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CERTIFICATE

I hereby certify that the thesis entitled "**Some Problems on Čech Closure Spaces**" is a bonafide work carried out by **Ms. Kavitha T.** under my guidance for the award of the degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**Some Problems on Čech Closure Spaces**" is based on the original work done by me under the supervision of **Dr. Ramachandran P. T.**, Associate Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

Chapter 0

Introduction

In this thesis we study some problems on Čech closure spaces. Čech closure operators were introduced by Edward Čech by weakening the idempotent condition of Kuratowski closure operators.

0.1 Motivation and Survey of Literature

Associated with the topological space (X, T) we can define a closure operator $c : P(X) \to P(X)$ where $c(A)$ is the smallest closed set containing A for each $A \in P(X)$. Then empty set is a fixed point of c, c is idempotent and c commutes with finite unions. Also if we are given a function $\theta : P(X) \to P(X)$ such that θ is an expansive operator, empty set is a fixed point of θ, θ is idempotent and θ commutes with finite unions, then there exists a unique topology T on X such that θ coincides with the closure operator associated with T. By relaxing the idempotent condition of a closure operator associated with a topology Čech E. introduced a Čech closure operator. According to him, a Čech closure operator V on a set X is an operator on $P(X)$ such that empty set is a fixed point of V, V is expansive and V commutes with finite unions [11]. An ordered pair (X, V) where V is a Čech closure operator is called a Čech closure space.

Every closure operator associated with a topology is a Čech closure operator but, not every Čech closure operator is a closure operator associated with a topology. A closure operator associated with a topology is also called Kuratowski closure operator. A Čech closure operator is a Kuratowski closure operator if and only if it is idempotent. This way we can regard Čech closure space as a generalization of topological spaces. For convenience, we call Čech closure operators as closure operators and Čech closure spaces as closure spaces. The theory of closure spaces was extensively studied by Čech E. in [11].

We can define a partial order in the set of all closure operators on a set X as $V_1 \leq V_2$ if and only if $V_2(A) \subseteq V_1(A)$ for every $A \subseteq X$ [11]. The set of all closure operators on a fixed non empty set X forms a complete lattice $L(X)$ under this partial order [11]. The indiscrete closure operator I is the smallest element and discrete closure operator D is the largest element of $L(X)$. Ramachandran P.T. determined atoms and dual atoms in the lattice of closure operators [32]. He studied the reflexive relation associated with a closure operator in [32]. Also proved that the lattice $L(X)$ of closure operators on a set X is complemented if and only if X is finite [33, 37]. Moreover $L(X)$ is a dually atomic and modular lattice [32].

Agashe P. and Levine N. introduced the concept of immediate successor and immediate predecessor in the lattice of topologies [2]. They proved that a completely normal first countable T_1 topology cannot have an immediate successor in the lattice of topologies. Also some properties of adjacent topologies were investigated in [2]. Analogous to the concept of immediate successors and immediate predecessors, Kunheenkutty M. introduced the concept of upper neighbours and lower neighbours of closure operators in the lattice of closure operators [24]. The study of upper neighbours and lower neighbours helps us to understand the lattice structure and to locate a closure operator. He determined the adjacency of co-finite closure operators in $L(X)$. The co-finite closure operator has a lower neighbour whereas it has no upper neighbour in the lattice of closure operators [24].

Alas O. T. and Wilson R. G. gave a characterization of those topologies which have an upper neighbour in the lattice of T_1 topologies on a set and showed that certain classes of compact and countably compact topologies do not have an upper neighbour [3]. In [4], authors discussed different sub posets of the lattice of T_1 topologies and decided which elements have and which do not have an upper neighbour.

In an atomic modular lattice every element other than the greatest element has an upper neighbour and in a dually atomic modular lattice every element other than the smallest element has a lower neighbour [19]. Thus every element of $L(X)$ other than the Indiscrete closure operator has a lower neighbour in $L(X)$. Also every element of $L(X)$ other than the discrete closure operator has an upper neighbour when X is a finite set.

The concept of simple extensions in the lattice of topologies were introduced by Levine N. [29]. This is a method of constructing strong topologies from a given topology. For a topology T on a set X, the simple extension of T by $A \subseteq X$ is the smallest topology containing A and is denoted by $T(A)$. Levine N. compared certain properties like regularity, normality, second countability and connectedness of topology with its simple extensions [29].

Ramachandran P. T. discussed the problem of representing permutation groups as the group of homeomorphisms of topological spaces [32,34]. He proved that if $X = \{a_1, a_2, \ldots, a_n\}$, $n \geq 3$, then the permutation group on X generated by the cycle $(a_1, a_2, \ldots, a_n)$ cannot be represented as the group of homeomorphisms of (X, T) for any topology T on X [32]. Then Sini P. and Ramachandran P. T. defined t-representability of permutation groups and studied t-representability of cyclic subgroups of the symmetric group $S(X)$ [39–41]. A permutation group K on X is said to be t-representable if there exists a topology T on X such that the group $H(X, T)$ of homeomorphisms of (X, T) is K [39]. In [40], it was proved that direct sum of t-representable finite permutation groups is t-representable on X. Kannan V. and Ramachandran P. T. gave several characterizations of hereditarily homogeneous topological spaces [21].

Császár considered the set of all maps $\gamma : P(X) \to P(X)$ such that γ is monotonic. A set $A \subseteq X$ is said to be γ-open if and only if $A \subset \gamma(A)$. Then the set of all γ-open sets forms a generalized topology on X [13]. Corresponding to a generalized topology on X, there exists a closure operator C on X such that

C is expansive, monotonic and idempotent. On the contrary if C is an operator on $P(X)$ such that C is expansive, monotonic and idempotent, then there exists a unique generalized topology on X. Tyagi B. K. and Choudhari R. introduced generalized closure operators on X and they studied generalized interior operator and generalized neighbourhood systems [44].

0.2 Organisation of the Thesis

Besides the Introduction, this Thesis contains 6 chapters. The introductory chapter, **Chapter 0** deals with the motivation and review of literature on adjacency in the lattice of closure operators, simple expansions of closure operators, t-representability of permutation groups and generalized closure operators. **Chapter 1** includes basic definitions and theorems we used in the forthcoming chapters.

In **Chapter 2**, we investigate the existence of upper neighbours of closure operators. We prove that the generalized form of the co-finite closure operator has no upper neighbour in $L(X)$. Upper neighbours of a T_1 closure operator is characterized. We establish that a first countable T_1 closure operator has no upper neighbour in the lattice of closure operators. We study some properties of upper neighbours of closure operators. Adjacency of a closure operator projectively and inductively generated by a function f is also explored in this chapter. If f is a bijection between two sets, then closure operators inductively generated by two adjacent closure operators and f are either equal or adjacent. Further we seek upper neighbours of the sum and finite product of closure operators.

Chapter 3 deals with simple expansions of closure operators. We determine when simple expansion of a topological closure operator becomes the closure operator associated with a topology. Then investigate some properties of the simple expansion of closure operators and compare certain properties of a closure operator with their simple expansion. We prove that the simple expansion of a normal closure operator V at a point x is normal if $V(\{x\}) = \{x\}$. We discuss the equality of simple expansions of closure operators by two distinct subsets at the same point x. Also we introduce countable expansion of closure operators in $L(X)$.

In **Chapter 4**, we investigate which permutation groups can be represented as the group of closure isomorphisms of closure spaces. In order to consider this problem, we define c-representability of permutation groups corresponding to t-representability of permutation groups. A permutation groups H is c-representable if there exists a closure operator V on X such that the group of closure isomorphisms of (X, V) is H. We investigate the c- representability of normal subgroups of the symmetric group $S(X)$. We prove that a proper non-trivial normal subgroup of $S(X)$ is c-representable if and only if $|X| = 3$. We determine the c-representability of the direct sum of an arbitrary family of finite c-representable permutation groups. Here we characterize hereditarily homogeneous T_1 closure spaces.

The lattice of generalized closure operators are studied in **Chapter 5**. We determine atoms and dual atoms in the lattice of generalized closure operators.

Here we study generalized Čech closure operators on a set. A comparison between the lattice of generalized closure operators and the lattice of generalized Čech closure operators on a set X is conducted. Also we introduce adjacency in the lattice of generalized Čech closure operators and prove that co-finite closure operator has no upper neighbour in the same. Simple expansions of generalized closure operators is also introduced.

The **Chapter 6** is the conclusion and some unsolved problems which we met during our study.

Chapter 1

Preliminaries

1.1 Introduction

This chapter deals with some basic definitions and preliminary results in set theory, group theory and Čech closure spaces in order to make the reading of this thesis simpler.

1.2 Set Theory

First let us go through the definition of a partially ordered set.

Definition 1.2.1. [7] Let X be a set and $\leq$ be a binary relation on X. A partially ordered set or a poset is an ordered pair $(X, \leq)$ such that the binary relation $\leq$ satisfies the following conditions:

1. $x \leq x$ for all $x \in X$,

2. If $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in X$,

3. If $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in X$.

Example 1.2.2. Let $\mathbb{Z}^+$ denotes the set of positive integers; and let $x \leq y$ if x divides y. Then $(\mathbb{Z}^+, \leq)$ is a partially ordered set.

Definition 1.2.3. [17] Let $(X, \leq)$ be a partially ordered set, A be a nonempty subset of X. An element $a \in A$ such that $a \leq x$ for all $x \in A$ is called a least element. A greatest element of A is an element $b \in A$ such that $x \leq b$ for all $x \in A$.

Definition 1.2.4. [17] A partially ordered set $(X, \leq)$ is said to be *well-ordered* if for every non empty subset A of X, there exists $x_0 \in A$ such that $x_0 \leq x$ for every $x \in A$.

Remark 1.2.5. Every element in a well-ordered set has an immediate successor except the greatest element.

Theorem 1.2.6. *Well-ordering Theorem [18]*

Every set can be well-ordered.

Theorem 1.2.7. *Principle of transfinite induction [18]*

Let W be a well-ordered set and V a subset of W in which, for every element $x \in W$, satisfies the following condition:

If every predecessor of x belongs to V, then x belongs to V.

Then $V = W$.

Notation. We write $x \vee y$ in place of the supremum of $\{x, y\}$ when it exists and $x \wedge y$ in place of the infimum of $\{x, y\}$ when it exists. Similarly we write $\bigvee S$ and $\bigwedge S$ for supremum and infimum of the set S respectively.

Lattice and complete lattice are defined as follows.

Definition 1.2.8. [7] Let $(X, \leq)$ be a non empty partially ordered set.

1. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$, then $(X, \leq)$ is called a lattice.

2. If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq X$, then $(X, \leq)$ is called a complete lattice.

Remark 1.2.9. The set of natural numbers given in Example 1.2.2 is a lattice but not a complete lattice because the set $\{2, 4, 6, \ldots\}$ has no supremum.

Note 1.2.10. Let $(L, \leq)$ be a lattice. For $a, b \in L$ we say a is an upper neighbour of b or a covers b if $b \leq a$, $a \neq b$ and for every $c \in L$ with $b \leq c \leq a$, either $c = b$ or $c = a$.

An atom of a lattice is an element which covers the least element 0 if it exists. A lattice is atomic if every element other than the least element can be written as the join of atoms. An anti-atom is an element which is covered by the greatest element 1 in the lattice. An anti-atom is also called a dual atom. A lattice is anti-atomic(dually atomic) if every element other than the greatest element can be written as the meet of anti-atoms.

Definition 1.2.11. [7] A lattice $(L, \leq)$ is called a distributive lattice if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$. This is equivalent to saying that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for every $a, b, c \in L$.

Definition 1.2.12. [7] A lattice L is called modular if for any $a, b, c \in L$, $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Remark 1.2.13. [7] Every distributive lattice is modular. But not every modular lattice is distributive.

Definition 1.2.14. [7] The lattice L is called semi-modular if for any $a, b \in L$ with $a \neq b$, and if a and b cover $a \wedge b$, then $a \vee b$ covers a and b.

Definition 1.2.15. [7] Two lattices L and $L^{'}$ are said to be isomorphic, if there exists a function $\phi : L \to L^{'}$ such that ϕ is one-one, onto and $\phi(a \vee b) = \phi(a) \vee \phi(b))$, $\phi(a \wedge b) = \phi(a) \wedge \phi(b))$ for every $a, b \in L$.

An isomorphism of a lattice onto itself is called an automorphism. A lattice is called self dual if it is isomorphic to its dual lattice.

1.3 Group Theory

We know that a binary operation $*$ on a set S is a function from $S \times S$ into S. More generally, for any sets A, B and C, we can view a map $* : A \times B \longrightarrow C$ as defining a multiplication of an element $a \in A$ with an element $c \in C$.

Definition 1.3.1. [16] Let X be a set and $(G, .)$ be a group. An action of

G on X is a function $* : G \times X \longrightarrow X$ such that

1) $ex = x$ for all $x \in X$.

2) $(g_1 g_2)(x) = (g_1)(g_2 x)$ for all $x \in X$ and for all $g_1, g_2 \in X$.

Under these conditions, X is called a G-set.

Definition 1.3.2. [16] A group G is transitive on a G-set X if for all $x, y \in X$, there exists $g \in G$ such that $gx = y$.

Definition 1.3.3. [16] Let X be a G-set, $x \in X$, $g \in G$. Then define $X_g = \{x \in X : gx = x\}$, $G_x = \{g \in G : gx = x\}$. Here G_x is a subgroup of G and is called the isotropy subgroup of X.

Definition 1.3.4. [16] Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Then $\sim$ is an equivalence relation on X. Each cell in the partition of the equivalence relation $\sim$ is an orbit in X under G. If $x \in X$, the cell containing x is the orbit of x, denoted by Gx. Then $Gx = \{gx : \ g \in G\}$.

Theorem 1.3.5. *[16] Let X be a G-set and $x \in X$. Then $|Gx| = (G; G_x)$. If $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.*

Theorem 1.3.6. Burnside's formula: *[16] Let G be a finite group and X be a finite G-set. If r is the number of orbits of X, then $r|G| = \Sigma_{g \in G}|X_g|$.*

Definition 1.3.7. [16] A permutation of a set X is a function $\phi : X \to X$ that is both one-one and onto.

The function composition $\circ$ is a binary operation on the collection of all permutations of a set A. This operation is called permutation multiplication.

Theorem 1.3.8. *[15] The set of all permutations of a set X forms a group under permutation multiplication, denoted by $S(X)$.*

Definition 1.3.9. [15] We write S_n to denote the group $S(X)$ when n is a positive integer and $X = \{1, 2, \ldots, n\}$.

Definition 1.3.10. [15] A permutation group is a subgroup of the symmetric group $S(X)$.

Each permutation p of a set X determines a natural partition of X into cells with the property that $x, y \in X$ are in the same cell if and only if $y = p^n(x)$ for some $n \in \mathbb{Z}$.

Definition 1.3.11. [16] Let p be a permutation of a set X. The equivalence classes in X determined by the equivalence relation $\sim$ given by $x \sim y$ if and only if $y = p^n(x)$ for some $n \in \mathbb{Z}$ are the orbits of p.

Now let us define a cycle.

Definition 1.3.12. [16] A permutation $p \in S_n$ is a cycle if it has at most one orbit containing more than one element. The length of a cycle is the number of elements in its largest orbit.

Note that every permutation p of a finite set is a product of disjoint cycles [16].

Remark 1.3.13. [16] While permutation multiplication in general is not commutative, it is readily seen that multiplication of disjoint cycles is commutative. Since the orbits of a permutation are unique, the representation of a permutation as a product of disjoint cycles, none of which is the identity permutation, is unique up to the order of factors.

Definition 1.3.14. [16] A cycle of length 2 is a transposition.

Note that any permutation of a finite set of at least two elements is a product of transpositions.

Definition 1.3.15. [16] A permutation of a finite set is even or odd according to whether it can be expressed as a product of an even number of transpositions or the product of an odd number of transpositions, respectively.

Definition 1.3.16. [6] Let G be any group. If $x, g \in G$, the element $g^{-1}xg$ is known as the conjugate of x by g and the set $\{g^{-1}xg : g \in G\}$ is called the conjugacy class of x in G.

Theorem 1.3.17. *[16] A subgroup H is the conjugate of a subgroup K of a group G if there exists an element $g \in G$ such that $gHg^{-1} = K$.*

Theorem 1.3.18. *[6] Let X be an infinite set. If $\alpha = |X|$, the cardinality of X, then $|S(X)| = 2^{\alpha}$.*

1.4 Čech Closure Spaces

Let X be a set and $P(X)$ denotes the power set of X.

Definition 1.4.1. [11] A Čech closure operator on a set X is a function $V : P(X) \to P(X)$ satisfying $V(\emptyset) = \emptyset,\ A \subseteq V(A)$ and $V(A \cup B) = V(A) \cup V(B)$ for every $A,\ B \in P(X)$. For convenience, we call V a closure operator on X and the pair (X, V) a closure space.

A subset A of a closure space (X, V) is said to be closed if $V(A) = A$, and is said to be open if its complement is closed. The collection of all open sets in a closure space (X, V) is a topology on X, called the topology associated with V. A closure operator V is said to be topological if and only if $V(V(A)) = V(A)$ for every $A \subseteq X$.

Examples 1.4.2. [11]

1. Let $(X, \leq)$ be a well ordered set. For each $A \subseteq X$, let $V(A)$ be the subset of X consisting of all points of A and the successors of all $x \in A$. The relation $A \to V(A)$ is a closure operator on X. If X contains at least three elements, then V is not a topological closure operator.

2. Let $I : P(X) \to P(X)$ be given by

$$I(A) = \begin{cases} \emptyset & ; \quad \text{if } A = \emptyset, \\ X & ; \quad \text{otherwise.} \end{cases}$$

Then I is a closure operator on X. This closure operator is the topological closure operator associated with the indiscrete topology on X and is called the indiscrete closure operator.

3. The closure operator D on X given by $D(A) = A$ for all $A \in P(X)$, is the topological closure operator associated with the discrete topology on X, called the discrete closure operator.

Definition 1.4.3. [11] Let (X, V) be a closure space. A subset A of X is said to be dense if $V(A) = X$.

Definition 1.4.4. [11] Let (X, V) be a closure space. Then interior operation is a function $int_v : P(X) \to P(X)$ such that $int_v(A) = X - V(X - A)$. Thus int_vA is called the interior of A in (X, V).

Theorem 1.4.5. *[11] In a closure space (X, V), we have the following:*

1. *$int_vX = X$.*

2. *For each $A \subseteq X$. $int_vA \subseteq A$.*

3. *For each $A, B \subseteq X$, $int_v(A \cap B) = int_vA \cap int_vB$.*

Theorem 1.4.6. *[11] A subset A of X is open if and only if $intA = A$.*

Definition 1.4.7. [11] A neighbourhood of a subset A of a space (X, V) is any subset U of X containing A in its interior. Thus U is a neighbourhood of A if and only if $A \subseteq intU$. By a neighbourhood of a point x of X we mean a neighbourhood of the singleton set $\{x\}$.

Remark 1.4.8. [11] Let $\mathscr{U}$ be the neighbourhood system of a subset A of a closure space (X, V). Then $\mathscr{U}$ is a filter on X, the intersection of which contains A that is every element of $\mathscr{U}$ contains A, $\mathscr{U}$ is closed under finite intersections and if $A \in \mathscr{U}$ and $A \subseteq B$, then $B \in \mathscr{U}$.

Definition 1.4.9. [11] Let $\mathscr{U}$ be the neighbourhood system of a subset A of a closure space (X, V). Then $\mathscr{U}$ is a filter on X. The base or sub-base of this filter is called, respectively a base or a sub-base of the neighbourhood system of A in X. The terms a local base at x and a local sub-base at x is used instead of a base and a sub-base of the neighbourhood system of the point x.

Theorem 1.4.10. *[11] Let U and V be two closure operators on a set X such that $U \leq V$. Then every U neighbourhood of $A \subseteq X$ is a V neighbourhood of A.*

For any set X the non empty collection $\mathscr{S}$ of subsets of X forms a stack if $\emptyset \notin \mathscr{S}$ and $A \in \mathscr{S}$, $B \supset A$ implies that $B \in \mathscr{S}$ [38].

Definition 1.4.11. [11] A closure space (X, V) is said to be first countable at a point x if there exists a countable local base at x and is said to be first countable if it is first countable at each point $x \in X$.

Let V_1, V_2 be two closure operators on set X. Then V_1 is said to be coarser than V_2 if $V_2(A) \subseteq V_1(A)$ for every $A \in P(X)$ and is denoted by $V_1 \leq V_2$ [11]. This relation in the set of all closure operators on X is a partial order. The set of all closure operators on X forms a lattice under this partial order and is denoted by $L(X)$ [11]. Also the meet of two closure operators V_1 and V_2 is given by $V_1 \wedge V_2(A) = V_1(A) \cup V_2(A)$ for each $A \subseteq X$ [11]. The least element of this lattice is the indiscrete closure operator I and the greatest element is the discrete closure operator D [11].

Let $\{V_a; a \in \mathscr{A}\}$ where $\mathscr{A}$ is some indexing set, be a non empty collection of closure operators on $L(X)$. A closure operator V which is the infimum of $\{V_a\}$ in $L(X)$ is given by $V(A) = \underset{a\in\mathscr{A}}{\cup} V_a(A)$ for $A \in P(X)$ [11]. If U is the supremum of a non empty collection $\{V_a\}$ in $L(X)$ and $A \in P(X)$, then $x \in U(A)$ if and only if for each finite cover $\{A_1, A_2, \ldots, A_n\}$ of A, there exists an A_i such that $x \in V_a(A_i)$ for each $a \in \mathscr{A}$ [11].
Ramachandran P. T. determined atoms and dual atoms in the lattice of closure operators [32].

Definition 1.4.12. [32] Let $a, b \in X$, $a \neq b$. Define $V_{a,b} : P(X) \to P(X)$ as

$$V_{a,b}(S) = \begin{cases} \emptyset & ; \text{ if } S = \emptyset, \\ X - \{b\} & ; \text{ if } S = \{a\}, \\ X & ; \text{ otherwise.} \end{cases}$$

Then $V_{a,b}$ is a closure operator on X.

Atoms in $L(X)$ are closure operators of the form $V_{a,b}$, $a, b \in X$, $a \neq b$ [32]. Dual atoms in $L(X)$ are closure operators associated with the ultra topologies in the lattice of topologies [32]. Let $P(X - \{a\}) \cup \mathscr{U}$ is an ultra topology on X. Define $V : P(X) \to P(X)$ as $V(A) = \begin{cases} A & ; \text{ if } x \in A \text{ or } X - A \in \mathscr{U} \\ A \cup \{x\} & ; \text{ otherwise.} \end{cases}$
Then V is called an ultra closure operator on X.

Definition 1.4.13. [24, 26] Let X be a set and α be an infinite cardinal number such that $|X| > \alpha$. Define $C_\alpha : P(X) \to P(X)$ as

$$C_\alpha(A) = \begin{cases} A & ; \text{ if } |A| < \alpha, \\ X & ; \text{ otherwise.} \end{cases}$$

Then C_α is a closure operator on X and is the closure operator associated with the topology $\{A \subseteq X : |X - A| < \alpha\} \cup \{\emptyset\}$.

Definition 1.4.14. [11] A semi-pseudometric for a set X is a function $d : X \times X \to \mathbf{R}$ which fulfills the conditions:

1 . For each $x \in X$, $d(x, x) = 0$,

2 . For all $x, y \in X$, $d(x, y) = d(y, x) \geq 0$.

Definition 1.4.15. [11] Let d be a semi-pseudometric for a set X. The relation $V(A) = \{x \in X \,:\, \text{dist}(x, A) = 0\}$, $A \subseteq X$ is called the closure induced by d. A closure operation V is said to be semi-pseudometrizable if V is induced by a semi-pseudometric.

Now let us look at the separation axioms in closure spaces.

Definition 1.4.16. [11] A closure space (X, V) is said to be T_0 if $x \in V(\{y\})$, $y \in V(\{x\})$ implies that $x = y$, and is said to be T_1 if every singleton subset of X is closed in X. Two subsets S_1 and S_2 of X are said to be separated if there exists neighbourhoods U_1 of S_1 and U_2 of S_2 such that $U_1 \cap U_2 = \emptyset$. A closure space (X, V) is said to be separated if any two distinct points of X are separated.

Definition 1.4.17. [11] A closure space (X, V) is said to be regular if for each point $x \in X$ and a subset S of X, such that $x \notin V(S)$, there exists neighbourhoods U_1 of x and U_2 of S such that $U_1 \cap U_2 = \emptyset$.

Definition 1.4.18. [11] A subset S of a closure space (X, V) is said to be connected if S is not the union of two non empty semi separated subsets of (X, V). That is $S = S_1 \cup S_2$, $(V(S) \cup S_2) \cup (S_1 \cap V(S_2)) = \emptyset$ implies that $S_1 = \emptyset$ or $S_2 = \emptyset$.

Definition 1.4.19. [11] A closure space (X, V) is said to be separable if it has a countable dense subset.

Definition 1.4.20. [24, 26] Let U and V be two closure operators on X such that $U < V$. Then V is called an upper neighbour of U if for any closure operator W on X such that $U \leq W \leq V$, then either $W = U$ or $W = V$. Then U and V are said to be adjacent.

Next we look at the definition of closure isomorphism from a closure space $(X,\ V)$.

Definition 1.4.21. [33] A closure isomorphism from a closure space $(X,\ V)$ to another closure space $(Y,\ V')$ is a bijection $f : X \longrightarrow Y$ such that $f(V(A)) = V'(f(A))$ for all $A \in P(X)$.

Remark 1.4.22. If $(X,\ V)$ is a closure space, then the set $CI(X,\ V)$ of all closure isomorphisms from $(X,\ V)$ onto itself is a group and is called the group of closure isomorphisms of $(X,\ V)$. Note that $CI(X,\ V) \leq S(X)$.

Chapter 2

Adjacency in the Lattice of Closure Operators

2.1 Introduction

In this chapter we study upper neighbours and lower neighbours of closure operators. We have the lattice of Čech closure operators on a set X is dually atomic and distributive. Hence every element of $L(X)$ other than the Indiscrete closure operator has a lower neighbour [19]. When X is a finite set, every element in $L(X)$ other than D has an upper neighbour. Our aim is to investigate the existence of upper neighbours in $L(X)$. We mainly concentrate on T_1 closure operators on an infinite set X. Here we study some properties of upper neighbours of closure operators too.

2.2 Adjacent Closure Operators

Here we make a platform for proving the non existence of upper neighbours of a first countable T_1 closure operator. In this section we generalize the definition of closure operators of the form $V_{A,x}$ and seek the existence of their upper neighbour. Recall the definition of upper neighbour of a closure operator.

Definition 2.2.1. [24, 26] Let U and V be two closure operators on a set X such that $U < V$. Then V is said to be an upper neighbour of U if for closure operator W on X such that $U \leq W \leq V$, either $W = U$ or $W = V$.

Remark 2.2.2. If V is an upper neighbour of U, then U is called a lower neighbour of V. In this case, U and V are said to be adjacent.

Examples 2.2.3. (a) Closure operators of the form $V_{a,b}$ and the indiscrete closure operator I are adjacent.

(b) The discrete closure operator D and ultra closure operators are adjacent.

(c) Let $X = \{a_1, a_2, a_3\}$. Define $U : P(X) \to P(X)$ as $U(\emptyset) = \emptyset$, $U(\{a_1\}) = \{a_1, a_2\}$, $U(\{a_2\}) = \{a_2\}$, $U(\{a_3\}) = \{a_3, a_1\}$ and $U(A) = \underset{a_i \in A}{\cup} U(\{a_i\})$ where $A \subseteq X$. Then U is a closure operator on X. Define $V : P(X) \to P(X)$ as $V(\emptyset) = \emptyset$, $V(\{a_1\}) = \{a_1\}$, $V(\{a_2\}) = \{a_2\}$, $V(\{a_3\}) = \{a_3, a_1\}$ and $V(A) = \underset{a_i \in A}{\cup} V(\{a_i\})$, $A \subseteq X$. Then V is a closure operator on X and V is an upper neighbour of U.

(d) Let X be an infinite set and let $A \subseteq X$ such that $|A| \geq 2$. Define V :

$P(X) \to P(X)$ as

$$V(S) = \begin{cases} \emptyset & ; \quad S = \emptyset, \\ A & ; \quad \text{if } S \subseteq A, S \neq \emptyset, \\ X & ; \quad \text{otherwise.} \end{cases}$$

Then V is a topological closure operator on X.

Let $a, b \in A$. Define $V' : P(X) \to P(X)$ as

$$V'(S) = \begin{cases} \emptyset & ; \quad S = \emptyset, \\ A \setminus \{a\} & ; \quad \text{if } S = \{b\}, \\ A & ; \quad \text{if } S \subseteq A, \ S \neq \emptyset \ \text{ and } S \neq \{b\}, \\ X & ; \quad \text{otherwise.} \end{cases}$$

Then $V'(S) \subseteq V(S)$ for every $S \subseteq X$. Also $V < V'$ since, $V'(\{b\}) = A \setminus \{a\} \subset V(\{b\}) = A$. Moreover $V'(V'(\{b\})) = V'(A \setminus \{a\}) = A \neq V'(A)$. Hence V' is not a topological closure operator. Suppose W is a closure operator on X such that $V \leq W \leq V'$. Then $V'(\{b\}) \subseteq W(\{b\}) \subseteq V(\{b\})$. Thus $A \setminus \{a\} \subseteq W(A) \subseteq A$. If $W \neq V$, then $W = V'$ and if $W \neq V'$, then $W = V$. That is V' is an upper neighbour of V.

(e) Let X be any set and fix some nonempty $A \subseteq X$ such that $|A| \geq 2$ and some $x \notin A$. Define $U_{A,x} : P(X) \to P(X)$ such that

$$U_{A,x}(S) = \begin{cases} \emptyset & ; \quad \text{if } S = \emptyset, \\ A \cup \{x\} & ; \quad \text{if } S \subseteq A, S \neq \emptyset, \\ X & ; \quad \text{otherwise.} \end{cases}$$

Then $U_{A,x}$ is a closure operator on X. Now $U_{A,x}(U_{A,x}(A)) = U_{A,x}(A \cup \{x\}) = X \neq U_{A,x}(A)$. Thus $U_{A,x}$ is not a topological closure operator. Now let us find an upper neighbour for $U_{A,x}$. Let $a \in A$. Then define $U : P(X) \to P(X)$ as

$$U(S) = \begin{cases} \emptyset & ; \quad \text{if } S = \emptyset, \\ A & ; \quad \text{if } S = \{a\}, \\ A \cup \{x\} & ; \quad \text{if } S \subseteq A, S \neq \emptyset \text{ and } S \neq \{a\}, \\ X & ; \quad \text{otherwise.} \end{cases}$$

U is not a topological closure operator on X. Also U is an upper neighbour of $U_{A,x}$. The following closure operator is a topological upper neighbour of $U_{A,x}$,

$$V(S) = \begin{cases} \emptyset & ; \quad S = \emptyset, \\ A & ; \quad \text{if } S \subseteq A, \ S \neq \emptyset, \\ X & ; \quad \text{otherwise.} \end{cases}$$

Ramachandran P. T. introduced closure operators of the form $V_{A,x} \in [C_0, D]$ where A is an infinite subset of X such that $x \notin A$. Kunheenkutty M. studied adjacency of $V_{A,x}$ and proved that $V_{A,x}$ has no upper neighbour in $L(X)$ [24].

Definition 2.2.4. [35] Let $x \in X$. Suppose A is an infinite subset of X

not containing x. Define $V_{A,x} : P(X) \to P(X)$ as

$$V_{A,x}(S) = \begin{cases} S & ; \text{ if } S \text{ is finite,} \\ X \setminus \{x\} & ; \text{ if } S \text{ is infinite, } S \setminus A \text{ finite and } x \notin S, \\ X & ; \text{ otherwise.} \end{cases}$$

Then $V_{A,x}$ is a closure operator on X and $C_0 < V_{A,x}$.

Lemma 2.2.5. *[24] Let $A \subseteq X$ such that $x \notin A$. Then $V_{A,x} \leq V$ if and only if $x \notin V(A)$.*

Theorem 2.2.6. *[24, 26] Let A be an infinite subset of X and $x \in X \setminus A$. Then $V_{A,x}$ has no upper neighbour in $L(X)$.*

Theorem 2.2.7. *Suppose A and B are two infinite subsets of X such that $x \notin A \cup B$ and $A \cap B$ is an infinite set. Then $V_{A,x} \wedge V_{B,x} = V_{A \cap B,x}$.*

Proof. Suppose $x \notin V_{A,x} \wedge V_{B,x}(S)$. Then $x \notin V_{A,x}(S)$ and $x \notin V_{B,x}(S)$. This implies that $S \setminus A$ and $S \setminus B$ are finite. This implies that $(S \setminus A) \cup (S \setminus B)$ is finite. That is $S \setminus (A \cap B)$ is finite and therefore $x \notin V_{A \cap B,x}(S)$. Hence $V_{A \cap B,x}(S) \subseteq V_{A,x} \wedge V_{B,x}(S)$. That means $V_{A,x} \wedge V_{B,x} \leq V_{A \cap B,x}$. In order to prove the reverse inclusion, it is enough to prove that $x \notin V_{A,x} \wedge V_{B,x}(A \cap B)$ by Lemma 2.2.5. We have $V_{A,x} \wedge V_{B,x}(A \cap B) = V_{A,x}(A \cap B) \cup V_{B,x}(A \cap B)$. Since $V_{A,x}(A \cap B) \subseteq V_{A,x}(A) = X \setminus \{x\}$, $x \notin V_{A,x}(A \cap B)$. Similarly $V_{B,x}(A \cap B) \subseteq V_{B,x}(B) = X \setminus \{x\}$. Then $x \notin V_{B,x}(A \cap B)$. Hence $x \notin V_{A,x} \wedge V_{B,x}(A \cap B)$. Thus $V_{A,x} \wedge V_{B,x} = V_{A \cap B,x}$. □

Remark 2.2.8. We can generalize the Theorem 2.2.7 for a finite collection of subsets $A_1, A_2, \ldots, A_n$ not containing x such that $A_1 \cap A_2 \cap \ldots \cap A_n$ is infinite.

Theorem 2.2.9. *If $A_1, A_2, \ldots, A_n$ is a finite collection of infinite subsets of X such that $x \notin A_i$ for any $i \in \{1, 2, \ldots, n\}$, then*
$V_{A_1,x} \bigvee V_{A_2,x} \bigvee \ldots \bigvee V_{A_n,x} = V_{A_1 \cup A_2 \cup \ldots \cup A_n, x}$.

Proof. Suppose $x \notin V_{A_i,x}(S)$. This implies that $S \setminus A_i$ is finite. We have $S \setminus (A_1 \cup A_2 \cup \ldots \cup A_n) \subseteq S \setminus A_i$ for each i. Therefore $S \setminus A_i$ is finite implies that $S \setminus (A_1 \cup A_2 \cup \ldots \cup A_n)$ is finite. That is $x \notin V_{A_1 \cup A_2 \cup \ldots \cup A_n, x}(S)$. Hence $V_{A_i,x} \leq V_{A_1 \cup A_2 \cup \ldots \cup A_n, x}$ for $i = 1, 2, \ldots, n$. Hence $V_{A_1,x} \bigvee V_{A_2,x} \bigvee \ldots \bigvee V_{A_n,x} \leq V_{A_1 \cup A_2 \cup \ldots \cup A_n, x}$. To prove $V_{A_1 \cup A_2 \cup \ldots \cup A_n, x} \leq V_{A_1,x} \bigvee V_{A_2,x} \bigvee \ldots \bigvee V_{A_n,x}$, it is enough to prove that $x \notin V_{A_1,x} \bigvee V_{A_2,x} \bigvee \ldots \bigvee V_{A_n,x}(A_1 \cup A_2 \cup \ldots \cup A_n)$ by Lemma 2.2.5. We have $x \notin V(A_i, x)(A_i)$ for $i = 1, 2, \ldots, n$. Therefore we get
$x \notin V_{A_1,x} \bigvee V_{A_2,x} \bigvee \ldots \bigvee V_{A_n,x}(A_1 \cup A_2 \cup \ldots \cup A_n)$.
Hence $V_{A_1 \cup A_2 \cup \ldots \cup A_n, x} \leq V_{A_1,x} \bigvee V_{A_2,x} \bigvee \ldots \bigvee V_{A_n,x}$. □

Remark 2.2.10. Theorem 2.2.9 is not true for an infinite collection of subsets of X. For example, let $X = \mathbb{Z}$ and A_k $k = 1, 2, \ldots$ denotes the set of all integers n not equal to 1 and 2^k does not divide n. Then $1 \in \bigvee_{k=1}^{\infty} V_{A_k,1}(A_2 \cup 8\mathbb{Z})$, but $1 \notin V_{\bigcup_{k=1}^{\infty} A_k, 1}(A_2 \cup 8\mathbb{Z})$. Hence $\bigvee_{k=1}^{\infty} V_{A_k,1} \neq V_{\bigcup_{k=1}^{\infty} A_k, 1}$.

Now we generalize Definition 2.2.4 for an infinite collection of infinite subsets of X as $V_{A_1, A_2, \ldots, x}$ and prove that $V_{A_1, A_2, \ldots, x}$ has no upper neighbour in $L(X)$.

Definition 2.2.11. Let $A_1, A_2, \ldots$ be infinite subsets of X such that $A_1 \subseteq A_2 \subseteq \ldots$ and $x \notin A_i$ for any positive integer i, then define

$$V_{A_1,A_2,\ldots,x}(S) = \begin{cases} S & ; \text{ if S is finite,} \\ X \setminus \{x\} & ; \text{ if S is infinite, } x \notin S \text{ and } S \setminus A_i \text{ finite for} \\ & \quad \text{some positive integer } i, \\ X & ; \text{ otherwise.} \end{cases}$$

Then $V_{A_1,A_2,\ldots,x}$ is a closure operator on X such that $C_0 < V_{A_1,A_2,\ldots,x}$.

Remark 2.2.12.

1. If $x \notin V_{A_i,x}(S)$, then $S \setminus A_i$ is finite. Therefore $x \notin V_{A_1,A_2,\ldots,x}(S)$. Thus $V_{A_i,x} \leq V_{A_1,A_2,\ldots,x}$ for each $i = 1, 2, \ldots$.

2. Suppose $A_1, A_2, \ldots$ be a disjoint collection of infinite subsets of X such that $x \notin \underset{i \in I}{\cup} A_i$. Then $V_{A_1,A_2,\ldots,x}(A_1) = X \setminus \{x\}$, $V_{A_1,A_2,\ldots,x}(A_2) = X \setminus \{x\}$. But $V_{A_1,A_2,\ldots,x}(A_1 \cup A_2) = X \neq V_{A_1,A_2,\ldots,x}(A_1) \cup V_{A_1,A_2,\ldots,x}(A_2)$. Hence in this case $V_{A_1,A_2,\ldots,x}$ is not a closure operator on X. Therefore the condition that $A_1 \subseteq A_2 \subseteq \ldots$ cannot be dropped from the Definition 2.2.

Let us look at when $V_{A_1,A_2,\ldots,x}$ becomes topological.

Theorem 2.2.13. *Let $A_1, A_2, \ldots$ be a monotonically increasing sequence of infinite subsets of X and $x \notin A_i$ for any positive integer i. Then a closure operator $V_{A_1,A_2,\ldots,x}$ is topological if and only if $X \setminus A_i$ is finite for some positive integer i.*

Proof. Suppose that $V_{A_1,A_2,\ldots,x}$ is a topological closure operator. Then for every positive integer j, we have $V_{A_1,A_2,\ldots,x}(V_{A_1,A_2,\ldots,x}(A_j)) = V_{A_1,A_2,\ldots,x}(A_j)$. This implies that $V_{A_1,A_2,\ldots,x}(X \setminus \{x\}) = X \setminus \{x\}$. Thus $X \setminus A_i$ is finite for some positive integer i.

Now suppose that $V_{A_1,A_2,\ldots,x}$ is not topological. Then there exists an infinite subset S of X such that $V_{A_1,A_2,\ldots,x}(V_{A_1,A_2,\ldots,x}(S)) = X$ but $V_{A_1,A_2,\ldots,x}(S) = X\setminus\{x\}$. Then $X \setminus A_i$ is infinite for every i. Hence if $X \setminus A_i$ is finite for some positive integer i, then $V_{A_1,A_2,\ldots,x}$ is topological. □

Remark 2.2.14. When $X \setminus A_i$ finite for some i, then $V_{A_1,A_2,\ldots,x}(V_{A_1,A_2,\ldots,x}(X \setminus \{x\})) = X \setminus \{x\}$. That is $\{x\}$ is an open set. Therefore if $V_{A_1,A_2,\ldots,x}$ is topological, then the topology associated with $V_{A_1,A_2,\ldots,x}$ is given by the topology generated by $C_0 \cup \{x\}$ where C_0 is the co-finite topology on X.

Theorem 2.2.15. *Let $A_1, A_2, \ldots$ be a monotonically increasing sequence of infinite subsets of X and $x \notin A_i$ for any positive integer i. Then $V_{A_1,A_2,\ldots,x} = \bigvee_{i=1}^{\infty} V_{A_i,x}$.*

Proof. If S is finite, then we have $V_{A_1,A_2,\ldots,x}(S) = S = \bigvee_{i=1}^{\infty} V_{A_i,x}(S)$. Now suppose that S is an infinite subset of X and $x \notin V_{A_i,x}(S)$. Then $S \setminus A_i$ is finite and $x \notin V_{A_1,A_2,\ldots,x}(S)$. Hence $V_{A_1,A_2,\ldots,x}(S) \subseteq V_{A_i,x}(S)$. That is $V_{A_i,x} \leq V_{A_1,A_2,\ldots,x}$. Then $\bigvee_{i=1}^{\infty} V_{A_i,x} \leq V_{A_1,A_2,\ldots,x}$.

Conversely suppose that $x \in \bigvee_{i=1}^{\infty} V_{A_i,x}(S)$, $x \notin S$. Then for every finite cover $\{S_1, S_2, \ldots S_n\}$ of S, there exists S_i such that $x \in V_{A_j,x}(S_i)$ for $j = 1, 2, \ldots$.

Then $S_i \setminus A_j$ is infinite for $j = 1, 2, \ldots$. Thus $S \setminus A_j$ is infinite for $j = 1, 2, \ldots$. Hence $x \in V_{A_1,A_2,\ldots,x}(S)$. It follows that $\bigvee_{i=1}^{\infty} V_{A_i,x}(S) \subseteq V_{A_1,A_2,\ldots,x}(S)$. Hence we get $V_{A_1,A_2,\ldots,x} \leq \bigvee_{i=1}^{\infty} V_{A_i,x}$. □

Lemma 2.2.16. *Let V be a closure operator on X such that $C_0 < V$. Then $V_{A_1,A_2,\ldots,x} \leq V$ if and only if $x \notin V(A_i)$ for any positive integer i.*

Proof. If $V_{A_1,A_2,\ldots,x} \leq V$, then $V(A_i) \subseteq V_{A_1,A_2,\ldots,x}(A_i) = X \setminus \{x\}$ for every positive integer i. Thus $x \notin V(A_i)$ for any positive integer i. Conversely suppose that $x \notin V(A_i)$ for any positive integer i. If $x \notin V(A_i)$ then $V_{A_i,x} \leq V$ by Lemma 2.2.5. That is $V_{A_i,x} \leq V$ for $i = 1, 2, \ldots$. Hence $V_{A_1,A_2,\ldots,x} \leq V$ by Theorem 2.2.15. □

Lemma 2.2.17. *If $A_i \Delta B_j$ is finite for every i, j, then $V_{A_1,A_2,\ldots,x} = V_{B_1,B_2,\ldots,x}$.*

Proof. Suppose $A_i \Delta B_j$ is finite for every i, j. Then $x \notin V_{A_1,A_2,\ldots,x}(S) \Rightarrow S \setminus A_i$ is finite for some i. That is if and if $S \setminus B_j$ is finite for some j. That is if and only if $x \notin V_{B_1,B_2,\ldots,x}(S)$. □

Remark 2.2.18. Let X be a set and $x \in X$. Suppose that $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are monotonically increasing sequences of infinite subsets of X such that $x \notin A_i$ and $x \notin B_i$ for any positive integer i.

1. $V_{A_1,A_2,\ldots,x} \leq V_{B_1,B_2,\ldots,x}$ implies that $x \notin V_{B_1,B_2,\ldots,x}(A_i)$ for every positive integer i. That is $V_{A_1,A_2,\ldots,x} \leq V_{B_1,B_2,\ldots,x}$ implies that for every A_i there exists a B_j such that $A_i \setminus B_j$ is finite. Similarly $V_{B_1,B_2,\ldots,x} \leq V_{A_1,A_2,\ldots,x}$

implies that for every B_k there exists A_l such that $B_k \setminus A_l$ is finite. Thus $V_{A_1,A_2,\ldots,x} = V_{B_1,B_2,\ldots,x}$ implies that for every A_i, there exists a B_j and for every B_j there exists A_k such that $A_i \setminus B_j$ and $B_j \setminus A_k$ are finite for $i,j,k = 1,2,\ldots$.

Theorem 2.2.19. *Let X be any set and $A_1, A_2, \ldots$ be infinite subsets of X such that $A_1 \subseteq A_2 \subseteq \ldots$ and $x \notin A_i$ for any positive integer i. Let B be any subset of X such that $B\Delta A_i$ is infinite for every positive integer i. Then $V_{A_1,A_2,\ldots,x} \bigvee V_{B,x} = V_{A_1\cup B,A_2\cup B,\ldots,x}$.*

Proof. Let $x \notin [V_{A_1,A_2,\ldots,x} \bigvee V_{B,x}](S)$. Then by the definition of join of closure operators, either $x \notin V_{A_1,A_2,\ldots,x}(S)$ or $x \notin V_{B,x}(S)$. But

$$\begin{aligned} x \notin V_{A_1,A_2,\ldots,x}(S) \text{ or } x \notin V_{B,x}(S) &\Rightarrow (S \setminus A_i) \text{ is finite for some } i = 1,2,\ldots \text{ or} \\ &\qquad (S \setminus B) \text{ is finite.} \\ &\Rightarrow (S \setminus A_i) \cap (S \setminus B) \text{ is finite for some} \\ &\qquad i = 1,2,\ldots \\ &\Rightarrow S \setminus (A_i \cup B) \text{ is finite for some } i = 1,2,\ldots \\ &\Rightarrow x \notin V_{A_1\cup B,A_2\cup B,\ldots,x}(S). \end{aligned}$$

This implies that $V_{A_1\cup B,A_2\cup B,\ldots,x}(S) \subseteq [V_{A_1,A_2,\ldots,x} \bigvee V_{B,x}](S)$ for $S \subseteq X$. Hence $[V_{A_1,A_2,\ldots,x} \bigvee V_{B,x}] \leq V_{A_1\cup B,A_2\cup B,\ldots,x}$.

In order to prove the reverse inclusion it is enough to prove that

$x \notin (V_{A_1,A_2,\ldots,x} \bigvee V_{B,x})(A_i \cup B)$ for any i. We have $x \notin V_{A_1,\ldots,x}(A_i)$ for any positive integer i and $x \notin V_{B,x}(B)$. Thus $x \notin (V_{A_1,A_2,\ldots,x} \bigvee V_{B,x})(A_i)$ and $x \notin (V_{A_1,A_2,\ldots,x} \bigvee V_{B,x})(B)$.

That is $x \notin (V_{A_1,A_2,\ldots,x} \bigvee V_{B,x})(A_i \cup B)$ for $i = 1, 2, \ldots$. Hence by Lemma 2.2.16, we have $V_{A_1 \cup B, A_2 \cup B, \ldots, x} \leq [V_{A_1,A_2,\ldots,x} \bigvee V_{B,x}]$. Therefore $[V_{A_1,A_2,\ldots,x} \bigvee V_{B,x}] = V_{A_1 \cup B, A_2 \cup B, \ldots, x}$. □

Theorem 2.2.20. *Let X be any set and $A_1, A_2, \ldots$ be infinite subsets of X such that $A_1 \subseteq A_2 \subseteq \ldots$ and $x \notin A_i$ for any positive integer i. Let B be any subset of X such that $B \Delta A_i$ is infinite for every positive integer i and $A_i \cap B$ is infinite for every i. Then $V_{A_1,A_2,\ldots,x} \wedge V_{B,x} = V_{A_1 \cap B, A_2 \cap B, \ldots, x}$.*

Proof. Let $x \notin [V_{A_1,A_2,\ldots,x} \wedge V_{B,x}](S)$. Then $x \notin V_{A_1,A_2,\ldots,x}(S) \cup V_{B,x}(S)$. Then either $x \notin V_{A_1,A_2,\ldots,x}(S)$ and $x \notin V_{B,x}(S)$. But

$$
\begin{aligned}
x \notin V_{A_1,A_2,\ldots,x}(S) \text{ and } x \notin V_{B,x}(S) &\Rightarrow (S \setminus A_i) \text{ is finite for some } i = 1, 2, \ldots \text{ and} \\
&\qquad (S \setminus B) \text{ is finite.} \\
&\Rightarrow (S \setminus A_i) \cup (S \setminus B) \text{ is finite for some} \\
&\qquad i = 1, 2, \ldots \\
&\Rightarrow S \setminus (A_i \cap B) \text{ is finite for some } i = 1, 2, \ldots \\
&\Rightarrow x \notin V_{A_1 \cap B, A_2 \cap B, \ldots, x}(S).
\end{aligned}
$$

This implies that $V_{A_1 \cap B, A_2 \cap B, \ldots, x}(S) \subseteq [V_{A_1,A_2,\ldots,x} \wedge V_{B,x}](S)$ for $S \subseteq X$.

Hence $[V_{A_1,A_2,\ldots,x} \wedge V_{B,x}] \leq V_{A_1 \cap B, A_2 \cap B, \ldots, x}$.

In order to prove that $V_{A_1\cap B,A_2\cap B,\ldots,x} \leq V_{A_1,A_2,\ldots,x} \wedge V_{B,x}$, it is enough to prove that $x \notin (V_{A_1,A_2,\ldots,x} \wedge V_{B,x})(A_i \cap B)$ for each i. We have $(V_{A_1,A_2,\ldots,x} \wedge V_{B,x})(A_i \cap B) = V_{A_1,A_2,\ldots,x}(A_i \cap B) \bigcup V_{B,x}(A_i \cap B)$. As $A_i \cap B \subseteq A_i$ for each i, clearly $V_{A_1,\ldots,x}(A_i \cap B) \subseteq V_{A_1,\ldots,x}(A_i)$. Since $x \notin V_{A_1,\ldots,x}(A_i)$ for any positive integer i, $x \notin (V_{A_1,A_2,\ldots,x}(A_i \cap B)$. Similarly $A_i \cap B \subseteq B$ implies that $V_{B,x}(A_i \cap B) \subseteq V_{B,x}(B)$. Since $x \notin V_{B,x}(B)$, we have $x \notin V_{B,x}(A_i \cap B)$. Thus we get $x \notin V_{A_1,A_2,\ldots,x}(A_i\cap B) \bigcup V_{B,x}(A_i\cap B)$. This implies that $x \notin (V_{A_1,A_2,\ldots,x}\wedge V_{B,x})(A_i\cap B)$ for any positive integer i. Therefore by Lemma 2.2.16, we have $V_{A_1\cap B,A_2\cap B,\ldots,x} \leq [V_{A_1,A_2,\ldots,x} \wedge V_{B,x}]$. Hence $[V_{A_1,A_2,\ldots,x} \wedge V_{B,x}] = V_{A_1\cap B,A_2\cap B,\ldots,x}$. □

We conclude this section by proving that $V_{A_1,A_2,\ldots,x}$ has no upper neighbour in the lattice of closure operators.

Theorem 2.2.21. *Let $A_1, A_2, \ldots$ be infinite subsets of X such that $A_1 \subseteq A_2 \subseteq \ldots$. Suppose $x \notin A_i$ for any positive integer i and $A_{i+1} \setminus A_i$ is infinite for every positive integer i. Then a closure operator of the form $V_{A_1,A_2,\ldots,x}$ has no upper neighbour in $L(X)$.*

Proof. Let A_1, $A_2, \ldots$ be infinite subsets of X such that $x \notin A_i$ for any positive integer i. So $V_{A_1,x} \leq V_{A_1,A_2,\ldots,x}$. Suppose that $V_{A_1,A_2,\ldots,x}$ has an upper neighbour V in $L(X)$. For a finite set S, $V_{A_1,A_2,\ldots,x}(S) = V(S)$. Since $V \neq V_{A_1,A_2,\ldots,x}$, there exists an infinite subset A of X such that $V(A) = X\setminus\{x\}$ but $V_{A_1,A_2,\ldots,x}(A) = X$. Thus $A \setminus A_i$ is infinite for every i. We can choose a countable infinite subset B_i of the infinite sets $A\setminus A_i$ for each positive integer i. Let $B_i = \{a_{i1}, a_{i2}, \ldots\}$. Now consider a subset

$C_i = \{a_{in} : n \text{ is an even positive integer}\}$. Then $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$

Let $V' = V_{C_1^c \setminus \{x\}, C_2^c \setminus \{x\}, \dots, x}$. Then $V'(C_1) = X$, but $V(C_1) \subseteq V(A) = X \setminus \{x\}$ and $V'(C_1^c \setminus \{x\}) = X \setminus \{x\}$, but $V_{A_1, A_2, \dots, x}(C_1^c \setminus \{x\}) = X$.

Note that

$$C_i \subseteq A \setminus A_i \Rightarrow X \setminus (A \cap A_i^c) \subseteq X \setminus C_i \Rightarrow A_i \subseteq C_i^c, \text{ for } i = 1, 2, \dots.$$

Since $A_i \subseteq C_i^c \setminus \{x\}$, for $i = 1, 2, \dots$, $S \setminus (C_i^c \setminus \{x\}) \subseteq S \setminus A_i$ for $i = 1, 2, \dots$. Thus we have the following implications.

$$\begin{aligned} x \notin V_{A_1, A_2, \dots, x}(S) &\Rightarrow (S \setminus A_i) \text{ is finite for some } i. \\ &\Rightarrow S \setminus (C_i^c \setminus \{x\}) \text{ is finite for some } i. \\ &\Rightarrow x \notin V^{'}(S). \end{aligned}$$

Hence $V^{'}(S) \subseteq V_{A_1, A_2, \dots, x}(S)$ for $S \subseteq X$. That is $V_{A_1, A_2, \dots, x} \leq V^{'}$.

Also $V_{B_1^c \setminus \{x\}, B_2^c \setminus \{x\}, \dots x} \leq V'$ and $V_{A_1, A_2, \dots, x} \leq V_{B_1^c \setminus \{x\}, B_2^c \setminus \{x\}, \dots x}$. That is $V_{A_1, A_2, \dots, x} \leq V_{B_1^c \setminus \{x\}, B_2^c \setminus \{x\}, \dots x} \leq V'$. Note that $V'(C_1) = X$ but $V(C_1) = X \setminus \{x\}$. Let $x \notin V_{A_1, A_2, \dots, x}(S)$. Then $S \setminus A_i$ is finite for some i. Choose sufficiently large j such that $i \leq j$. Then $S \setminus (A_j)$ is finite. Consider the closure operator $V' \wedge V_{A_j, x}$. Then $V_{A_1, A_2, \dots, x} \leq V' \wedge V_{A_j, x} \leq V$. This completes the proof of the theorem. □

2.3 On First Countable T_1 Closure Operators

In this section we characterize upper neighbours of T_1 closure operators and prove that a first countable T_1 closure operator has no upper neighbour in $L(X)$. First of all we decompose a T_1 closure operator by defining closure operators of the form V^x for each $x \in X$.

Definition 2.3.1. Let V be a T_1 closure operator on a set X. Let $x \in X$. Define V^x on $P(X)$ by

$$V^x(A) = \begin{cases} A & ; \text{ if } A \text{ is finite,} \\ X \setminus \{x\} & ; \text{ if } A \text{ is infinite, } x \notin V(A), \\ X & ; \text{ otherwise.} \end{cases}$$

Then V^x is a closure operator on X.

Remark 2.3.2. We have $x \notin V^x(A) \Rightarrow x \notin V(A)$. Hence $V^x \leq V$ for each $x \in A$.

Lemma 2.3.3. *Let V be a T_1 closure operator on a set X. Then* $V = \bigvee_{x \in X} V^x$.

Proof. Let $V^{'} = \bigvee_{x \in X} V^x$. We have by definition of V^x, $V(A) \subseteq V^x(A)$ for every subset A of X. Then $V^x \leq V$. Hence $V^{'} \leq V$.

Now we have to prove that $V \leq V^{'}$. Suppose not. Then there exists an infinite subset A of X such that $V^{'}(A) \nsubseteq V(A)$. Thus there exists $y \in V^{'}(A)$ such that $y \notin V(A)$. Since $y \notin V(A)$, $V^y(A) = X \setminus \{y\}$. Now $y \in V^{'}(A)$

implies that for every finite cover $\{A_1, A_2, \ldots, A_n\}$ of A, there exists A_i such that $y \in V^x(A_i)$ for every $x \in X$. In particular for the cover $\{A\}$, $y \in V^y(A)$. Hence $V^y(A) = X$. This is a contradiction to the fact that $V^y(A) = X \setminus \{y\}$. Hence $V \leq V'$. □

Next is a characterization theorem for upper neighbours of T_1 closure operators.

Theorem 2.3.4. *A T_1 closure operator V on a set X has an upper neighbour on $L(X)$ if and only there exists an $x \in X$ such that V^x has an upper neighbour.*

Proof. Suppose V has an upper neighbour W on $L(X)$. Then $V < W$. Then there exists some $S \subseteq X$ such that $W(S) \subset V(S)$. Thus we can find $x \in X$ such that $x \notin W(S)$ but $x \in V(S)$. We have $W = \bigvee\limits_{x \in X} W^x$ by Lemma 2.3.3. We have the following sequence of implications for an infinite subset A:

$$x \in W^x(A) \Rightarrow x \in W(A) \Rightarrow x \in V(A) \Rightarrow V^x(A) = X \Rightarrow x \in V^x(A).$$

Thus $W^x(A) \subseteq V^x(A)$ and hence $V^x \leq W^x$. Also $x \notin W(S) \Rightarrow x \notin W^x(S)$. But $x \in V(S) \Rightarrow x \in V^x(S)$. Hence $V^x < W^x$. Now let U be a closure operator on X such that $V^x \leq U \leq W^x$. Then

$$V = \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee V^x \leq \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee U \leq \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee W^x \leq W.$$

Since W is an upper neighbour of V, either $V^x = U$ or $U = W^x$. Thus W^x is an upper neighbour of V^x.

Conversely assume that W is an upper neighbour of V^x for some $x \in X$. Consider $U = \bigvee_{y \in X \setminus \{x\}} V^y \bigvee W$. We have $V = \bigvee_{x \in X} V^x$ and $W = \bigvee_{x \in X} W^x$ by Lemma 2.3.3. Since $V^x < W$, $V < U$. Suppose that there exists a closure operator U' on X such that $V \leq U' \leq U$. Then $x \notin V^x(S) \Rightarrow x \notin V(S) \Rightarrow x \notin U'(S) \Rightarrow x \notin U'^x(S)$. Thus $V^x \leq U'^x$. Similarly we can see that $U'^x \leq U$. Also $x \notin U(S) \Leftrightarrow x \notin \bigvee_{y \in X \setminus \{x\}} V^y(S)$ or $x \notin W(S)$. Since $V^y(S) = X \setminus \{y\}$ and $y \neq x$, we have $x \in \bigvee_{y \in X \setminus \{x\}} V^y(S)$. Therefore $x \notin W(S)$. Hence $U'^x = W$. □

Theorem 2.3.5. *Let V be a first countable T_1 closure operator on a set X. Then V has no upper neighbour in $L(X)$.*

Proof. Let V be a first countable T_1 closure operator on X. Then $V = \bigvee_{x \in X} V^x$, by Lemma 2.3.3. Let $x \in X$. Suppose $L = \{A_1, A_2, \ldots\}$ denotes a countable local base at x. Let $A_1' = A_1$, $A_2' = A_1 \cap A_2$, $A_3' = A_1 \cap A_2 \cap A_3$, ... Thus $A_1' \supseteq A_2' \supseteq A_3' \supseteq \ldots$. That is $L' = \{A_1', A_2', \ldots\}$ is a monotonically decreasing countable local base at x.

Take $B_1 = A_1'$. Let B_2 be the first A_i', $i > 1$ such that $A_1' \setminus A_i'$ is infinite. Now choose B_3 as first $A_j', j > i$ such that $B_2 \setminus A_j'$ is infinite. Continuing this process we arrive at one of the two cases.

Case 1: Above process terminates after a finite stage. That is there exists a positive integer k such that $A_k' \setminus A_{k+j}'$ is finite for $j = 1, 2, \ldots$.

Observe that $x \notin V(X \setminus A'_m)$, hence $x \notin V^x(X \setminus A'_m)$. It follows that from Lemma 2.2.5 that $V_{X\setminus A'_m,x} \leq V^x$. Suppose that $V^x \nleq V_{X\setminus A'_m,x}$. Then there exists an infinite subset $M \subseteq X$ such that $V_{X\setminus A'_m,x}(M) \nsupseteq V^x(M)$. Thus $V_{X\setminus A'_m,x}(M) = X$ and $V^x(M) = X \setminus \{x\}$. From the definition of $V_{X\setminus A'_m,x}$, it follows that $M \setminus (X \setminus A'_m) = M \cap A'_m$ is infinite. Since $x \notin V^x(M)$, we obtain $x \notin V(M)$, hence $X \setminus M$ is a neighbourhood of x. There exists $l \geqslant m$ such that $A'_l \subseteq X \setminus M$. Then $A'_l \subseteq A'_m$ and $M \cap A'_m = M \cap A'_l \cup (A'_m) \setminus A'_l \subseteq (M \cap A'_l) \cup (A'_m \setminus A'_l) = A'_m \setminus A'_l$, a contradiction, since $M \cap A'_m$ is infinite and $A'_m \setminus A'_l$ is finite. Since $B_k \setminus A'_m$ is infinite and $B_k \setminus A'_m \subseteq X \setminus A'_m$, we have $X \setminus A'_m$ is infinite. Thus $V^x = V_{X\setminus A'_m,x}$. So V^x has no upper neighbour in $L(X)$.

Case 2: Above process can be continued and we get an infinite collection $\{B_1, B_2, \ldots\}$ of subsets of X such that $B_i \setminus B_{i+1}$ is infinite for $i = 1, 2, \ldots$.

In this case we prove that $V^x = V_{B_1^c,\, B_2^c,\ldots,x}$. We have $A'_i \in L'$. Since L' is a local base at x, $x \notin V(X \setminus A'_i)$ for $i = 1, 2, \ldots$. When S is a finite subset of X, then $V^x(S) = S = V_{B_1^c,\, B_2^c,\ldots,x}(S)$.

Now suppose A is an infinite subset of X. If $x \notin V^x(A)$ then $x \notin V(A)$ by the definition of V^x. Then $X \setminus A$ is a neighbourhood of x in (X, V). Hence there exists a $A'_m \in L'$ such that $A'_m \subseteq X \setminus A$, for some positive integer m. That is $A \subseteq X \setminus A'_m$. We can choose B_k from the infinite collection $\{B_1, B_2, \ldots\}$ so that $B_k \subseteq A'_m$. Then $A \subseteq X \setminus A'_m \subseteq X \setminus B_k$. Thus $A \subseteq B_k^c$, for the positive integer k. Thus $A \setminus B_k^c = \phi$. So $V_{B_1^c,\, B_2^c,\ldots,x}(A) = X \setminus \{x\}$ by definition. In other words $x \notin V_{B_1^c,\, B_2^c,\ldots,x}(A)$. So $V_{B_1^c,\, B_2^c,\ldots,x}(A) \subseteq V^x(A)$ for every infinite subset A of X.

Hence $V^x \leq V_{B_1^c,\, B_2^c,\ldots,x}$.

Next we have to prove that $V_{B_1^c,\, B_2^c,\ldots,x} \leq V^x$. We have $V_{B_1^c,\, B_2^c,\ldots,x} \leq V^x$ if and only if $x \notin V^x(B_i^c)$ for $i = 1, 2, \ldots$, by Lemma 2.2.16. If $x \in V^x(B_i^c)$ for some i, then $x \in V(B_i^c)$. Thus $x \in V(X \setminus A_j^{'})$ for some positive integer j. This is a contradiction since $A_j^{'}$ is a neighbourhood of x with respect to V. Hence $V_{B_1^c,B_2^c,\ldots,x} = V^x$.

By Theorem 2.2.21, $V_{B_1^c,\, B_2^c,\ldots,x}$ has no upper neighbour and therefore V^x has no upper neighbour. In either case, V^x has no upper neighbour. Now V is first countable at each $x \in X$. Hence by Theorem 2.2.6 and 2.2.21, we have V^x has no upper neighbour in $L(X)$ for any $x \in X$. Thus by Lemma 2.3.4, V has no upper neighbour in $L(X)$. □

Remark 2.3.6. Suppose V is a first countable T_1 closure operator on a set X. If $Y \subseteq X$, then $V|_Y$ has no upper neighbour on $L(Y)$, since first countability and T_1 properties are hereditary properties.

Corollary 2.3.7. *A metrizable closure operator has no upper neighbour in the lattice of closure operators.*

Proof. A metrizable space is first countable and separated. Hence by Theorem 2.3.5, V has no upper neighbour. □

2.4 Upper Neighbours and Lower Neighbours of C_α

Kunheenkutty M. proved that co-finite closure operator C_0 has no upper neighbour in $L(X)$ [24]. Here we examine the existence of upper and lower neighbours of generalized form of the co-finite closure operator on the sub lattice $[C_0, D]$ of $L(X)$.

Definition 2.4.1. Let X be any set. Let $\alpha \leq |A|$ and x an element of $(X \setminus A)$. Then define $C^{\alpha}_{A,x} : P(X) \longrightarrow P(X)$ as,

$$C^{\alpha}_{A,x}(S) = \begin{cases} S & ; \text{ if } |S| < \alpha, \\ X \setminus \{x\} & ; \text{ if } |S| \geq \alpha, |S \setminus A| < \alpha \text{ and } x \notin S, \\ X & ; \text{ otherwise.} \end{cases}$$

Then $C^{\alpha}_{A,x}$ is an element of $L(X)$ and $C_\alpha < C^{\alpha}_{A,x}$ and thus $C^{\alpha}_{A,x} \in (C_\alpha, D)$.

Remark 2.4.2. If $B \subseteq A \subseteq X$ such that $\alpha \leq |B|$. Then $x \notin C^{\alpha}_{B,x}(S) \Rightarrow |S \setminus A| < \alpha$ and $x \notin S \Rightarrow |S \setminus A| < \alpha$ and $x \notin S \Rightarrow x \notin C^{\alpha}_{A,x}(S)$. Thus $C^{\alpha}_{A,x}(S) \subseteq C^{\alpha}_{B,x}(S)$ for every $S \subseteq X$. Hence $C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x}$.

Lemma 2.4.3. *The closure operator $C^{\alpha}_{A,x}$ is topological if and only if $|X \setminus A| < \alpha$.*

Proof. Suppose $C^{\alpha}_{A,x}$ is topological. Then $C^{\alpha}_{A,x}(C^{\alpha}_{A,x}(S)) = C^{\alpha}_{A,x}(S)$ for every subset S of X. Now $C^\alpha_{A,x}(C^\alpha_{A,x}(A)) = C^\alpha_{A,x}(X \setminus \{x\})$. This implies that $|(X \setminus \{x\}) \setminus A| < \alpha$. That is $|X \setminus A| < \alpha$.

Conversely suppose that $|X \setminus A| < \alpha$. If $|S| < \alpha$, then $C^\alpha_{A,x}(C^\alpha_{A,x}(S)) = C^\alpha_{A,x}(S)$. If $|S| \geq \alpha$, $|S \setminus A|$ finite and $x \notin S$, then $C^\alpha_{A,x}(C^\alpha_{A,x}(S)) = C^\alpha_{A,x}(X \setminus \{x\}) = X \setminus \{x\} = C^\alpha_{A,x}(S)$. Also if $|S| \geq \alpha$ and $|S \setminus A|$ infinite, then $C^\alpha_{A,x}(C^\alpha_{A,x}(S)) = C^\alpha_{A,x}(X) = X = C^\alpha_{A,x}(S)$. □

Remark 2.4.4. We denote $C^\alpha_{A,x}$ by C^α_x when $|X \setminus A| < \alpha$. When $|X \setminus A| < \alpha$, $C^\alpha_{A,x}(X \setminus \{x\}) = X \setminus \{x\}$. That is $\{x\}$ is open in $(X, C^\alpha_{A,x})$. Thus the topology associated with $C^\alpha_{A,x}$ when $|X \setminus A| < \alpha$ is given by $C_\alpha \cup \{\{x\}\}$.

Theorem 2.4.5. *If A and B are two distinct subsets of X such that $\alpha \leq |A|$ and $\alpha \leq |B|$ and $x \in X \setminus (A \cup B)$, then $C^\alpha_{A,x}$ and $C^\alpha_{B,x}$ are equal if and only if $|A \Delta B| < \alpha$.*

Proof. Suppose $|A \Delta B| < \alpha$. Then $|A \setminus B| < \alpha$ and $|B \setminus A| < \alpha$. Thus we have $|S \setminus A| < \alpha$ if and only if $|S \setminus B| < \alpha$. Thus $C^\alpha_{A,x}$ and $C^\alpha_{B,x}$ are equal, by definition.

Conversely assume that $C^\alpha_{A,x} = C^\alpha_{B,x}$. Then $C^\alpha_{A,x}(A) = C^\alpha_{B,x}(A) = X \setminus \{x\}$. Hence $|A \setminus B| < \alpha$. Also $C^\alpha_{A,x}(B) = C^\alpha_{B,x}(B) = X \setminus \{x\}$. Then $|B \setminus A| < \alpha$. Thus $|A \Delta B| < \alpha$. □

Lemma 2.4.6. *If $A \subset X$, $|A| \geq \alpha$, and $x \in X \setminus A$, then there is a proper subset D of A such that $C^\alpha_{D,x} < C^\alpha_{A,x}$.*

Proof. Since $|A| \geq \alpha$, there exists a subset B of A with $|B| = \alpha$. Since $\alpha + \alpha = \alpha$, B has a subset D with $|D| = \alpha$ and $|B \Delta D| = \alpha$. Hence by Theorem 2.4.5, $C^{\alpha}_{D,x} < C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x}$. □

Lemma 2.4.7. *Let V be a closure operator on X such that $C_\alpha < V$. Then $C^{\alpha}_{A,x} \leq V$ if and only if $x \notin V(A)$.*

Proof. Suppose $C^{\alpha}_{A,x} \leq V$. Then $V(A) \subseteq C^{\alpha}_{A,x}(A) = X \setminus \{x\}$. That is $x \notin V(A)$. Conversely suppose that $x \notin V(A)$. In order to prove $C^{\alpha}_{A,x} \leq V$, it is enough to prove that $x \notin C^{\alpha}_{A,x}(S)$ implies that $x \notin V(S)$ for every $S \subseteq X$. Suppose $x \notin C^{\alpha}_{A,x}(S)$. Then $|S| \geq \alpha$, $|S \setminus A| < \alpha$ and $x \notin S$. We have $S = (S \setminus A) \cup (S \cap A)$. Therefore $V(S) = V(S \setminus A) \cup V(S \cap A)$. Since $S \cap A \subseteq A$ and $x \notin V(A)$, $x \notin V(S \cap A)$. Since $|S \setminus A| < \alpha$, $V(S \setminus A) = S \setminus A$. We have $x \notin S$, therefore $x \notin V(S \setminus A)$. Hence $x \notin V(S)$ and therefore $C^{\alpha}_{A,x} \leq V$. □

Theorem 2.4.8. *The closure operator C_α has no upper neighbour in the lattice of closure operators.*

Proof. Let V be any closure operator in $L(X)$ satisfying $C_\alpha < V$. When $|S| < \alpha$, $C_\alpha(S) = S$. Choose $A \subset X$ with $|A| \geq \alpha$ and $V(A) \subset C_\alpha(A) = X$. Thus there is $x \in X \setminus V(A)$. Hence $C^{\alpha}_{A,x} \leq V$ by Lemma 2.4.7. Now by Lemma 2.4.6, there are closure operators $C^{\alpha}_{D,x}$ and $C_{B,x}$ such that $C^{\alpha}_{D,x} < C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x}$. Hence $C_\alpha < C^{\alpha}_{D,x} < C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x} \leq V$. □

Remark 2.4.9. If $\alpha = \aleph_0$, then C_α is the co-finite closure operator and thus Theorem 2.4.8 says that co-finite closure operator has no upper neighbour in the lattice of closure operators.

Theorem 2.4.10. *Let V be any closure operator on X such that $C_\alpha < V$. Then V is the join of all closure operators of the form $C^\alpha_{A,x} \leq V$ where $A \subseteq X$ such that $|A| \geq \alpha$ and $x \in X \setminus A$.*

Proof. Let U be the join of all closure operators of the form $C^\alpha_{A,x} \leq V$. Then we have $U \leq V$. If $U \neq V$, then there is a $S \subseteq X$ and $x \in U(S)$, but $x \notin V(S)$. That is $x \notin S$. If $|S| > \alpha$, then $V(S) = U(S) = S$, therefore $|S| \geq \alpha$. Since $x \notin V(S)$, $C^\alpha_{S,x} \leq V$. Consequently, we have that $C^\alpha_{S,x} \leq U$. So by Lemma 2.4.6, we get $x \notin U(S)$, which is a contradiction. Hence the result. □

Theorem 2.4.11. *Let $x \in X$. Then C^α_x has no upper neighbour in $L(X)$.*

Proof. Let $V \in L(X)$ such that $C^\alpha_x < V$. Then there is $A \subset X$ such that $V(A) \subset C^\alpha_x(A)$. Then there is a $y \in C^\alpha_x(A)$ such that $y \notin V(A)$. Then $y \neq x$. If $y = x$, then we have $x \notin A$ and $x \in C^\alpha_x(A) = X$. That is $|X \setminus A| \geq \alpha$. Further we claim that A is infinite. For, if A is finite, $y \notin C^\alpha_x(A) = A$, a contradiction. Then $C^\alpha_{A,y} \leq V$, since $y \notin V(A)$. There is an infinite proper subset C of A such that $C^\alpha_{D,y} < C^\alpha_{A,y}$ by Lemma 2.4.6. Let $V' = C^\alpha_x \bigvee C^\alpha_{D,y}$. Then $C^\alpha_x < V$ and $C^\alpha_{D,y} < C^\alpha_{A,y} \leq V$ implies that $V' = C^\alpha_x \bigvee C^\alpha_{D,y} < V$. Also $C^\alpha_x < C^\alpha_x \bigvee C^\alpha_{D,y} = V'$. Thus C^α_x does not have an upper neighbour in $L(X)$. □

Theorem 2.4.12. *Let $A \subseteq X$ such that $|A| \geq \alpha$ and $x \in X \setminus A$. Then $C^\alpha_{A,x}$ has no upper neighbour in $L(X)$.*

Proof. Let V be an upper neighbour of $C^\alpha_{A,x}$. Then there exists $B \subset X$ such that $|B| \geq \alpha$ and $V(B) \subset C^\alpha_{A,x}(B)$. This implies that we can find $y \in C^\alpha_{A,x}(B)$ such that $y \notin V(B)$. Suppose $y \neq x$. Then $y \notin V(B)$ we get that $C^\alpha_{B,y} \leq V$ by Lemma 2.4.7. Again by Lemma 2.4.6, there exists a proper subset D of B such that $C^\alpha_{D,y} < C^\alpha_{B,y} \leq V$. Let $V' = C^\alpha_{A,x} \bigvee C^\alpha_{D,y}$. Since $C^\alpha_{A,x} < V$ and $C^\alpha_{D,y} < V$, we get $V' < V$. That is $C^\alpha_{A,x} < V' < V$, which is a contradiction.
Now suppose $y = x$. Then $x \in C^\alpha_{A,x}(B)$. This implies that $|B \setminus A| \geq \alpha$. Let D be a proper subset of $B \setminus A$ such that $|D| = \alpha$. Let D_1 be a subset of D such that $|D_1| = |D|$. Then $C^\alpha_{D_1,x} < C^\alpha_{B,x}$. But $x = y \notin V(B)$.
Then $C^\alpha_{B,x} \leq V$. Thus $C^\alpha_{D_1,x} < C^\alpha_{B,x} \leq V$. Consider $C^\alpha_{D_1,x} \bigvee C^\alpha_{A,x}$.
That is, $C^\alpha_{A,x} < C^\alpha_{D_1,x} \bigvee C^\alpha_{A,x}$ and $C^\alpha_{D_1,x} \leq C^\alpha_{D_1,x} \bigvee C^\alpha_{A,x}$. Thus $C^\alpha_{A,x} \leq C^\alpha_{D_1,x} \bigvee C^\alpha_{A,x} \leq V$. This is a contradiction to the fact that V is an upper neighbour of $C^\alpha_{A,x}$. Hence we can conclude that $C^\alpha_{A,x}$ has no upper neighbour in $L(X)$. □

Let us conclude this section by examining the existence of lower neighbours of C_α.

Definition 2.4.13. Let $a, b \in X$ with $a \neq b$. Define a function $C^\alpha_{a,b} : P(X) \to P(X)$ as,

$$C^\alpha_{a,b}(S) = \begin{cases} S & ; \quad \text{if } |S| < \alpha \; a \notin S, \\ S \cup \{b\} & ; \quad \text{if } |S| < \alpha, a \in S, \\ X & ; \quad \text{if } |S| \geq \alpha. \end{cases}$$

Then $C^\alpha_{a,b}$ is a closure operator on X and $C^\alpha_{a,b} < C_\alpha$.

Theorem 2.4.14. *The closure operator $C^\alpha_{a,b}$ is a lower neighbour of C_α and any lower neighbour of C_α is of the form $C^\alpha_{a,b}$.*

Proof. Let V be any closure operator satisfying $C^\alpha_{a,b} < V \leq C_\alpha$. Thus $C_\alpha(S) \subseteq V(S) \subset C^\alpha_{a,b}(S)$. Then for any S with $|S| \geq \alpha$, $V(S) = X$. If $x \in X$, $x \neq a$, then $\{x\} \subseteq V(\{x\}) \subseteq C^\alpha_{a,b}(\{x\})$. Thus $V(\{x\}) = \{x\}$ for every $x \in X$, $x \neq a$. Also since $C^\alpha_{a,b} < V$, we have $\{a\} \subseteq V(\{a\}) \subset C^\alpha_{a,b}(\{a\}) = \{a, b\}$. So we get $V(\{a\}) = \{a\}$. Hence $V = C_\alpha$ and $C^\alpha_{a,b}$ is a lower neighbour of C_α.
Now suppose that $V \in L(X)$ and $V < C_\alpha$, then there is some $a \in X$ with $V(\{a\}) \neq \{a\}$. If $|S| \geq \alpha$, since $V < C_\alpha$, we have $C^\alpha_{a,b}(S) = C_\alpha(S) = X \subseteq V(S)$. Thus $V(S) = X$. If $S \subset X$, $|S| < \alpha$ and $a \notin S$, then $S = C^\alpha_{a,b}(S) = C_\alpha(S) \subseteq V(S)$. If $|S| < \alpha$ and $a \in S$, then $b \in V(S)$ and $C^\alpha_{a,b}(S) = S \cup \{b\}$. Hence $C^\alpha_{a,b}(S) = S \cup \{b\} \subseteq V(S)$. Thus $V \leq C^\alpha_{a,b}$. □

Remark 2.4.15. If $\alpha = \aleph_0$, Kunheenkutty M. determined lower neighbours of C_0, the co-finite closure operator, denoted by $W_{a,b}$ and proved that any lower neighbour of C_0 is of the form $W_{a,b}$, where $W_{a,b}$ is given by

$$W_{a,b}(S) = \begin{cases} S & ; \text{ if S is finite and } a \notin S, \\ S \cup \{b\} & ; \text{ if S is finite and } a \in S, \\ X & ; \text{ if S is infinite.} \end{cases}$$

2.5 Properties of Adjacent Closure Operators

We are concerned with the problem of existence of upper neighbours so far. Now, assuming the existence of adjacent closure operators we check some properties of adjacent closure operators. Adjacency of adjacent closure operators projectively and inductively generated by a function is explored in this section.

2.5.1 Projectively Generated Closure Operators

In this section we investigate adjacency of projectively generated closure operators.

Definition 2.5.1. [11] Let X and Y be any two sets and $f : X \to Y$ be a function. A closure operator U for a set X is said to be projectively generated by f if U is the coarsest closure operator on X such that f is continuous. Then U is given by $U(A) = f^{-1}(V(f(A)))$, for all $A \subseteq X$ where V is the closure operator on Y.

Lemma 2.5.2. *Let X and Y be sets. Let V and V' be closure operators on Y. Let $f : X \to Y$ be any function. If $V \leq V'$, then the closure operators U, U' projectively generated by $f : X \to (Y, V)$ and $f : X \to (Y, V')$ are such that $U \leq U'$.*

Proof. Suppose U is projectively generated by $f : X \to (Y, V)$. Then $U(A) = f^{-1}(V(f(A)))$ for all $A \subseteq X$. We have $V'(f(A)) \subseteq V(f(A))$ for all $A \subseteq X$.

Hence $U^{'}(A) \subseteq U(A)$ for all $A \subseteq X$. Hence $U \leq U^{'}$. $\square$

Example 2.5.3. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Let $f : X \rightarrow Y$ be defined as $f(1) = a$, $f(2) = b = f(3)$ and $f(4) = c$. Define $V : P(Y) \rightarrow P(Y)$ as $V(\emptyset) = \emptyset$, $V(\{a\}) = \{a, b\}$, $V(\{b\}) = \{b, c\}$ and $V(\{c\}) = \{a, c\}$ and $V(S) = \underset{s \in S}{\cup} V(\{s\})$ for $S \subseteq Y$. Consider another closure operator V' on Y defined as $V'(\emptyset) = \emptyset$, $V'(\{a\}) = \{a\}$, $V'(\{b\}) = \{b, c\}$ and $V'\{c\} = \{a, c\}$ and $V(S) = \underset{s \in S}{\cup} V'(\{s\})$ where $S \subseteq Y$. Then the closure operator projectively generated by V is given by $U(\emptyset) = \emptyset$, $U(\{1\}) = \{1, 2, 3\}$, $U(\{2\}) = \{2, 3, 4\}$, $U(\{3\}) = \{2, 3, 4\}$ and $U(\{4\}) = \{1, 4\}$ and $U(S) = \underset{s \in S}{\cup} U(\{s\})$, $S \subseteq X$. And the closure operator U' projectively generated by V' is given by $U'(\emptyset) = \emptyset$, $U'(\{1\}) = \{1\}$, $U'(\{2\}) = \{2, 3, 4\}$, $U'(\{3\}) = \{2, 3, 4\}$, $U'(\{4\}) = \{1, 4\}$ and $U'(S) = \underset{s \in S}{\cup} U'(\{s\})$, $S \subseteq X$. Thus the closure operators projectively generated by $f : X \rightarrow (Y, V)$ and $f : X \rightarrow (Y, V')$ are not adjacent even though V and V' are adjacent.

Example 2.5.4. Let $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$. Let $f : X \rightarrow Y$ is given by $f(a) = a,\ f(b) = c,\ f(c) = d$. Suppose $V : P(Y) \rightarrow P(Y)$ be given by $V(\phi) = \phi$, $V(\{a\}) = \{a, b, c\}$, $V(\{b\}) = \{b, c\}$, $V(\{c\}) = \{c\}$, $V(\{d\}) = \{a, d\}$ and $V(S) = \underset{s \in S}{\cup} V(\{s\})$ for $S \subseteq Y$. Then the closure operator U projectively generated by V is given by $U(\phi) = \phi$, $U(\{a\}) = \{a, b\}$, $U(\{b\}) = \{b\}$, $U(\{c\}) = \{a, c\}$ and $U(S) = \underset{s \in A}{\cup} U(\{s\})$ for $S \subseteq X$. We have $V' : P(Y) \rightarrow P(Y)$ given by $V'(\phi) = \phi$, $V'(\{a\}) = \{a, b\}$,$V'(\{b\}) = \{b, c\}$, $V'(\{c\}) = \{c\}$, $V'(\{d\}) = \{a, d\}$ and $V(S) = \underset{s \in S}{\cup} V'(\{s\})$ for $S \subseteq Y$. Then the closure operator U' projectively generated by V' is given by $U'(\phi) = \phi$, $U'(\{a\}) = \{a\}$, $U'(\{b\}) = \{b\}$, $U'(\{c\}) = \{c\}$ and $U'(S) = \underset{s\in S}{\cup} U'(\{s\})$ for $S \subseteq X$. It follows that U' is not an upper neighbour of U even though V' is an upper neighbour of V and f is a one-one function.

Theorem 2.5.5. *Suppose $f : X \to Y$ be a bijection. Let V_1 and V_2 are two closure operators on X such that V_2 is an upper neighbour of V_1. Then the closure operators U_1 and U_2 on Y which are projectively generated by $f : X \to (Y, V_1)$ and $f : X \to (Y, V_2)$ respectively are either equal or adjacent.*

Proof. Since $V_1 < V_2$, we have $U_1 \leq U_2$ by Lemma 2.5.2. Assume that $U_1 \neq U_2$. Suppose that there exists a closure operator U on X such that $U_1 \leq U \leq U_2$. Define $V : P(Y) \to P(Y)$ as $V(S) = S \cup f(U(f^{-1}(S)))$ for $S \subseteq Y$. We prove that $V_1 \leq V \leq V_2$. We have $U(f^{-1}(S)) \subseteq U_1(f^{-1}(S))$ for every $S \subseteq Y$. Then $V(S) = S \cup f(U(f^{-1}(S)) \subseteq f(U_1(f^{-1}(S)))$ and $f(U_1(f^{-1}(S)) = f(f^{-1}(V_1(f(f^{-1}(S))))) = V_1(S)$, since f is onto. That is $V(S) \subseteq V_1(S)$ and $V_1 \leq V$. Similarly note that $U_2(f^{-1}(S)) \subseteq U(f^{-1}(S))$ for each subset S of Y. This implies that $S \cup f(U_2(f^{-1}(S))) \subseteq S \cup f(U(f^{-1}(S))) = V(S)$. But, $f(U_2(f^{-1}(S))) = f(f^{-1}(V_2(f(f^{-1}(S))))) = V_2(S)$. Then $V_2(S) \subseteq V(S)$ and $V \leq V_2$. Since V_2 is an upper neighbour of V_1, we have either $V = V_1$ or $V = V_2$. Then $V = V_1$ implies that $V(f(S)) = V_1(f(S))$ for each subset S of X. Then $f(S) \cup f(U(f^{-1}(f(S)))) = V_1(f(S)) \Rightarrow U(S) = f^{-1}(V_1(f(S))) = U_1(S)$. Similarly we can prove that $U(S) = U_2(S)$ for each $S \subseteq X$ if $V = V_2$. Thus U_2 is an upper neighbour of U_1. $\square$

Remark 2.5.6. Converse of the Theorem 2.5.5 is not true.

Example 2.5.7. Let $X = Y = \{a, b, c\}$. Let $f : X \to Y$ defined as $f(a) = f(b) = a$, $f(c) = c$. Then f is not a bijection. Let V be a closure operator on Y given by $V(\emptyset) = \emptyset$, $V(\{a\}) = \{a, b\}$, $V(\{b\}) = \{b, c\}$, $V(\{c\}) = \{c\}$ and $V(S) = \underset{s\in S}{\cup} V(\{s\})$ for $S \subseteq Y$. Then the closure operator U on X projectively generated by V is given by $U(\emptyset) = \emptyset$, $U(\{a\}) = \{a, b\}$, $U(\{b\}) = \{a, b\}$, $U(\{c\}) = \{c\}$ and $V(S) = \underset{s\in S}{\cup} V(\{s\})$ for $S \subseteq X$. Now $V' : P(Y) \to P(Y)$ given by $V'(\emptyset) = \emptyset$, $V'\{a\} = \{a, b\}$, $V'\{b\} = \{b\}$, and $V'\{c\} = \{c\}$ and $V'(S) = \underset{s\in S}{\cup} V'(\{s\})$ for $S \subseteq Y$ is an upper neighbour of V on $L(Y)$. Then the closure operator U' projectively generated by $f : X \to (Y, V')$ is the closure operator U itself.

2.5.2 Inductively Generated Closure Operators

This sub section deals with adjacency properties of closure operators inductively generated by a function f.

Definition 2.5.8. [11] Let V be a closure operator on X. A closure operator inductively generated by f is the finest closure operator on Y which makes f continuous [11]. Let $f : (X, V) \to Y$ be the given function. Then the closure operator U which is inductively generated by f is given by
$U(A) = A \cup f(V(f^{-1}(A)))$ where $A \subseteq Y$.

Lemma 2.5.9. *Let X and Y be sets. Let V_1 and V_2 be closure operators on X. Let $f : X \to Y$ be any function. If $V_1 \leq V_2$, then the closure operators $U, U^{'}$ inductively generated by $f : (X, V_1) \to Y$ and $f : (X, V_2) \to Y$ are such that $U \leq U^{'}$.*

Proof. Suppose $V_1 \leq V_2$. Then $V_2(S) \subseteq V_1(S)$ for every $S \subseteq X$. Then $V_2(f^{-1}(S)) \subseteq V_1(f^{-1}(S))$ which implies that $S \cup f(V_2(f^{-1}(S)) \subseteq S \cup f(V_1(f^{-1}(S))$. This implies that $U_1 \leq U_2$. □

Remark 2.5.10. Two closure operators U and V are adjacent does not imply that a closure operator $U^{'}$ inductively generated by the function $f : (X, U) \to Y$ and a closure operator $V^{'}$ inductively generated by $f : (X, V) \to Y$ are adjacent.

Example 2.5.11. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b\}$. Let f be a function from $X \to Y$ defined as $f(1) = f(2) = a$ and $f(3) = f(4) = b$. Let $V : P(X) \to P(X)$ be defined as $V(\emptyset) = \emptyset$ $V(\{1\}) = \{1, 2\}$, $V(\{2\}) = \{2, 3\}$, $V(\{3\}) = \{3, 4\}$, $V(\{4\}) = \{4, 1\}$, $V(S) = \underset{s \in S}{\cup} V(\{s\})$, $S \subseteq X$. Then the closure operator inductively generated by V is given by $U : P(Y) \to P(Y)$ as $U(\emptyset) = \emptyset$, $U(\{a\}) = \{a\} \cup f(V(f^{-1}(\{a\}))) = \{a\} \cup f(V(\{1, 2\})) = \{a\} \cup f(\{1, 2, 3\}) = \{a, b\}$ and $U(\{b\}) = \{b\} \cup f(V(f^{-1}\{b\})) = \{b\} \cup f(\{3, 4, 1\}) = \{a, b\}$. Now define $V^{'} : P(X) \to P(X)$ as $V^{'}(\emptyset) = \emptyset$, $V^{'}\{1\} = \{1\}$, $V^{'}\{2\} = \{2, 3\}$, $V^{'}\{3\} = \{3, 4\}$, $V(\{4\}) = \{4, 1\}$ and $V(S) = \underset{s \in S}{\cup} V(\{s\})$, $S \subseteq X$. Then $V^{'}$ is an upper neighbour of V on $L(X)$. Then the closure operator inductively generated by $V^{'}$ is given by $U^{'} : P(Y) \to P(Y)$ by $U^{'}\{a\} = \{a, b\}$ and $U^{'}\{b\} = \{a, b\}$. Thus $U^{'} = U$.

The following theorem says that when f is a bijection, then the closure operators inductively generated by adjacent closure operators are either equal or adjacent.

Theorem 2.5.12. *Suppose $f : X \to Y$ be a bijection. Suppose V_1 and V_2 are two closure operators on X which are adjacent. Then the closure operators U_1 and U_2 on Y which are inductively generated by V_1 and V_2 respectively are either equal or adjacent.*

Proof. We are given V_1 and V_2 are adjacent. Let $V_1 < V_2$. Then $U_1 \leq U_2$ by Lemma 2.5.9. Suppose $U_1 \neq U_2$. Now suppose that W is a closure operator on Y such that $U_1 \leq W \leq U_2$. Let $A \subseteq X$, define $V'(A) = A \cup f^{-1}(W(f(A)))$. Then V' is a closure operator on X. We prove that $V_1 \leq V' \leq V_2$. Note that $W(f(A)) \subseteq U_1(f(A))$. Then $f^{-1}(W(f(A))) \subseteq f^{-1}(U_1(f(A)))$. But $U_1(f(A)) = f(A) \cup f(V_1(f^{-1}(f(A))))$. Since f is one-one we have, $V'(A) \subseteq V_1(A)$ for every $A \subseteq X$. Hence $V_1 \leq V'$. Now $U_2(f(A)) \subseteq W(f(A))$ for every $A \subseteq X$. Hence $f^{-1}(U_2(f(A))) \subseteq f^{-1}(W(f(A)))$. Using the fact that $U_2(f(A)) = f(A) \cup f(V_2(f^{-1}(f(A))))$ and f is one-one we arrive at $V_2(A) \subseteq V'(A)$. Thus
$V_1 \leq V^{'} \leq V_2$ and hence either $V_1 = V^{'}$ or $V_2 = V^{'}$. That is $V_1(A) = A \cup f^{-1}(W(f(A)))$ or $V_2(A) = A \cup f^{-1}(W(f(A)))$.
If $V_1(A) = A \cup f^{-1}(W(f(A)))$, then $U_1(f(A)) = f(A) \cup f(V_1(f^{-1}(f(A)))) = f(A) \cup f(V_1(A)) = f(A) \cup f(A \cup f^{-1}(W(f(A)))) = f(A) \cup W(f(A)) = W(f(A))$.

Also we have $W(f(A)) \subseteq U_1(f(A))$. Hence $U_1 = W$. Similarly if $V_2(A) = A \cup f^{-1}(W(f(A)))$, then

$$\begin{aligned} U_2(f(A)) &= f(A) \cup f(V_2(f^{-1}(f(A)))) \\ &= f(A) \cup f(V_2(A)) \\ &= f(A) \cup f(A \cup f^{-1}(W(f(A))) \\ &= f(A) \cup W(f(A)), \text{ since } f \text{ is a bijection.} \\ &= W(f(A)). \end{aligned}$$

Thus either $W = U_1$ or $W = U_2$. Hence U_1 and U_2 are adjacent. □

2.6 Sum of Closure Spaces

In this section we investigate upper neighbour of sum of closure operators.

Definition 2.6.1. The sum of a family of closure spaces$\{(X_a, V_a), a \in \mathscr{A}\}$ is defined to be the space (X, V) where X is the sum of the family of sets $\{X_a\}$ and V is the sum closure sometimes denoted by $\sum\{V_a\}$ defined by $V(\sum_{a\in\mathscr{A}}\{X_a\}) = \sum_{a\in\mathscr{A}}\{V_a(X_a)\}$ for each subset $\sum_{a\in A}\{X_a\}$ of X.

Now we discuss sum of adjacent closure operators.

Lemma 2.6.2. *Suppose $X = \sum_{a\in\mathscr{A}} X_a$ where $\mathscr{A}$ is some indexing set. Let U and V be two closure operators on a set X. Then $U \leq V$ if and only if $U|_{X_a} \leq V|_{X_a}$*

for each $a \in \mathscr{A}$.

Proof. Suppose $U \leq V$. Let $S \subseteq \sum_{a\in\mathscr{A}} X_a$. Then $S = \sum_{a\in\mathscr{A}} S_a$ where $S_a \subseteq X_a$. Thus $V(S) \subseteq U(S)$ for each $S \subseteq \sum_{a\in\mathscr{A}} X_a$. That is $\sum_{a\in\mathscr{A}} V_a(S_a) \subseteq \sum_{a\in\mathscr{A}} (U_a(S_a))$. Now $V_a(S_a) \subseteq \sum_{a\in\mathscr{A}} V_a(S_a) \subseteq \sum_{a\in\mathscr{A}} U_a(S_a)$, and $U_a(S_a) \subseteq \sum_{a\in\mathscr{A}} U_a(S_a)$. Suppose $V_a(S_a) \nsubseteq U_a(S_a)$. Then there exists $y \in V_a(S_a)$ such that $y \notin U_a(S_a)$. This is not possible since $\sum_{a\in\mathscr{A}} V_a(S_a) \subseteq \sum_{a\in\mathscr{A}} (U_a(S_a))$. Thus $V_a(S_a) \subseteq U_a(S_a)$ for each $a \in \mathscr{A}$. Hence $U|_{X_a} \leq V|_{X_a}$ for each $a \in \mathscr{A}$.

Conversely suppose that $U|_{X_a} \leq V|_{X_a}$ for each $a \in \mathscr{A}$. Then $V_a(S_a) \subseteq U_a(S_a)$ for each $a \in \mathscr{A}$. Thus $\sum_{a\in\mathscr{A}} V_a(S_a) \subseteq \sum_{a\in\mathscr{A}} U_a(S_a)$. Hence $U \leq V$. □

Example 2.6.3. Let $X_1 = \{1, 2, 3\}$ and $X_2 = \{4, 5, 6\}$. Then $X_1 + X_2 = \{1, 2, 3, 4, 5, 6\}$. Let us define a closure operator V_1 on X_1 as $V_1(\emptyset) = \emptyset$, $V_1\{1\} = \{1, 2\}$, $V_1\{2\} = \{2, 3\}$, $V_1\{3\} = \{3, 1\}$ and $V_1(A) = \underset{a\in A}{\cup} V_1(\{a\})$ for each $A \subseteq X_1$. Similarly define a closure operator V_2 on X_2 as $V_2(\emptyset) = \emptyset$, $V_2(\{4\}) = \{4, 5\}$, $V_2(\{5\}) = \{5, 6\}$, $V_2(\{6\}) = \{6, 4\}$, $V_2(A) = \underset{a\in A}{\cup} V_2(\{a\})$ for $A \subseteq X_2$. Let $V = V_1 + V_2$. Then $V(A) = V_1(A_1) + V_2(A_2)$ where $A = A_1 + A_2$, $A_1 \subseteq X_1$, $A_2 \subseteq X_2$.

Now define V_1' on X_1 as $V_1'(\emptyset) = \emptyset$, $V_1'(\{1\}) = \{1\}$, $V_1'\{2\} = \{2, 3\}$, $V_1'(\{3\}) = \{3, 1\}$ and $V_1'(A) = \underset{a\in A}{\cup} (V_1'(\{a\}))$. Then V_1' is an upper neighbour of V_1. Also the sum $V' = V_1' + V_2$ is an upper neighbour of V.

Theorem 2.6.4. *Let* $X = \sum_{a\in\mathscr{A}} X_a$. *Let* $\{(X_a, U_a) : a \in \mathscr{A}\}$, $\{(X_a, V_a) : a \in \mathscr{A}\}$ *be closure spaces. Then* $(\sum X_a, \sum V_a)$ *is an upper neighbour of* $(\sum X_a, \sum U_a)$

if and only if V_α *is an upper neighbour of* U_α *on* $L(X_\alpha)$ *for some* $\alpha \in \mathscr{A}$ *and* $V_\beta = U_\beta$ *for all* $\beta \neq \alpha$.

Proof. Suppose that $(\sum X_a, \sum V_a)$ is an upper neighbour of $(\sum X_a, \sum U_a)$. Then we have to prove that V_α is an upper neighbour of U_α for some $\alpha \in \mathscr{A}$ and $V_\beta = U_\beta$ for all $\beta \neq \alpha$. We have by Lemma 2.6.2, $U \leq V$ if and only if $U_a \leq V_a$ for each $a \in \mathscr{A}$. Suppose V_α is an upper neighbour of U_α, V_β is an upper neighbour of U_β, $\beta \neq \alpha$ and $V_\gamma = U_\gamma$ for every $\gamma \neq \alpha, \beta$ where $\alpha, \beta, \gamma \in \mathscr{A}$. We have $\sum U_a \leq \sum V_a$. Consider $\sum\limits_{a\in\mathscr{A}} V_\alpha = \sum\limits_{\gamma\neq\alpha,\beta} V_\gamma + V_\beta + V_\alpha$. We have $V_\alpha(A) \subseteq U_\alpha(A)$ for every $A \subseteq X_\alpha$ and $V_\beta(B) \subseteq U_\beta(B)$ for all $B \subseteq X_\beta$. Then $(V_\alpha + V_\beta)(A) \subseteq (U_\alpha + U_\beta)(A)$ for all $A \subseteq (X_\alpha + X_\beta)$. Then $(\sum\limits_{\gamma\neq\alpha,\beta} U_\gamma + V_\alpha + V_\beta)(S) \subseteq (\sum\limits_{\gamma\neq\alpha} U_\gamma + V_\alpha)(S) \subseteq \sum\limits_{\gamma} U_\gamma(S)$. Thus $\sum\limits_{a\in\mathscr{A}} V_a$ is not an upper neighbour of $\sum\limits_{a\in\mathscr{A}} U_a$.

Conversely suppose that V_α is an upper neighbour of U_α and $V_\beta = U_\beta$ for every $\beta \neq \alpha$. We have to prove that $\sum V_\alpha$ is an upper neighbour of $\sum U_\alpha$. Then $V_\alpha(S_\alpha) \subseteq U_\alpha(S_\alpha)$ for every $S_\alpha \subseteq X_\alpha$. Then $\sum V_\alpha(S) \subseteq \sum U_\alpha(S)$ for every $S \subseteq \sum X_\alpha$. Then $\sum U_\alpha \leq \sum V_\alpha$. Suppose $\sum U_\alpha \leq \sum W_\alpha \leq \sum V_\alpha$. Then $U_\alpha \leq W_\alpha \leq V_\alpha$ and $W_\beta = U_\beta$ for all $\beta \neq \alpha$. Thus $W_\alpha = U_\alpha$ or $W_\alpha = V_\alpha$ since V_α is an upper neighbour of U_α. Hence the theorem. □

2.7 Product of Closure Spaces

Here we consider the problem of existence of upper neighbours of the product of a finite number of closure operators.

Definition 2.7.1. [11] Let $(X_1, V_1), (X_2, V_2), \dots (X_n, V_n)$ be closure spaces. Then the product of closure operators $\prod_{i=1}^{n} V_i$ on $X = X_1 \times X_2 \times \dots \times X_n$ is defined as $(\prod_{i=1}^{n} V_i)(A) = \prod_{i=1}^{n} (V_i(A_i))$ where $A = A_1 \times A_2 \times \dots A_n$.

Theorem 2.7.2. *Let $(X_1, V_1), (X_2, V_2), \dots (X_n, V_n)$ be disjoint closure spaces. Then $\prod_{i=1}^{n} V_i'$ is an upper neighbour of $\prod_{i=1}^{n} V_i$ on $L(X)$ if and only if V_j' is an upper neighbour of V_j on $L(X_j)$ for some j and $V_i = V_i'$ for all $i \neq j$.*

Proof. Suppose $\prod V_i'$ is an upper neighbour of $\prod V_i$. Then $(\prod_{i=1}^{n} V_i')(A) \subseteq (\prod_{i=1}^{n} V_i)(A)$ for every $A \subseteq \prod X_i$. We have $(\prod_{i=1}^{n} V_i)(A) = V_1(A_1) \times V_2(A_2) \times \dots \times V_n(A_n)$ where $A_i \subseteq X_i$ for $i = 1, 2, \dots, n$. Then $\prod_{i=1}^{n} V_i'(A_i) \subseteq \prod_{i=1}^{n} V_i(A_i)$. This implies that $V_i'(A_i) \subseteq V_i(A_i)$ for $i = 1, 2, \dots, n$. Then $V_i \leq V_i'$ for $i = 1, 2, \dots, n$. Suppose V_i' is an upper neighbour of V_i and V_j' is an upper neighbour of V_j for $i \neq j$ and $V_k = V_k'$ for every $k \neq i, j$. Then

$$\begin{aligned}
(\prod_{i=1}^{n} V_i')(A) &= V_1'(A_1) \times \dots \times V_i'(A_i) \times \dots \times V_j'(A_j) \times \dots V_n'(A_n) \\
&\subseteq V_1(A_1) \times \dots \times V_i'(A_i) \times \dots \times V_j(A_j) \times \dots \times V_n(A_n) \\
&\subseteq V_1(A_1) \times V_2(A_2) \times \dots \times V_n(A_n).
\end{aligned}$$

Thus $V_1 \times V_2 \times \dots \times V_n \leq V_1 \times V_2 \times V_i' \times \dots \times V_n \leq V_1' \times V_2' \times \dots \times V_n'$. Thus $\prod_{i=1}^{n} V_i'$ is not an upper neighbour of $\prod_{i=1}^{n} V_i$.
Conversely Suppose that V_j' is an upper neighbour of V_j and $V_i = V_i'$ for all $i \neq j$. Then $V_j'(A_j) \subseteq V_j(A_j)$ for all $A_j \subseteq X_j$. Let $S \subseteq \prod_{i=1}^{n} X_i$. Then $S = S_1 \times S_2 \times \dots \times S_n$. Then $V_1'(S_1) \times V_2'(S_2) \times \dots \times V_n'(S_n) \subseteq V_1(S_1 \times V_2(S_2) \times \dots \times V_n(S_n)$.

Thus $\prod_{i=1}^{n} V_i \leq \prod V_i'$. If $\prod_{i=1}^{n} V_i \leq \prod_{i=1}^{n} W_i \leq \prod_{i=1}^{n} V_i'$, then $V_i = W_i$ for $i \neq j$ and $V_j \leq W_j \leq V_j'$. Since V_j' is an upper neighbour of V_j, for $j \neq i$. Thus $V_j' = W_j$ or $V_j = W_j$ for every $j \neq i$. Hence $\prod_{i=1}^{n} V_i'$ is an upper neighbour of $\prod_{i=1}^{n} V_i$. □

Example 2.7.3. Take X_1, X_2, V_1 and V_2 as in the Example 2.6.3. Then $X = X_1 \times X_2 = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$. Let $V = V_1 \times V_2$. Then $V(\emptyset) = \emptyset$, for $A = A_1 \times A_2$, $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$, $V(A) = V_1(A_1) \times V_2(A_2)$. Now consider $V_1'(\{1\}) = \{1,2\}$, $V_1'(\{2\}) = \{2\}$, $V_1'(\{3\}) = \{3,1\}$ and $V_1'(A) = \underset{a \in A}{\cup} V_1'(\{a\})$. Let $V' = V_1' \times V_2$. Then $V_1'(A) \subseteq V(A)$ for all $A \subseteq X$. Also there exists no closure operators in between V' and V. Thus V' is an upper neighbour of V.

Chapter 3

Simple Expansions of Closure Operators

3.1 Introduction

Simple Extensions of topologies were introduced by Levine N. in [29]. The simple extension of a topology τ on X by $A \subseteq X$ is the smallest topology on X containing A and τ [29]. Analogous to this concept Kunheenkutty M. introduced the concept of simple expansions of Čech closure operators [24]. We note that some properties of a closure operator V need not be shared with the simple expansion of V. In this chapter we investigate the conditions under which certain properties of closure operators are preserved under simple expansion. Here we introduce countable expansions of closure operators in $L(X)$.

3.2 Simple Expansions of Closure Operators

Simple expansions of a closure operator is defined and discussed in [24]. Kunheenkutty M. introduced simple expansions of a closure operator. Before going to the properties of simple expansion of closure operators, we need the following closure operator defined in [24].

Definition 3.2.1. [24] Let A be a non empty proper subset of X and $x \in A$. Define, $V_{(A,x)} : P(X) \to P(X)$ by,

$$V_{(A,x)}(S) = \begin{cases} \emptyset & ; \text{ if } S = \emptyset, \\ X \setminus \{x\} & ; \text{ if } S \neq \emptyset \text{ and } S \subseteq X \setminus A, \\ X & ; \text{ otherwise.} \end{cases}$$

Then $V_{(A,x)}$ is a closure operator on X.

Remark 3.2.2. If $B \subseteq A$, then $x \notin V_{(A,x)}(S) \Rightarrow S \subseteq (X \setminus A) \Rightarrow S \subseteq (X \setminus B) \Rightarrow x \notin V_{(B,x)}(S)$. Hence $V_{(A,x)} \leq V_{(B,x)}$. Conversely suppose $V_{(A,x)} \leq V_{(B,x)}$. Then by similar arguments as above we can prove that $B \subseteq A$.

First of all we study some properties of $V_{(A,x)}$ which will help us to understand simple expansion of closure operators well.

Lemma 3.2.3. *Let W be a closure operator on X and A be a subset of X containing x. Then $V_{(A,x)} \leq W$ if and only if $x \notin W(X \setminus A)$.*

Proof. Suppose $V_{(A,x)} \leq W$. Then $W(X \setminus A) \subseteq V_{(A,x)}(X \setminus A) = X \setminus \{x\}$. That is

$x \notin W(X \setminus A)$. Conversely let $x \notin W(X \setminus A)$ and $S \subseteq X$. Suppose $S \subseteq (X \setminus A)$, then $W(S) \subseteq W(X \setminus A)$. Thus $x \notin W(X \setminus A) \Rightarrow x \notin W(S) \Rightarrow W(S) \subseteq X \setminus \{x\} \subseteq V_{(A,x)}(S)$. Thus in this case $V_{(A,x)} \leq W$. If $S \nsubseteq X \setminus A$, then $W(S)$ is always a subset of $V_{(A,x)}(S)$ since $V_{(A,x)}(S) = X$. In this case also $V_{(A,x)} \leq W$. □

Remark 3.2.4. [24, 26] Let $X = \{a, b\}$, $V_{(\{x\},x)} = V_{x,y}$, the infra closure operator. Then it is not a topological closure operator. But when $|X| \neq 2$, then $V_{(A,x)}$ is topological if and only if $A = \{x\}$.

Next we prove a characterization theorem for upper neighbours of $V_{(A,x)}$.

Theorem 3.2.5. *Let A and B be subsets of X such that $B \in A$ and let $x \in X$ such that $x \in B$. Let $y \in X$, $x \neq y$. Then $V_{(B,x)}$ is an upper neighbour of $V_{(A,x)}$ if and only if $A \setminus B$ is a singleton subset of X.*

Proof. Suppose $A = B \cup \{y\}$. Since $B \subseteq A$, then $x \notin V_{(A,x)}(S) \Rightarrow S \subseteq X \setminus A \subseteq X \setminus B \Rightarrow S \subseteq X \setminus B \Rightarrow x \notin V_{(B,x)}(S)$. That is $V_{(A,x)} \leq V_{(B,x)}$. Let W be a closure operator on X such that $V_{(A,x)} \leq W \leq V_{(B,x)}$. Then $V_{(B,x)}(X \setminus A) \subseteq W(X \setminus A) \subseteq V_{(A,x)}(X \setminus A)$. Since $B \subseteq A$, $X \setminus A \subseteq X \setminus B$. Thus $W(X \setminus A) \subseteq W(X \setminus B)$ and this implies that $V_{(A,x)}(X \setminus A) = X \setminus \{x\}$. Also $x \notin V_{(B,x)}(X \setminus A) = X \setminus \{x\}$. Thus $x \notin W(X \setminus B)$. Then by Lemma 3.2.3, $V_{(B,x)} \leq W$. Hence $W = V_{(B,x)}$.

Now suppose $V_{(B,x)}$ is an upper neighbour of $V_{(A,x)}$. Then $V_{(A,x)} < V_{(B,x)}$. Then $V_{(B,x)}(X \setminus A) \subseteq V_{(A,x)}(X \setminus A) = X \setminus \{x\}$. Thus $x \notin V_{(B,x)}(X \setminus A)$. This implies that $(X \setminus A) \subseteq (X \setminus B)$. That is $B \subseteq A$. Suppose $A \setminus B$ is a non-empty set such that its cardinality is greater than or equal to 2. Then we can find a $C \subseteq X$

such that $B \subset C \subset A$. Then $V_{(A,x)} \leq V_{(C,x)} \leq V_{(B,x)}$. Then either $V_{(C,x)} = V_{(A,x)}$ or $V_{(C,x)} \leq V_{(B,x)}$. This is a contradiction. Hence $A \setminus B$ is a singleton set. □

Theorem 3.2.6. *Suppose A and B are two subsets of X such that $x \in A \cap B$, then $V_{(A,x)} \bigvee V_{(B,x)} = V_{(A \cap B,x)}$.*

Proof. We have $V_{(A,x)} \leq V_{(A\cap B,x)}$ and $V_{(B,x)} \leq V_{(A\cap B,x)}$. Thus $V_{(A,x)} \bigvee V_{(B,x)} \leq V_{(A\cap B,x)}$. In order to prove $V_{(A\cap B,x)} \leq V_{(A,x)} \bigvee V_{(B,x)}$, it is enough to prove that $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X \setminus (A \cap B))$ by Lemma 3.2.3. $V_{(A,x)} \bigvee V_{(B,x)}(X \setminus (A \cap B)) = V_{(A,x)} \bigvee V_{(B,x)}((X \setminus A) \cup (X \setminus B)) = [V_{(A,x)} \bigvee V_{(B,x)}((X \setminus A)] \cup [V_{(A,x)} \bigvee V_{(B,x)}((X \setminus B)]$. We have $x \notin V_{(A,x)}(X \setminus A)$ and $x \notin V_{(B,x)}(X \setminus B)$.
Since $V_{(A,x)} \leq V_{(A,x)} \bigvee V_{(B,x)}$ and $V_{(B,x)} \leq V_{(A,x)} \bigvee V_{(B,x)}$, $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X \setminus A)$ and $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X \setminus B)$. Hence $x \notin V_{(A,x)} \bigvee V_{(B,x)}(X \setminus (A \cap B))$. □

We can extend Theorem 3.2.6 for a finite collection of subsets of X.

Corollary 3.2.7. *Suppose $A_1, A_2, \ldots, A_n$ be a finite collection of subsets of X such that $x \in A_1 \cap A_2 \cap \ldots A_n$. Then $\bigvee\limits_{i=1}^{n} V_{(A_i,x)} = V_{(A_1 \cap A_2 \cap \ldots \cap A_n,x)}$.*

Proof. Proof is clear from Theorem 3.2.6. □

Remark 3.2.8. Theorem 3.2.6 is not true for an infinite collection of subsets of X. For example $X = \mathbb{Z}$. Let $A_n = 2\mathbb{Z} \cup \{2n \setminus 1\}$, $n = 1, 2, \ldots$. Let $A = \{1, 3, 5, \ldots\}$. Then $\bigvee\limits_{i=1}^{\infty} V_{(A_i,2)}(A) = X$. But $V_{(\bigcap\limits_{i=1}^{\infty} A_i,2)}(A) = X \setminus \{2\}$.

Theorem 3.2.9. *Suppose A and B are two subsets of X such that $x \in A \cap B$, then $V_{(A,x)} \wedge V_{(B,x)} = V_{(A\cup B,x)}$.*

Proof. Suppose S is a nonempty subset of X such that $x \notin V_{(A \cup B, x)}(S) \Rightarrow S \subseteq (X \setminus (A \cup B)) = (X \setminus A) \cap (X \setminus B) \subseteq X \setminus A \Rightarrow S \subseteq X \setminus A \Rightarrow x \notin V_{(A,x)}(S)$. Thus $V_{(A,x)}(S) \subseteq V_{(A \cup B, x)}(S)$ for every $S \subseteq X$. Hence $V_{(A \cup B, x)} \leq V_{(A,x)}$. Similarly we can prove that $V_{(A \cup B, x)} \leq V_{(B,x)}$. Therefore $V_{(A \cup B, x)} \leq V_{(A,x)} \wedge V_{(B,x)}$. To prove the converse inequality assume that $x \notin V_{(A,x)} \wedge V_{(B,x)}(S)$ for some $S \subseteq X$. Then we have the following implications.

$$\begin{aligned}
x \notin V_{(A,x)} \wedge V_{(B,x)}(S) &\Rightarrow x \notin V_{(A,x)}(S) \cup V_{(B,x)}(S) \\
&\Rightarrow x \notin V_{(A,x)}(S) \text{ and } x \notin V_{(B,x)}(S) \\
&\Rightarrow S \subseteq X \setminus A \text{ and } S \subseteq X \setminus B \\
&\Rightarrow S \subseteq (X \setminus A) \cap (X \setminus B) \\
&\Rightarrow S \subseteq X \setminus (A \cup B) \\
&\Rightarrow x \notin V_{(A \cup B, x)}(S).
\end{aligned}$$

Thus $V_{(A \cup B, x)}(S) \subseteq V_{(A,x)} \wedge V_{(B,x)}(S)$ for every $S \subseteq X$. Therefore $V_{(A,x)} \wedge V_{(B,x)} \leq V_{(A \cup B, x)}$ and hence $V_{(A,x)} \wedge V_{(B,x)} = V_{(A \cup B, x)}$. □

Corresponding to every closure operator V on X we can define a closure operator denoted by $V^{(x)}$ at each point $x \in X$.

Definition 3.2.10. Let V be a closure operator on X. Let $x \in X$. Define

$V^{(x)} : P(X) \to P(X)$ as

$$V^{(x)}(S) = \begin{cases} \emptyset & ; \quad \text{if } S = \emptyset \\ X \setminus \{x\} & ; \quad \text{if } x \notin V(S) \\ X & ; \quad \text{otherwise.} \end{cases}$$

Then $V^{(x)}$ is a closure operator on X.

Proposition 3.2.11. $V = \bigvee_{x \in X} V^{(x)}$.

Proof. When $x \notin V^{(x)}(S), x \notin V(S) \Rightarrow V(S) \subseteq V^{(x)}(S)$. Hence $V^{(x)} \leq V$ and $\bigvee_{x \in X} V^{(x)} \leq V$. Suppose $x \notin V(A)$. Then $x \notin V^{(x)}(A)$. Thus $x \in \bigvee_{x \in X} V^{(x)}(A)$. Hence the result. □

Let $\mathscr{A}$ be the collection of $A \subseteq X$ such that $x \notin V^{(x)}(X \setminus A)$ for each $x \in X$. Next we prove that $V^{(x)}$ can be written as the join of closure operators of the form $V_{(A,x)}$ where $A \in \mathscr{A}$.

Theorem 3.2.12. *Let V be a closure operator on X. Let $x \in X$. Then $V^{(x)} = \bigvee_{A \in \mathscr{A}} V_{(A,x)}$, where $\mathscr{A}$ is the collection of $A \subseteq X$ such that $x \notin V^{(x)}(X \setminus A)$.*

Proof. Let $S \subseteq X$. Suppose $x \notin \bigvee_{A \in \mathscr{A}} V_{(A,x)}(S)$. Then there exists a finite cover $\{S_1, S_2, \ldots, S_n\}$ of S and $A \in \mathscr{A}$ such that $x \notin V_{(A,x)}(S_i)$ for $i = 1, 2, \ldots, n$. By the definition of $V_{(A,x)}$, we have $S_i \subseteq X \setminus A$ for $i = 1, 2, \ldots, n$. Also $x \notin V^{(x)}(X \setminus A)$ since $A \in \mathscr{A}$. By the definition of $V^{(x)}$, we have $x \notin V(X \setminus A)$. Since $S_i \subseteq (X \setminus A)$ for $i = 1, 2, \ldots, n$, $x \notin V(S_i)$ for $i = 1, 2, \ldots, n$. Hence $x \notin V^{(x)}(S)$. Thus

$V^{(x)}(S) \subseteq \bigvee_{A\in\mathscr{A}} V_{(A,x)}(S)$. That is $\bigvee_{A\in\mathscr{A}} V_{(A,x)} \leq V^{(x)}$.

Conversely assume that $x \notin V^{(x)}(S)$. We have to prove that $x \notin \bigvee_{A\in\mathscr{A}} V_{(A,x)}(S)$. It is enough to prove that there exists a finite cover $\{S_1, S_2, \ldots, S_n\}$ of S and $A \in \mathscr{A}$ such that $x \notin V_{(A,x)}(S_i)$ for $i = 1, 2, \ldots, n$. We have $x \notin V^{(x)}(S)$ implies that $X \setminus S \in \mathscr{A}$. Also $V_{(X\setminus S,x)}(S) = X \setminus \{x\}$ by the definition of $V_{(A,x)}$. Thus $\{S\}$ itself a finite cover of S and $X \setminus S \in \mathscr{A}$ such that $x \notin V_{(X\setminus S,x)}(S)$. So $x \notin \bigvee_{A\in\mathscr{A}} V_{(A,x)}(S)$. Hence $V^{(x)} \leq \bigvee_{A\in\mathscr{A}} V_{(A,x)}$. □

Chacko B. determined the group of automorphisms of $L(X)$ when X is finite [5]. Also he determined the same for the sub lattice $[I, C_0]$ of $L(X)$ and the lattice of all quasi discrete closure operators on $L(X)$. Let X be a non empty set. Let $p \in S(X)$. For a closure operator on X, define $T_pV(A) = p^{-1}V(p(A))$, for $A \subseteq X$. Then $T_p : L(X) \to L(X)$ is an automorphism of $L(X)$ [5].

Proposition 3.2.13. *The automorphism T_p maps the closure operator $V_{(A,x)}$ into $V_{(p^{-1}(A),p^{-1}(x))}$.*

Proof. Let $A \subseteq X$ containing x. Then by definition of $V_{(A,x)}$, we get $V_{(A,x)} : P(X) \to P(X)$ by,

$$V_{(A,x)}(p(S)) = \begin{cases} \emptyset & ; \text{ if } p(S) = \emptyset, \\ X \setminus \{x\} & ; \text{ if } p(S) \neq \emptyset \text{ and } p(S) \subseteq X \setminus A, \\ X & ; \text{ otherwise.} \end{cases}$$

$$= \begin{cases} \emptyset & ; \text{ if } S = \emptyset, \\ X \setminus \{x\} & ; \text{ if } S \neq \emptyset \text{ and } S \subseteq X \setminus p^{-1}A, \\ X & ; \text{ otherwise.} \end{cases}$$

Thus

$$\begin{aligned} T_p(V_{(A,x)})(S) &= p^{-1}(V_{(A,x)}(p(S))) \\ &= \begin{cases} \emptyset & ; \text{ if } S = \emptyset, \\ X \setminus \{p^{-1}(x)\} & ; \text{ if } S \neq \emptyset \text{ and } S \subseteq X - p^{-1}(A), \\ X & ; \text{ otherwise.} \end{cases} \\ &= V_{(p^{-1}(A),p^{-1}(x))} \end{aligned}$$

Thus $T_p(V_{(A,x)}) = V_{(p^{-1}(A),p^{-1}(x))}$ □

Now let us go through the definition of the simple expansion of a closure operator.

Definition 3.2.14. [26] Let V be any closure operator on X and A be a subset of X such that $x \in A$. The closure operator $V_A^x = V \bigvee V_{(A,x)}$ is called a simple expansion of V by A at x. The closure operator V_A^x is given by,

$$V_A^x(S) = \begin{cases} V(S) \setminus \{x\} & ; \text{ if } S \cap (X \setminus A) \neq \emptyset \text{ and } x \notin V(S \cap A) \\ V(S) & ; \text{ otherwise.} \end{cases}$$

Remark 3.2.15. 1. If A is closed in (X, V), then A is closed in (X, V_A^x).

2. By the simple expansion of a closure operator V on X, we mean the simple expansion of a closure operator V other than the discrete closure operator D at some point $x \in X$.

Lemma 3.2.16. *Let (X, V) be a closure space. Suppose A and B are two subsets of X such that $x \in A \cap B$. Then $V_A^x \leq V_{A\cap B}^x$ and $V_B^x \leq V_{A\cap B}^x$.*

Proof. Let $S \subseteq X$. Then

$$\begin{aligned} x \in V_{A\cap B}^x(S) &\Rightarrow S \subseteq A \cap B \text{ or } x \in V(S \cap A \cap B) \\ &\Rightarrow S \subseteq A \text{ or } x \in V(S \cap A) \\ &\Rightarrow x \in V_A^x(S). \end{aligned}$$

Then $V_{A\cap B}^x(S) \subseteq V_A^x(S)$. Hence $V_A^x \leq V_{A\cap B}^x$. Similarly we can prove that $V_B^x \leq V_{A\cap B}^x$. □

Lemma 3.2.17. *Let (X, V) be a closure space. Suppose A and B are two subsets of X such that $x \in A \cup B$. Then $V_{A\cup B}^x \leq V_A^x$ and $V_{A\cup B}^x \leq V_B^x$.*

Proof. Let S be a subset of X. Then

$$\begin{aligned} x \in V_A^x(S) &\Rightarrow S \subseteq A \text{ or } x \in V(S \cap A) \\ &\Rightarrow S \subseteq A \cap B \text{ or } x \in V(S \cap (A \cup B)) \text{ since } S \cap A \subseteq S \cap (A \cup B) \\ &\Rightarrow x \in V_{A\cup B}^x(S). \end{aligned}$$

Thus $V^x_{A\cup B} \leq V^x_A$. Similarly we can prove that $V^x_{A\cup B} \leq V^x_B$. □

Remark 3.2.18. We have $V \leq V^x_{A\cup B} \leq V^x_A \leq V^x_{A\cap B}$ by Lemma 3.2.16 and Lemma 3.2.17.

Lemma 3.2.19. *Let A and B are subsets of X such that $A \cup B \neq X$. Let $x \in A \cap B$. Suppose V is a closure operator on X such that $x \notin V((X \setminus B) \cap A)$. Then $V^x_A \vee V^x_B = V^x_{A\cap B}$.*

Proof. We have by Lemma 3.2.16, $V^x_A \leq V^x_{A\cap B}$ and $V^x_B \leq V^x_{A\cap B}$. Therefore we get $V^x_A \bigvee V^x_B \leq V^x_{A\cap B}$. Now we have to prove that $V^x_{A\cap B} \leq V^x_A \bigvee V^x_B$. That is to prove that $(V^x_A \bigvee V^x_B)(S) \subseteq V^x_{A\cap B}(S)$ for every S subset of X. Suppose $x \notin V^x_{A\cap B}(S)$. Then by the definition of $V^x_{A\cap B}$, we have $S \cap (X \setminus (A \cap B)) \neq \phi$ and $x \notin V(S \cap A \cap B)$. Since $x \notin V(S \cap A \cap B)$ and $S \cap A \cap B \subseteq S \cap A$, we have $x \notin V(S \cap A)$. Similarly $x \notin V(S \cap B)$. If $S \cap (X \setminus A) \neq \phi$, we get $x \notin V^x_A(S)$. Similarly if $S \cap (X \setminus B) \neq \phi$, $x \notin V^x_B(S)$. This implies that $x \notin (V^x_A \bigvee V^x_B)(S)$. Hence $V^x_A \bigvee V^x_B = V^x_{A\cap B}$. □

Remark 3.2.20.

1. The Lemma 3.2.19 can be extended to a finite collection of subsets $A_1, A_2, \ldots, A_n$ of X containing x provided $x \notin V((X \setminus A_i) \cap A_j$ for $i, j = 1, 2, \ldots n$.
2. The condition that $x \notin V((X \setminus B) \cap A)$ is necessary to the Lemma 3.2.19 be true.

Example 3.2.21. Let $X = \{1, 2, 3, 4, 5\}$. Define $V : P(X) \to P(X)$ by

$V(\emptyset) = \emptyset$, $V(\{1\}) = \{1,2,5\}$, $V(\{2\}) = \{1,2,3\}$, $V(\{3\}) = \{2,3,4\}$, $V(\{4\}) = \{3,4,5\}$, $V(\{5\}) = \{4,5,6\}$ and $V(S) = \underset{s\in S}{\cup} V(\{s\})$. Let $A = \{1,2,3\}$, $B = \{3,4,5\}$ and $x = 3$. We have $V((X\setminus B)\cap A) = V(\{1,2\})$. That is $3 \in V((X\setminus B)\cap A)$. Let $S = \{2,4\}$. Then $V_A^x(S) = V(S) = \{1,2,3,4,5\}$ since $3 \in V(S\cap A) = V(\{2\})$. Similarly $V_B^x(S) = V(S) = \{1,2,3,4,5\}$ since $3 \in V(S \cap B) = V(\{4\})$. We have $A \cap B = \{3\}$. Now $V_{A\cap B}^x(S) = V(S)\setminus\{3\}$, since $S\cap(X\setminus(A\cap B)) \neq \emptyset$ and $3 \in V(S\cap A\cap B)$. That is $V_{A\cap B}^x(S) \neq (V_A^x \bigvee V_B^x)(S)$.

Theorem 3.2.22. *Let A and B are subsets of X such that $A\cup B \neq X$. Let $x \in A\cap B$. Then $V_A^x \wedge V_B^x = V_{A\cup B}^x$.*

Proof. We have $V_{A\cup B}^x \leq V_A^x$ and $V_{A\cup B}^x \leq V_B^x$ by Lemma 3.2.16. Hence $V_{A\cup B}^x \leq V_{A,x} \wedge V_{B,x}$. Now suppose $x \notin V_{(A\cup B)}^x(S) \Rightarrow S\cap X\setminus(A\cup B) \neq \emptyset$ and $x \notin V(S\cap(A\cup B)) \Rightarrow S\cap(X\setminus A\cup X\setminus B) \neq \emptyset$ and $x \notin V(S\cap A)\cup V(S\cap B)$. That is $S\cap X\setminus A \neq \emptyset$, $S\cap X\setminus B \neq \emptyset$, $x \notin V(S\cap A)$ and $x \notin V(S\cap B)$. Then $x \notin V_A^x(S)$ and $x \notin V_B^x(S)$. That is $x \notin V_A^x(S)\cup V_B^x(S)$. Hence $V_{A\cup B}^x = V_A^x \wedge V_B^x$. □

Remark 3.2.23. Theorem 3.2.22 is not true for a finite collection of subsets of X containing x. That is for subsets $A_1, A_2, \ldots, A_n$ of X containing x, $V_{A_1}^x \wedge V_{A_2}^x \wedge \ldots \wedge V_{A_n}^x = V_{A_1\cup A_2\cup\ldots\cup A_n}^x$.

Next we find out the relation between simple expansion of closure operators and upper neighbours of closure operators.

Theorem 3.2.24. *Let U and V be two closure operators on a set X such that*

U is an upper neighbour of V. Then U is a simple expansion of V at some point $x \in X$.

Proof. Suppose that U is an upper neighbour of V. Then there exists a subset A of X such that $U(A) \subset V(A)$. Let $x \in A$. Consider the simple expansion of V by A at x. We have $V \leq V_A^x$. We prove that $V_A^x \leq U$. That is to prove that $U(S) \subseteq V_A^x(S)$ for each $S \subseteq X$.

Case (i) $S \subseteq X$ or $x \in V(S \cap A)$.

In this case $V_A^x(S) = V(S)$. Therefore $U(S) \subseteq V_A^x(S)$.

Case (ii) $S \cap (X \setminus A) \neq \phi$ and $x \notin V(S \cap A)$.

In this case $V_A^x(S) = V(S) \setminus \{x\}$. In order to prove $U(S) \subseteq V_A^x(S)$, it is enough to prove that $x \notin U(S)$. Since $x \notin V(S \cap A)$ and $V < U$, we have

$$x \notin U(S \cap A) \tag{3.1}$$

Since A is a V-neighbourhood of x, A is a U-neighbourhood of x. Therefore $A \cup (X \setminus S)$ is a U-neighbourhood of x. This implies that $x \notin U(X \setminus (A \cup (X \setminus S)))$. That is

$$x \notin U(S \cap (X \setminus A)) \tag{3.2}$$

Hence from 3.1 and 3.2, $x \notin U(S)$. Therefore we get $U(S) \subseteq V_A^x(S)$ for each $S \subseteq X$. Thus $V \leq V_A^x \leq U$. Now by the definition of upper neighbours either $V = V_A^x$ or $U = V_A^x$. Since $V \neq V_A^x$, we have $U = V_A^x$. This completes the proof. $\square$

Remark 3.2.25. Converse of the Theorem 3.2.24 is not true.

Example 3.2.26. Let $X = \{a, b, c, d\}$. Define $V : P(X) \to P(X)$ as $V(\emptyset) = \emptyset$, $V(\{a\}) = \{a, b\}$, $V(\{b\}) = \{a, b, c\}$, $V(\{c\}) = \{c\}$, $V(\{d\}) = \{a, d\}$ and $V(S) = \underset{s \in S}{\cup} V(\{s\})$. Now let $A = \{a\}$ and consider V_A^a. Then $V_A^a(\{a\}) = \{a, b\}$, $V_A^a(\{b\}) = \{b, c\}$, $V_A^a(\{c\}) = \{c\}$ and $V_A^a(\{d\}) = \{d\}$. Thus V_A^a is not an upper neighbour of V, since $V \leqslant U \leqslant V_A^x$ where U is given by $U(\emptyset) = \emptyset$. $U(\{a\}) = \{a, b\}$, $U(\{b\}) = \{a, b, c\}$, $U(\{c\}) = \{c\}$, $U(\{d\}) = \{d\}$ and $U(B) = \underset{b \in B}{\cup} V(\{b\})$.

Remark 3.2.27. If V is a topological closure operator, then simple expansion of V by A at a point x need not be a topological closure operator.

Example 3.2.28. Let $X = \mathbb{R}$, $A = \mathbb{Z}$ and $x = 1$. Then $C_0(\mathbb{Z}) = \mathbb{R}$. Now $C_{0A}^x(\mathbb{R} \setminus \mathbb{Z}) = \mathbb{R} \setminus \{1\}$. But $C_{0A}^x(C_{0A}^x(\mathbb{R} \setminus \mathbb{Z})) = C_{0A}^x(\mathbb{R} \setminus \{1\}) = C_0(\mathbb{R} \setminus \{1\}) = \mathbb{R}$. Here C_0 is a topological closure operator, but C_{0A}^x is not a topological closure operator.

Now we check when simple expansion of a topological closure operator becomes a topological closure operator.

Theorem 3.2.29. *Let V be a closure operator on X and let $A \subseteq X$ with $x \in A$ such that $V(A) = A$ and $V(X \setminus \{x\}) = X \setminus \{x\}$. Then if V is a topological closure operator, then V_A^x is a topological closure operator.*

Proof. Suppose V is a topological closure operator. Then $V(V(A)) = V(A)$ for every $A \subseteq X$. Now suppose that $S \subseteq X$.

Case (i):$S \cap (X \setminus A) = \emptyset$.

That is $S \subseteq A$. In this case $V_A^x(S) = V(S)$. Since $S \subseteq A$, we have $V(S) \subseteq V(A) = A$. Thus $V_A^x(V(S)) = V(S)$ since $V(S) \subseteq A$. Now $V_A^x(V_A^x(S)) = V_A^x(V(S)) = V(V(S)) = V(S) = V_A^x(S)$.

Case (ii): $x \in V(S \cap A)$.

In this case $V_A^x(S) = V(S)$. We have $S \subseteq V(S) \Rightarrow S \cap A \subseteq V(S) \cap A \Rightarrow V(S \cap A) \subseteq V(V(S) \cap A)$. Hence $x \in V(V(S) \cap A)$ and therefore $V_A^x(V_A^x(S)) = V_A^x(S)$.

Case (iii): $S \cap (X \setminus A) \neq \emptyset$ and $x \notin V(S \cap A)$.

Then $V_A^x(S) = V(S) \setminus \{x\}$. Since $S \subseteq V_A^x(S)$, $S \cap (X \setminus A) \neq \emptyset \Rightarrow V_A^x(S) \cap (X \setminus A) \neq \emptyset$ and $x \notin V(V_A^x(S) \cap A)$. Hence $x \notin V_A^x(S) \Rightarrow x \notin V_A^x(V_A^x(S))$. Thus $V_A^x(V_A^x(S)) \subseteq V_A^x(S)$. If V is a topological closure operator, V_A^x is a topological closure operator. □

Remark 3.2.30. There exists a closure operator V which is not a topological closure operator but it has a simple expansion V_A^x where $V(A) = A$ which is topological.

Example 3.2.31. Let $X = \{a, b, c\}$. Let $V : P(X) \to P(X)$ be defined as $V(\{a\}) = \{a\}$, $V(\{b\}) = \{b, c\}$, $V(\{c\}) = \{c, a\}$ and $V(A) = \underset{s \in S}{\cup} V(\{s\})$. Then V is a closure operator on X which is not a topological closure operator. Now consider the simple expansion of V by $A = \{a\}$ at a. We have $V_A^x(\{a\}) = V(\{a\}) = \{a\}$, $V_A^x(\{b\}) = \{b, c\}$, $V_A^x(\{c\}) = V(\{c\}) \setminus \{a\} = \{c\}$ and $V_A^x(S) = \underset{s \in S}{\cup} V(\{s\})$. Then V_A^x is a topological closure operator.

3.3 Equality of Simple Expansions

In this section we try to solve the equality of simple expansions of a closure operator by two sets at the same point.

Proposition 3.3.1. *Let V be a closure operator on a set X. Let A and B be subsets of X such that $x \in A \cap B$. If $V_A^x \leq V_B^x$, then $A \cup B \neq X$ and $x \notin V(B \cap (X \setminus A))$.*

Proof. Suppose $V_B^x(S) \subseteq V_A^x(S)$ for each $S \subseteq X$. Then $V_B^x(X \setminus A) \subseteq V_A^x(X \setminus A) = V(X \setminus A) \setminus \{x\}$. That is $x \notin V_B^x(X \setminus A)$. This implies that $(X \setminus A) \cap (X \setminus B) \neq \emptyset$ and $x \notin V(B \cap (X \setminus A))$. That is $A \cup B \neq X$ and $x \notin V(B \cap (X \setminus A))$. □

Theorem 3.3.2. *If $V_A^x = V_B^x$, then $x \notin V(A \Delta B)$.*

Proof. Since $V_A^x(X \setminus A) = V_B^x(X \setminus A) \Rightarrow x \notin V_B^x(X \setminus A)$. Then $X \setminus A \cap X \setminus B \neq \emptyset$ and $x \notin V(X \setminus A \cap B)$. Also $V_A^x(X \setminus B) = V_B^x(X \setminus B) \Rightarrow x \notin V_A^x(X \setminus B)$. Then $X \setminus A \cap X \setminus B \neq \emptyset$ and $x \notin V(X \setminus B \cap A)$. That is $X \setminus A \cap X \setminus B \neq \emptyset$, $x \notin V(X \setminus A \cap B)$ and $x \notin V(X \setminus B \cap A)$. Thus $x \notin V((X \setminus A) \cap B) \cap ((X \setminus B) \cap A)$. Hence $x \notin V(A \Delta B)$. □

Remark 3.3.3. The converse of the Theorem 3.3.2 is not true.

Example 3.3.4. Let $X = \mathbb{R}$, $A = 2\mathbb{Z}$, $B = 2\mathbb{Z} \cap \{1, 3, \ldots, 11\}$. Let $x = 2$.

Then $x \notin C_0(A \Delta B)$. We know that

$$C^x_{0A}(S) = \begin{cases} C_0(S) \setminus \{2\} & ; \quad \text{if } S \cap (X \setminus 2\mathbb{Z}) \neq \emptyset, x \notin C_0(S \cap 2\mathbb{Z}) \\ C_0(S) & ; \quad \text{otherwise.} \end{cases}$$

and

$$C^x_{0B}(S) = \begin{cases} C_0(S) \setminus \{2\} & ; \quad \text{if } S \cap (X \setminus B) \neq \emptyset, x \notin C_0(S \cap B) \\ C_0(S) & ; \quad \text{otherwise.} \end{cases}$$

We have $C^x_{0A}(X \setminus B) = C_0(B) = X$, since $2 \in C_0(X \setminus (2\mathbb{Z} \cap \{1, 3, \ldots, 11\}) \cap 2\mathbb{Z})$. But $C^x_{0B}(X \setminus B) = C_0(X \setminus B) \setminus \{2\}$. That is $C^x_{0A}(X \setminus B) \neq C^x_{0B}(X \setminus B)$. Hence $C^x_{0A} \neq C^x_{0B}$.

Note 3.3.5. We consider the simple expansion of an infra closure operator $V_{a,b}$, $a \neq b$. We have

$$V_{a,b}(S) = \begin{cases} \emptyset & ; \quad \text{if } S = \emptyset; \\ X \setminus \{b\} & ; \quad S = \{a\}; \\ X & ; \quad \text{otherwise.} \end{cases}$$

We know that if $V^x_A = V^x_B$, then $x \notin V(A \Delta B)$. Thus if $x \in V(A \Delta B)$, then

$V_A^x \neq V_B^x$ by Theorem 3.3.2. We have

$$V_{a,b}(A\Delta B) = \begin{cases} \emptyset & ; \quad A\Delta B = \emptyset \\ X \setminus \{b\} & ; \quad \text{if } A\Delta B = \{a\}, \\ X & ; \quad \text{otherwise.} \end{cases}$$

Thus $x \in V_{a,b}(A\Delta B)$ for every subsets of X A and B with $A\Delta B \neq \emptyset$. If $A\Delta B = \{a\}$, then $x \in V_{a,b}(A\Delta B)$ for every $x \neq b$. Thus no two simple expansions of $V_{a,b}$ by distinct sets A and B by $x \neq b$ are equal if $A\Delta B \neq \{a\}$.

If $A\Delta B = \{a\}$, then $V^x_{a,b\,A}$ and $V^x_{a,b\,B}$ are not equal if $x = b$. Also $V^b_{a,b\,A} \leq V^b_{a,b\,B}$ or $V^b_{a,b\,B} \leq V^b_{a,b\,A}$ according as $A = B \cup \{a\}$ or $B = A \cup \{a\}$.

Theorem 3.3.6. *Simple expansions of a closure operator V other than the discrete closure operator D by every subsets of X at every point of X are equal if and only if it is an ultra closure operator.*

Proof. If V is an ultra closure operator, then simple expansions of V by every subsets of X at every point of X are equal to the discrete closure operator D. Conversely suppose that simple expansions of V by every subsets of X at every point of X are equal. If V is not an ultra closure operator, then there exists V' such that $V \leq V' \leq D$. That is there exists $A \subseteq X$ such that $V'(A) \subseteq V(A)$. That is there exists $x \in V(A)$ such that $x \notin V'(A)$. We have $V^x_{X\setminus A}(A) = V(A)$. Choose a superset B of A such that $x \in X \setminus B$. Then $A \cap B \neq \emptyset$ and $x \notin V(A \cap (X \setminus B))$. Thus $V^x_{X\setminus B}(A) = V(A) \setminus \{x\}$. Since simple expansions are equal, $V(A) = V(A) \setminus \{x\}$. This implies that $x \notin V(A)$, which is a contradiction. Hence V is an ultra closure operator. $\square$

3.4 Properties of Simple Expansions of Closure Operators

Note that any expansion of a T_0(respectively T_1 and T_2) space has the same separation property. But in general properties of a closure operator need not be shared with its simple expansions. In this section we check under what conditions do various properties of closure operators like regularity, normality, separability and connectedness hold for its simple expansion.
Recall the definition of a regular closure space.

Definition 3.4.1. [11] A closure space (X, V) is said to be regular if for each point $x \in X$ and a subset S of X, such that $x \notin V(S)$, there exists neighbourhoods U_1 of x and U_2 of S such that $U_1 \cap U_2 = \emptyset$.

Levine N. proved that for a regular topological space (X, τ) and $A \notin \tau$, simple expansion of τ, $(X, \tau(A))$ is regular.

Next we analyze regularity property of the simple expansion of a closure operator.

Theorem 3.4.2. *Suppose (X, V) is a regular closure space and let $A \subseteq X$ such that $V(A) = A$. Let $x \in A$. Then (X, V_A^x) is a regular closure space.*

Proof. Suppose (X, V) is a regular closure space and $A \subseteq X$ such that $V(A) = A$. Let $S \subseteq X$. If $S \subseteq A$, then $V_A^x(S) = V(S)$. Since $V \leq V_A^x$, every V neighbourhood is a V_A^x neighbourhood of S therefore (X, V_A^x) is regular.

If $y \neq x$, then $y \notin V_A^x(S)$ implies that $y \notin V(S)$. Since V is regular, there exists neighbourhoods U_1 of x and U_2 of S such that $U_1 \cap U_2 = \emptyset$. Again, since every V neighbourhood is a V_A^x neighbourhood, V_A^x is regular. Now suppose that $x \notin V_A^x(S)$, by definition of V_A^x, $S \cap (X \setminus A) \neq \emptyset$ and $x \notin V(S \cap A)$. Now $x \notin V(S \cap A)$ implies that there exists a neighbourhood U of x and W of $S \cap A$ such that $U \cap W = \emptyset$.

We have U and A are V_A^x-neighbourhood of x implies that $U \cap A$ is a neighbourhood of x. Now $U \cap A \subseteq A$ and therefore $V_A^x(U \cap A) \subseteq V_A^x(A) = V(A) = A$. Also $U \cap A \subseteq U$ implies that $V_A^x(U \cap A) \subseteq V(U)$. Thus $V_A^x(U \cap A) \subseteq V(U) \cap A$. We have $U \cap W = \emptyset$ implies that $U \subseteq X \setminus W$. Then $V(U) \subseteq V(X \setminus W)$. Now $X \setminus V(X \setminus W) \subseteq X \setminus V(U)$.
$V(U) \cap A \subseteq V(U)$ then $X \setminus V(U) \subseteq X \setminus (V(U) \cap A)$. Thus we have

$$\begin{aligned} S \cap A &\subseteq X \setminus V(X \setminus W)) \\ &\subseteq X \setminus V(U) \\ &\subseteq X \setminus (V(U) \cap A). \end{aligned}$$

As mentioned above we have $V_A^x(U \cap A) \subseteq V(U) \cap A$. Thus $X \setminus V(U) \cap A \subseteq X \setminus V_A^x(U \cap A)$. Therefore $S \cap A \subseteq X \setminus V_A^x(U \cap A)$. Now $V_A^x(U \cap A) = V(U \cap A) \subseteq$

$V(A) = A$. Then $S \cap (X \setminus A) \subseteq (X \setminus A) \subseteq X \setminus (V_A^x(U \cap A))$. Hence $X \setminus (U \cap A)$ is a neighbourhood of S. Thus $U \cap A$ is a neighbourhood of x and $X \setminus (U \cap A)$ is a neighbourhood of S. Hence (X, V_A^x) is a regular closure space. $\square$

Remark 3.4.3. Let V be a regular closure operator. Suppose A and B are closed subsets of X. Then $A \cup B$ is closed. Then $V_{A \cup B, x}$ is regular. We have $V(A) = A$, $V(B) = B$. Then $A \cap B \subseteq A \Rightarrow V(A \cap B) \subseteq V(A)$ and $A \cap B \subseteq B \Rightarrow V(A \cap B) \subseteq V(B)$. Thus $V(A \cap B) \subseteq V(A) \cap V(B) = A \cap B$. Hence $V(A \cap B) = A \cap B$. Hence by Theorem 3.4.2 and by Lemma 3.2.19, we have $V_A^x \bigvee V_B^x$ is regular.

Next we consider the simple expansion of a normal closure operator. The following is a characterization theorem for a simple extension of a normal topological space.

Theorem 3.4.4. *[29] Let (X, τ) be normal and $A \notin \tau$, $X \setminus A \in \tau$. Then $(X, \tau(A))$ is normal if and only if $(X \setminus A, \tau \cap (X \setminus A))$ is normal.*

The following is a characterization theorem for normal closure spaces in [11].

Theorem 3.4.5. *[11] A closure space (X, V) is said to be normal if and only if the following conditions are satisfied.*

(i). *If S_1, S_2 are subsets of X such that $V(S_1) \cap V(S_2) = \emptyset$, then S_1 and S_2 are separated.*

(ii). *If $x \in X$ and $S \subseteq X$ are such that $V(\{x\}) \cap V(S) \neq \emptyset$, then $x \in V(S)$.*

Theorem 3.4.6. *Suppose (X, V) be a normal closure space. Let $A \subseteq X$ such that $x \in A$ and $V(\{x\}) = \{x\}$. Consider the simple expansion V_A^x. Then (X, V_A^x) is a normal closure space.*

Proof. Suppose (X, V) is a normal closure space. Then we have $V(S_1) \cap V(S_2) = \emptyset$, implies that S_1 and S_2 are separated. Now suppose that $V_A^x(S_1) \cap V_A^x(S_2) = \emptyset$. Then $V_A^x(S_1) \subseteq X \setminus V_A^x(S_2)$ and $V_A^x(S_2) \subseteq X \setminus V_A^x(S_1)$. Then $X \setminus S_1$ is a neighbourhood of S_2 and $X \setminus S_2$ is a neighbourhood of S_1. Hence S_1 and S_2 are separated. Now we have to prove that $V_A^x(\{y\}) \cap V_A^x(S) \neq \emptyset$ implies that $y \in V_A^x(S)$. Suppose $y \neq x$, $V_A^x(\{y\}) \cap V_A^x(S) \neq \emptyset$, $S \subseteq X$. We have $V_A^x(\{y\}) = V(\{y\})$ or $V_A^x(\{y\}) = V(\{y\}) \setminus \{x\}$. Since V is normal, this implies that $y \in V(S)$. Next consider the case when $y = x$. Then $V_A^x(\{x\}) = V(\{x\})$ by definition and by assumption $V_A^x(\{x\}) = \{x\}$. If $V_A^x(\{x\}) \cap V_A^x(S) \neq \emptyset$ implies that $\{x\} \cap V_A^x(S) \neq \emptyset$. Then $x \in V_A^x(S)$. Hence (X, V_A^x) is normal. □

Theorem 3.4.7. *[29] Let (X, τ) be separable and $A \notin \tau$. Then $(X, \tau(A))$ is separable if and only if $(A, \tau \cap A)$ is separable.*

Theorem 3.4.8. *Let (X, V) be a closure space. Let $x \in X$ and $A \subseteq X$ such that $x \in A$. Then (X, V) is separable if and only if (X, V_A^x) is separable.*

Proof. Suppose (X, V_A^x) is separable. Then there exists a countable set S of X such that $V_A^x(S) = X$. Since $V_A^x(S) \subseteq V(S)$, we have $V(S) = X$. Hence (X, V) is separable.

Conversely let (X, V) be separable. Let $S \subseteq X$ such that S is countable and $V(S) = X$. If $S \subseteq A$, then $V_A^x(S) = V(S) = X$ and if $x \in V(S \cap A)$, then $V_A^x(S) = V(S) = X$. Now suppose that $S \cap X \setminus A \neq \emptyset$ and $x \notin V(S \cap A)$, then $V_A^x(S) = V(S) \setminus \{x\} = X \setminus \{x\}$. Now consider the set $S \cup \{x\}$. We have $V_A^x(S \cup \{x\}) = V_A^x(S) \cup V_A^x(\{x\}) = X \setminus \{x\} \cup V(\{x\}) = X$. Thus (X, V_A^x) is separable. □

Next we check simple expansion of a connected closure space.

Definition 3.4.9. [11] A subset S of a closure space (X, V) is said to be connected if S is not the union of two non empty semi separated subsets of (X, V). That is $S = S_1 \cup S_2$, $(V(S) \cap S_2) \cup (S_1 \cap V(S_2)) = \emptyset$ implies that $S_1 = \emptyset$ or $S_2 = \emptyset$.

Example 3.4.10. Let $X = \mathbb{Z}$, $A = \{1, 2, \ldots, n\}$ and $x = 2$. Then $C_{0A}^2(\mathbb{Z} \setminus \{2\}) = \mathbb{Z} \setminus \{2\}$ since $\mathbb{Z} \setminus \{2\} \cap \mathbb{Z} \setminus \{1, 2, \ldots, n\} \neq \emptyset$ and $2 \notin C_0(\mathbb{Z} \setminus \{2\} \cap A)$. Then $\mathbb{Z} = \{2\} \cup \mathbb{Z} \setminus \{2\}$. And $\{2\} \cap C_{0A}^2(\mathbb{Z} \setminus \{2\}) = \emptyset$ and $C_0(\{2\}) \cap (\mathbb{Z} \setminus \{2\}) = \emptyset$. That is (X, C_0) is connected but (X, C_{0A}^2) is not connected.

Theorem 3.4.11. *Let (X, V) be a closure space. Let $A \subseteq X$ be such that $x \in X$. If $(X \setminus A, V|_{X \setminus A})$ is connected, then (X, V_A^x) is connected.*

Proof. suppose that $(X \setminus A, V|_{X \setminus A})$ is connected. Assume that $X = X_1 \cup X_2$ such that $V_A^x(X_1) \cap X_2 = \emptyset$ and $X_1 \cap V_A^x(X_2) = \emptyset$. We have $V_A^x(S) = V(S)$ or $V_A^x(S) = V(S) \setminus \{x\}$ for every $S \subseteq X$.

Case (i): $V_A^x(X_1) = V(X_1)$ and $V_A^x(X_2) = V(X_2)$.

In this case $V(X_1) \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) = \emptyset$. This implies that $V(X_1) \cap X \setminus A \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) \cap X \setminus A = \emptyset$. Thus by assumption $X_1 = \emptyset$ or $X_2 = \emptyset$.

Case (ii): $V_A^x(X_1) = V(X_1) \setminus \{x\}$ and $V_A^x(X_2) = V(X_2)$.

In this case $V(X_1) \setminus \{x\} \cap X \setminus A \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) \cap X \setminus A = \emptyset$. This implies that $V(X_1) \cap X \setminus A \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) \cap X \setminus A = \emptyset$, since $x \in A$. Hence by assumption $X_1 = \emptyset$ or $X_2 = \emptyset$.

Case (iii): $V_A^x(X_1) = V(X_1)$ and $V_A^x(X_2) = V(X_2) \setminus \{x\}$.

Then $V(X_1) \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) \setminus \{x\} = \emptyset$, which implies that $V(X_1) \cap X \setminus A \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) \setminus \{x\} \cap X \setminus A = \emptyset$. Since $x \in A$, we have $V(X_1) \cap X \setminus A \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) \cap X \setminus A = \emptyset$. Thus $X_1 = \emptyset$ or $X_2 = \emptyset$.

Case (iv): $V_A^x(X_1) = V(X_1) \setminus \{x\}$ and $V_A^x(X_2) = V(X_2) \setminus \{x\}$.

In this case also since $x \in A$, $V(X_1) \cap X \setminus A \cap X_2 = \emptyset$ and $X_1 \cap V(X_2) \cap X \setminus A = \emptyset$. So $X_1 = \emptyset$ or $X_2 = \emptyset$ by assumption. □

3.5 Countable Expansions of Closure Operators

Borges C. J. R. introduced the concept of infinite extensions of topologies and proved that most of the results which hold for simple extensions also hold for countably infinite extensions [9]. Now let us look at the expansion of a closure operator by a countable collection of subsets of X.

Definition 3.5.1. [9] Let (X, τ) be a topological space and $\mathscr{F} = \{\tau(A_\alpha) : \alpha \in L\}$ be a family of simple extensions of τ. Then Λ is the $\mathscr{F}$-extension of τ if Λ is the smallest topology on X which contains $\tau(A_\alpha)$ for each $\alpha \in L$.

Before going to the countable expansions of a closure operator we have to define a closure operator of the form $V_{(A_1,A_2,\ldots,x)}$, where $A_1, A_2, \ldots$ are subsets of X and $x \in A_i$ for each i. In section 3.2, we remarked that Theorem 3.2.6 is not true for an infinite collection of subsets of X. This is the motivation for defining $V_{(A_1,A_2,\ldots,x)}$.

Definition 3.5.2. Let A_1, A_2, $\ldots$ be a collection of subsets of X such that $x \in A_n$ for every $n \in \mathbb{N}$. Define

$$V_{(A_1,A_2,\ldots,x)}(S) = \begin{cases} \emptyset & ; \quad S = \emptyset \\ X \setminus \{x\} & ; \quad S \neq \emptyset \text{ and } S \subseteq X \setminus A_i \text{ for some } i \\ X & ; \quad \text{otherwise.} \end{cases}$$

Then $V_{(A_1,A_2,\ldots,x)}$ is a closure operator on X.

Remark 3.5.3. Suppose $x \notin V_{(A_i,x)}(S)$. Then $S \subseteq X \setminus A_i$. Hence $x \notin V_{(A_1,A_2,\ldots,x)}(S)$. Thus $V_{(A_1,A_2,\ldots,x)}(S) \subseteq V_{(A_i,x)}(S)$. So $V_{(A_i,x)} \leq V_{(A_1,A_2,\ldots,x)}$.

Theorem 3.5.4. *Let A_1, A_2, $\ldots$ be a collection of subsets of X such that $x \in A_n$ for every $n \in \mathbb{N}$. Then $V_{(A_1,A_2,\ldots,x)} = \bigvee_{i=1}^{\infty} V_{(A_i,x)}$.*

Proof. We have $V_{(A_i,x)} \leq V_{(A_1,A_2,\ldots,x)}$ for $i = 1, 2, \ldots$. Therefore $\bigvee_{i=1}^{\infty} V_{(A_i,x)} \leq V_{(A_1,A_2,\ldots,x)}$. Let $x \in \bigvee_{i=1}^{\infty} V_{(A_i,x)}(S)$. Then for every finite sub cover $\{S_1, S_2, \ldots, S_n\}$ of S there exists S_j such that $x \in V_{(A_i,x)}(S_j)$, $i = 1, 2, \ldots$. Thus S is not contained in $(X \setminus A_i)$ for any A_i. Therefore $x \in V_{(A_1,A_2,\ldots,x)}(S)$. □

Definition 3.5.5. Let (X, V) be a closure space. Let $x \in X$ and $A_1, A_2, \ldots$ be a countable collection of subsets containing x. Then the countable expansion of V by $A_1, A_2, \ldots$ is given by $V \bigvee V_{(A_1,A_2,\ldots,x)}$ and is denoted by $V^x_{A_1,A_2,\ldots}$.

Remark 3.5.6. Suppose $x \notin V_{(A_i,x)}(S) \Rightarrow S \subseteq X \setminus A_i$. Thus $x \notin V_{(A_1,A_2,\ldots,x)}(S)$. Thus $V_{(A_1,A_2,\ldots,x)}(S) \subseteq V_{(A_i,x)}(S)$. Therefore $V_{(A_i,x)} \leq V_{(A_1,A_2,\ldots,x)}$, which implies that $V^x_{A_i} \leq V^x_{A_1,A_2,\ldots}$.

Theorem 3.5.7. *Suppose $A_1, A_2, \ldots$ be subsets of X such that $x \in A_i$ for $i = 1, 2, \ldots$. Then $V^x_{A_1,A_2,\ldots} = V$ if and only if $x \notin V(X \setminus A_i)$ for any i.*

Proof. $V^x_{A_1,A_2,\ldots} = V \Rightarrow V(S) \cap V_{(A_1,A_2,\ldots,x)}(S) = V(S) \Rightarrow V(S) \subseteq V_{(A_1,A_2,\ldots,x)}(S)$ for each $S \subseteq X$. Then $V(X \setminus A_i) \subseteq V_{(A_1,A_2,\ldots,x)}(X \setminus A_i) = X \setminus \{x\}$ for each i. Thus $x \notin V(X \setminus A_i)$ for any i. Conversely suppose that $x \notin V(X \setminus A_i)$ for any i. Then $V(X \setminus A_i) \subseteq V^x_{A_1,A_2,\ldots}(X \setminus A_i)$. □

Next we check when countable expansion of a topological closure operator becomes a topological closure operator. For that first we have to prove the following Proposition.

Proposition 3.5.8. *Let V_1 and V_2 be two topological closure operators on a set X. Then $V_1 \vee V_2$ is a topological closure operator on X.*

Proof. Since V_1 and V_2 are topological, $V_1(V_1(A)) = V_1(A)$ and $V_2(V_2(A)) = V_2(A)$. We have $V_1 \vee V_2(V_1 \vee V_2(S)) = V_1 \vee V_2(V_1(S) \cap V_2(S)) =$

$V_1(V_1(S)\cap V_2(S))\cap V_2(V_1(S)\cap V_2(S)) \subseteq V_1((V_1(S)\cap V_2(S))) \subseteq V_1(V_1(S)) = V_1(S)$. Similarly $V_1 \vee V_2(V_1 \vee V_2(S)) \subseteq V_2(V_1(S) \cap V_2(S)) \subseteq V_2(V_2(S)) = V_2(S)$. Thus $V_1 \vee V_2(V_1 \vee V_2(S)) \subseteq V_1(S) \cap V_2(S) = (V_1 \vee V_2)(S)$ for each $S \subseteq X$. Also $(V_1 \vee V_2)(S) \subseteq V_1 \vee V_2(V_1 \vee V_2(S))$. Hence $V_1 \vee V_2$ is topological closure operator.

□

Theorem 3.5.9. *Suppose $A_1, A_2, \ldots$ be a countable collection of subsets of X such that $x \in A_i$ for each i. Then $V_{(A_1,A_2,\ldots,x)}$ is topological if and only if $A_i = \{x\}$ for some i.*

Proof. Suppose $V_{(A_1,A_2,\ldots,x)}$ is topological. Then $V_{(A_1,A_2,\ldots,x)}(V_{(A_1,A_2,\ldots,x)}(S)) = V_{(A_1,A_2,\ldots,x)}(S)$. If $S \subseteq X \setminus A_i$ for some i, then $V_{(A_1,A_2,\ldots,x)}(V_{(A_1,A_2,\ldots,x)}(S)) = V_{(A_1,A_2,\ldots,x)}(X \setminus \{x\}) = X \setminus \{x\}$. This implies that $X \setminus \{x\} \subseteq A_i$ for some i. That is $A_i = \{x\}$ for some i. Conversely suppose that $A_i = \{x\}$ for some i. Then $V_{(A_1,A_2,\ldots,x)}(X \setminus \{x\}) = X \setminus \{x\}$. Hence $V_{(A_1,A_2,\ldots,x)}(V_{(A_1,A_2,\ldots,x)}(S) = V_{(A_1,A_2,\ldots,x)}(S)$. □

Theorem 3.5.10. *Suppose that V is a topological closure operator on X. Let $A_1, A_2, \ldots$ be subsets of X such that $x \in A_i$ for each i. Then the countable expansion of V by $A_1, A_2, \ldots$ at x is a topological closure operator if $A_i = \{x\}$ for some i.*

Proof. Suppose $A_i = \{x\}$ for some i. Then by Theorem 3.5.9, we have $V_{(A_1,A_2,\ldots,x)}$ is topological. The countable expansion $V^x_{A_1,A_2,\ldots}$ of V by $A_1, A_2, \ldots$ at x is

$V \vee V_{(A_1,A_2,\dots,x)}$. Since V and $V_{(A_1,A_2,\dots,x)}$ is topological, $V^x_{A_1,A_2,\dots}$ is topological by Proposition 3.5.8. □

Proposition 3.5.11. *Suppose $V^x_{A_i}$ is regular for each $i = 1, 2, \dots$. Then $V^x_{A_1,A_2,\dots}$ is regular.*

Proof. Suppose $x \notin V^x_{A_1,A_2,\dots}(S) \Rightarrow x \notin V_{(A_1,A_2,\dots,x)}(S)$ or $x \notin V(S) \Rightarrow x \notin V_{(A_i,x)}(S)$ or $x \notin V(S) \Rightarrow x \notin V^x_{A_i}(S)$ or $x \notin V(S)$. Since $V^x_{A_i}$ is regular, there exist a neighbourhood U of x and a neighbourhood W of S such that $U \cap W = \emptyset$. By Remark 3.5.6 and since every $V^x_{A_i}$-neighbourhood is a $V^x_{A_1,A_2,\dots}$-neighbourhood, we can conclude that $V^x_{A_1,A_2,\dots}$ is a regular closure operator. □

Corollary 3.5.12. *Suppose V is a regular closure operator on X and $A_1, A_2, \dots$ are closed subsets of X containing x, then $V^x_{A_1,A_2,\dots}$ is regular.*

Proof. By Theorem 3.4.2, each $V^x_{A_i}$ is regular. Then by Theorem 3.5.11, $V^x_{A_1,A_2,\dots}$ is regular. □

Next we compare the separability property of a closure operator with its countable expansion.

Theorem 3.5.13. *Let (X, V) be a closure space. Let $A_1, A_2, \dots$ are subsets of X containing x. Then (X, V) is separable if and only if $(X, V^x_{A_1,A_2,\dots})$ is separable.*

Proof. Suppose $(X, V^x_{A_1,A_2,\dots})$ is separable. Then there exists a countable set S of X such that $V^x_{A_1,A_2,\dots}(S) = X$. Since $V^x_{A_1,A_2,\dots}(S) \subseteq V(S)$, we have $V(S) = X$. Hence (X, V) is separable.

Conversely let (X, V) be separable. Let $S \subseteq X$ such that S is countable and $V(S) = X$. $V^x_{A_1,A_2,\ldots}(S) = V(S) \cap V_{(A_1,A_2,\ldots,x)}(S)$. If $V^x_{A_1,A_2,\ldots}(S) = V(S)$, then S itself is a countable dense subset of X and therefore $(X, V^x_{A_1,A_2,\ldots})$ is separable. We have $x \notin V^x_{A_1,A_2,\ldots}(S)$ if and only if $S \subseteq X \setminus A_i$ for some i. Then consider the set $S \cup \{x\}$. Now $V^x_{A_1,A_2,\ldots}(S \cup \{x\}) = X$. Thus $S \cup \{x\}$ is a countable dense subset of X. Therefore $(X, V^x_{A_1,A_2,\ldots})$ is separable. $\square$

Chapter 4

Group of Closure Isomorphisms

4.1 Introduction

In this chapter we investigate some problems related to the group of closure isomorphisms of Čech closure spaces. Ramachandran P. T. proved that no proper nontrivial normal subgroups of the group of all permutations of a set X can be represented as the group of homeomorphisms of (X, T) for any topology T on X [32,34]. A permutation group K on X is said to be t-representable on X if there exists a topology T on X such that the group $H(X, T)$ of homeomorphisms of (X, T) is K [40]. In [39–41], Sini P. studied the t-representability of permutation groups in detail. Here we consider an analogous problem in Čech closure spaces.

4.2 On c-representability of Subgroups of $S(X)$

In this section we define c-representability of a permutation group and determine c-representability of some subgroups of $S(X)$. We know that the set of all open subsets of (X, V) form a topology on X called the topology associated with the closure operator V. If T is the topology associated with V, then every closure isomorphism of (X, V) is a homeomorphism of (X, T). That is the group of closure isomorphisms of (X, V) denoted by $CI(X,\ V)$ is a subgroup of $H(X,\ T)$, where $H(X,\ T)$ is the group of all homeomorphisms of $(X,\ T)$ and the inclusion is proper.

Example 4.2.1. Let $X = \{1, 2, \ldots\}$. Define $V : P(X) \to P(X)$ by $V(\emptyset) = \emptyset$, $V(\{k\}) = \{1, 2, 3, \ldots, k, k+1\}$ and $V(A) = \bigcup_{k \in A} V(\{k\})$ for every $A \subseteq X$. Then $T = \{\emptyset, X\}$ is the topology associated with V. Here $CI(X,\ V) = \{I\}$ and $H(X,\ T) = S(X)$.

Analogous to the concept of t-representability of permutation groups we define the c-representability of permutation groups as follows.

Definition 4.2.2. A subgroup H of $S(X)$ is called c-representable on X if there exists a closure operator V on X such that $CI(X,\ V) = H$.

Theorem 4.2.3. *Let X be a set. If a permutation group H is t-representable on X, then H is c-representable on X.*

Proof. Suppose H is a t-representable permutation group on X. Then there exists a topology T on X such that $H(X,T) = H$. Consider the closure operator V associated with T. That is $V(A) = \bar{A}$, where $\bar{A}$ denote the topological closure of A, $A \subseteq X$. Let $h \in CI(X,V)$. Then $h(V(A)) = V(h(A))$ for every $A \subseteq X$. Since $h(V(A)) = h(\bar{A}) \subseteq \overline{h(A)} = V(h(A))$. Hence h is continuous on (X,T). Similarly $V(h(A)) \subseteq h(V(A))$ for every $A \subseteq X \Rightarrow h^{-1}V(h(A)) \subseteq V(h^{-1}(h(A)))$. Hence h^{-1} is continuous. Thus $CI(X,V) \subseteq H$. Now let h be a homeomorphism on (X,T). Then h is continuous on (X,T). Then $h(V(A)) \subseteq V(h(A))$ and since h^{-1} is continuous on X and so $h^{-1}(V(h(A))) \subseteq V(h^{-1}(h(A)))$. Hence $V(h(A)) \subseteq h(V(A))$. Thus $h(V(A)) = V(h(A))$ for every $A \subseteq X$. Hence h is a closure isomorphism. That is $H \subseteq CI(X,V)$. Thus $CI(X,V) = H$. □

Remark 4.2.4. The converse of the Theorem 4.2.3 is not true. A permutation group which is c-representable on a set X need not be t-representable on X.

Example 4.2.5. Let $X = \{1,2,3\}$. Define $V : P(X) \to P(X)$ by $V(\emptyset) = \emptyset$, $V(\{1\}) = \{1,2\}$, $V(\{2\}) = \{2,3\}$, $V(\{3\}) = \{3,1\}$ and $V(A) = \bigcup_{x \in A} V(\{x\})$ for each $A \subset X$. Then $CI(X,V) = \{I,(1,2,3),(1,3,2)\} = A(X)$ and hence $A(X)$ is c-representable on X. We know that there is no topology T on X such that $H(X,T) = A(X)$ [32,34].

Next we prove a property of c-representable permutation groups.

Theorem 4.2.6. *Let X be any set and H be a subgroup of $S(X)$. Then H is c-representable if and only if its conjugate is c-representable on X.*

Proof. Let H be a c-representable permutation group on X. Then there exists a closure operator V on X such that $CI(X,V) = H$. Let $p \in S(X)$. Define $V' : P(X) \to P(X)$ as $V'(A) = p^{-1}(V(p(A)))$, $A \subseteq X$. Let $h \in CI(X,V)$ and $p \in S(X)$. Then $h(V(A)) = V(h(A))$ for every $A \subseteq X$.

Then

$$\begin{aligned} h \in CI(X,V) &\Rightarrow h(V(p(A))) = V(h(p(A))) \text{ for every } A \subseteq X \\ &\Rightarrow pp^{-1}h(V(p(A))) = V(h(p(A))) \text{ for every } A \subseteq X \\ &\Rightarrow p^{-1}hV(p(A)) = p^{-1}Vh(p(A)) \text{ for every } A \subseteq X \\ &\Rightarrow p^{-1}hp[p^{-1}V(p(A))] = p^{-1}(V(p(p^{-1}hp(A)))) \text{ for every } A \subseteq X \\ &\Rightarrow p^{-1}hp(V'(A)) = V'(p^{-1}hp(A)) \text{ for every } A \subseteq X \\ &\Rightarrow p^{-1}hp \in CI(X,V'). \end{aligned}$$

That is $p^{-1}Hp \subseteq CI(X,V')$.

Suppose that $h' \in CI(X,V')$. To prove that $h' \in p^{-1}Hp$. That is to prove that there exists $h \in H$ such that $h' = p^{-1}hp$. Thus it is enough to prove that $ph'p^{-1}$ is a closure isomorphism of (X,V). Since $h' \in CI(X,V')$, we have $h'V'(p^{-1}(A)) = V'h'(p^{-1}(A))$. From the definition of V',

$$\begin{aligned} h'V'(p^{-1}(A)) = V'h'(p^{-1}(A)) &\Rightarrow h'p^{-1}V(p(p^{-1}(A))) = p^{-1}V(p(h'(p^{-1}(A)))) \\ &\Rightarrow ph'p^{-1}V(A) = V(p(h'(p^{-1}(A)))) \\ &\Rightarrow ph'p^{-1} \in CI(X,V). \end{aligned}$$

Thus $CI(X, V') = p^{-1}Hp$. Hence the theorem. □

Remark 4.2.7. Theorem 4.2.6 says that while determining the c-representability of subgroups of $S(X)$, it is enough to consider the c-representability of conjugacy classes of subgroups of $S(X)$.

Let us look at the definition of direct sum of permutation groups.

Definition 4.2.8 ([40]). Let $\{X_i\}_{i\in I}$ be an arbitrary family of mutually disjoint sets and H_i be a subgroup of $S(X_i)$ for every $i \in I$. Then the *direct sum* of permutation groups $\{H_i,\, i \in I\}$ is the permutation group $\bigoplus\limits_{i\in I} H_i$ on X = $\bigcup\limits_{i\in I} X_i$ whose elements are $\underset{i\in I}{\oplus} h_i$ where $h_i \in H_i$ and the action of $\underset{i\in I}{\oplus} h_i$ is given by $\underset{i\in I}{\oplus} h_i(x) = h_i(x)$ if $x \in X_i$, $i \in I$.

Theorem 4.2.9. *Let X be any set and $Y \subseteq X$. If H is a c-representable permutation group on Y, then the permutation group $\{I_{X\setminus Y}\} \oplus H$ is c-representable on X, where $I_{X\setminus Y}$ denotes the identity permutation on $X \setminus Y$.*

Proof. Since H is c-representable on Y, there exists a closure operator V_1 on Y such that $CI(X, V_1) = H$. Let $Z = X \setminus Y$. If $Z = \emptyset$, there is nothing to prove. Suppose $Z \neq \emptyset$. By the well ordering theorem, well order the set Z by the order relation $<$. We can use the ordinals to index the members of Z. Let x_0 be the first element of Z and x_1 be the first element of $Z \setminus \{x_0\}$. In general x_α denotes the first element of $Z \setminus \{x \in Z : x < x_\alpha\}$ provided $\{x \in Z : x < x_\alpha\}$ is non-empty. Then define $V_2 : P(Z) \to P(Z)$ as $V_2(A) = \underset{x_\alpha\in A}{\cup} V_2(x_\alpha)$ for $A \subseteq Z$

where $V_2(x_\alpha) = Z \setminus \{x \in Z : x < x_\alpha\}$. Then V_2 is a closure operator on Z. Consider X as $X = Y \cup Z$. Let $A \subseteq X$. then $A = A_1 \cup A_2$ where $A_1 = A \cap Y$ and $A_2 = A \cap Z$. Define $V : P(X) \to P(X)$ as follows:

$$V(A) = \begin{cases} \emptyset & ; \quad \text{if } A = \emptyset \\ V_1(A_1) & ; \quad \text{if } A_2 = \emptyset \\ Y \cup V_2(A_2) & ; \quad \text{if } A_2 \neq \emptyset. \end{cases}$$

We have to prove that V is a closure operator on X.

Let $A \subseteq X$. If $A = \emptyset$, then there is nothing to prove. Now suppose that $A \neq \emptyset$. We have $A = A_1 \cup A_2$. If $A_2 = \emptyset$, then $V(A) = V_1(A)$ and hence $A \subseteq V(A)$. If $A_2 \neq \emptyset$, $V(A) = Y \cup V_2(A_2)$. Then clearly $A \subseteq V(A)$.

Let $A, B \subseteq X$. $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, where $A_1, B_1 \subseteq Y$ and $A_2, B_2 \subseteq Z$.

Case (i): $A_2 = \emptyset$, $B_2 = \emptyset$

In this case A, $B \subseteq Y$ and hence $V(A) = V_1(A_1)$ and $V(B) = V_1(B_1)$. Then $V(A \cup B) = V_1(A_1 \cup B_1) = V_1(A_1) \cup V_1(B_1) = V(A) \cup V(B)$.

Case (ii): $A_2 \neq \emptyset$, $B_2 = \emptyset$ Then $V(A) = Y \cup V_2(A_2)$, $V(B) = V_1(B_1)$.

Now

$$\begin{aligned} V(A) \cup V(B) &= Y \cup V_2(A_2) \cup V_1(B_1) \\ &= Y \cup V_2(A_2), \text{ since } V_1(B_1) \subseteq Y. \end{aligned}$$

and

$$\begin{aligned} V(A \cup B) &= V[(A_1 \cup B_1) \cup (A_2 \cup B_2)] \\ &= Y \cup V_2(A_2 \cup B_2) \\ &= Y \cup V_2(A_2) \end{aligned}$$

Hence $V(A \cup B) = V(A) \cup V(B)$.

Case (iii): $A_2 = \emptyset$, $B_2 \neq \emptyset$.

Similar to Case (ii).

Case (iv): $A_2 \neq \emptyset$, $B_2 \neq \emptyset$

Here $V(A) = Y \cup V_2(A_2)$ and $V(B) = Y \cup V_2(B_2)$. Then

$$\begin{aligned} V(A \cup B) &= Y \cup V_2(A_2 \cup B_2) \\ &= Y \cup V_2(A_2) \cup V_2(B_2) \\ &= [Y \cup V_2(A_2)] \cup [Y \cup V_2(B_2)] \\ &= V(A) \cup V(B). \end{aligned}$$

Thus V is a closure operator on X.

Next we claim that $CI(X, V) = \{I_Z\} \oplus Y$. Let $f = I_Z \oplus h \in \{I_Z\} \oplus H$ and $A \subseteq X$. Then we have to show that $V(f(A)) = f(V(A))$. Now $V(f(A)) = V(f(A_1 \cup A_2)) = V(A_1 \cup h(A_2)) = V(A_1 \cup h(A_2))$. Since $A = A_1 \cup A_2$, we consider the following cases.

Case (i): $A_2 = \emptyset$

Then $V(f(A)) = V(h(A_1)) = V_1(h(A_1))$. Now $f(V(A)) = f(V_1(A_1)) = h(V_1(A_1)) = V_1(h(A_1))$. Hence $V(f(A)) = f(V(A))$.

Case (ii): $A_2 \neq \emptyset$

Then $V(f(A)) = V(h(A_1) \cup A_2) = Y \cup V_2(A_2)$.

$$\begin{aligned} \text{Now } f(V(A)) &= f(V(A_1 \cup A_2)) \\ &= f(Y \cup V_2(A_2)) \\ &= h(Y) \cup V_2(A_2) = Y \cup V_2(A_2). \end{aligned}$$

Thus $f(V(A)) = V(f(A))$, for every $A \subseteq X$. Hence $\{I_Z\} \oplus H \subseteq CI(X, V)$.

Now let $f \in CI(X, V)$. Then $V(X \setminus \{x_0\}) = X \setminus \{x_0\}$. Hence $\{x_0\}$ is open in X. Then $f(\{x_0\})$ is open in X. Since the only one point set open in X is $\{x_0\}$, $f(x_0) = x_0$. Also $V(X \setminus \{x_0, x_1\}) = X \setminus \{x_0, x_1\}$. That is $\{x_0, x_1\}$ is open in X. Therefore $f(\{x_0, x_1\})$ is open in X. Since the only two point set which is open in X is $\{x_0, x_1\}$, we have $f(\{x_1\})) = x_1$. Let x_α be any element of Z such that $f(x) = x$ for every $x < x_\alpha$. If x_α has no immediate successor, then x_α is the last element of Z. Since $V(Y) = V_1(Y) = Y$, we have Z is open in X and hence $f(Z)$ is open in X. Thus $f(Z) = (Z \setminus \{x_\alpha\}) \cup \{f(x_\alpha)\}$ which implies that $f(x_\alpha) = x_\alpha$.

If x_α has an immediate successor x_β, then $V(X \setminus \{x \in Z : x < x_\beta\}) = X \setminus \{x \in Z : x < x_\beta\}$. This implies that $U = \{x \in X \setminus Y : x < x_\beta\}$ is an open set. Then $f(U) = \{x \in Z : x < x_\beta\} \cup \{f(x_\alpha)\}$. By the definition of V, $f(U) = U$ and hence $f(x_\alpha) = x_\alpha$. Thus we get $f|_Z = I_Z$.

Since f is a closure isomorphism, $f(V(A)) = V(f(A))$ for every $A \subseteq X$. If $A \subseteq Y$, then $f(V(A)) = f(V_1(A)) = f|_Y(V_1(A))$. Since f is a bijection on X and $f|_Z = I_Z$, we have $f(A) \subseteq Y$ and hence $V(f(A)) = V_1(f(A)) = V_1(f|_Y(A))$. Therefore $f|_Y(V_1(A)) = V_1(f|_Y(A))$. Thus we have $f|_Y \in H$. That is $f = I_Z \oplus h$, where $h = f|_Y \in H$. Since $Z = X \backslash Y$ it follows that $CI(X, V) = \{I_{X \backslash Y}\} \oplus H$. □

Remark 4.2.10. By Theorem 4.2.9, in order to determine the c-representability of permutation group H on a set X, we have to consider only the c-representability on the set of all points which are moved by the permutations of H.

4.3 c-representability of Dihedral Group D_n

Sini P. proved that the Dihedral group D_n is not t-representable for $n \geq 5$ [40]. Here we check the c-representability of the Dihedral group D_n.

Definition 4.3.1. [16] For $n \geq 3$, the Dihedral group D_n is defined as the rigid motions of the plane preserving a regular n-gon with the operations being composition. The order of the Dihedral group D_n is $2n$.

Theorem 4.3.2. *The Dihedral group D_n is c-representable.*

Proof. Let $X = \{a_1, a_2, \ldots, a_n\}$. Define $V : P(X) \to P(X)$ as $V(a_k) =$

$\{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}$, $V(A) = \underset{a_k \in A}{\cup} V(\{a_k\})$ for each $A \subseteq X$. The generators of the Dihedral group D_n on $X = \{a_1, a_2, \ldots, a_n\}$ are the cycle $p = (a_1, a_2, \ldots, a_n)$ and

$$s = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots a_k \ldots & a_{n-1} & a_n \\ a_1 & a_n & a_{n-1} & \ldots a_{n+2-k} \ldots & a_3 & a_2 \end{pmatrix}.$$

We have $p(V(a_1)) = p(\{a_n, a_1, a_2\}) = \{a_1, a_2, a_3\}$. Also $V(p(\{a_1\})) = V(\{a_2\}) = \{a_1, a_2, a_3\}$. That is $p(V(a_1)) = V(p(\{a_1\}))$.

Similarly $p(V(a_k)) = p(\{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}) = \{a_{k\oplus 1}, a_{k\oplus 2}, a_{k\oplus n}\}$ and $V(p(\{a_k\})) = V(\{a_{k\oplus 1}\}) = \{a_{k\oplus 1}, a_{k\oplus 2}, a_{k\oplus n}\}$. Thus $p(V(a_k)) = V(p(\{a_k\}))$ for $k = 1, 2, \ldots, n$. Thus p is a closure isomorphism of (X, V). Next we prove that s is a closure isomorphism. Then $s(V(\{a_1\})) = s(\{a_1, a_2, a_n\}) = \{a_1, a_n, a_2\}$ and $V(s(\{a_1\})) = V(\{a_1\}) = \{a_1, a_2, a_n\}$. That is $s(V(\{a_1\})) = V(s(\{a_1\}))$. Now $s(V(\{a_k\})) = s(\{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}) = \{a_{n+2-k}, a_{n+2-(k\oplus 1)}, a_{n+2-(k\oplus(n-1))}\}$ and $V(s(\{a_k\})) = V(\{a_{n+2-k}) = \{a_{n+2-k}, a_{n+2-(k\oplus 1)}, a_{n+2-(k\oplus(n-1))}\})$. Hence $s \in CI(X, V)$. Then every element of D_n is a closure isomorphism. That is

$$D_n \subseteq CI(X, V) \tag{4.1}$$

Now suppose that $h \in CI(X, V)$. Then $h(V(\{a_1\}) = V(h(\{a_1\}))$. Suppose $h(a_1) = a_k$. Then $h(V(\{a_1\}) = h(\{a_1, a_2, a_n\}) = \{a_k, h(a_2), h(a_k)$. And $V(h(\{a_1\})) = V(\{a_k\}) = \{a_k, a_{k\oplus 1}, a_{k\oplus(n-1)}\}$. Then $h(a_2)$ is either $a_{k\oplus 1}$ or $a_{k\oplus(n-1)}$, and $h(a_n)$ is either $a_{k\oplus 1}$ or $a_{k\oplus(n-1)}$

Case (i): $h(a_2) = a_{k\oplus 1}$ and $h(a_n) = a_{k\oplus(n-1)}$.

Since h is a closure isomorphism, $V(h(a_2)) = h(V(\{a_2\}))$. But $V(\{a_{k\oplus 1}\}) = \{a_{k\oplus 1}, a_k, a_{k\oplus 2}\}$ and $h(V(\{a_2\})) = h(\{a_1, a_2, a_3\})$. This implies that $h(a_3) = a_{k\oplus 2}$ and $h(a_{n-1}) = a_{k\oplus n-2}$. That is

$h = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_k & a_{k\oplus 1} & a_{k\oplus 2} & \dots & a_{k\oplus(n-2)} & a_{k\oplus(n-1)} \end{pmatrix} = p^{k-1}$. Hence $h \in D_n$.

Case (ii): $h(a_2) = a_{k\oplus(n-1)}$ and $h(a_n) = a_{k\oplus 1}$.

In this case $h(a_3) = a_{k\oplus(n-2)}$ and $h(a_{n-1}) = a_{k\oplus 2}$.

Hence $h = (a_1, a_k)(a_2, a_{k\oplus(n-1)}) \cdots (a_n, a_{k\oplus 1})(a_{n-1}, a_{k\oplus 2})$. Then $h = p^{n-k}s \in D_n$.

Hence

$$CI(X, V) \subseteq D_n \tag{4.2}$$

From 4.1 and 4.2, $D_n = CI(X, V)$. This completes the proof. □

4.4 On c-representability of Direct Sum of Permutation Groups

In this section we investigate the c-representability of a direct sum of finite permutation groups .

Theorem 4.4.1. *Let $\{(X_i, V_i)\}_{i\in I}$ be an arbitrary family of disjoint closure spaces where each X_i is finite and H_i be c-representable subgroup of $S(X_i)$ for $i \in I$. Then $\bigoplus_{i\in I} H_i$ is c-representable on $X = \bigcup_{i\in I} X_i$.*

Proof. By assuming axiom of choice, well order the set I. For each $A \subseteq X$, let $A_i = A \cap X_i$. Then $A = \bigcup_{i \in I} A_i$. Let $I' = \{i \in I : A_i \neq \emptyset\}$ and i_0 be the first element of I'. In order to define a closure operator V on X, we define $V(A)$, $A \subseteq X$ as follows.

$$V(A) = \begin{cases} \emptyset & ; \quad \text{if } A = \emptyset \\ (\underset{i > i_0}{\cup} X_i) \cup V_{i_0}(A_{i_0}) & ; \quad \text{if } A \neq \emptyset. \end{cases}$$

First of all we have to verify that V is a closure operator on X. Let $x \in A$, then $x \in A_i$ for some i. Then $i_0 \leq i$. If $i \neq i_0$, $X_i \subset V(A)$, then $x \in V(A)$. If $i = i_0$, $x \in A_{i_0}$. Then $x \in V_{i_0}(A_{i_0})$ and hence $x \in V(A)$. Thus $A \subseteq V(A)$.

Let $A, B \subseteq X$. We have $A = \underset{i \in I}{\cup} A_i$ and $B = \underset{j \in I}{\cup} B_j$. Let $I'_A = \{i \in I : A_i \neq \emptyset\}$ and $I'_B = \{j \in I : B_j \neq \emptyset\}$. Let i_0 and j_0 be first element of I'_A and I'_B respectively.

Case (i): $i_0 = j_0$

$V(A) = (\underset{i > i_0}{\cup} X_i) \cup V_{i_0}(A_{i_0})$ and $V(B) = (\underset{i > i_0}{\cup} X_i) \cup V_{i_0}(B_{i_0})$. Then

$$\begin{aligned} V(A \cup B) &= (\underset{i > i_0}{\cup} X_i) \cup V_{i_0}(A_{i_0} \cup B_{i_0}) \\ &= (\underset{i > i_0}{\cup} X_i) \cup (V_{i_0}(A_{i_0}) \cup V_{i_0}(B_{i_0})) \\ &= ((\underset{i > i_0}{\cup} X_i) \cup V_{i_0}(A_{i_0})) \bigcup ((\underset{i > i_0}{\cup} X_i) \cup V_{i_0}(B_{i_0})) \\ &= V(A) \cup V(B). \end{aligned}$$

Case (ii): $i_0 \neq j_0$

Then either $i_0 > j_0$ or $i_0 < j_0$. Suppose that $i_0 > j_0$. $V(A) = (\underset{i>i_0}{\cup} X_i) \cup V_{i_0}(A_{i_0})$ and $V(B) = (\underset{j>j_0}{\cup} X_j) \cup V_{j_0}(B_{j_0})$. Then $V(A) \subset V(B)$ and so $V(A) \cup V(B) = V(B)$. Now

$$\begin{aligned} V(A \cup B) &= \underset{j>j_0}{\cup} X_j \cup V_{j_0}(A_{j_0} \cup B_{j_0}) \\ &= \underset{j>j_0}{\cup} X_j \cup (V_{j_0}(A_{j_0}) \cup V_{j_0}(B_{j_0})) \\ &= \underset{j>j_0}{\cup} X_j \cup V_{j_0}(B_{j_0}) \\ &= V(B). \end{aligned}$$

Hence $V(A \cup B) = V(A) \cup V(B)$.

Thus V is a closure operator on X.

Next we have to prove that $CI(X,V) = \bigoplus_{i \in I} H_i$. Let $h \in \underset{i \in I}{\oplus} H_i$. For $A \neq \emptyset$,

$$\begin{aligned} h(V(A) &= h((\underset{i>i_0}{\cup} X_i) \cup V_{i_0}(A_{i_0})) \\ &= h(\underset{i>i_0}{\cup} X_i) \cup h(V_{i_0}(A_{i_0})) \\ &= (\underset{i>i_0}{\cup} X_i) \cup h_{i_0}(V_{i_0}(A_{i_0})) \\ &= \underset{i>i_0}{\cup} X_i \cup V_{i_0}(h_{i_0}(A_{i_0})) \\ &= V(h(A)). \end{aligned}$$

Hence $\bigoplus_{i \in I} H_i \subseteq CI(X,V)$.

Now let h be a closure isomorphism on X. Suppose $h(X_{\alpha_0}) \neq X_{\alpha_0}$ where α_0 is the first element of I. Then there exists $x \in X_{\alpha_0}$ such that $h(x) \notin X_{\alpha_0}$ or there exists $x \notin X_{\alpha_0}$ such that $h(x) \in X_{\alpha_0}$. Without loss of generality we can assume that $x \in X_{\alpha_0}$ such that $h(x) \notin X_{\alpha_0}$. Then $h(x) \in X_j$ for some $j > \alpha_0$. We have $V(\underset{i>\alpha_0}{\cup} X_i) = \underset{i>\alpha_0}{\cup} X_i$. Therefore X_{α_0} is open in X. Since h is a closure isomorphism, $h(X_{\alpha_0})$ is open in X. We have $h(x) \in h(X_{\alpha_0})$ but $h(x) \notin X_{\alpha_0}$. Then $|X_{\alpha_0}| < |h(X_{\alpha_0})|$, since X_{α_0} is finite. This is a contradiction, since h is injective. So $h(X_{\alpha_0}) = X_{\alpha_0}$.

Now assume that $h(X_j) = X_j$ for every $j \in I$ such that $j < i$. To show that $h(X_i) = X_i$. Suppose $h(X_i) \neq X_i$. Then there exists $x \in X_i$ such that $h(x) \notin X_i$. This implies that $h(x) \in X_l$ for some $l > i$. Now consider $\underset{k>i}{\cup} X_k$. Since $V(\underset{k>i}{\cup} X_k) = \underset{k>i}{\cup} X_k$, then $\underset{k>i}{\cup} X_k$ is closed. Then $\underset{k\leq i}{\cup} X_k$ is open in X and hence $h(\underset{k\leq i}{\cup} X_k)$ is open in X. Now $h(\underset{k\leq i}{\cup} X_k) = \underset{k<i}{\cup} X_k \cup h(X_i)$. Since $x \in X_i$ and $h(x) \in h(X_i)$ and X_i is finite, we have $|X_i| < |h(X_i)|$. This is a contradiction. It follows that $h(X_i) = X_i$. Hence by the principle of transfinite induction, $h(X_i) = X_i$ for all $i \in I$. Then $h|X_i = h_i$ is a closure isomorphism of (X_i, V_i) for all $i \in I$. Therefore $h_i = \underset{i\in I}{\oplus} h_i$. Hence $CI(X, V) = \bigoplus_{i\in I} H_i$. $\square$

4.5 c-representability of Cyclic Subgroups of $S(X)$

Sini P. investigated the t-representability of permutation groups generated by a product of disjoint cycles [39,41]. Here we investigate c-representability of such permutation groups. She proved that a permutation group on X generated by an arbitrary product of disjoint cycles having length n, $n > 2$ is t- representable on X [39, 41]. Hence they are c-representable by Theorem 4.2.3.

Theorem 4.5.1. *Let X be a finite set $\{a_1, a_2, \ldots, a_n\}$ and H be the group of permutations of X generated by the cycle $p = (a_1, a_2, \ldots, a_n)$. Then H is c-representable on X.*

Proof. Define $V : P(X) \to P(X)$ by $V(\emptyset) = \emptyset$, $V(\{a_i\}) = \{a_i, a_{i\oplus 1}\}$ where $\oplus$ denotes addition modulo n and $V(A) = \bigcup_{a_i \in A} V(\{a_i\})$ for every $A \subset X$. Given $p = (a_1, a_2, \ldots, a_n)$. Then $p(V(A)) = V(p(A))$ for every $A \subseteq X$. So p is a closure isomorphism and hence all powers of p are closure isomorphisms. Thus we get $H \subseteq CI(X, V)$. Conversely suppose that $h \in CI(X, V)$ such that $h(\{a_1\}) = a_j$ for some j. Then $V(h(\{a_i\}) = h(V(\{a_i\}))$. Then $\{a_j, a_{j\oplus 1}\} = h(\{a_1, a_2\})$. Thus h maps a_2 to $a_{j\oplus 1}$. Now suppose that $h(a_m) = a_k$, $1 \leq m, k \leq n$. So $h(\{a_m, a_{m\oplus 1}\}) = \{a_k, a_{k\oplus 1}\}$ and thus $h(a_{m\oplus 1}) = a_{k\oplus 1}$. This implies that $h(a_i) = a_{i\oplus(j-1)}$ for $i = 1, 2, \ldots, n$ and hence $h = p^{j-1}$. Thus $CI(X, V) \subseteq H$. This completes the proof. □

Remark 4.5.2. When X is an infinite set, infinite cycles on X are t-representable on X [37], hence are c-representable on X.

Corollary 4.5.3. *Let X be a set and p be a cycle on X. Then the permutation group generated by p is c-representable on X.*

Proof. The proof is obvious from Theorem 4.5.1 and Theorem 4.2.9. □

Next we investigate the c-representability of the permutation group which is generated by a product of two disjoint cycles having equal length. Let p be a permutation on X which is a product of two disjoint cycles having equal length. Sini P. proved that the cyclic group generated by p is not t-representable on X [41].

Theorem 4.5.4. *Let X be a set. Let p be a permutation which is a product of two disjoint finite cycles having equal lengths. Then the cyclic group generated by p is c-representable on X.*

Proof. Let $p = (a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$ be a permutation on X. Let H be the cyclic group generated by p. Suppose $Y = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. By Theorem 4.2.9, it is enough to prove that H is c-representable on Y. Let $X_1 = \{a_1, a_2, \ldots, a_n\}$ and $X_2 = \{b_1, b_2, \ldots, b_n\}$. Define $V : P(Y) \to P(Y)$ as $V(\emptyset) = \emptyset$, $V(\{a_j\} = \{a_j, a_{j\oplus 1}, b_j\})$ and $V(\{b_j\}) = \{b_j, b_{j\oplus 1}\}$, $j = 1, 2, \ldots, n$ and $V(A) = \bigcup\limits_{a\in A} V(\{a\})$, $A \subseteq Y$. Then $p(V(\{a_i\}) = p(\{a_i, a_{i\oplus 1}, b_i\}) = \{a_{i\oplus 1}, a_{i\oplus 2}, b_{i\oplus 1}\}$. Now $V(p\{a_i\}) = V(\{a_{i\oplus 1}\}) = \{a_{i\oplus 1}, a_{i\oplus 2}, b_{i\oplus 1}\} = p(V(\{a_i\})$ for $i = 1, 2, \ldots, n$. Also $p(V(\{b_i\}) = p(\{b_i, b_{i\oplus 1}\}) = \{b_{i\oplus 1}, b_{i\oplus 2}\}$ and $V(p(\{b_i\})) = V(\{b_{i\oplus 1}\}) = \{b_{i\oplus 1}, b_{i\oplus 2}\}$. That is $p(V(\{b_i\}) = V(p(\{b_i\}))$ for each $i = 1, 2, \ldots, n$. Thus p is a closure isomorphism on Y.

Now let h be a closure isomorphism of (Y, V). Then $h(V(A)) = V(h(A))$ for every $A \subseteq Y$. If $h(a_i) = b_k$, then we have $h(V(\{a_i\})) = V(h(\{a_i\})) \Rightarrow h(\{a_i, a_{i\oplus 1}, b_i\}) = V(b_k) \Rightarrow \{h(a_i), h(a_{i\oplus 1}), h(b_i)\} = \{b_k, b_{k\oplus 1}\}$. This is not possible. Thus $h(a_i) \in X_1$. Now suppose that $h(a_i) = a_k$. Then $V(h(a_i)) =$

$V(a_k) = \{a_k, a_{k\oplus 1}, b_k\} = \{h(a_i), h(a_{i\oplus 1}), h(b_i)\}$. This implies that $h(b_i) = b_k$. Thus $h(X_2) = X_2$. Now let $h(a_1) = a_k$ and $h(b_1) = b_k$.Then $V(h(b_1)) = V(b_k) = \{b_k, b_{k\oplus 1}\}$ and $V(h(a_1)) = V(a_k) = \{a_k, a_{k\oplus 1}, b_k\}$. We have $h(V(\{b_1\})) = \{h(b_1), h(b_2)\}$ and $h(V(\{a_1\})) = \{h(a_1), h(a_2), h(b_1)\}$. Since h is a closure isomorphism, $h(V(\{a_1\})) = V(h(a_1))$ and $h(V(\{b_1\})) = V(h(b_1))$. Thus $\{a_k, a_{k\oplus 1}, b_k\} = \{h(a_1), h(a_2), h(b_1)\}$ and $\{b_k, b_{k\oplus 1}\} = \{h(b_1), h(b_2)\}$. Hence $h(a_2) = a_{k\oplus 1}$ and $h(b_2) = b_{k\oplus 1}$. Now suppose that $h(b_m) = b_j$ and $h(a_m) = a_j$ where $1 < m, j < n$. Then $V(h(b_m)) = V(b_j) = \{b_j, b_{j\oplus 1}\}$ and $V(h(a_m)) = V(a_j) = \{a_j, a_{j\oplus 1}, b_j\}$. Also $h(V(\{b_m\})) = h(\{b_m, b_{m\oplus 1}\})$ and $h(V(\{a_m\})) = h(\{a_m, a_{m\oplus 1}, b_m\})$. Thus $h(b_{m\oplus 1}) = b_{j\oplus 1}$ and $h(a_{m\oplus 1}) = a_{j\oplus 1}$. Thus $h = p^{j-1}$. Hence $h \in H$. Thus $CI(Y, V) = H$. □

Theorem 4.5.5. *[39, 41] If p is a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n where $n > 2$, then the group generated by p is $t-$representable on X.*

By Theorem 4.5.5 and 4.2.3 we have the group generated by p where p is a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n where $n > 2$ is c-representable on X.

Theorem 4.5.6. *Let X be any set and p be the permutation which is an arbitrary product of disjoint cycles having equal length. Then the cyclic group generated by p is c-representable on X.*

Proof. Proof follows from Theorem 4.5.1, 4.5.3, 4.5.4 and 4.5.5. □

Corollary 4.5.7. *Every permutation group of prime order is c-representable.*

Proof. Let X be any set and H be a permutation group on X having order n, where n is a prime number. Then H is a cyclic group generated by a permutation p which is of order n. This implies that p is a product of disjoint cycles having equal length. So by Theorem 4.5.6, H is c-representable on X. □

We proved that direct sum of c-representable finite permutation groups are c-representable on X. From this result we can deduce that the permutation group generated by two disjoint cycles having lengths m and n where $gcd(m, n) = 1$ is c-representable on X.

Theorem 4.5.8. *A group generated by a permutation on a finite set X which is a product of two disjoint cycles having lengths n and m respectively where $gcd(n, m) = 1$ is c-representable.*

Proof. Let $X = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$. Let $p = p_1p_2$ where $p_1 = (a_1, a_2, \ldots, a_n)$ and $p_2 = (b_1, b_2, \ldots, b_m)$. Let H be the group generated by p. Treat X as $X_1 \cup X_2$ where $X_1 = \{a_1, a_2, \ldots, a_n\}$ and $X_2 = \{b_1, b_2, \ldots, b_m\}$. By Theorem 4.5.1, H_1 is c-representable on X_1 and H_2 is c-representable on X_2. Since m and n are relatively prime, $H = H_1 \oplus H_2$. Hence H is c-representable on X by Theorem 4.4.1. □

4.6 On c-representability of Normal Subgroups of $S(X)$

Here we characterize c-representable normal subgroups of the symmetric group $S(X)$. First of all let us have a look at what are the normal subgroups of $S(X)$.

Note 4.6.1. [36]

If $|X| = n$, the Alternating group $A(X)$ is the only non-trivial proper normal subgroup of $S(X)$ except when $n = 4$ where $A(X) = \{g \in S(X) : g \text{ is even}\}$. When $X = \{a, b, c, d\}$, $S(X)$ has another normal subgroup
$H = \{I, (a, b)(c, d), (a, c)(b, d), (a, d)(b, c)\}$. Let X be an infinite set, $g \in S(X)$ and $\aleph_0 \leq k \leq |X|$, then support of g is

$$supp(g) = \{x \in X : g(x) \neq x\}.$$

and the bounded symmetric group is given by

$$BS(X, k) = \{g \in S(X) : |supp(g)| < k\}.$$

Then $A(X)$ can be defined as $A(X) = \{g \in BS(X, \aleph_0) : g \text{ is an even permutation on } X\}$.

Theorem 4.6.2. *[36] If $|X| = \alpha$ where α is an infinite cardinal number and N is a normal subgroup of $S(X)$, then N is one of the group in the chain $\{I\} \leq$*

$A(X) \leq BS(X, \aleph_0) \leq BS(X, \aleph_1) \leq \ldots \leq BS(X, \alpha) \leq S(X)$.

Clearly the improper subgroup $S(X)$ are c-representable on a nonempty set X. Since the trivial group $\{I\}$ is t-representable on a set X, it is c-representable on X. Now we consider the c-representability of non trivial proper normal subgroup of $S(X)$. It was proved that no proper non trivial normal subgroups of $S(X)$ are t-representable [32, 34]. When $|X| = 3$, we have by Theorem 4.5.1, $A(X)$ is c-representable.

Now we investigate the c-representability of normal subgroups of $S(X)$ when $|X| \neq 3$.

Lemma 4.6.3. *Let (X, V) be a closure space such that $A(X)$ is contained in $CI(X, V)$ and $|X| \neq 3$. Then V is a topological closure operator.*

Proof. When $|X| < 3$, there is nothing to prove. Now suppose that $|X| > 3$ and V is not a topological closure operator. Then everu subset of X is either closed or dense in X. Suppose not. Then there exists a subset A of X which is neither closed nor dense in X. Let $x \in V(A) \setminus A$, $y \in X \setminus V(A)$. If A is not a singleton, choose $a, b \in A$ such that $a \neq b$. Then $p = (x, y)(a, b)$ is a permutation on X such that $p \in A(X)$. Then p is a closure isomorphism on (X, V) and hence $V(p(A)) = p(V(A))$. Then $y \in V(A)$, which is a contradiction. Hence either $V(A) = A$ or $V(A) = X$. Now if A is a singleton, we can choose $z \in X \setminus A$, $x \neq z \neq y$ since $|X| \geq 4$. Let $p = (x, y, z)$. Then $p \in A(X)$ and $p(V(A)) = V(p(A))$. Then $y \in V(A)$, which is also a contradiction. In either case $V(V(A)) = V(A)$. Thus V is a topological closure operator on X. □

Remark 4.6.4. The condition $A(X)$ is a subset of $CI(X, V)$ is not necessary in Lemma 4.6.3. That is we can find a topological closure operator V on X such that $A(X) \nsubseteq CI(X, V)$.

Example 4.6.5. Take $X = \{1, 2, 3, 4\}$. Let $V : P(X) \to P(X)$ be defined as $V(\emptyset) = \emptyset$, $V(\{1\}) = \{1, 2\} = V(\{2\})$ and $V(\{3\}) = \{3, 4\} = V(\{4\})$, and $V(A) = \underset{i \in A}{\cup} V(\{i\})$. Then V is a topological closure operator on X. Here
$CI(X, V) = \{I, (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2), (1, 2)(3, 4), (1, 4)(2, 3), (1, 3),$
$(2, 4)\}$ and $A(X) \nsubseteq CI(X, V)$.

Remark 4.6.6. As noted above, when $X = \{a, b, c, d\}$, $S(X)$ has a normal subgroup given by $H = \{I, (a, b)(c, d), (a, c)(b, d), (a, d)(b, c)\}$. This normal subgroup is not t-representable [32, 34].

Lemma 4.6.7. *Let* $X = \{a, b, c, d\}$*. Then*
$H = \{I, (a, b)(c, d), (a, c)(b, d), (a, d)(b, c)\}$ *is not c-representable on* X*.*

Proof. Suppose H is c-representable on X. Then there exists a closure operator V on X such that $CI(X, V) = H$. Now we prove V is a topological closure operator. Suppose not, there exists a subset A of X which is neither closed nor dense in X. Then we can choose two distinct points a and b such that $a \in V(A) \setminus A$ and $b \in X \setminus V(A)$. Then $A = \{c, d\}$, $\{c\}$ or $\{d\}$.

Case (i) $A = \{c, d\}$

Since $p = (a, b)(c, d)$ is a closure isomorphism, we have $p(V(A)) = V(p(A))$.

Since $a \in V(A)$, $p(a) \in p(V(A)) = V(p(A)) = V(A)$. This implies that $b \in V(A)$, which is a contradiction. Hence V is a topological closure operator on X.

Case (ii) $A = \{c\}$.

We have $a \in V(\{c\})$ and $b \notin V(\{c\})$. Then either $V(\{c\}) = \{a, c, d\}$ or $V\{c\} = \{a, c\}$. If $V\{c\} = \{a, c\}$, then the transposition (a, c) is a closure isomorphism, which is a contradiction. Now let $V(\{c\}) = \{a, c, d\}$, then $V\{b\} = \{a, b, d\}$, $V\{a\} = \{a, b, c\}$ and $V\{d\} = \{a, b, d\}$. Here the cycle (a, b, d, c) is a closure isomorphism, which is a contradiction.

Case (iii) $A = \{d\}$

Similar to **Case (ii)**.

Hence V is a topological closure operator. This implies that H is t-representable on X, which is a contradiction to Remark 4.6.6. Thus H is not c-representable on X. □

Lemma 4.6.8. *Let (X, V) be a closure space. If $|X| \neq 3$, then no proper non-trivial normal subgroup K of $S(X)$ is c-representable on X.*

Proof. Clearly $S(X)$ has no proper normal subgroups when $|X| < 3$. Let $|X| = 4$. Suppose $X = \{a, b, c, d\}$, then $H = \{I, (a, b)(c, d), (a, c)(b, d), (a, d)(b, c)\}$ and $A(X)$ are two proper normal subgroups. Then H is not c-representable by Theorem 4.6.7. Now $A(X)$ is not c-representable on X by Lemma 4.6.3 and the fact that $A(X)$ is not t-representable on X.

Now suppose that $|X| > 4$. Then any proper non trivial normal subgroups of $S(X)$ is either $A(X)$ or it contains $A(X)$ by Note 2.2.4. Now if K is c-representable on X, there exists a closure operator V on X such that $CI(X, V) = K$. Then by Lemma 4.6.3, V is a topological closure operator on X. This is a contradiction since H is not t-representable on X. □

Theorem 4.6.9. *Let X be any set and H be a proper normal subgroup of $S(X)$. Then H is c-representable on X if and only if $|X| = 3$.*

Proof. Let X be any set such that $|X| \neq 3$, then H is not c-representable on X by Lemma 4.6.8. If $|X| = 3$, then the only proper normal subgroup is the cyclic group generated by the three cycle in X, which is c-representable on X by Theorem 4.5.1. □

4.7 Hereditarily Homogeneous Closure Spaces

In this section we discuss hereditarily homogeneous closure spaces. Kannan V. and Ramachandran P. T. gave several characterizations of hereditarily homogeneous topological spaces in [21].

Definition 4.7.1. A closure space $(X,\ V)$ is homogeneous if for all $a,\ b \in X$ there exists a closure isomorphism $h : X \longrightarrow X$ such that $h(a) = b$.

Remark 4.7.2. The subspace of a homogeneous space need not be homogeneous.

Example 4.7.3. Let $X = \{a_1, a_2, \ldots, a_n\}$, $n \geq 5$. Define $V : P(X) \to P(X)$ as $V(\emptyset) = \emptyset$, $V(\{a_i\}) = \{a_i, a_{i\oplus 1}\}$ where $\oplus$ denotes addition modulo n and $V(A) = \bigcup_{a_i \in A} V(\{a_i\})$ for every $A \subset X$. Then $CI(X, V)$ is the cyclic group generated by the cycle $(a_1, a_2, \ldots, a_n)$ which is a transitive permutation group. Let $A = \{a_1, a_2, a_3, a_4\}$ and $V' = V|_A$. Then (X, V) is homogeneous but (A, V') is not homogeneous.

Definition 4.7.4. A closure space $(X,\ V)$ is said to be hereditarily homogeneous if every subspace of it is homogeneous.

Clearly discrete closure spaces and indiscrete closure spaces are hereditarily homogeneous closure spaces.

Definition 4.7.5. [32] A closure space $(X,\ V)$ is called completely homogeneous if the group of closure isomorphisms $(X,\ V)$ is the symmetric group $S(X)$.

P. T. Ramachandran characterized completely homogeneous closure spaces in [32]. He proved that a closure space (X, V) is completely homogeneous if and only if V is a closure operator associated with a completely homogeneous topology on X [32].

Proposition 4.7.6. *Suppose that (X, V) is a completely homogeneous closure space. Then (X, V) is hereditarily homogeneous.*

Proof. Suppose (X, V) is a completely homogeneous closure space. Then $CI(X, V) =$

$S(X)$. Let $Y \subseteq X$ and $a, b \in Y, a \neq b$. We have $V'(A) = V(A) \cap Y$ is the induced closure operator on Y. Since $CI(X, V) = S(X)$, the transposition $p = (a, b)$ is a closure isomorphism of (X, V). Then $p(V'(A)) = p(V(A) \cap Y) = p(V(A)) \cap Y = V(p(A)) \cap Y = V'(p(A))$. Hence p is a closure isomorphism of (Y, V'). That is (Y, V') is homogeneous. Hence (X, V) is hereditarily homogeneous. □

Lemma 4.7.7. *Let (X, V) be a hereditarily homogeneous closure space. Then the closure operator V is either T_1 or indiscrete.*

Proof. Let (X, V) be a hereditarily homogeneous space which is not T_1. Then there exists an element $a \in X$ such that $V(\{a\}) \neq \{a\}$. Let $b \in X, b \neq a$ such that $b \in V(\{a\})$. Now consider the subspace $Y = \{a, b\}$. Let $V^{'} = V|_Y$. Since X is hereditarily homogeneous, $(Y, V^{'})$ is homogeneous. If $V^{'}(\{a\}) = \{a\}$, then $V(\{a\}) = \{a\}$ which is a contradiction. Also if $V^{'}\{a\} = \{a, b\}$, then there exists no closure isomorphism which maps a in to b. Hence $(Y, V^{'})$ is indiscrete. Suppose A is a non empty subset of X such that $V(X \setminus A) = X \setminus A$. Let $c \in X$ such that $c \neq a, b$. Consider the subspace $\{a, b, c\}$ of X. Let $V'' = V|\{a, b, c\}$. Then $\{a, b, c\}$ is homogenenous. If $a, b \in A$ then $c \in X \setminus A$ and if $a, b \notin A$ then $c \in A$.

Case (i): $a, b \in A$ and $c \in X \setminus A$.

Let $V'' = V|\{a, b, c\}$. Then $V''\{a\} = \{a, b, c\} \cap V(\{a\}) = \{a, b\}$. But $V''\{c\} = \{a, b, c\} \cap V(\{c\}) = \{c\}$. This is a contradiction to the fact that $\{a, b, c\}$ is homogeneous.

Case (ii): $a, b \notin A$ and $c \in A$.

$V''(\{a,b\}) = \{a,b\}$ and $V''(\{a,c\}) = \{a,b,c\} \cap V(\{a,c\}) = \{a,b,c\}$, which is a contradiction since $\{a,b,c\}$ is homogeneous. Thus (X,V) is indiscrete. □

Lemma 4.7.8. *Suppose that every transpositions on a closure space (X,V) is a closure isomorphism, then V is a topological closure operator.*

Proof. Suppose that there exists a subset A of X such that A is neither closed nor dense in X. Then we have $A \neq V(A) \neq X$. Let $x \in V(A) \setminus A$ and $y \in X \setminus V(A)$. Now consider the transposition $p = (x,y)$ on X, which permutes x and y and fixes all other elements of X. Since every transposition is a closure isomorphism, p is a closure isomorphism. Thus $p(V(A)) = V(p(A))$. Then we have $x \in V(A) \Rightarrow p(x) \in p(V(A)) = V(p(A)) = V(A) \Rightarrow y \in V(A)$, which is a contradiction, since $y \notin V(A)$. Thus every subset of X is either closed or dense in X. That is either $V(A) = A$ or $V(A) = X$ for every $A \subseteq X$. In either case $V(V(A)) = V(A)$. Hence V is topological. □

Lemma 4.7.9. *Suppose (X,V) is a hereditarily homogeneous closure space. Then every transposition of X is a closure isomorphism of (X,V) onto itself.*

Proof. Let $x,y \in X$ such that $x \neq y$. Consider the transposition $p = (x,y)$. We have to prove that p is a closure isomorphism. Suppose $A \subseteq X$ such that $x,y \notin A$. Consider the subspace $A \cup \{x,y\}$. Since (X,V) is hereditarily homogeneous, $A \cup \{x,y\}$ is homogeneous. Then there exists a closure isomorphism $h : X \to Y$ such that $h(x) = y$. Then $h(A) = A$ or $h(A) = (A \setminus h(\{y\})) \cup \{x\}$ according as $h(y) = x$ or $h(y) \neq x$.

If $h(A) = A$ then $x \in V(A) \Leftrightarrow y = h(x) \in h(V(A)) = V(h(A)) = V(A)$. Thus $x \in V(A) \Leftrightarrow y \in V(A)$. Let $h(A) = (A \setminus h(\{y\})) \cup \{x\}$. Then $x \in V(A) \Leftrightarrow y = h(x) \in h(V(A)) = V(h(A)) = V(A \setminus \{h(y)\} \cup \{x\}) = V(A \setminus h(y)) \cup V(\{x\}) \subseteq V(A) \cup \{x\} \Rightarrow y \in V(A)$.

Case (i): Let $x \notin A, y \notin A$.

Then $x \in V(A) \Leftrightarrow y \in V(A)$. Thus $p(V(A)) = V(p(A))$.

Case (ii): Let $x \in A, \in y \in A$.

Then $p(V(A)) = V(A) = V(p(A))$.

Case (iii): Let $x \in A$ and $x \notin A$.

Then $p(A) = A \setminus \{x\} \cup \{y\}$. Then $V(p(A)) = V(A \setminus \{x\} \cup \{y\}) = V(A \setminus \{x\}) \cup V(\{y\}) = A \setminus \{x\} \cup \{y\}$, since V is T_1. Now $p(V(A)) = p(V(A \setminus \{x\}) \cup \{y\}) = V(p(A))$.

Case (iv): Let $x \notin A$ and $y \in A$.

Similar to **Case (iii)**. □

Lemma 4.7.10. *Let (X, V) be a hereditarily homogeneous closure space. Then V is a topological closure operator.*

Proof. Now we prove that V is topological. Suppose there exists a subset A of X such that A is neither closed nor dense in X. Then we have $A \neq V(A) \neq X$. Let $x \in V(A) \setminus A$ and $y \in X \setminus V(A)$. Now consider the transposition $p = (x, y)$ on X, which permutes x and y and fixes all other elements of X. Since (X, V) is a hereditarily homogeneous space the transposition p is a closure isomorphism.

Thus $p(V(A)) = V(p(A))$. Then

$$\begin{aligned} x \in V(A) \quad &\Rightarrow p(x) \in p(V(A)) = V(p(A)) = V(A) \\ &\Rightarrow y \in V(A). \end{aligned}$$

This is a contradiction, since $y \notin V(A)$. Hence every subset of X is either closed or dense in X. Hence either $V(A) = A$ or $V(A) = X$ for every $A \subseteq X$. In either case $V(V(A)) = V(A)$. Hence V is topological. □

Theorem 4.7.11. *Let (X, V) be a T_1 Closure space. Then the following are equivalent.*

(i) *(X, V) is hereditarily homogeneous.*

(ii) *(X, V) is the closure space associated with a hereditarily homogeneous topological space.*

(iii) *All transpositions in X are closure isomorphisms of (X, V) onto itself.*

(iv) *Every permutation of X which moves only a finite number of elements of X is a closure isomorphism of (X, V).*

Proof. We have (ii) follows from (i) by Lemma 4.7.10. Now suppose that (ii). Then clearly (X, V) is hereditarily homogeneous. Thus (i) and (ii) are equivalent. Now assume (iii). Suppose p is a permutation of X which moves only a finite number of elements of X, then p is product of a finite number of transpositions.

Hence p is a closure isomorphism. Suppose (iv). Then clearly (iii) holds. Thus (iii) and (iv) are equivalent. Now assume (iii). Suppose $Y \subseteq X$, let $a, b \in Y$ and $a \neq b$. Then by assumption, $p = (a, b)$ is a closure isomorphism. Let $A \subseteq Y$. We have $p(V'(A)) = p(V(A) \cap Y) = p(V(A)) \cap Y = V(p(A)) \cap Y = V'(p(A))$. Hence p is a closure isomorphism of (Y, V'). That is (Y, V') is homogeneous. Then (X, V) is hereditarily homogeneous. Hence (i) holds. $\square$

Chapter 5

Generalized Closure Operators

5.1 Introduction

In this chapter we discuss the lattice of generalized closure operators and the lattice of generalized Čech closure operators. We prove that the lattice of generalized closure operators on a fixed non-empty set is a complete lattice. Here we determine atoms and dual atoms of the lattice of generalized closure operators and of the lattice of generalized Čech closure operators. Simple expansion of a generalized closure operator is also introduced.

5.2 Preliminaries

First let us go through the definition of the generalized topology on a set X.

Definition 5.2.1. [13] Let X be a set. A collection μ of subsets of X is said to be a generalized topology on X if $\emptyset \in \mu$ and arbitrary union of elements in μ is again in μ. The ordered pair (X, μ) is called a generalized topological space.

Let (X, μ) be a generalized topological space. The elements of μ are called μ open sets or simply open sets [13]. A subset A of X is said to be closed set if $X \setminus A$ is open. Note that a generalized topology is said to be strong if $X \in \mu$ [13].

An operator on C on $P(X)$ which maps $g \in \mu$ to the smallest closed set containing g is a closure operator on (X, μ). It satisfies the conditions $A \subseteq C(A)$, $A \subseteq B \Rightarrow C(A) \subseteq C(B)$ and $C(C(A)) = C(A)$. By relaxing the idempotent condition of C, a generalized closure operator is defined as follows [44].

Definition 5.2.2. [44] Let X be a set. A function $C : P(X) \to P(X)$ satisfying the conditions $A \subseteq C(A)$ and $A \subseteq B \Rightarrow C(A) \subseteq C(B)$ for every $A, B \subseteq X$ is called a generalized closure operator. The ordered pair (X, C) is called a generalized closure space.

A subset A of X is said to be closed if $C(A) = A$ and is said to be open if its complement is closed. The set of all open subsets of X forms a generalized topology on X called the generalized topology associated with the generalized closure operator C [44].

Let μ be a generalized topology on X. Then the operator on X which maps A into the smallest closed set containing A is a generalized closure operator on X. This is called the generalized closure operator associated with μ. A generalized closure operator is said to be strong if $C(C(A)) = C(A)$ for every $A \subseteq X$ [44]. The generalized closure operator associated with μ is strong.

Example 5.2.3. Let $X = \{a, b, c\}$. Define $C : P(X) \to P(X)$ as $C(\emptyset) = \emptyset$, $C(\{a\}) = \{a\}$, $C(\{b\}) = \{b, c\}$, $C(\{c\}) = \{c\}$, $C(\{a, b\}) = C(\{b, c\}) = C(\{c, a\}) = C(X) = X$. Then $C(C(\{b\})) = C(\{b, c\}) = X \neq C(\{b\})$. Thus C is a generalized closure operator on X which is not a strong generalized closure operator.

5.3 Lattice of Generalized Closure Operators

Let C_1, C_2 be two generalized closure operators on a set X. Then we say $C_1 \leq C_2$ if and only if $C_2(A) \subseteq C_1(A)$ for every $A \subseteq X$. Then $\leq$ is a partial order on the set of all generalized closure operators. Define $(C_1 \wedge C_2)(A) = C_1(A) \cup C_2(A)$ and $(C_1 \vee C_2)(A) = C_1(A) \cap C_2(A)$. Then $C_1 \wedge C_2$ and $C_1 \vee C_2$ are generalized closure operators on X. Then the set of all generalized closure operators on X forms a lattice under this partial order and is denoted by $LG(X)$.

Theorem 5.3.1. *Let X be a set. The lattice of generalized closure operators on X is a complete lattice. Let $\{C_i : i \in \mathscr{I}\}$ where $\mathscr{I}$ is some indexing set, be a nonempty family of generalized closure operators on X. Then the greatest lower bound*

$inf\{C_i | i \in \mathscr{I}\}(S) = \bigcup\limits_{i \in \mathscr{I}} \{C_i(S)\}$, for each $S \subseteq X$.

Proof. Let C denotes the greatest lower bound $inf\{C_i | i \in \mathscr{I}\}$. First of all we prove that $C(S) = \underset{i \in \mathscr{I}}{\cup}\{C_i(S)\}$ for $S \subseteq X$ is a generalized closure operator on X. We have $S \subseteq C_i(S)$ for every $i \in \mathscr{I}$ and $S \subseteq X$. Thus $S \subseteq C(S)$ for every $S \subseteq X$. Suppose $S_1 \subseteq S_2$. Then $C_i(S_1) \subseteq C_i(S_2)$ for each $i \in \mathscr{I}$. Hence $\underset{i \in \mathscr{I}}{\cup}\{C_i(S_1)\} \subseteq \underset{i \in \mathscr{I}}{\cup}\{C_i(S_2)\}$. Thus $C(S_1) \subseteq C(S_2)$. Thus C is a generalized closure operator. Let C' be a generalized closure operator on X such that $C' \leq C_i$ for every $i \in \mathscr{I}$. Then $C_i(S) \subseteq C'(S)$ for every $i \in \mathscr{I}$. This implies that $\underset{i \in \mathscr{I}}{\cup}\{C_i(S)\} \subseteq C'(S)$. Hence $C' \leq C$ and therefore $inf\{C_i | i \in \mathscr{I}\}(S) = \underset{i \in \mathscr{I}}{\cup}\{C_i(S)\}$, for each $S \subseteq X$. That is every subset of $LG(X)$ has a meet. Thus $LG(X)$ is a complete lattice. □

Note that the generalized closure operator defined on X as $I_g(A) = X$ for all $A \subseteq X$ is the smallest element of $LG(X)$. The discrete closure operator defined by $D(A) = A$ for all $A \subseteq X$ is the largest element of $LG(X)$.

Remark 5.3.2. $(\bigvee_{i \in \mathscr{I}} C_i)(S) = \underset{i \in \mathscr{I}}{\cap}\{C_i(S)\}$ for each $S \subseteq X$. The map $C = \{S \to \underset{i \in \mathscr{I}}{\cap}\{C_i(S) | S \subseteq X\}$ is a generalized closure operator on X. For, we have $S \subseteq C_i(S)$ for each $i \in \mathscr{I}$ and each $S \subseteq X$. Now suppose that $S_1 \subseteq S_2$. Then $C_i(S_1) \subseteq C_i(S_2)$ for each $i \in \mathscr{I}$. Then $\underset{i \in \mathscr{I}}{\cap} C_i(S_1) \subseteq \underset{i \in \mathscr{I}}{\cap} C_i(S_2)$. Thus $C(S_1) \subseteq C(S_2)$. Hence C is a generalized closure operator on X.

Since $(\bigvee_{a \in \mathscr{A}} C_a)(S) = \underset{a \in \mathscr{A}}{\cap}\{C_a(S)\}$ and $(\bigwedge_{a \in \mathscr{A}} C_a)(S) = \underset{a \in \mathscr{A}}{\cup}\{C_a(S)\}$ for each $S \subseteq X$, we conclude that the lattice $LG(X)$ is distributive hence modular.

Next we are trying to find out atoms and dual atoms of $LG(X)$.

Definition 5.3.3. Let X be any set and $x \in X$. Define $C_x : P(X) \to P(X)$

as

$$C_x(A) = \begin{cases} X \setminus \{x\} & ; \text{if } A = \emptyset \\ X & ; \text{if } A \neq \emptyset \end{cases}$$

Then C_x is a generalized closure operator on X for each $x \in X$.

Atoms in the lattice of generalized closure operators are generalized closure operators of the form C_x.

Theorem 5.3.4. *A generalized closure operator on X is an infra generalized closure operator if and only if it is of the form C_x for some $x \in X$.*

Proof. Assume that C is an infra generalized closure operator on X. Then there exists a subset $A \subseteq X$ such that $C(A) \subset I_g(A) = X$. That is there exists $x \in X$ such that $x \notin C(A)$. Thus $x \notin C(\emptyset)$. In other words $C(\emptyset) \subseteq X \setminus \{x\}$ and $C(A) \subseteq X \setminus \{x\}$. Hence $I_g \leq C_x \leq C$. Since C is an infra closure operator and $C \neq I_g$ we get $C = C_x$.

Conversely we have to prove that C_x is an infra generalized closure operator. If A is a non-empty subset of X, then $I_g(A) = C_x(A)$. We have $C_x(\emptyset) = X \setminus \{x\} \subset I_g(\emptyset) = X$. Hence $I_g \leq C_x$. Suppose $I_g \leq C' < C_x$. Then $C_x(\emptyset) \subset C'(\emptyset) \subseteq I_g(\emptyset)$. Then $X \setminus \{x\} \subset C'(\emptyset) \subseteq X$. Hence $C'(\emptyset) = X$. Thus $C' = I_g$. □

We have dual atoms in the lattice of generalized topologies are of the form $P(X) \setminus \{\{x\}\}$, $x \in X$ [14]. The generalized closure operator associated with the dual atom $P(X) \setminus \{\{x\}\}$ is given by

$$C(A) = \begin{cases} X & ; \text{if } A = X \setminus \{x\} \\ A & ; \text{otherwise.} \end{cases}$$

Then there exists no generalized closure operator other than C which is strictly larger than C and strictly smaller than D.

Theorem 5.3.5. *A generalized closure operator on X is an ultra generalized closure operator if and only if it is the generalized closure operator associated with some ultra generalized topology on X.*

Proof. Let V be the ultra generalized closure operator. Then there exists no generalized closure operator V' such that $V < V' < D$. Since V is the generalized closure operator associated with the ultra generalized topology $P(X)\setminus\{x\}$, then $V(X \setminus \{x\}) \neq X \setminus \{x\}$ and $V(S) = S$ for every $S \neq X \setminus \{x\}$. But $X \setminus \{x\} = D(X \setminus \{x\}) \subseteq V(X \setminus \{x\})$. This implies that $V(X \setminus \{x\}) = X$ and $V(S) = S$ for every $S \neq X \setminus \{x\}$. Hence V is the generalized closure operator associated with the ultra generalized topology on X.

Now suppose that V is the generalized closure operator associated with the ultra generalized topology $P(X)\setminus\{x\}$. Let V' be the generalized closure operator on X such that $V < V' \leq D$. Then $D(S) \subseteq V'(S) \subset V(S)$ for every $S \subseteq X$. Then $V' = D$. Hence V is the ultra generalized closure operator on X. □

5.4 Generalized Čech Closure Operators

Corresponding to a generalized topology, we have the strong generalized closure operator. Corresponding to a strong generalized topology on a set X, we can find a closure operator C on X which satisfies the conditions $C(\phi) = \phi$, $A \subseteq C(A)$, $A \subseteq B \Rightarrow C(A) \subseteq C(B)$ and $C(C(A)) = C(A)$. There is defined a closure operator by weakening the idempotent condition of the above mentioned closure operator in [8]. In this section we study such closure operators.

Definition 5.4.1. [8] A function $Cl : P(X) \to P(X)$ is said to be a generalized Čech closure operator if it satisfies the following conditions, $Cl(\emptyset) = \emptyset$, $A \subseteq Cl(A)$ and if $A \subseteq B$, then $Cl(A) \subseteq Cl(B)$ for every $A, B \subseteq X$. The ordered pair (X, Cl) is called the generalized Čech closure space.

Every generalized Čech closure operator is a generalized closure operator and every Čech closure operator is a generalised Čech closure operator. Converse is not true.

Example 5.4.2. Let $X = \{1, 2, 3\}$. Define $C : P(X) \to P(X)$ as $C(\emptyset) = \emptyset$, $C(\{1\}) = \{1\}$, $C(\{2\}) = \{2\}$, $C(\{3\}) = \{2, 3\}$, $C(\{2, 3\}) = \{2, 3\}$ and $C(\{1, 2\}) = C(\{1, 3\}) = C(X) = X$. Thus C is a generalised Čech closure operator which is not a Čech closure operator, since $C(\{1, 2\}) \neq C(\{1\}) \cup C(\{2\})$.

A subset $A \subseteq X$ is said to be closed if $Cl(A) = A$ and is said to be open if its complement is open. The set of all open sets in a generalized Čech closure space (X, Cl) forms a strong generalized topology and is called the generalized topology associated with the generalized Čech closure operator Cl. Now consider the converse situation. Let (X, μ) be a strong generalized topology on X. Then the operator on $P(X)$ which maps A in to the smallest closed set containing A is a generalized Čech closure operator and is called the generalized Čech closure operator associated with the strong generalised topology μ on X.

With any generalized Čech closure operator, there is associated an interior operation denoted by int and is defined as below.

Definition 5.4.3. Let (X, Cl) be a generalized Čech closure space. Then $int : P(X) \to P(X)$ defined by $int(S) = X \setminus Cl(X \setminus S)$. The set $int(S)$ is called interior of S in (X, Cl).

From the definition of interior operator and generalized Čech closure operator we have the following proposition.

Proposition 5.4.4. *In a generalized Čech closure space we have the following:*

(a). $intX = X$.

(b). For each $S \subseteq X$, $intS \subseteq S$.

(c). If $A \subseteq B$, $intA \subseteq intB$.

Example 5.4.5. $int(A \cap B) \neq intA \cap intB$. For example Let $X = \{a, b, c\}$. $Cl(\emptyset) = \emptyset$, $Cl(\{a\}) = \{a\}$, $Cl(\{b\}) = \{b\}$, $Cl(\{c\}) = \{c\}$, $Cl(\{a, b\}) = Cl(\{b, c\}) = Cl(\{a, c\}) = Cl(X) = X$. Then Cl is a generalized Čech closure operator on X. Then $int(\{a\}) = int(\{b\}) = int(\{c\}) = \emptyset$, $int(\{a,b\}) = \{a,b\}$, $int\{b,c\} = \{b,c\}$. Then $\emptyset = int(\{a,b\} \cap \{b,c\}) \neq int(\{a,b\}) \cap int\{b,c\} = \{b\}$.

Proposition 5.4.6. *Let (X, Cl) be a generalized Čech closure space. Then $S \subseteq X$ is open if and only if $int\, S = S$.*

Proof. $int\, S = S \Leftrightarrow X \setminus Cl(X \setminus S) = S \Leftrightarrow X \setminus S = Cl(X \setminus S) \Leftrightarrow S$ is open. □

Definition 5.4.7. A neighbourhood of a subset S of a generalized Čech closure operator is any subset N containing S in its interior. Thus N is a neighbourhood of S if and only if $S \subseteq int(N)$. We say N is a neighbourhood of an element $x \in X$, if N is a neighbourhood of the singleton set $\{x\}$.

Proposition 5.4.8. *Let (X, Cl) be a generalized Čech closure space. A subset N of X is a neighbourhood of a subset S of X if and only if N is a neighbourhood of each point of X. Also a subset S of X is open if and only if it is a neighbourhood of each of its points.*

Proof. Proof is clear from the definition of neighbourhood and the Proposition 5.4.6. □

Theorem 5.4.9. *Let $\mathscr{N}$ be the neighbourhood system of a subset S of a generalized Čech closure space (X, Cl). Then every member of $\mathscr{N}$ contains S and if $X \supset N_1 \supset N_2 \in \mathscr{N}$, then $N_2 \in \mathscr{N}$.*

Proof. This result is obvious from the definition of a neighbourhood of a subset S of X. □

Remark 5.4.10. Let N_1 and N_2 be a neighbourhood of a point $x \in X$. Then $N_1 \cap N_2$ need not be a neighbourhood of x. Consider the Example 5.4.5. $\{a, b\}$ and $\{a, c\}$ is a neighbourhood of $\{a\}$. But $\{a\}$ is not a neighbourhood of $\{a\}$. Thus the neighbourhood of a point x forms a stack on X, that is $\emptyset \notin \mathscr{S}$ and $A \in \mathscr{S}$, $B \supset A$ implies that $B \in \mathscr{S}$.

Theorem 5.4.11. *Let (X, Cl) be a generalized Čech closure space. Then a point x is in the closure of a subset S of X if and only if every neighbourhood of x meets S.*

Proof. Suppose $x \notin Cl(S)$, Then by the definition of neighbourhoods, $X \setminus S$ is a neighbourhood of S. That is there exists a neighbourhood of x which does not meet S. Thus every neighbourhood of x meets S implies that $x \in Cl(S)$. Conversely suppose that N is a neighbourhood of x which does not meet S. Then $x \in int\, N$ and $S \cap N = \emptyset$. Thus $x \in int\, N \subseteq int\,(X \setminus S) = X \setminus Cl(S)$. This implies that $x \notin Cl(S)$. Hence x is in the closure of a subset S of X implies that every neighbourhood of x meets S. □

Theorem 5.4.12. *Let Cl_1 and Cl_2 be two generalized closure operators on a set X. Then $Cl_1 \leq Cl_2$ if and only if for each $x \in X$, every Cl_1-neighbourhood of x is a Cl_2-neighbourhood of x.*

Proof. Suppose $Cl_1 \leq Cl_2$ and let U be a Cl_1-neighbourhood of x. Then $x \notin Cl_1(X \setminus U)$. Then clearly $x \notin Cl_2(X \setminus U)$. This implies that U is a Cl_2-neighbourhood of x. Conversely assume that every Cl_1-neighbourhood of x is a

Cl_2-neighbourhood of x. Let $x \notin Cl_1(S)$. Then there exists a Cl_1-neighbourhood U of x which does not intersects S. Then U is a Cl_2-neighbourhood of x which does not intersects S. Hence $x \notin Cl_2(S)$. Thus $Cl_1 \leq Cl_2$. □

Join and meet of a non void arbitrary collection of generalized Čech closure operators are same as the join and meet of a non void arbitrary collection of generalized closure operators. Thus the lattice of generalized Čech closure operators, $LGC(X)$ is distributive and modular. We can prove that atoms in the lattice of generalized Čech closure operators is same as the atoms in the lattice of Čech closure operators.

Theorem 5.4.13. *Atoms in the lattice of generalized Čech closure operators are same as the atoms in the lattice of Čech closure operators.*

Proof. Let C be a generalized Čech closure operator such that $I \leq C < V_{a,b}$. Then $V_{a,b}(\{a\}) \subset C(\{a\}) \subseteq I(\{a\})$. Then $X \setminus \{b\} \subset C(\{a\}) \subseteq X$. Then $C(\{a\}) = X$. Hence $C = I$. □

5.5 Adjacent Generalized Closure Operators

In this section we discuss adjacency properties of generalized closure operators. We define upper neighbours of a generalized closure operators.

Definition 5.5.1. Let C_1 and C_2 be two generalized closure operators on a set X such that $C_1 < C_2$. Then C_2 is an upper neighbour of C_1 if there exists a generalized closure operator $C^{'}$ on X such that $C_1 \leq C^{'} \leq C_2$, then either $C^{'} = C_1$ or $C^{'} = C_2$. Then we say that C_1 and C_2 are adjacent and C_1 is a lower neighbour of C_2.

Theorem 5.5.2. *[19] In a dually atomic modular lattice every element has a lower neighbour.*

Remark 5.5.3. The lattice of generalized closure operators is dually atomic and modular. Therefore by Theorem 5.5.2, we have every element of $LGC(X)$ other than I_g has a lower neighbour.

Example 5.5.4.

(1) I_g and C_x are adjacent in $LG(X)$.

(2) Discrete closure operator D and the generalized closure operator associated with ultra generalized topology are adjacent.

(3) Let $X = \{a, b, c\}$. Define $C : P(X) \to P(X)$ as $C(\emptyset) = \{a\}$, $C(\{a\}) = \{a, b\}$, $C(\{b\}) = \{a, b\}$, $C(\{c\}) = \{c, a\}$, $C(\{a, b\}) = C(\{b, c\}) = C(\{a, c\}) = C(X) = X$. Then C is a generalized closure operator on X. Define $C^{'} : P(X) \to P(X)$ as $C'(\emptyset) = \emptyset$ and $C^{'}(A) = C(A)$ for every $A \subseteq X$, $A \neq \emptyset$. Then $C^{'}$ is an upper neighbour of C. Now define $C^{"}(\{a, b) = \{a, b\}$ and $C^{"}(A) = C(A)$ for every $A \subseteq X$, $A \neq \{a, b\}$. Then $C^{''} \neq C^{'}$, they are incomparable and both are upper neighbours of C.

(4) Let $X = \{a, b, c\}$. Define $C : P(X) \to P(X)$ as $C(\emptyset) = \{a, b\}$, $C(\{a\}) =$

$\{a, b\}$, $C(\{b\}) = \{a, b\}$, $C(\{c\}) = X$, $C(\{a, b\}) = \{a, b\}$, $C(\{b, c\}) = C(\{a, c\}) = C(X) = X$. Then C is a generalized closure operator on X. Now define $C'(\emptyset) = \{a\}$, $C'(A) = C(A)$ for every $A \neq \emptyset$. Then C' is the upper neighbour of C.

Now we define adjacency in the lattice of generalized Čech closure operators and seek upper neighbours of co-finite closure operators in $LGC(X)$.

Definition 5.5.5. Let Cl_1 and Cl_2 be two generalized Čech closure operators on a set X such that $Cl_1 < Cl_2$. Then Cl_2 is an upper neighbour of Cl_1 if for a generalized closure operator Cl' on X such that $Cl_1 \leq Cl' \leq Cl_2$, either $Cl' = Cl_1$ or $Cl' = Cl_2$. Then we say that Cl_1 and Cl_2 are adjacent and Cl_1 is a lower neighbour of Cl_2.

Examples 5.5.6.

(1) I_g and closure operators of the form $V_{a,b}$ are adjacent in $LGC(X)$.

(2) Discrete closure operator D and the generalized closure operator associated with ultra generalized topology are adjacent in $LGC(X)$.

In order to find out upper neighbours of co-finite closure operators we need the following generalized Čech closure operator.

Definition 5.5.7. Let A be an infinite subset of a set X. Let $x \in X$ be such that $x \notin A$. Define a function $C_{A,x} : P(X) \to P(X)$ such that

$$C_{A,x}(S) = \begin{cases} S & ; \text{ if } S \text{ is finite} \\ X \setminus \{x\} & ; \text{ if } S \text{ is infinite and } S \subseteq A \\ X & ; \text{ otherwise.} \end{cases}$$

Then $C_{A,x}$ is a generalized Čech closure operator on X and $C_0 < C_{A,x}$.

Remark 5.5.8.

1. Note that $C_{A,x}$ is not a Čech closure operator on X. For, let S_1 be a finite subset of X not containing x such that $S_1 \nsubseteq A$. Let S_2 be an infinite subset of X such that $S_2 \subseteq A$. Then $C_{A,x}(S_1) = S_1$, $C_{A,x}(S_2) = X \setminus \{x\}$ and $C_{A,x}(S_1) \cup C_{A,x}(S_2) = S_1 \cup (X \setminus \{x\}) = X \setminus \{x\}$. We have $S_1 \cup S_2$ is an infinite subset of X and is not contained in A, therefore $C_{A,x}(S_1 \cup S_2) = X$. Thus $C_{A,x}(S_1 \cup S_2) \neq C_{A,x}(S_1) \cup C_{A,x}(S_2)$. Thus $C_{A,x}$ does not preserve finite union.

2. Also $x \notin C_{A,x}(S) \Rightarrow S$ infinite subset of X contained in $A \Rightarrow S \backslash A$ is finite $\Rightarrow$ $x \notin C_{A,x}(S)$. Hence $C_{A,x} \leq V_{A,x}$.

Theorem 5.5.9. *Let Cl be a generalized Čech closure operator on X and A be an infinite subset of X not containing $x \in X$. Then $C_{A,x} \leq Cl$ if and only if $x \notin Cl(A)$.*

Proof. Suppose $C_{A,x} \leq Cl$. Then $Cl(A) \subseteq C_{A,x}(A) = X \setminus \{x\} \Rightarrow x \notin Cl(A)$. Conversely suppose that $x \notin Cl(A)$. If $x \notin C_{A,x}(S)$, then S is an infinite subset of X such that $S \subseteq A$ by definition of $C_{A,x}$. Then $Cl(S) \subseteq Cl(A)$. But $x \notin Cl(A)$.

Then $Cl(S) \subseteq C_{A,x}(S)$ for every $S \subseteq X$. Hence $C_{A,x} \leq Cl$. □

Theorem 5.5.10. *We have $C_{A,x}$ is a strong generalized closure operator if and only if $A = X \setminus \{x\}$.*

Proof. Suppose $C_{A,x}$ is a strong generalized closure operator on X. Then $C_{A,x}(C_{A,x}(S)) = C_{A,x}(S)$ for each $S \subseteq X$. For $S \subseteq A$, we have $C_{A,x}(C_{A,x}(S)) = C_{A,x}(S) \Rightarrow C_{A,x}(X \setminus \{x\}) = X \setminus \{x\} \Rightarrow X \setminus \{x\} \subseteq A$. This implies that $A = X \setminus \{x\}$. Conversely suppose $A = X \setminus \{x\}$. Then for $S \subseteq X$ not containing x, $S \subseteq A$ and hence $C_{A,x}(C_{A,x}(S)) = C_{A,x}(S)$. If S is an infinite subset of X containing x, then $C_{A,x}(S) = X = C_{A,x}(C_{A,x}(S))$. Thus $C_{A,x}$ is strong. □

Proposition 5.5.11. *Let A and B are two infinite subsets of a set X such that $x \notin A \cap B$. Then $C_{A,x} \vee C_{B,x} = C_{A \cup B,x}$.*

Proof. We have $x \notin (C_{A,x} \vee C_{B,x})(S) \Leftrightarrow x \notin C_{A,x}(S)$ or $x \notin C_{B,x})(S) \Leftrightarrow S \subseteq A$ or $S \subseteq B \Leftrightarrow S \subseteq A \cup B \Leftrightarrow x \notin C_{A \cup B,x}$. Thus $C_{A,x} \vee C_{B,x} = C_{A \cup B,x}$. □

Lemma 5.5.12. *If A is an infinite subset of X, then there is a proper infinite subset B of A such that $C_{B,x} < C_{A,x}$.*

Proof. Suppose B is a countable proper subset of A. Then $x \notin C_{B,x}(S) \Rightarrow S$ is infinite and $S \subseteq B$. Then $S \subseteq A$. Hence $x \notin C_{A,x}(S)$. Hence $C_{B,x} \leq C_{A,x}$. Also $C_{B,x}(A) = X$ whereas $C_{A,x}(A) = X \setminus \{x\}$. Hence $C_{B,x} < C_{A,x}$. □

Theorem 5.5.13. *The closure operator C_0 has no upper neighbour in the lattice of generalized Čech closure operators.*

Proof. Let Cl be any generalized Čech closure operator satisfying $C_0 < Cl$. If S is finite, then $C_0(S) = S$. Since $C_0 < Cl$, we have an infinite subset S of X such that $Cl(S) \subset C_0(S)$. Then there exists $x \in X$ such that $x \notin Cl(S)$. Then by Lemma 5.5.9, $C_{S,x} \leq Cl$. We have $C_0 \leq C_{S,x} \leq Cl$. Again by Lemma 5.5.12, there exists $S^{'} \subset S$ such that $C_{S^{'},x} < C_{S,x}$. Thus we get $C_0 \leq C_{S^{'},x} < C_{S,x} \leq Cl$. Thus C_0 has no upper neighbour in the lattice of generalized Čech closure operators. □

Theorem 5.5.14. *The generalized closure operator of the form $C_{A,x}$ has no upper neighbour in the lattice of generalized Čech closure operators.*

Proof. Suppose Cl is an upper neighbour of $C_{A,x}$ in $LGC(X)$. Then there is an infinite subset S of X such that $Cl(S) \subset C_{A,x}(S)$. Let $y \in X$ be such that $y \notin Cl(S)$, but $y \in C_{A,x}(S)$.

Case (i): $y \neq x$

Since $y \notin Cl(S)$, by Lemma 5.5.9, $C_{S,y} \leq Cl$. Again by Lemma 5.5.12, $C_{S',y} < C_{S,y} \leq Cl$. Consider $Cl' = C_{A,x} \vee C_{S',y}$. We have $C_{A,x} \leq Cl'$, $C_{A,x} < Cl$ and $C_{S',y} < Cl$. Thus $C_{A,x} < Cl' < Cl$.

Case (ii): $y = x$

$x \notin Cl(S)$ implies that $C_{S,x} \leq Cl$. Also $x \in C_{A,x}(S) \Rightarrow S \cap (X \setminus A) \neq \emptyset$. Let $S' \subseteq (S \setminus A)$. Then $C_{S',x} < C_{S,x} \leq Cl$. Let $C' = C_{A,x} \vee C_{S',x}$. We have $C_{A,x} < Cl$, $C_{S',x} < Cl$ implies that $C' < C$. Thus $C_{A,x} < C' < Cl$. Hence in both cases we arrive at a contradiction. Hence $C_{A,x}$ has no upper neighbour in $LGC(X)$. □

Now we define simple expansion of a generalized closure operator to produce finer generalized closure operators.

Definition 5.5.15. Let (X, V) be a generalized closure space. Let A be a subset of X such that $x \in A$. Then define $C_{(A,x)} : P(X) \to P(X)$ such that

$$C_{(A,x)}(S) = \begin{cases} X \setminus \{x\} & \text{if } S \subseteq X \setminus A, \\ X & \text{otherwise.} \end{cases}$$

Definition 5.5.16. Let (X, V) be a generalized closure space. Let A be a subset of X such that $x \in A$. Then the simple expansion of C by A at x is given by $C_A^x = C \vee C_{(A,x)}$.

Proposition 5.5.17. *$C_A^x = C$ if and only if $x \notin C(X \setminus A)$.*

Proof. Suppose $C_A^x = C$. Then $C(X \setminus A) = C_{(A,x)}(X \setminus A) = X \setminus \{x\}$. That is $x \notin C(X \setminus A)$. Now suppose $x \notin C(X \setminus A) \Rightarrow C(X \setminus A) \subseteq X \setminus \{x\} = C_{A,x}(X \setminus A)$. □

Proposition 5.5.18. *Let C and C' be two adjacent generalized closure operators on X. Then C' is a simple expansion of C by some subset of X.*

Proof. Suppose $C < C'$. Then there exists a non-empty subset A of X such that $C'(A) \subset C(A)$. Let $x \in A$. Consider the simple expansion of C by A at x. Then $C \leq C_A^x$. Also $C_A^x(A) = C(A)$. It follows that $C'(A) \subset C_A^x(A) = C(A)$. Thus $C \leq C_A^x \leq C'$. Hence either $C = C_A^x$ or $C_A^x = C'$. Since $C \neq C_A^x$, $C' = C_A^x$. This completes the proof. □

Remark 5.5.19. Converse of the Proposition 5.5.18 is not true.

Example 5.5.20. Let $X = \{1, 2, 3\}$. $C : P(X) \to P(X)$ can be defined as $C(\phi) = \{1\}$, $C(\{1\}) = C(\{2\}) = \{1, 2\}$, $C(\{3\}) = \{1, 3\}$, $C(\{1, 2\}) = C(\{2, 3\}) = C(\{3, 1\}) = C(X) = X$. Consider the subset $A = \{1, 2\}$, $x = 1$. Then $C^x_A(\phi) = \phi$, $C^x_A(\{1\}) = \{1, 2\}$, $C^x_A(\{2\}) = \{1, 2\}$, $C^x_A(\{3\}) = \{3\}$, $C^x_A(\{1, 2\}) = X$, $C^x_A(\{2, 3\}) = \{2, 3\}$, $C^x_A(\{1, 3\}) = X = C^x_A(X)$. Then C^x_A is not an upper neighbour of C. Since $C' : P(X) \to P(X)$ defined by $C'(\phi) = \phi$, $C'(\{1\}) = \{1, 2\}$, $C'(\{2\}) = \{1, 2\}$, $C'(\{3\}) = \{3\}$, $C'(\{1, 2\}) = C'(\{2, 3\}) = C'(\{1, 3\}) = X = C'(X)$ is such that $C \leq C' \leq C^x_A$.

Chapter 6

Conclusion

This thesis is a study of some problems related to Čech closure spaces. Here we investigated adjacency properties of closure operators. We continued the work of Kunheenkutty M. [24] by seeking the existence of upper neighbours of some closure operators on a set X. We proved that a first countable T_1 closure operator has no upper neighbour in the lattice of closure operators. Adjacency properties of sum of closure operators and product of closure operators were discussed. We compared certain properties like regularity, normality and connectedness of closure operators with their simple expansions. We discussed when simple expansions of a closure operator V on a set X by two subsets of X at the same point become equal.

We investigated c-representability of a permutation groups analogous to the concept of $t-$representability of permutation groups in [40]. We proved that no normal subgroup of $S(X)$ is $c-$representable if $|X| \neq 3$. Also we studied the $c-$representability of the direct sum of closure operators. We characterized hereditarily homogeneous T_1 closure spaces.

We studied generalized form of closure operators and studied their lattice. We compared the lattice of generalized closure operator and generalized Čech closure operators. We found atoms and dual atoms of their lattices. We proved that co-finite closure operator has no upper neighbour in the lattice of generalized Čech closure operators. Also we introduced simple expansions in the lattice of generalized closure operators.

6.1 Further Scope of Research

Here we would like to mention some open problems which arose during our study. To find out more closure operators having an upper neighbour and to characterize the existence of upper neighbours of closure operators is an open problem. There are several conditions for two simple extensions of topologies to be equal. An analogous problem in closure spaces is still open. We determined the c-representability of some subgroups of $S(X)$. It would be interesting to determine completely the c-representability of permutation groups generated by a permutation.

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Appendix

List of Publications

1. Kunheenkutty M., Kavitha T., Ramachandran P. T., *Adjacency in the Lattice of Čech Closure Operators*, International Journal of Pure and Applied Mathematics, 105(1), 73−86 (2015). ISSN: 1311-8080(print version), ISSN: 1314-3395(online version). doi:10.12732/ijpam.v105i1.7

2. Kavitha T., Sini P., Ramachandran P. T., *Closure Isomorphisms of Čech Closure Spaces* Journal of Advanced Studies in Topology, Vol. 7, No. 3, 132 − 136(2016). (ISSN: 2090-8288(print version), ISSN: 2090-388X(online version). doi:10.20454/jast.2016.1061.

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