*, **15**, 867-873.

42. **Cordeiro, G.M. and de Castro, M.** (2011). A new family of generalized distributions, *Journal of Statistical Computation and Simulation*, **81**, 883-893.

43. **Cordeiro, G.M., Ortega, E.M.M. and Nadarajah, S.** (2010). The Kumaraswamy Weibull distribution with application to failure data, *Journal of the Franklin Institute*, **347**, 1399-1429.

44. **Cordeiro, G.M., Ortega, E.M.M. and Silva, G.O.** (2014). The Kumaraswamy modified Weibull distribution: theory and applications, *Journal of Statistical Computation and Simulation*, **84**, 1387-1411.

45. **Cordeiro, G.M., Silva, G.O. and Ortega, E.M.M.** (2016). An extended-G geometric family, *Journal of Statistical Distributions and Applications*, **3**, 1-16.

46. **Csorgo, S. and Welsh, A.** (1989). Testing for exponential and Marshall-Olkin distributions, *Journal of Statistical Planning and Inference*, **23**, 287-300.

47. **David, H.A. and Nagaraja, H.N.** (2003). Order Statistics, *John Wiley and Sons, New York.*

48. **Eissa, F.H.** (2017). The exponentiated Kumaraswamy-Weibull distribution with application to real data, *International Journal of Statistics and Probability*, **6**, 167-182.

49. **Elbatal, I. and Aryal, G.** (2013). On the transmuted additive Weibull distribution, *Austrian Journal of Statistics*, **42**, 117-132.

50. **Elbatal, I., Mansour, M.M. and Ahsanullah, M.** (2016). The additive Weibull-geometric distribution: theory and applications, *Journal of Statistical Theory and Applications*, **15**, 125-141.

51. **El-Damcese, M., Mustafa, A. and Eliwa, M.** (2015). Bivariate exponentiated generalized Weibull-Gompertz distribution, *arXiv:1501.02241.*

52. **El-Gohary, A., El-Bassiouny, A.H. and El-Morshedy, M.** (2016). Bivariate exponentiated modified Weibull extension distribution, *Journal of Statistics Applications and Probability*, **5**, 67-78.

53. **El-Sherpieny, E.A, Ibrahim, S.A. and Bedar, R.E.** (2013). A new bivariate generalized Gompertz distribution, *Asian Journal of Applied Sciences*, **1**, 141-150.

54. **Eugene, N., Lee, C. and Famoye, F.** (2002). Beta-normal distribution and its application, *Communications in Statistics - Theory and Methods*, **31**, 497-512.

55. **Famoye, F., Lee, C. and Olumolade, O.** (2005). The beta-Weibull distribution, *Journal of Statistical Theory and Applications*, **4**, 121-136.

56. **Farlie, D.J.G.** (1960). The performance of some correlation coefficients for a general bivariate distribution, *Biometrika*, **47**, 307-323.

57. **Ferguson, T.S.** (1996). A course in large sample theory, *Chapman and Hall, London.*

58. **Fisher, R.A. and Tippett, L.H.C.** (1928). Limiting forms of the frequency distribution of the largest and smallest member of a sample, *Proceedings of the Cambridge Philosophical Society*, **24**, 180-190.

59. **Galambos, J. and Kotz, S.** (1978). Characterizations of probability distributions. A unified approach with an emphasis on exponential and related models, *Lecture Notes in Mathematics, 675. Berlin, Germany: Springer.*

60. **Ghitany, M.E., Al-Awadgi, F.A. and Alkhalfan, I.A.** (2007). Marshall-Olkin extended Lomax distribution and its application to censored data, *Communications in Statistics-Theory and Methods*, **36**, 1855-1866.

61. **Ghitany, M.E., Al-Hussaini, E.K. and Al-Jaralla, R.A.** (2005). Marshall-Olkin extended Weibull distribution and its application to censored data, *Jour-*

nal of Applied Statistics, **32**, 1025-1034.

62. **Glanzel, W.** (1987). A characterization theorem based on truncated moments and its application to some distribution families, *Mathematical Statistics and Probability Vol.B, D.Reidel Publishing Company Dordrecht-Holland*, 75-84.

63. **Glanzel, W.** (1990). Some consequences of a characterization theorem based on truncated moments, *Statistics*, **21**, 613-618.

64. **Gomes, A.E., da Silva, C.Q. and Cordeiro, G.M.** (2015). The exponentiated-G Poisson model, *Communications in Statistics-Theory and Methods*, **44**, 4217-4240.

65. **Gómez-Déniz, E.** (2010). Another generalization of the geometric distribution, *Test*, **19**, 399-415.

66. **Gumbel, E.J.** (1958). Statistics of extremes, *Columbia University Press*, New York.

67. **Gupta, R.C. and Ahsanullah, M.** (2006). Some characterization results based on the conditional expectation of truncated order statistics (record values), *Journal of Statistical Theory and Applications*, **5**, 391-402.

68. **Gupta, P.L., Gupta, R.C. and Tripathi, R.C.** (1997). On the monotonic properties of the discrete failure rate, *Journal of Statistical Planning and Inference*, **65**, 225-268.

69. **Gupta, R.D. and Kundu, D.** (1999). Generalized exponential distributions, *Australian and New Zealand Journal of Statistics*, **41**, 173-188.

70. **Hamedani, G.G.** (2010). Characterizations of continuous univariate distributions based on the truncated moments of functions of order statistics, *Studia Scientiarum Mathematicarum Hungarica*, **47**, 462-484.

71. **Hamedani, G.G. and Ahsanullah, M.** (2005). Characterizations of univariate continuous distributions based on hazard function II, *Journal of Statistical Theory and Applications*, **4**, 218-238.

72. **Hanagal, D.D.** (1996). A multivariate Weibull distribution, *Economic Quality Control*, **11**, 193-200.

73. **Huang, W.J. and Su, N.C.** (2012). Characterizations of distributions based on moments of residual life, *Communications in Statistics - Theory and Methods*, **41**, 2750-2761.

74. **Inusah, S. and Kozubowski, T.J.** (2006). A discrete analogue of the Laplace distribution, *Journal of Statistical Planning and Inference*, **136**, 1090-1102.

75. **Jafari, A.A. and Tahmasebi, S.** (2016). Gompertz-power series distributions, *Communications in Statistics-Theory and Methods*, **45**, 3761-3781.

76. **Janardan, K.G.** (1978). A new functional equation analogous to Cauchy-Pexider functional equation and its application, *Biomedical Journal*, **20**, 323-

328.

77. **Janardan, K.G. and Schaeffer, D.J.** (1978). Another characterization of the Weibull distribution, *The Canadian Journal of Statistics*, **6**, 77-78.

78. **Janardan, K.G. and Taneja, V.S.** (1979a). Characterization of the Weibull distribution by properties of order statistics, *Biomedical Journal*, **21**, 3-9.

79. **Janardan, K.G. and Taneja, V.S.** (1979b). Some theorems concerning characterization of the Weibull distribution, *Biomedical Journal*, **21**, 139-144.

80. **Jayakumar, K. and Babu, M.G.** (2015). Some generalizations of Weibull distribution and related processes, *Journal of Statistical Theory and Applications*, **14**, 425-434.

81. **Jayakumar, K. and Babu, M.G.** (2017). T-transmuted X family of distributions, *Statistica*, **77**, 251-276.

82. **Jayakumar, K. and Babu, M.G.** (2018). Discrete Weibull geometric distribution and its properties, *Communications in Statistics-Theory and Methods*, **47**, 1767-1783.

83. **Jayakumar, K. and Babu, M.G.** (2019a). Discrete additive Weibull geometric distribution, *Journal of Statistical Theory and Applications*, **18**, 33-45.

84. **Jayakumar, K. and Babu, M.G.** (2019b). A new generalization of Fréchet distribution: properties and applications, *Statistica* **79**, 267-289.

85. **Jayakumar, K. and Babu, M.G.** (2019c). Discrete type I half-logistic Weibull distribution and its properties, *Pakistan Journal of Statistics and Operation Research.* (Under review).

86. **Jayakumar, K. and Sankaran, K.K.** (2016). On a generalisation of uniform distribution and its properties, *Statistica*, **76**, 83-91.

87. **Jayakumar, K. and Sankaran, K.K.** (2017). Generalized exponential truncated negative binomial distribution, *American Journal of Mathematical and Management Sciences*, **36**, 98-111.

88. **Jayakumar. K. and Sankaran, K.K.** (2018). A generalisation of discrete Weibull distribution, *Communications in Statistics - Theory and Methods*, **47**, 6064-6078.

89. **Jayakumar, K. and Thomas, M.** (2008). On a generalization Marshall-Olkin scheme and its application to Burr type XII distribution, *Statistical Papers*, **49**, 421-439.

90. **Jazi, M.A., Lai, C.D. and Alamatsaz, M.H.** (2010). A discrete inverse Weibull distribution and estimation of its parameters, *Statistical Methodology*, **7**, 121-132.

91. **Jose, K.K., Naik, S.R. and Ristić, M.M.** (2010). Marshall-Olkin q-Weibull distribution and max-min processes, *Statistical Papers*, **51**, 837-851.

92. **Jose, K.K.** (2011). Marshall-Olkin family of distributions and their applications in reliability theory, time series modelling and stress-strength analysis, *Int. Statistical Inst. Proc. 58th World Statistical Congress, Dublin (Session CPS005)*, 39183923.

93. **Kamel, B.I., Youssef, S.E.A. and Sief, M.G.** (2016). The uniform truncated negative binomial distribution and its properties, *Journal of Mathematics and Statistics*, **21**, 139-144.

94. **Kemp, A.W.** (1997). Characterization of a discrete normal distribution, *Journal of Statistical Planning and Inference*, **63**, 223-229.

95. **Kemp, A.W.** (2008). The discrete half-normal distribution, *Advances in mathematical and statistical modeling*, 353-365.

96. **Khan, A.H. and Beg, M.I.** (1987). Characterization of the Weibull distribution by conditional variance, *Sankhya: The Indian Journal of Statistics, Series A*, 268-271.

97. **Khan, M.S.** (2010). The beta inverse Weibull distribution, *International Transcations in Mathematical Sciences and Computer*, **3**, 113-119.

98. **Khan, M.S.A., Khalique, A. and Abouammoh, A.M.** (1989). On estimating parameters in a discrete Weibull distribution, *IEEE Transactions on Reliability*, **38**, 348-350.

99. **Khan, M.S. and King, R.** (2013a). Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution, *European Journal of Pure and Applied Mathematics*, **6**, 66-88.

100. **Khan, M.S. and King, R.** (2013b). Transmuted generalized inverse Weibull distribution, *Journal of Applied Statistical Sciences*, **20**, 15-32.

101. **Kotz, S., Lumelskii, Y. and Pensky, M.** (2003). The stress-strength model and its generalizations: theory and applications, *World Scientific Co., Singapore.*

102. **Kotz, S. and Shanbhag, D.N.** (1980). Some new approaches to probability distributions. *Advances in Applied Probability*, **12**, 903-921.

103. **Kozubowski, T.J. and Inusah, S.** (2006). A skew Laplace distribution on integers, *Annals of the Institute of Statistical Mathematics*, **58**, 555-571.

104. **Krishna, H. and Pundir, P.S.** (2009). Discrete Burr and discrete Pareto distributions, *Statistical Methodology*, **6**, 177-188.

105. **Krishna, H. and Singh, P.** (2009). A bivariate geometric distribution with applications to reliability, *Communications in Statistics - Theory and Methods*, **38**, 1079-1093.

106. **Kulasekera, K.B.** (1994). Approximate MLEs of the parameters of a discrete Weibull distribution with type I censored data, *Microelectronics Reliability*, **34**, 1185-1186.

107. **Kundu, D. and Dey, A.K.** (2009). Estimating the parameters of the Marshall-Olkin bivariate Weibull distribution by EM algorithm, *Computational Statistics and Data Analysis*, **53**, 956-965.

108. **Kundu, D. and Gupta, A.K.** (2014). On bivariate Weibull-geometric distribution, *Journal of Multivariate Analysis*, **123**, 19-29.

109. **Kundu, D. and Gupta, R.D.** (2009). Bivariate generalized exponential distribution, *Journal of Multivariate Analysis*, **100**, 581-593.

110. **Kundu, D. and Gupta, R.D.** (2010). Modified Sarhan-Balakrishnan singular bivariate distribution, *Journal of Statistical Planning and Inference*, **140**, 526-538.

111. **Kus, C.** (2007). A new lifetime distribution, *Computational Statistics and Data Analysis*, **51**, 4497-4509.

112. **Lai, C.D., Murthy, D.N.P. and Xie, M.** (2011). Weibull distributions, *Wiley Interdisciplinary Reviews: Computational Statistics*, **3**, 282-287.

113. **Lai, C.D., Xie, M. and Murthy, D.N.P.** (2003). A modified Weibull distribution, *IEEE Transactions on Reliability*, **52**, 33-37.

114. **Lee, L. and Krutchkoff, R. G.** (1980). Mean and variance of partially-truncated distributions, *Biometrics*, **36**, 531-536.

115. **Lee, E.T. and Wang, J.** (1992). Statistical methods for survival data analysis, *John Wiley and Sons, New York*.

116. **Lee, E.T. and Wang, J.** (2003). Statistical methods for survival data analysis, *John Wiley and Sons, New York.*

117. **Lemonte, A.J., Cordeiro, G.M. and Ortega, E.M.M.** (2014). On the additive Weibull distribution, *Communications in Statistics - Theory and Methods*, **43**, 2066-2080.

118. **Lewis, P.A.W. and McKenzie, E.** (1991). Minification process and their transformations, *Journal Applied Probability*, **28**, 45-57.

119. **Lin, G.D., Dou, X., Kuriki, S. and Huang, J.S.** (2014). Recent developments on the construction of bivariate distributions with fixed marginals, *Journal of Statistical Distributions and Applications*, **1**, 1-14.

120. **Lisman, J.H.C. and Van Zuylen, M.C.A.** (1972). Note on the generation of the most probable frequency distribution, *Statistica Neerlandica*, **26**, 19-23.

121. **Mardia, K.V.** (1967). Some contributions to contingency type bivariate distributions, *Biometrika*, **54**, 235-249.

122. **Marshall, A.W. and Olkin, I.** (1967). A multivariate exponential distribution, *Journal of the American Statistical Association*, **62**, 30-44.

123. **Marshall, A.W. and Olkin, I.** (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families, *Biometrika*, **84**, 641-652.

124. **Merovci, F. and Elbatal, I.** (2014). Transmuted Weibull-geometric distribution and its applications, *Scientia Magna*, **10**, 68-82.

125. **Morais, A.L. and Barreto-Souza, W.** (2011). A compound class of Weibull and power series distributions, *Computational Statistics and Data Analysis*, **55**, 1410-1425.

126. **Mudholkar, G.S. and Srivastava, D.K.** (1993). Exponentiated Weibull family for analyzing bathtub failure rate data, *IEEE Transactions on Reliability*, **42**, 299-302.

127. **Muhammed, H.Z.** (2016). Bivariate inverse Weibull distribution, *Journal of Statistical Computation and Simulation*, **86**, 2335-2345.

128. **Murthy, D.N.P., Xie, M. and Jiang, R.** (2004). Weibull models, *John Wiley and Sons, New Jersey.*

129. **Nadarajah, S., Cordeiro, G.M. and Ortega, E.M.M.** (2015). The exponentiated-G-geometric family of distributions, *Journal of Statistical Computation and Simulation*, **85**, 1634-1650.

130. **Nadarajah, S., Jayakumar, K. and Riśtic, M.M.** (2013). A new family of lifetime models, *Journal of Statistical Computation and Simulation*, **83**, 1389-1404.

131. **Nair, N.U. and Sudheesh, K.K.** (2010). Characterization of continuous distributions by properties of conditional variance, *Statistical Methodology*, **7**,

30-40.

132. **Nakagawa, T. and Osaki, S.** (1975). The discrete Weibull distribution, *IEEE Transactions on Reliability*, **24**, 300-301.

133. **Nanda, A.K. (2010)**. Characterization of distributions through failure rate and mean residual life functions, *Statistics and Probability Letters*, **80**, 752-755.

134. **Nelson, R.B.** (2007). An introduction to copulas, *Springer, Berlin.*

135. **Nekoukhou, V., Alamatsaz, M.H. and Bidram, H.** (2012). A discrete analogue of the generalized exponential distribution, *Communications in Statistics - Theory and Methods*, **41**, 2000-2013.

136. **Noughabi, M.S., Roknabady, A.H.R. and Borzadaran, G.R.M.** (2011). Discrete modified Weibull distribution, *Metron*, **69**, 207-222.

137. **Padgett, W.J. and Spurrier, J.D.** (1985). Discrete failure models, *IEEE Transactions on Reliability*, **34**, 253-256.

138. **Pham, H. and Lai, C.D.** (2007). On recent generalizations of the Weibull distribution, *IEEE Transactions on Reliability*, **56**, 454-458.

139. **Plackett, R.L.** (1965). A class of bivariate distributions, *Journal of the American Statistical Association*, **60**, 516-522.

140. **Popović, B.V., Ristić, M.M. and Cordeiro, G.M.** (2016). A two-parameter

distribution obtained by compounding the generalized exponential and exponential distributions, *Mediterranean Journal of Mathematics*, **13**, 2935-2949.

141. **Prudnikov, A.P., Brychkov, Y.A. and Marichev, O.I.** (1986). Integrals and Series Vol.1, *Gordon and Breach Sciences, Amsterdam, Netherlands.*

142. **Rényi, A.** (1961). On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, **1**, 547-561.

143. **Rinne, H.** (2009). The Weibull distribution: A hand book, *CRC Press, New York.*

144. **Rodrigues, C., Cordeiro, G.M., Demetrio, C.G.B. and Ortega, E.M.M.** (2011). The Weibull negative binomial distribution, *Advances and Applications in Statistics*, **22**, 25-55.

145. **Rohatgi, V.K. and Saleh, E.A.K.** (2001). An introduction to probability and statistics, 2nd edition, *John Wiley and Sons, New York.*

146. **Rosin, P. and Rammler, E.** (1933). The laws governing the fineness of powdered coal, *Journal of the Institute of Fuels*, **6**, 29-36.

147. **Roy, D.** (2003). The discrete normal distribution, *Communications in Statistics - Theory and Methods*, **32**, 1871-1883.

148. **Roy, D.** (2004). Discrete Rayleigh distribution, *IEEE Transactions on Reliability*, **53**, 255-260.

149. **Roy, D. and Gupta, R. P.** (1999). Characterizations and model selections through reliability measures in the discrete case, *Statistics and Probability Letters*, **43**, 197-206.

150. **Saboor, A., Elbatal, I. and Cordeiro, G.M.** (2016). The transmuted exponentiated Weibull geometric distribution: theory and applications, *Hacettepe Journal of Mathematics and Statistics* , **45**, 973-987.

151. **Saboor, A., Kamal, M. and Ahmad, M.** (2015). The transmuted exponentiated - Weibull distribution with applications, *Pakistan Journal of Statistics*, **31**, 229-250.

152. **Salvia, A.A. and Bollinger, R.C.** (1982). On discrete hazard functions, *IEEE Transactions on Reliability*, **31**, 458-459.

153. **Sarhan, A.M., and Balakrishnan, N.** (2007). A new class of bivariate distributions and its mixture, *Journal of Multivariate Analysis*, **98**, 1508-1527.

154. **Sarhan, A.M., Hamilton, D.C., Smith, B. and Kundu, D.** (2011). The bivariate generalized linear failure rate distribution and its multivariate extension, *Computational Statistics and Data Analysis*, **55**, 644-654.

155. **Sarhan, A.M. and Zaindin, M.** (2009). Modified Weibull distribution, *Applied Sciences*, **11**, 123-136.

156. **Sato, H., Ikota, M., Aritoshi, S. and Masuda, H.** (1999). A new defect distribution meteorology with a consistent discrete exponential formula and its

applications, *IEEE Transactions on Semiconductor Manufacturing*, **12**, 409-418.

157. **Scholz, F.W.** (1990). Characterization of the Weibull distribution. *Computational Statistics and Data Analysis*, **10**, 289-292.

158. **Shahbaz, M.Q., Shahbaz, S. and Butt, N.S.** (2012). The Kumaraswamy-inverse Weibull distribution, *Pakistan Journal of Statistics and Operation Research*, **8**, 479-489.

159. **Shaked, M., Shanthikumar, J.G. and Valdez-Torres, J.B.** (1995). Discrete hazard rate functions, *Computers and Operations Research*, **22**, 391-402.

160. **Shannon, C.E.** (1948). A mathematical theory of communication, *The Bell System Technical Journal*, **27**, 379-423.

161. **Shaw, W.T. and Buckley, I.R.** (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, *Research report.*

162. **Shimizu, R. and Davies, L.** (1981). General characterization theorems for the Weibull and the stable distributions, *Sankhya: The Indian Journal of Statistics, Series A*, **43**, 282-310.

163. **Silva, R.B., Barreto-Souza, W. and Cordeiro, G.M.** (2010a). The beta modified Weibull distribution, *Lifetime data analysis*, **16**, 409-430.

164. **Silva, R.B., Barreto-Souza, W. and Cordeiro, G.M.** (2010b). A new distribution with decreasing, increasing and upside-down bathtub failure rate, *Computational Statistics and Data Analysis*, **54**, 935-944.

165. **Sim, C.H.** (1986). Simulation of Weibull and gamma autoregressive stationary process, *Communications in Statistics - Simulation and Computation*, **15**, 1141-1146.

166. **Singh, R.K., Yadav, A.S., Singh, S.K. and Singh, U.** (2016). Marshall-Olkin extended exponential distribution: different method of estimations, *Journal of Advanced Computing*, **5**, 12-28.

167. **Sklar, A.** (1959). Fonctions de répartition á n dimensions et leurs marges, *Publications de lInstitut de Statistique de lUniversite de Paris*, **8**, 229-231.

168. **Stein, W.E. and Dattero, R.** (1984). A new discrete Weibull distribution, *IEEE Transactions on Reliability*, **33**, 196-197.

169. **Su, J.C. and Huang, W.J.** (2000). Characterizations based on conditional expectations. *Statistical Papers*, **41**, 423-435.

170. **Tahir, M.H. and Cordeiro, G.M.** (2016). Compounding of distributions: a survey and new generalized classes, *Journal of Statistical Distributions and Applications*, **3**, 1-35.

171. **Tahir, M.H., Zubair, M., Cordeiro, G.M., Alzaatreh, A. and Mansoor, M.** (2016). The Poisson-X family of distributions, *Journal of Statistical*

Computation and Simulation, **86**, 2901-2921.

172. **Tavares, L.V.** (1980). An exponential Markovian stationary process, *Journal of Applied Probability*, **17**, 1117-1120.

173. **Tojeiro, C., Louzada, F., Roman, M. and Borges, P.** (2014). The complementary Weibull geometric distribution, *Journal of Statistical Computation and Simulation*, **84**, 1345-1362.

174. **Trivedi, P.K. and Zimmer, D.M.** (2007). Copula modeling: an introduction for practitioners, *Foundations and Trends in Econometrics, Now Publishers, Boston.*

175. **Wang, F.K.** (2000). A new model with bathtub-shaped failure rate using an additive Burr XII distribution, *Reliability Engineering and System Safety*, **70**, 305-312.

176. **Wang, M. and Elbatal, I.** (2015). The modified Weibull geometric distribution, *Metron*, **73**, 303-315.

177. **Weibull, W.** (1939). A statistical theory of the strength of materials, *Ingeniors Vetenskaps Akademiens Handligar*, Report No. 15, Stockholm.

178. **Xie, M. and Lai, C.D.** (1995). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function, *Reliability Engineering and System Safety*, **52**, 87-93.

179. **Xie, M., Tang, Y. and Goh, T.N.** (2002). A modified Weibull extension with bathtub shaped failure rate function, *Reliability Engineering and System Safety*, **76**, 279-285.

180. **Yamachi, C.Y., Romana, M., Louzada, F., Franco, M.A.P. and Cancho, V.G.** (2013). The exponentiated complementary exponential geometric distribution, *Journal of Modern Mathematics Frontier*, **2**, 78-83.

181. **Zografos, K. and Balakrishnan, N.** (2009). On families of beta and generalized gamma-generated distributions and associated inference, *Statistical Methodology*, **6**, 344-362.

LIST OF PUBLISHED WORKS

PUBLISHED IN JOURNALS:

1. **Jayakumar, K. and Babu, M.G.** (2015). Some generalizations of Weibull distribution and related processes, *Journal of Statistical Theory and Applications*, **14(4)**, 425-434.

2. **Babu, M.G.** (2016). On a generalization of Weibull distribution and its applications, *International Journal of Statistics and Applications*, **6(3)**, 168-176.

3. **Jayakumar, K. and Babu, M.G.** (2017). T-transmuted X family of distributions, *Statistica*, **77(3)**, 251-276.

4. **Jayakumar, K. and Babu, M.G.** (2018). Discrete Weibull geometric distribution and its properties, *Communications in Statistics - Theory and Methods*, **47(7)**, 1767-1783.

5. **Babu, M.G. and Jayakumar, K.** (2018a). A new bivariate distribution with modified Weibull distribution as marginals, *Journal of the Indian Society for Probability and Statistics*, **19(2)**, 271-297.

6. **Babu, M.G. and Jayakumar, K.** (2018b). Characterizations of Weibull truncated negative binomial distribution, *Proceedings of the National Conference on Advances in Statistical Methods, Kannur University*, 105-118.

7. **Jayakumar, K. and Babu, M.G.** (2019a). Discrete additive Weibull Geometric distribution, *Journal of Statistical Theory and Applications*, **18(1)**, 33-45.

8. **Jayakumar, K. and Babu, M.G.** (2019b). A new generalization of Fréchet distribution: properties and applications, *Statistica* **79(3)**, 267-289.

PRESENTATIONS IN CONFERENCES/SEMINARS:

1. **Some generalizations of Weibull distribution and related processes**, presented in the National Conference on Statistics for Twenty-First Century (NCSTS)-2014, in connection with International year of Statistics Celebrations during March 20-22, 2014 organized by the Department of Statistics, University of Kerala, Thiruvananthapuram, Kerala.

2. **Analysis and forecasting of Bombay stock exchange (BSE) index**, presented in the International Conference on Statistics and Information Technology for a growing Nation, in conjunction with XXXIV Annual Convention of Indian Society for Probability and Statistics (ISPS) and in association with Indian Statistical Institute, Kolkata during 30th November to 2nd December, 2014 organized by the Department of Statistics, S.V.University, Tirupathi, Andra Pradesh.

3. **Some generalizations of Weibull-X family of distributions**, presented in the International Conference on Statistics for Twenty-First Century (ICSTC)-

2015, during December 17-19, 2015 organized by the Department of Statistics, University of Kerala, Thiruvananthapuram, Kerala.

4. **Weibull distribution and its generalizations**, presented in the National Seminar and Annual Conference of the Kerala Statistical Association (KSA) during February 12-13, 2016 organized by the Department of Statistics, Nirmala College, Moovattupuzha, Kerala.

5. **A new generalization of Weibull distribution and its applications in lifetime modelling**, presented in the Second International Conference on Statistics for Twenty-First Century (ICSTC)-2016, during December 21-23, 2016 organized by the Department of Statistics, University of Kerala, Thiruvananthapuram, Kerala.

6. **On Weibull geometric distribution**, presented in the National Conference on Advances in Statistical Sciences and Annual Conference of the Kerala Statistical Association (KSA) during February 17-18, 2017 organized by the Department of Statistical Sciences, Kannur University, Kannur, Kerala.

7. **On discrete additive Weibull geometric distribution and its properties**, presented in the National Seminar on Recent Trends in Statistical Theory and Applications (NSSTA)-2017 during June 28-30, 2017 organized by the Department of Statistics, University of Kerala, Thiruvananthapuram, Kerala.

8. **On a discrete analogue of T-X family of distributions**, presented in

the Third International Conference on Statistics for Twenty-First Century (ICSTC)-2017, during December 14-16, 2017 organized by the Department of Statistics, University of Kerala, Thiruvananthapuram, Kerala.

9. **Discrete type I half-logistic Weibull distribution and its properties**, presented in the International Conference on Theory and Applications of Statistics and Information Sciences (TASIS)-2018, in conjunction with XXXVII Annual Convention of Indian Society for Probability and Statistics (ISPS) and in association with Indian Bayesian Society (IBS) during January 5-7, 2018 organized by the Department of Statistics, Bharathiar University, Coimbatore, Tamil Nadu.

10. **On odds X-Weibull family of distributions**, presented in the National Seminar on Innovative Approaches in Statistics in conjunction with the Annual Conference of Kerala Statistical Association (KSA), during February 15-17, 2018 organized by the Department of Statistics, St. Thomas College (Autonomous), Thrissur, Kerala.

11. **On a class of distributions related to T-X family**, presented in the National Seminar during March 13-15, 2018 organized by the Department of Statistics, University of Calicut, Kerala.

12. **Characterizations of Weibull truncated negative binomial distribution**, presented in the National Seminar during November 08-10, 2018 organized by the Department of Statistical Sciences, Kannur University, Kannur,

Kerala.

13. **A new generalization of discrete Weibull distribution**, presented in the Fourth International Conference on Statistics for Twenty-First Century-2018 (ICSTC-2018) during December 13-15, 2018 organized by the Department of Statistics, University of Kerala, Thiruvananthapuram, Kerala.

****