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SOME PROBLEMS ON GENERALIZED TOPOLOGIES AND FUZZY GENERALIZED TOPOLOGIES

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CERTIFICATE

I hereby certify that the thesis entitled "**Some problems on generalized topologies and fuzzy generalized toplogies**" is a bonafide work carried out by **Ms. Dhanya P. M.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**Some problems on generalized topologies and fuzzy generalized toplogies**" is based on the original work done by me under the supervision of Dr. Ramachandran P. T., Associate Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Chapter 0

Introduction

0.1 Motivation and survey of literature

In 1963 [32], Levine introduced semi-open sets in topological spaces. Since then, many papers were devoted to many weak forms of open sets, namely pre-open sets [34], α-open sets [38], β-open sets [1], feebly open sets [33] etc. These open sets can be defined using some combinations of interior operators and closure operators of a topology. Császár Á. observed the similarities of these generalized open sets and pointed out that these can be defined using more generalized class of functions.

A collection μ of subsets of a set X is said to form a generalized topology on X if $\emptyset \in \mu$ and arbitrary union of elements in μ is again in μ and the pair (X, μ) is called the generalized toplogical space [9].

Császár defined a map $\gamma : P(X) \to P(X)$, from the powerset of the underlying set X into itself possessing the property of monotonicity, i.e., $A \subseteq B \Rightarrow \gamma(A) \subseteq \gamma(B)$ for every $A, B \in P(X)$. A subset $A \subseteq X$ is γ-open [8] if and only if $A \subseteq \gamma(A)$. Then if τ is a topology on X and we denote the interior of $A \subseteq X$ with respect to τ by iA and the closure of A by cA, we obtain as important particular cases the collection of all open sets ($\gamma = i$), the collection of all semi-open sets ($\gamma = ci$), the collection of all pre-open sets ($\gamma = ic$), the collection of all β-open sets ($\gamma = cic$) and the collection of all α-open sets($\gamma = ici$) [9]. Thus these generalized forms of open sets can be generalized to γ-open sets and the collection of all γ-open sets in X constitute a generalized topology in X [8].

A generalized topology need not contain X and need not be closed under finite intersection. Note that every topology is a generalized topology and a generalized topology need not be a topology. Hence we get a bigger arena to explore.

Many articles have been published in the topic related to the properties of generalized topologies such as compactness, countability, separation axioms, product, quotient etc. For more details, see [13, 35, 40, 43]. Discussion on generalized topology and preorders can be seen in [24, 42]. Generalized topological spaces has applications in evolutionary theory and combinatorial chemistry [41].

In this dissertation we consider the collection of all generalized topologies on a set X denoted by $LGT(X)$. Comparison of different topologies on the same basic set has been an interesting problem ever. In 1963, Garett Birkhoff, in his paper, "On the combination of topologies", compared different topologies by ordering the family of all topologies on a given set and considering the resulting lattice, $LT(X)$. Birkhoff used the usual order of set inclusion. Orders other than set inclusion are defined in [39] and [45]. But here in $LGT(X)$, we use only set inclusion as the order.

0.2 Organisation of the thesis

This dissertation comprises of 5 chapters. The introductory chapter, **Chapter 0** deals with the motivation and review of literature of the study of generalized topologies and in **Chapter 1** preliminary definitions and results for the development of the thesis are given.

Basakaran, Murugalingam and Sivaraj [4] proved that the family of all generalized topologies on a nonempty set is a lattice, neither distributive nor complemented. They use the notation $\mathcal{G}(X)$ for the lattice of generalized topologies on a set X. They proved a characterization theorem for the existence of complement of a generalized topology on a set X. Also the direct sum of two generalized topologies is discussed in [4] and characterized the same. As an extension of this paper we discuss in **Chapter 2** some properties of $LGT(X)$ and define simple expansion of a generalized topology [2]. Simple expansion of topologies has been studied previously by many mathematicians and the similar concept can be generalized to generalized topologies.

Let X be a non empty set, $\mu \in LGT(X)$ and A be a subset of X which does not belong to μ. Then the simple expansion of μ by A, denoted by $\mu(A)$, is defined as

$$\mu(A) = \mu \cup \{G \cup A : G \in \mu\}.$$

We can prove easily that a simple expansion of μ forms a generalized topology. Also it is obvious from the definition that $\mu(A)$ is the smallest generalized topology containing μ and A.

We prove several equivalent conditions for a simple expansion of generalized topology by a subset A of X to be an upper neighbor of the generalized topology. Using these we compare $LT(X)$ and $LGT(X)$ and we answer the following problem: given a generalized topology on X, when does it possess a topological upper neighbor and vice versa. We provide examples for generalized topologies which do not possess upper neighbors. Given a generalized topological space (X, μ) with a property P, when will a simple expansion of (X, μ) possess the same property P, we discuss this in the same chapter. The main result we prove in **Chapter 2** is the determination of automorphism group of $LGT(X)$.

Determination of automorphism group is interesting and important in the lattice of topologies, $LT(X)$. It is proved that if X contains one or two elements or X is infinite, the group of automorphisms of $LT(X)$ is isomorphic to the symmetric group on X. If X is finite and contains more than two elements, the group of automorphisms of $LT(X)$ is isomorphic to the direct product of the symmetric group on X with the two element group [17, 22]. This result is important because this has the following consequence. If X is infinite, then the only lattice automorphisms of $LT(X)$ are elements of permutation group

$S(X)$, i.e., those which permute elements of X. Therefore if P is any topological property then a topology possessing the property P can be identified from the lattice structure of $LT(X)$.

Fuzzy set theory was introduced by Zadeh in 1965 [47]. According to him a fuzzy set is defined as a class of objects with a continuum of grades of membership. He assigns a grade of membership ranging from 0 to 1. Generalizing the lattice $[0, 1]$, Gougen [20] introduced the concept of L-fuzzy sets, where L can be a semigroup, a poset, a lattice or a boolean ring. Using fuzzy sets, Chang [7] introduced a new branch of mathematics called fuzzy topology as a generalization of ordinary topology and Heba I. M. [23] introduced fuzzy generalized topology as a generalization of generalized topology.

The L-fuzzy generalized topological space is defined in the following way. Let X be a nonempty ordinary set, L an F-lattice and $\mu \subseteq L^X$. Then μ is called an L-fuzzy generalized topology or fuzzy generalized topology on X, and (L^X, μ) is called an L-fuzzy generalized topological space or fuzzy generalized topological space, if μ satisfies the following conditions:

1. $\underline{0} \in \mu$,
2. $\forall \mathcal{A} \subseteq \mu, \bigvee\limits_{A \in \mathcal{A}} A \in \mu$.

For $L = \{0, \frac{1}{2}, 1\}$, Baby Chacko [6] determined the automorphism group of the lattice $LFT(X, L)$ of all L-fuzzy topological spaces on X. Madhavan Namboothiri [37] determined the group of automorphisms of the lattice $LFT(X, L)$

in the cases when L is a finite chain and when L is the diamond-type lattice. We consider the same problem in L-fuzzy generalized topological space.

We determine the automorphism group of lattice of fuzzy generalized topologies, $LFGT(X, L)$ on a set X and when L is a finite chain and L is the diamond-type lattice in **Chapter 3**.

Many investigations have been done in the study of topological property homogeneity in topological spce. John Ginsburg in his paper [19] proved a simple representation theorem for finite topological spaces which are homogeneous. In **Chapter 4** we discuss homogeneity in generalized topological spaces and in L-fuzzy generalized topological spaces. In the first section of this chapter, we characterize completely homogeneous generalized topological spaces. In the following sections we discuss homogeneous generalized topological spaces in a cyclic ordered set and completely homogeneous L-fuzzy generalized topological spaces. We try to find out new homogeneous generalized topological spaces by considering the join of homogeneous generalized topologies and discusses the properties.

We conclude the thesis with **Chapter 5**, some unsolved problems are discussed in this chapter and a bibliography is provided.

Chapter 1

Preliminaries

1.1 Introduction

This chapter deals with basic definitions and preliminary results in lattice theory, generalized topology, fuzzy set theory and fuzzy generalized topology, which would make the reading of the thesis simpler.

1.2 Lattice theory

First let us go through the definition of a partially ordered set.

Definition 1.2.1. *[12] Let P be a set. A partial order on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,*

1. $x \leq x$,
2. $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

3. $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$.

These conditions are referred to, respectively, as reflexivity, antisymmetry, and transitivity. A set P *equipped with a partial order relation* $\leq$ *is said to be a partially ordered set or poset and is denoted by* $(P, \leq)$.

Chain. Let $(P, \leq)$ be a partially ordered set. Then $(P, \leq)$ is a chain, if for all $x, y \in P$, either $x \leq y$ or $y \leq x$(that is any two elements of P are comparable).

Definition 1.2.2. *[12] Let* $(P, \leq)$ *be a partially ordered set and let* $S \subseteq P$. *An element* $x \in P$ *is an upper bound of* S *if* $s \leq x$ *for all* $s \in S$. *A lower bound is defined dually.* x *is the least upper bound or supremum of* S *if*

1. x *is an upper bound of* S, *and*
2. $x \leq y$ *for all upper bounds* y *of* S.

Dually greatest lower bound or infimum of a set can be defined.

Notation. We write $x \vee y$ in place of supremum of $\{x, y\}$ when it exists and $x \wedge y$ in place of infimum of $\{x, y\}$ when it exists. Similarly we write $\bigvee S$ and $\bigwedge S$ for supremum of the set S and infimum of S respectively. Lattice and complete lattice are defined as follows.

Definition 1.2.3. *[12] Let* $(P, \leq)$ *be a non empty partially ordered set.*

1. *If* $x \vee y$ *and* $x \wedge y$ *exist for all* $x, y \in P$, *then* $(P, \leq)$ *is called a lattice.*
2. *If* $\bigvee S$ *and* $\bigwedge S$ *exist for all* $S \subseteq P$, *then* $(P, \leq)$ *is called a complete lattice.*

The set of natural numbers with usual order is a lattice but not a complete lattice because the set $\{2, 4, 6, \ldots\}$ has no supremum.
The following theorem [21] states that a lattice can be defined as an algebra and a lattice as an algebra and a lattice as a poset are equivalent concepts.

Theorem 1.2.1. *1. Let the poset $\mathfrak{L} = (L, \leq)$ be a lattice. Set*

$$a \wedge b = inf\{a, b\}, \ a \vee b = sup\{a, b\}.$$

Then the algebra $\mathfrak{L}^a = (L; \wedge, \vee)$ is a lattice.

2. Let the algebra $\mathfrak{L}^a = (L; \wedge, \vee)$ be a lattice. Set

$$a \leq b \ if \ and \ only \ if \ a \wedge b = a.$$

Then $\mathfrak{L}^p = (L; \leq)$ is a poset, and the poset $\mathfrak{L}^p$ is a lattice

3. Let the poset $\mathfrak{L} = (L; \leq)$ be a lattice. Then $(\mathfrak{L}^a)^p = \mathfrak{L}$.

4. Let the algebra $\mathfrak{L} = (L; \wedge, \vee)$ be a lattice. Then $(\mathfrak{L}^p)^a = \mathfrak{L}$.

From now on we use the notation L instead of precise notation $(L, \leq)$ or $(L, \wedge, \vee)$ for lattices and posets unless for some reason we want to be more exact. We also use the following definitions in the forthcoming chapters. For more details see [5, 21].

Let $(L, \leq)$ be a lattice with smallest element 0 and largest element 1. For $a, b \in L$, we say a is an upper neighbor of b or a covers b if $b \leq a$ and $a \neq b$ and for every $c \in L$ with $b \leq c \leq a$, we have either $c = b$ or $c = a$. An atom of the lattice L is an element which covers the smallest element 0. A lattice is atomic if every element other than the least element can be written as the join of atoms. An anti-atom is an element which is covered by the largest element 1 in the lattice. An anti-atom is also called dual atom. A lattice is anti-atomic if every element other than the largest element can be written as the meet of anti-atoms.

Definition 1.2.4. *A lattice L is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$. This is equivalent to $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.*

Definition 1.2.5. *A lattice L is called modular if for any $a, b, c \in L$, $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$.*

The following is a characterization theorem for modular lattice.

Theorem 1.2.2. *[21] A lattice L is modular if and only if it has no sublattice isomorphic to a pentagon(see Figure 1.1).*

Definition 1.2.6. *The lattice L is called semi-modular if for any $a, b \in L$ with $a \neq b$, and if a and b cover $a \wedge b$, then $a \vee b$ covers a and b.*

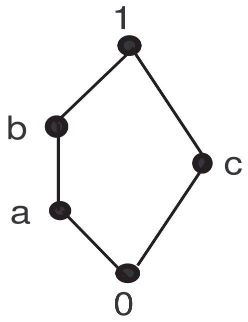

Figure 1.1: Pentagon

Definition 1.2.7. *Let L be a complete lattice. L is called infinitely distributive, if L satisfies both the following two conditions (IFD1) and (IFD2), called the 1st infinitely distributive law and the 2nd infinitely distributive law respectively:*
(IFD1) $\forall a \in L, \forall B \subseteq L, a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b)$,
(IFD2) $\forall a \in L, \forall B \subseteq L, a \vee \bigwedge B = \bigwedge_{b \in B} (a \vee b)$.

Definition 1.2.8. *[36] Let L be a complete lattice. L is called completely distributive, if L satisfies the following two conditions called completely distributive laws:*
$\forall \{\{a_{i,j} : j \in J_i\} : i \in I\} \subseteq P(L) \setminus \{\emptyset\}$, $I \neq \emptyset$, *where $P(L)$ denotes the powerset of L,*

1. $\bigwedge_{i \in I} (\bigvee_{j \in J_i} a_{i,j}) = \bigvee_{\phi \in \prod_{i \in I} J_i} (\bigwedge_{i \in I} a_{i,\phi(i)})$,
2. $\bigvee_{i \in I} (\bigwedge_{j \in J_i} a_{i,j}) = \bigwedge_{\phi \in \prod_{i \in I} J_i} (\bigvee_{i \in I} a_{i,\phi(i)})$.

Definition 1.2.9. *[36] Let L be a lattice. A mapping $' : L \to L$ is called order reversing, if $\forall a, b \in L, a \leq b \Rightarrow a^{'} \geq b^{'}$; called involution on L, if $'' = id_L : L \to L$; called complementary operation, if $\forall a \in L$, $a^{'}$ is a complement of a.*

Definition 1.2.10. *[21] The lattices $(L_0, \leq)$ and $(L_1, \leq^{'})$ are isomorphic and the map $\phi : L_0 \to L_1$ is an isomorphism if and only if ϕ is one-to-one and onto and*

$$a \leq b \ in \ L_0 \ if \ and \ only \ if \ \phi(a) \leq^{'} \phi(b) \ in \ L_1$$

By Theorem 1.2.1, the previous definition of isomorphism is equivalent to the following result.

Proposition 1.2.1. *[21] The lattices $(L_0, \wedge, \vee)$ and $(L_1, \wedge, \vee)$ are isomorphic and the map $\phi : L_0 \to L_1$ is an isomorphism if and only if ϕ is one-to-one and onto and*

$$\phi(a \vee b) = \phi(a) \vee \phi(b)$$
$$\phi(a \wedge b) = \phi(a) \wedge \phi(b)$$

An isomorphism of a lattice with itself is called an automorphism. A lattice is called self dual if it is isomorphic to its dual lattice.

Definition 1.2.11. *[21] A poset P is called graded if we can define an integer-valued function h on P such that for $x, y \in P$ with $x \leq y$ we have that $h(x)+1 = h(y)$ if and only if y covers x.*

Example 1.2.1. *The following diagram represents a graded poset(1.2) and it can be easily seen that a pentagon(1.1) is not a graded poset.*

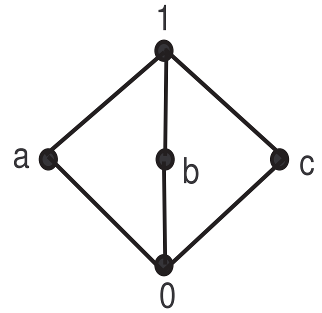

Figure 1.2: Poset

1.3 Generalized topology

Generalized topology has been extensively studied by Császár. For more details of his work, see [9–11]. Let us go through some basic definitions in generalized topology.

Definition 1.3.1. *[9] A collection μ of subsets of a set X is said to form a generalized topology on X if $\emptyset \in \mu$ and arbitrary union of elements in μ is again in μ and the pair (X, μ) is called a generalized toplogical space.*

Let (X, μ) be a generalized topological space. The elements of μ are called μ-open sets or simply open sets. A subset $H \subseteq X$ is said to be μ-closed or a closed set if the complement of H is in μ. A subset A of X with the generalized topology $\mu \cap A = \{G \cap A : G \in \mu\}$ is called a subspace of (X, μ). If A is open in (X, μ), then $(A, \mu \cap A)$ is called an open subspace and if A is closed in (X, μ), then $(A, \mu \cap A)$ is called a closed subspace. The union of all elements of μ will be denoted by M_μ. A non-empty μ-open subset A of a generalized topological space (X, μ) is called a minimal μ-open set if the only non-empty μ-open set which is contained in A is A. The collection of all minimal μ-open sets in (X, μ) will be denoted by $min(X, \mu)$ [18] and $min(X, \mu)$ is a base for (X, μ).

Let F be a subset of X. Then the closure of F with respect to μ, denoted by $\overline{F}$, is the smallest closed set in (X, μ) containing F. If $\overline{F}_\mu = M_\mu$, then A is said to be dense in (X, μ).

Let $\beta \subseteq \mu$, then β is said to be a base for the generalized topology (X, μ) if each and every element of μ can be written as the union of some elements of β [26]. A generalized topological space generated by the subfamily $\mathcal{A} \subseteq P(X)$ is the set $\cap\{\tau : \tau$ is a generalized topology on X containing $\mathcal{A}\}$. In fact it is the smallest generalized topology containing $\mathcal{A}$.

Definition 1.3.2. *Let $f : (X, \mu) \to (Y, \lambda)$ be a function on generalized topological space.*

1. *[9]f is said to be (μ, λ)-continuous if $B \in \lambda$ implies that $f^{-1}(B) \in \mu$.*
2. *[11]f is said to be (μ, λ)-open if $A \in \mu$ implies that $f(A) \in \lambda$.*
3. *[10]f is called a (μ, λ)-homeomorphism if f is bijective, (μ, λ)-continuous, and f^{-1} is (λ, μ)-continuous, equivalently if f is bijective, (μ, λ)-continuous, and (μ, λ)-open. If $f : (X, \mu) \to (Y, \lambda)$ is a (μ, λ)-homeomorphism, then we say that (X, μ) is homeomorphic to (Y, λ).*

Definition 1.3.3. *[18] A generalized topological space (X, μ) is said to be homogeneous if for any two points x, $y \in M_\mu$ there exists a (μ, μ)-homeomorphism $f : (X, \mu) \to (X, \mu)$ such that $f(x) = y$ and (X, μ) is called completely homogeneous if every bijection on X is a homeomorphism on (X, μ).*

Let us go through the definitions of separation axioms in generalized topologies which are taken from [46].

Definition 1.3.4. *A generalized topological space (X, μ) is said to be μ-T_o if for every $x, y \in X$ there exists a set $U \in \mu$ such that either $U \cap \{x, y\} = \{x\}$ or $U \cap \{x, y\} = \{y\}$.*

Definition 1.3.5. *A generalized topological space (X, μ) is said to be μ-T_1 if there exist sets $U, V \in \mu$ such that $U \cap \{x, y\} = \{x\}$ and $V \cap \{x, y\} = \{y\}$.*

Definition 1.3.6. *A generalized topological space (X, μ) is said to be μ-T_2 if for every $x, y \in X$ there exist disjoint open sets $U, V \in \mu$ such that $x \in U$ and $y \in V$.*

Definition 1.3.7. *A generalized topological space (X, μ) is said to be μ- regular if for every $x \in X$ and a closed set F, not containing x, there exist disjoint open sets U, V such that $x \in U$ and $F \subseteq V$.*

Definition 1.3.8. *A generalized topological space (X, μ) is said to be μ- normal if for every pair of disjoint closed sets F, H, there exist disjoint open sets U, V such that $F \subseteq U$ and $H \subseteq V$.*

Definition 1.3.9. *[3] A generalized topological space (X, μ) is said to be a μ-second countable if there is a countable base for the generalized topology μ and (X, μ) is said to be μ-separable if X contains a countable dense subset.*

If there is no confusion we call a (μ, λ)-continuous function(μ-open set, (μ, λ)-open, (μ, μ)-homeomorphism) simply continuous function(open set, open map and homeomorphism) on (X, μ).

1.4 Fuzzy set theory and fuzzy generalized topology

Fuzzy set was introduced by L. A. Zadeh in 1965 [47]. He described fuzzy sets using the unit interval $[0, 1]$ as the lattice. Goguen generalized this concept with general lattice L. Here first we define general L-fuzzy set and consider the fuzzy set based on $[0, 1]$ as a particular case.

Definition 1.4.1. *[36] Let X be a nonempty ordinary set, L be a complete lattice. An L-fuzzy subset on X is a mapping $A : X \to L$, i.e. the family of all the L-fuzzy subsets on X is just L^X consisting of all the mappings from X to L. L^X here is called L-fuzzy space.*

[36]An L-fuzzy set $A \in L^X$ is called a crisp subset on X, if there exists an ordinary subset $U \subseteq X$ such that $A = \chi_U : X \to \{0, 1\} \subseteq L$, i.e. if A is a characteristic function of some ordinary subset of X.
An L-fuzzy point on X is an L-fuzzy subset $x_a \in L^X$ where $a \in L, a \neq 0$ is defined as, for every $y \in X$,

$$x_a(y) = \begin{cases} a, & y = x, \\ 0, & y \neq x. \end{cases}$$

A constant function from X to L is also an $L-$fuzzy set and is denoted by $\underline{a}$ if every element of X takes the value $a \in L$.

Definition 1.4.2. *[36] Let L^X be an L-fuzzy space. Define the partial order $\leq$ in L^X by:*

$$\forall A, B \in L^X, A \leq B \Leftrightarrow \forall x \in X, A(x) \leq B(x).$$

Proposition 1.4.1. *[36] Let L^X be an L-fuzzy space. Then*

1. *L^X is a complete lattice and for every $\mathcal{A} \subseteq L^X$, the join $\bigvee \mathcal{A}$ and the meet $\bigwedge \mathcal{A}$ satisfy,*

$$\forall x \in X, (\bigvee \mathcal{A})(x) = \bigvee_{A \in \mathcal{A}} A(x), \text{ and } (\bigwedge \mathcal{A})(x) = \bigwedge_{A \in \mathcal{A}} A(x).$$

2. *L is distributive $\Leftrightarrow$ L^X is distributive.*
3. *L satisfies (IFD1)$\Leftrightarrow$ L^X satisfies (IFD1).*
4. *L satisfies (IFD2)$\Leftrightarrow$ L^X satisfies (IFD2).*
5. *L is completely distributive $\Leftrightarrow$ L^X is completely distributive.*

Definition 1.4.3. *[36] Let L^X, L^Y be L-fuzzy spaces. Let $f : X \to Y$ be an ordinary mapping. Based on $f : X \to Y$, define L-fuzzy mapping $f : L^X \to L^Y$ and its reverse mapping $f^{-1} : L^Y \to L^X$ by*

$$f(A)(y) = \vee\{A(x) : x \in X, f(x) = y\} \ \forall A \in L^X, \ \forall y \in Y \text{ and}$$
$$f^{-1}(B)(x) = B(f(x)) \ \forall B \in L^Y, \ \forall x \in X.$$

In this case, we say the ordinary mapping $f : X \to Y$ produces the correspondent L-fuzzy mapping $f : L^X \to L^Y$, or say $f : L^X \to L^Y$ is induced from $f : X \to Y$.

The definition of a fuzzy set and fuzzy topological space is defined in [36]. Before stating the definition of L-fuzzy generalized topological space let us go through some other basic definitions.

Definition 1.4.4. *[36] A completely distributive lattice L is called an F-lattice, if L has an order-reversing involution $' : L \to L$.*

Let X be a nonempty ordinary set, L an F-lattice and $'$ the order-reversing involution on L. $\forall A \in L^X, \forall \mathcal{B} \subseteq L^X$, use the order-reversing involution $'$ to define an operation $'$ on L^X by:

$$A^{'}(x) = (A(x))^{'}, \forall x \in X;$$

also define:

$$\mathcal{B}^{'} = \{B^{'} : B \in \mathcal{B}\}.$$

Call $' : L^X \to L^X$ the pseudo-complementary operation on L^X, $A^{'}$ the pseudo-complementary set of A in L^X.

Proposition 1.4.2. *[36] Let X be a nonempty ordinary set, L an F-lattice, then the pseudo-complementary operation $' : L^X \to L^X$ is an order reversing involution.*

Now a fuzzy generalized topology is defined as follows.

Definition 1.4.5. *[23] Let X be a nonempty ordinary set, L an F-lattice, $\mu \subseteq L^X$. Then μ is called an L-fuzzy generalized topology or fuzzy generalized topology on X, and (L^X, μ) is called an L-fuzzy generalized topological space or fuzzy generalized topological space, if μ satisfies the following conditions:*

1. $\underline{0} \in \mu$,

2. $\forall \mathcal{A} \subseteq \mu, \bigvee \mathcal{A} \in \mu$.

If the largest L-fuzzy set $\underline{1}$ also belongs to the L-fuzzy generalized topology μ, then it is called a strong L-fuzzy generalized topology on X. Every element in μ is called an open set in L^X, every pseudo-complementary set of an open set is called a closed set in L^X.

Example 1.4.1. *Let $X = \{a, b, c, d\}$ and $L = \{0, \frac{1}{2}, 1\}$ with order $0 < \frac{1}{2} < 1$. Then L is an F-lattice with order reversing involution $0' = 1$ and $\frac{1}{2}' = \frac{1}{2}$. Also $\{\underline{0}, \underline{1}, a_1, b_{\frac{1}{2}}, f\}$ is an L-fuzzy generalized topology where $f(a) = 1, f(b) = \frac{1}{2}, f(c) = f(d) = 0$.*

Definition 1.4.6. *[23] Let μ_1, μ_2 are L-fuzzy generalized topologies on X and Y respectively. Let $f : X \to Y$. Then f is called a continuous function if for every $A \in \mu_2$, $f^{-1}(A) \in \mu_1$, where f^{-1} is the L-fuzzy reverse mapping induced from f. f is called a homeomorphism if it is bijective and the induced L-fuzzy map, f and L-fuzzy reverse map, f^{-1} are continuous.*

1.5 Group theory

In the following chapters we use the terms symmetric group and cycles in group theory. Definitions of symmetric group and cycle are given in this section.

Definition 1.5.1. *[16] A permutation of a set A is a function $\phi : A \to A$, that is both one to one and onto.*

The function composition is a binary operation on the collection of all permutations of a set A. This operation is called permutation multiplication.

Theorem 1.5.1. *[16] Let A be a nonempty set and let $S(A)$ be the collection of all permutations of A. Then $S(A)$ is a group under permutation multiplication called the symmetric group on A.*

Definition 1.5.2. *[16] Let A be the finite set $\{1, 2, \ldots, n\}$. The group of all permutations of A is the symmetric group on n-letters and is denoted by S_n.*

Let A be a nonempty set and σ be a permutation of A. We define a relation on A. For $a, b \in A$, let $a \sim b$ if and only if $b = \sigma^n(a)$ for some $n \in \mathbb{Z}$. This relation is in fact an equivalence relation.

Definition 1.5.3. *[16] Let σ be a permutation of a set A. The equivalence classes in A determined by the above equivalence relation are the orbits of σ.*

Definition 1.5.4. *[16] A permutation $\sigma \in S_n$ is a cycle if it has atmost one orbit containing more than one element.*

Notation of a cycle: If $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 6 & 4 & 5 & 1 & 7 & 8 \end{pmatrix}$ is a permutation on 8 element set $\{1, 2, \ldots, 8\}$, then we use single cyclic notation for this, $(1\ 3\ 6)$.

Chapter 2

The lattice of generalized topologies

Baskaran, Murugalingam and Sivaraj proved that the collection, $LGT(X)$, of all generalized topologies on a set X forms a complete lattice and studied the properties of $LGT(X)$ in their paper [4]. In the first part of this chapter we discuss some simple properties of the lattice $LGT(X)$ and determine the automorphism group of $LGT(X)$. We try to study the lattice structure of $LGT(X)$ by introducing simple expansion in the following sections of this chapter.

2.1 Some properties of $LGT(X)$

Let X be any set. Consider the collection $LGT(X)$, of all generalized topologies on X. This is a partially ordered set under the order of set inclusion. Moreover it is a complete lattice [4]. Let $\mu_1, \mu_2 \in LGT(X)$. Then $\mu_1 \vee \mu_2$ is the smallest generalized topology containing $\mu_1 \cup \mu_2$ and $\mu_1 \wedge \mu_2 = \mu_1 \cap \mu_2$. Note that the smallest element and largest element of the lattice $LGT(X)$ is respectively

$\{\emptyset\}$ and $P(X)$, where $P(X)$ denotes the power set of X.

To start our study on the lattice of generalized topologies, first we verify the basic properties of the lattice, i.e., atomic, anti-atomic, distributive, modular and semi modular properties of the lattice. This will give a better understanding of the lattice structure of $LGT(X)$. We know that the lattice $LT(X)$ of topologies on a set X is both atomic and anti-atomic [28]. Here we prove that $LGT(X)$ is atomic but not anti atomic. Throughout this chapter X will denote a set(X can be empty also) unless otherwise specified and $P(X)$ denotes the power set of X.

Theorem 2.1.1. *The lattice $LGT(X)$ is an atomic lattice. If $X \neq \emptyset$ then the atoms are generalized topologies of the form $\{\emptyset, A\}$, where $\emptyset \subsetneq A \subseteq X$. If X is finite and if $|X| = n$, then $LGT(X)$ contain $2^n - 1$ atoms. If X is infinite and $|X| = \alpha$, then $LGT(X)$ contain 2^α atoms.*

Proof. If $X = \emptyset$, then obviously it is an atomic lattice. It can be seen that the atoms in $LGT(X)$ are precisely the generalized topologies of the form $\{\emptyset, A\}$, where A is a nonempty subset of X. Also given any generalized topology μ on X, we have $\mu = \bigvee_{\substack{A\in\mu \\ A\neq\phi}} \{\emptyset, A\}$. Thus $LGT(X)$ is an atomic lattice. Since we are considering every non empty subset A of X here, the total number of atoms in $LGT(X)$ is $2^n - 1$ where $n = |X|$ and number of atoms in $LGT(X)$ is 2^α if X is an infinite set of cardinality α. $\square$

Note 2.1.1. $LGT(X)$ possess anti-atoms. The anti-atoms are precisely the generalized topologies of the form $P(X) \setminus \{\{x\}\}$, $x \in X$. Because given any generalized topology μ on X with $\mu \neq P(X)$, there exists an element $x \in X$ such that $\{x\}$ does not belong to μ. This implies $\mu \subseteq P(X) \setminus \{\{x\}\}$ and also $P(X) \setminus \{\{x\}\}$ is a generalized topology since $\emptyset \in P(X) \setminus \{\{x\}\}$ and any arbitrary union of elements in $P(X) \setminus \{\{x\}\}$ is again in the same collection. Also it is obvious that there exists no proper subset of $P(X)$ between $P(X)$ and $P(X) \setminus \{\{x\}\}$ and hence no generalized topology exists between them for any $x \in X$. If $|X| = \alpha$, $LGT(X)$ contain α anti-atoms. But $LGT(X)$ is not anti-atomic since $\{\emptyset, \{x\}\}$ cannot be written as the meet of any collection of anti-atoms because every anti-atom contains X so is their intersection.

Now the question we face is whether there are generalized topologies which can be written as the meet of some dual atoms? Note that here meet is the set theoretic intersection since the order we are considering is the usual order of set inclusion. We answer this question in the following result.

Proposition 2.1.1. *A generalized topology μ on X can be written as meet of some dual atoms if and only if it contains all subsets A of X which has cardinality atleast* 2.

Proof. Let μ be a generalized topology on X such that $\mu = \bigwedge_{x \in K} (P(X) \setminus \{\{x\}\})$ for some $K \subseteq X$. Since $P(X) \setminus \{\{x\}\}$, $\forall x \in X$, contains every subset A of X such that $|A| \geq 2$, so is its intersection.

Conversely, let μ be a generalized topology which contain all subsets A of X such that $|A| \geq 2$, and let $L = \{y \in X : \{y\} \notin \mu\}$. Then μ can be written as $\mu = \bigwedge_{y \in L} (P(X) \setminus \{\{y\}\})$, where $P(X) \setminus \{\{y\}\}$ are dual atoms of $LGT(X)$ and hence the proof is complete. $\square$

Thus a generalized topology μ on X, which is the meet of some dual atoms has the form $P(X) \setminus \{\{x\}\}_{x \in K}$ for some $K \subseteq X$. Thus if $|X| = \alpha$, then there are exactly 2^α generalized topologies which can be written as the meet of some dual atoms.

It is known that every distributive lattice is modular [21]. But it is proved in [4] that $LGT(X)$ is not distributive for $|X| \geq 2$. In the next theorem we enquires when $LGT(X)$ is modular.

Recall from Chapter 1 that a lattice is modular if and only if it has no sublattice isomorphic to a pentagon.

Theorem 2.1.2. *$LGT(X)$ is modular if $|X| \leq 1$ and not modular if $|X| \geq 2$.*

Proof. If $|X| \leq 1$, we can see from the lattice diagrams below(2.1) that it has no sublattice isomorphic to a pentagon. Hence $LGT(X)$ is modular if $|X| \leq 1$.

But when $|X| = 2$, from the below lattice diagram, $\{\{\emptyset\}, \{\emptyset, \{b\}\}, \{\emptyset, \{b\}, X\}, \{\emptyset, \{a\}, X\}\}$ constitute a pentagon and hence it is not modular. Now let $|X| = n$ with $n \geq 3$. Then there exist elements $a, b, c \in X$ with $a \neq b \neq c$. Consider the generalized topologies G_i, $i = 1, 2, \ldots 5$ on X, where $G_1 = \{\emptyset\}$, $G_2 = \{\emptyset, \{a, b\}\}$, $G_3 = \{\emptyset, \{a, c\}\}$, $G_4 = \{\emptyset, \{a, b\}, \{a, b, c\}\}$ and $G_5 = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.

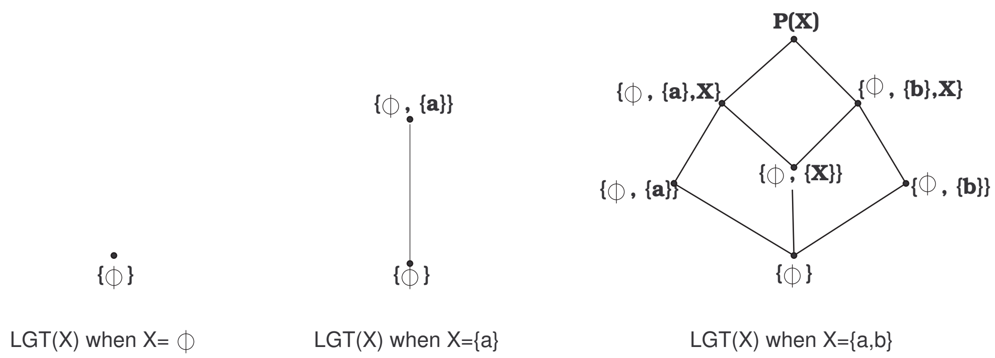

Figure 2.1:

It can be observed that in the lattice diagram of $LGT(X)$, the above five generalized topologies constitute a pentagon as below(2.2) and hence is not modular.

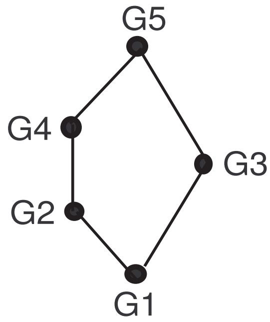

Figure 2.2:

□

Definition 2.1.1. *[21] Let L be a lattice with least element 0. We define the height function as follows: for $a \in L$, let $h(a)$ denotes the length of a longest maximal chain in $[0, a]$, where $[0, a] = \{l \in L : 0 \leq l \leq a\}$ if it exists and is finite; otherwise, put $h(a) = \infty$.*

Recall the definition of semi-modular lattice. A lattice L is called semi-modular if for any $a, b \in L$ with $a \neq b$, and if a and b cover $a \wedge b$, then $a \vee b$ covers a and b. Next theorem gives a necessary and sufficient condition for semi-modularity using height function of the lattice.

Theorem 2.1.3. *[21] Let L be a finite lattice. L is semi-modular if and only* *if* $h(a) + h(b) \geq h(a \wedge b) + h(a \vee b)$ *for all* a *and* b *in* L*, where* h *is the height function.*

We use the above theorem to prove that $LGT(X)$ is not semi-modular when X is finite with $|X| \geq 3$.

Theorem 2.1.4. *Let* X *be a finite set. Then* $LGT(X)$ *is semi-modular if and only if* $|X| \leq 1$*.*

Proof. When $|X| \leq 1$, we proved that it is modular and hence is semi-modular. Let $|X| = 2$. We investigate the diagram 2.1, for the height function. The values of the height function are 1,2,3,4 on the 1-st, 2-nd, 3-rd and 4-th levels of the diagram. The condition to be verified is in Theorem 2.1.3. This is invalid for $\mu_1 = \{\emptyset, \{a\}\}$ and $\mu_2 = \{\emptyset, \{b\}\}$, whose intersection and union are $\{\emptyset\}$ and $P(X)$. Then $h(\mu_1)+h(\mu_2) = 2+2 = 4$, while $h(\mu_1 \wedge \mu_2)+h(\mu_1 \vee \mu_2) = 1+4 = 5$. That is, for $|X| = 2$ we do not have semi-modularity. Then for any $|X| \geq 3$ we do not have semi-modularity either, since for $X_0 \subseteq X$, and $|X_0| = 2$ we have that $LGT(X_0)$ is a sublattice of $LGT(X)$. $\square$

The following example illustrates that $LGT(X)$ is in general not semi-modular, when X is infinite.

Example 2.1.1. *Consider* $\mathbb{R}$*, the set of real numbers and let* $a = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ *and* $b = \{\emptyset, \mathbb{R}, \{\pi\}\}$ *are generalized topologies on* $\mathbb{R}$*, where* $\mathbb{Q}$ *is the set of rational numbers. Then* a *and* b *cover* $a \wedge b$*, but* $a \vee b$ *does not cover* a *and* b*. Here semi-modularity fails.*

Theorem 2.1.5. $LGT(X)$ *is self dual if and only if* $|X| \leq 1$*.*

Proof. For $X = \emptyset$, we have that $LGT(X)$ contains one element, namely $\{\emptyset\}$. If $|X| = 1$, $LGT(X)$ contain only two elements, namely $\{\emptyset\}$ and $\{\emptyset, X\} = P(X)$. Hence for $|X| \leq 1$, $LGT(X)$ is obviously self dual.

Now assume $|X| = \alpha \geq 2$. If the lattice $LGT(X)$ is a self dual lattice, then there exists an isomorphism which map atoms onto anti-atoms and vice versa. But the number of atoms in $LGT(X)$ is atleast $2^\alpha - 1$ ad the number of anti-atoms are α. Hence $LGT(X)$ is not self dual. □

In general the lattice, $LGT(X)$, is not distributive, not modular and not even semi modular. This reveals a complicated structure of $LGT(X)$. Also the collection of all topological spaces on a set X is not a sublattice of $LGT(X)$ if $|X| \geq 3$, for if a, b and c are three distinct elements in X and let $\tau_1 = \{\emptyset, X, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{a, c\}\}$, then $\tau_1 \vee \tau_2 = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ is not a topology on X. For $|X| \leq 2$ it is a sublattice, as can be seen easily from the lattice diagrams from the proof of Theorem 2.1.2.

2.2 Automorphisms of the lattice of generalized topologies

In this section we prove, the automorphism group of lattice of generalized topologies on any set X is isomorphic to the symmetric group on X.

The following Lemma and Theorem are routine consequences of the isomorphism. We just formulate them. Throughout this section let $S(X)$ denotes the group of all bijections on X.

Lemma 2.2.1. *For $p \in S(X)$ and $\mu \in LGT(X)$ let $p(\mu) = \{p(G) : G \in \mu\}$ where $p(G) = \{p(x) : x \in G\}$. Then $p(\mu)$ is a generalized topology on X.*

Proof. $p(\emptyset) = \emptyset$ implies $\emptyset \in p(\mu)$. Consider an arbitrary collection of sets $\{G_i\}_{i \in I}$ in $p(\mu)$. Then for every $i \in I$, $G_i = p(U_i)$ for some $U_i \in \mu$. Also $\underset{i \in I}{\cup} G_i = \underset{i \in I}{\cup} p(U_i) = p(\underset{i \in I}{\cup} U_i) \in p(\mu)$ since $\underset{i \in I}{\cup} U_i \in \mu$. Thus $p(\mu)$ is a generalized topology on X. □

Next we show that each bijection in X naturally induces an automorphism in $LGT(X)$.

Theorem 2.2.1. *Let $p \in S(X)$, define a map A_p on $LGT(X)$ by $A_p(\mu) = p(\mu)$ for $\mu \in LGT(X)$. Then A_p is an automorphism of $LGT(X)$.*

Proof. Let $\mu, \tau \in LGT(X)$. $A_p(\mu) = A_p(\tau)$ implies $p(\mu) = p(\tau)$. Now

$$
\begin{aligned}
G \in \mu &\Leftrightarrow p(G) \in p(\mu) \\
&\Leftrightarrow p(G) \in p(\tau) \\
&\Leftrightarrow G \in \tau.
\end{aligned}
$$

This proves that $\mu = \tau$. Hence A_p is one-one. Let $\mu \in LGT(X)$ and take $\tau = \{p^{-1}(G) : G \in \mu\}$ where $p^{-1}(G) = \{x \in X : p(x) \in G\}$ and it is easy to see that τ is a generalized topology. Then $\tau = p^{-1}(\mu)$ and $A_p(\tau) = p(p^{-1}(\mu)) = \mu$ proving that A_p is onto.

Let $\mu \subseteq \tau$. That is

$$
\begin{aligned}
(G \in \mu \Rightarrow G \in \tau) &\Leftrightarrow (p(G) \in p(\mu) \Rightarrow p(G) \in p(T)) \\
&\Leftrightarrow p(\mu) \subseteq p(\tau) \\
&\Leftrightarrow A_p(\mu) \subseteq A_p(\tau).
\end{aligned}
$$

Hence A_p is an automorphism of $LGT(X)$. $\square$

Note 2.2.1. *An automorphism of $LGT(X)$ maps atoms of the lattice to atoms and dual atoms to dual atoms.*

Lemma 2.2.2. *An automorphism of $LGT(X)$ maps a generalized topology consisting of n elements to a generalized topology consisting of same number of elements.*

Proof. Let μ be a generalized topology consisting of n elements and A be an automorphism of $LGT(X)$. Then μ is larger than precisely $n-1$ atoms. Therefore $A(\mu)$ must be larger than precisely $n-1$ atoms. Hence $A(\mu)$ consists of n elements. $\square$

Another application of isomorphism is the following Lemma. Before that let us go through the definition of complement of an element in a lattice.

Definition 2.2.1. *In a lattice with smallest element* 0 *and largest element 1, the elements a and b are complements to each other if $a \wedge b = 0$ and $a \vee b = 1$.*

Lemma 2.2.3. *Let A be an automorphism of $LGT(X)$. Then $\mu, \tau \in LGT(X)$ are complements to each other if and only if $A(\mu)$ and $A(\tau)$ are compliments to each other.*

Proof. We have $\mu \vee \tau = P(X)$ and $\mu \wedge \tau = \{\emptyset\}$ where $P(X)$ and $\{\emptyset\}$ being the largest and smallest elements of $LGT(X)$. Also $A(\mu) \vee A(\tau) = A(\mu \vee \tau) = A(P(X)) = P(X)$ and $A(\mu) \wedge A(\tau) = A(\mu \wedge \tau) = A(\{\emptyset\}) = \{\emptyset\}$, since an automorphism always preserves the largest and smallest element of a lattice. Hence $A(\mu)$ and $A(\tau)$ are complements to each other. Similarly we can prove the converse also. $\square$

For $p \in S(X)$ recall the definition of map A_p on $LGT(X)$: $A_p(\mu) = p(\mu)$ for $\mu \in LGT(X)$ and $p(\mu) = \{p(G) : G \in \mu\}$ where $p(G) = \{p(x) : x \in G\}$. Then we have the following theorem.

Theorem 2.2.2. *The set of automorphisms of $LGT(X)$ is precisely $\{A_p : p \in S(X)\}$.*

Proof. In Theorem 2.2.1, it is proved that A_p is an automorphism of $LGT(X)$ for every $p \in S(X)$. Now let A be an automorphism of $LGT(X)$. Let $\mathcal{N}$ denotes the collection of all atoms of the form $I_x = \{\emptyset, \{x\}\}$ where $x \in X$.

Claim: A maps $\mathcal{N}$ onto itself.

Let $I_x \in \mathcal{N}$. Consider the dual atom $\mu = P(X) \setminus \{\{x\}\}$. As μ and I_x are complements to each other, $A(\mu)$ and $A(I_x)$ are complements to each other. Since $A(\mu)$ is also a dual atom there exists a $y \in X$ such that $A(\mu) = P(X) \setminus \{\{y\}\}$. Then $A(I_x)$ must contain $\{y\}$ since $A(\mu) \vee A(I_x) = P(X)$ and therefore $\{\emptyset, \{y\}\} \subseteq A(I_x)$. But $A(I_x)$ is an atom implying that $A(I_x) = \{\emptyset, \{y\}\}$. Thus A maps $\mathcal{N}$ into itself. Now take $I_z \in \mathcal{N}$. Consider $\delta = P(X) \setminus \{\{z\}\}$. Since A is onto, there exists a dual atom, say $\xi \in LGT(X)$ such that $A(\xi) = \delta$. Let $\xi = P(X) \setminus \{\{w\}\}$ and $I_w = \{\emptyset, \{w\}\}$. As $A(I_w) \vee \delta = A(I_w) \vee A(\xi) = A(I_w \vee \xi) = A(P(X)) = P(X)$, $\{z\}$ must belong to $A(I_w)$. Since $A(I_w)$ is an atom $A(I_w) = I_z$. Hence the claim.

Now define a map $p : X \to X$ such that $p(x) = y$ whenever $A(I_x) = I_y$. Since y is unique for a fixed x implying that p is well defined. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since A is injective, we have $A(I_x) \neq A(I_y)$. Hence $p(x_1) \neq p(x_2)$ implies that p is injective. Let $y \in X$ and consider $I_y = \{\emptyset, \{y\}\}$. Since A maps $\mathcal{N}$ onto itself, there exists an $x \in X$ such that $A(I_x) = I_y$ which implies $p(x) = y$. Hence p is onto.

Now consider the automorphism A_p on $LGT(X)$ induced by the bijection p on X.

Claim: $A = A_p$ on $\mathcal{N}$.

Let $I_x \in \mathcal{N}$. Then $A(I_x) = I_{p(x)} = \{\emptyset, \{p(x)\}\} = p(\{\emptyset, \{x\}\}) = p(I_x) = A_p(I_x)$. Hence $A = A_p$ on $\mathcal{N}$.

Let $\alpha = \{\emptyset, G\}$, $G \subseteq X$, be an atom which does not belong to $\mathcal{N}$. Let $A(\alpha) = \{\emptyset, H\}$ where $H \subseteq X$. Consider $A_p(\alpha) = \{\emptyset, p(G)\}$. We have to prove that $A_p(\alpha) = A(\alpha)$.

Let $x \in G$ and $A(I_x) = I_y$. Then $y = p(x) \in p(G)$. Now $A(\{\emptyset, \{x\}, G\})$ $= A(\{\emptyset, \{x\}\} \vee \{\emptyset, G\}) = A(\{\emptyset, \{x\}\}) \vee A(\{\emptyset, G\}) = \{\emptyset, \{y\}\} \vee \{\emptyset, H\} = \{\emptyset, \{y\}, H, H \cup \{y\}\}$. But A maps an n element set to n element set only, hence $H \cup \{y\} = H$. This implies $y \in H$. Since $x \in G$ is arbitrary $p(G) \subseteq H$.

To prove the reverse inclusion, we apply the above result to A^{-1}. Then p is replaced by p^{-1} and $A^{-1}(\{\emptyset, H\}) = \{\emptyset, G\}$, implying $p^{-1}(H) \subseteq G$, i.e., $H \subseteq p(G)$.

Hence $H = p(G)$ and consequently $A(\{\emptyset, G\}) = \{\emptyset, H\} = \{\emptyset, p(G)\} = A_p(\{\emptyset, G\})$. Thus we proved that $A = A_p$ on all atoms of $LGT(X)$. Since $LGT(X)$ is an atomic lattice, $A = A_p$ on $LGT(X)$. This completes the proof. □

2.3 Simple expansion of generalized topological space

The concepts of immediate predecessor(lower neighbor) and immediate successor(upper neighbor) in the lattice of topologies, $LT(X)$, are studied extensively by Pushpa Agashe and Norman Levine [2] and others. Simple expansion of topologies has been studied previously by many mathematicians and this concept can be generalized to generalized topologies.

Definition 2.3.1. *Let X be a non empty set, $\mu \in LGT(X)$ and A be a subset of X which does not belong to μ. Then the simple expansion of μ by A, denoted by $\mu(A)$, is defined as*

$$\mu(A) = \mu \cup \{G \cup A : G \in \mu\}$$

Remark 2.3.1. *We can prove easily that a simple expansion of μ is a generalized topology. Also it is obvious from the definition that $\mu(A)$ is the smallest generalized topology containing μ and A. Hence we have the following theorem.*

Theorem 2.3.1. *Let $\mu \in LGT(X)$ and $\mu(A)$ is a simple expansion of μ, where $A \subseteq X$ and $A \notin \mu$. Then $\mu(A) \in LGT(X)$ and $\mu(A) = \mu \vee \{\emptyset, A\}$.*

Let $\mu_1, \mu_2 \in LGT(X)$, we say the generalized topology μ_1 is finer than μ_2(or μ_2 is weaker than μ_1) if $\mu_2 \subseteq \mu_1$. It can be easily seen that $\mu(A)$ is finer than μ whenever $A \subseteq X$ and $A \notin \mu$. The following example shows that it need not always be an upper neighbor.

Example 2.3.1. *Consider the set $X = \{a, b, c, d\}$ and the generalized topology $\mu = \{\emptyset, X, \{a, b, c\}, \{d\}\}$ on X. Let $A = \{a\}$. Then $\mu(A) = \{\emptyset, X, \{a, b, c\}, \{d\}, \{a\}, \{a, d\}\}$. We can see that $\mu(A)$ is not an upper neighbor of μ since if we take $\tau = \{\emptyset, X, \{a, b, c\}, \{d\}, \{a, d\}\}$, $\tau \in LGT(X)$ and $\mu < \tau < \mu(A)$.*

But we can prove that every upper neighbor is a simple expansion.

Theorem 2.3.2. *Let $\mu \in LGT(X)$. If τ is an upper neighbor of μ in $LGT(X)$, then there exists a subset, $A \subseteq X$ such that $\tau = \mu(A)$.*

Proof. Let $A \in \tau \backslash \mu$. Then $\mu(A)$ is the smallest generalized topology containing μ and A. Thus $\mu < \mu(A) \leq \tau$. But τ is an upper neighbor of μ, hence $\mu(A) = \tau$. $\square$

Does every generalized topology possess an upper neighbor? The following lemma shows that the answer is affirmative when X is finite. The proof of this lemma provides a construction of an upper neighbor of a generalized topology.

Lemma 2.3.1. *Let μ be a generalized topology on a finite set X and $\mu \neq P(X)$. Then there exists a subset A of X that does not belong to μ such that $\mu(A) = \mu \cup \{A\}$ and $\mu(A)$ is an upper neighbor of μ.*

Proof. Let $|X| = n$. It is enough to show the existence of a set $A \subseteq X$ such that $\mu \cup \{A\}$ is a generalized topology on X.

If X does not belong to μ, take $A = X$, so that $\mu \cup \{X\} \in LGT(X)$. If $X \in \mu$, then consider the collection of all subsets of X with cardinality $n - 1$. Let us denote the collection by F_{n-1}. If $F_{n-1} \nsubseteq \mu$ then choose $A \in F_{n-1}$ such that $A \notin \mu$.
Claim: $\mu \cup \{A\}$ is a generalized topology on X.

Let $U, V \in \mu \cup \{A\}$. If $U, V \in \mu$, then obviously $U \cup V \in \mu$. If $U \in \mu$ and $V = A$, then either $U \cup V = X$ or $U \cup V = A$. In either case $U \cup V \in \mu \cup \{A\}$. Hence $\mu \cup \{A\}$ is closed under finite union. Note that we are considering generalized topology on finite set only. Also $\emptyset \in \mu$ implies that $\emptyset \in \mu \cup \{A\}$ and hence $\mu \cup \{A\} \in LGT(X)$.

If $F_{n-1} \subseteq \mu$, then consider F_{n-2}, which is the collection of all subsets of X with cardinality $n-2$. If $F_{n-2} \nsubseteq \mu$ then choose $A \in F_{n-2}$ such that $A \notin \mu$ and we can prove that $\mu \cup \{A\}$ is a generalized topology on X. Proceeding similarly if μ contain all 2 element sets and its supersets, since $\mu \neq P(X)$, there exists an $x \in X$ such that $\{x\}$ does not belong to μ, then take $A = \{x\}$. Then A will satisfy the required property. Hence the theorem. □

Thus every generalized topology on a finite set X other than $P(X)$ has an upper neighbor. The following theorem actually tells the form of an upper neighbor of a generalized topology also.

Theorem 2.3.3. *Let μ be a generalized topology on a finite set X, then every upper neighbor of μ is of the form $\mu \cup \{A\}$ for some set $A \subseteq X$.*

Proof. Let τ be an upper neighbor of μ with $|\tau| = |\mu| + k$ where $k \geq 2$. Let us write τ as $\mu \cup \{A_1, A_2, \ldots, A_k\}$ where $A_1, A_2, \ldots, A_k$ are distinct subsets of X which do not belong to μ. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_k\}$ and $I = \{1, 2, 3, \ldots k\}$.

Case 1 : Assume $A_i \nsubseteq A_j$ for every $i, j \in I$ with $i \neq j$.

Consider $\mu \cup \{A_1\}$. $\emptyset \in \mu \cup \{A_1\}$ since μ is a generalized topology. Let $U, V \in \mu \cup \{A_1\}$. If $U, V \in \mu$, then $U \cup V \in \mu$ since μ is a generalized topology. If $U \in \mu$ and $V = A_1$ then also $U \cup V \in \mu \cup \{A_1\}$, for otherwise if $U \cup V = A_k$ for some $k \neq 1$ then $A_1 \subseteq A_k$ which is not possible. Therefore $\mu \cup \{A_1\}$ is closed under finite union and hence it is a generalized topology since we are considering generalized topology on finite set only. Then $\mu < \mu \cup \{A_1\} < \tau$ which is a contradiction since τ is a cover of μ.

Case 2: $A_i \subseteq A_j$ for some $i, j \in I$, where $i \neq j$.

Fix a $j \in I$ such that $A_i \subseteq A_j$ for some $i \in I$, $i \neq j$. Let $\mathcal{F}_j = \{A_l : l \in I, l \neq j, A_l \subseteq A_j\}$. Consider $\mathcal{F} \setminus \mathcal{F}_j$. Assume the set $\mathcal{F} \setminus \mathcal{F}_j$ contain $s+1$ elements and rename the elements in $\mathcal{F} \setminus \mathcal{F}_j$ such that $\mathcal{F} \setminus \mathcal{F}_j = \{A_j, A_{j+1}, \ldots, A_{j+s}\}$. Let us denote $\mathcal{F} \setminus \mathcal{F}_j$ by $\mathcal{F}_j^c$. Note that $\mathcal{F}_j^c \neq \emptyset$ since $A_j \in \mathcal{F}_j^c$.

Claim: $\mu \cup \mathcal{F}_j^c$ is a generalized topology.

Since $\emptyset \in \mu$ $\emptyset \in \mu \cup \mathcal{F}_j^c$. Let $U, V \in \mu \cup \mathcal{F}_j^c$. If $U, V \in \mu$, then obviously $U \cup V \in \mu$. If $U \in \mu$ and $V \in \mathcal{F}_j^c$, then also $U \cup V \in \mu \cup \mathcal{F}_j^c$, for otherwise let $U \cup V \in \mathcal{F}_j$. Then $U \cup V = A_h$ for some $h \in \{1, 2, \ldots, j-1, s, s+1, \ldots, k\}$ so that $V \subseteq A_h \subseteq A_j$. Thus $V \in \mathcal{F}_j$ which is not possible since $V \in \mathcal{F}_j^c$. Similarly if $U, V \in \mathcal{F}_j^c$ then $U \cup V \in \mu \cup \mathcal{F}_j^c$ by the same argument. Hence the claim.

Thus we get $\mu < \mu \cup \mathcal{F}_j^c < \tau$, a contradiction since τ is an upper neighbor. Thus in both cases we proved that an upper neighbor of μ cannot have cardinality $|\mu| + k$ where $k \geq 2$. By Lemma 2.3.1 we have that there exists an upper neighbor with cardinality $|\mu| + 1$. Thus every upper neighbor of μ is of the form $\mu \cup \{A\}$ for some set $A \subseteq X$ and $A \notin \mu$. □

The following corollary reveals a pattern in the lattice structure of $LGT(X)$ when X is finite.

Corollary 2.3.1. *$LGT(X)$ is a graded poset when X is finite.*

Proof. Define a function

$$h : LGT(X) \to \mathbb{Z}^+$$

by $h(\mu) = |\mu|$ for every $\mu \in LGT(X)$. That is, h maps each generalized topology into its cardinal number. Then by the above theorem, if $\tau \in LGT(X)$ is an upper neighbor of $\mu \in LGT(X)$, then $|\tau| = |\mu| + 1$. Thus $LGT(X)$ is a graded poset. □

Now we give some results on expansions of generalized topologies by a collection of subsets of X.

Definition 2.3.2. *Let (X, μ) be a generalized topological space and $F = \{A_i \subseteq X : i \in I\}$ be a collection of subsets of X. Then the generalized topology on X which is the smallest generalized topology containing $\mu(A_i)$ for each $i \in I$ shall be denoted by $\mu(F)$.*

The following theorem is a direct consequence of above definition.

Theorem 2.3.4. *Let (X, μ) be a generalized topological space and A and B are nonempty subsets of X. Then $\mu(\{A, B\}) = (\mu(A))(B) = (\mu(B))(A)$.*

Theorem 2.3.5. *If μ_1 and μ_2 are generalized topologies on a set X, then $\mu_1 \subseteq \mu_2$ if and only if there exists a family $F \subseteq P(X)$ such that $\mu_2 = \mu_1(F)$.*

Proof. Let $F = \{A \subseteq X : A \in \mu_2 \setminus \mu_1\}$. Note that $\mu_1(F)$ is the smallest generalized topology containing μ_1 and F. But $\mu_1 \subseteq \mu_2$ and $F \subseteq \mu_2$. Therefore $\mu_1(F) \subseteq \mu_2$. Also since $\mu_2 \setminus \mu_1 = F$, we have $\mu_2 = \mu_1 \cup F \subseteq \mu_1(F)$. Hence $\mu_2 = \mu_1(F)$. □

2.4 Characterization of upper neighbors of $LGT(X)$

Here by comparing two simple expansions of a generalized topology, we prove several characterization theorems for a simple expansion to be an upper neighbor of a generalized topology. We had seen in the last section that every upper neighbor of μ is of the form $\mu \cup \{A\}$ for some set $A \subseteq X$. In this section we prove that this result holds in general, it doesn't matter whether X is finite or infinite.

Theorem 2.4.1. *Let $\mu, \mu^{'}$ are generalized topologies on X. Then $\mu^{'}$ is an upper neighbor of μ if and only if $\mu^{'} = \mu(A)$ for every $A \in \mu^{'} \backslash \mu$.*

Proof. Suppose $\mu^{'}$ is an upper neighbor of μ. Let $A \in \mu^{'} \backslash \mu$. Then $\mu(A)$ is the smallest generalized topology containing μ and A and hence $\mu(A) \subseteq \mu^{'}$. Thus $\mu \subseteq \mu(A) \subseteq \mu^{'}$. Since $\mu^{'}$ is an upper neighbor of μ, $\mu(A) = \mu^{'}$. Note that $A \in \mu^{'} \setminus \mu$ is arbitrary, therefore $\mu(A) = \mu^{'}$ for every $A \in \mu^{'} \backslash \mu$. Now assume $\mu^{'} = \mu(A)$ for every $A \in \mu^{'} \backslash \mu$. If $\mu^{'}$ is not an upper neighbor of μ, then there exists a generalized topology $\mu^{''}$ on X such that $\mu \subseteq \mu^{''} \subseteq \mu^{'}$, $\mu \neq \mu^{''}$ and $\mu^{''} \neq \mu'$. Then there exists a set $B \in \mu^{''}$ such that $B \notin \mu$. Consequently $B \in \mu^{'}$ and by assumption $\mu^{'} = \mu(B)$. Note that $\mu^{'}$ is the smallest generalized topology containing μ and B resulting $\mu^{''} = \mu(B) = \mu^{'}$, a contradiction to our assumption and thus proving $\mu^{'}$ is an upper neighbor of μ. $\square$

The following proposition point out some obvious covers of a generalized topology μ on X.

Proposition 2.4.1. *Let μ be a generalized topology on a set X. Then,*

1. *if $X \notin \mu$, then $\mu(X) = \mu \cup \{X\}$ is always an upper neighbor of μ.*
2. *if X is finite and if μ is a strong generalized topology on X, then for every $A \subseteq X$ such that $|A| = |X| - 1$, $\mu(A)$ is always an upper neighbor of μ, if $A \notin \mu$.*
3. *Let $A \subseteq X$ and $A \notin \mu$. Then if for every $G \in \mu$ suppose either $A \subseteq G$ or $G \subseteq A$, then $\mu(A)$ is an upper neighbor of μ.*

Proof. This can be easily verified by the reader. $\square$

Theorem 2.4.2. *Let μ be a generalized topology on a set X and let $A \subseteq X$ and $A \notin \mu$. Then for every $G \in \mu$, the simple expansion $\mu(A)$ is finer than $\mu(G \cup A)$.*

Proof. Let $G \in \mu$, then $G \cup A \in \mu(A)$. Also $\mu(G \cup A)$ is the smallest generalized topology containing μ and $G \cup A$, implying $\mu(G \cup A) \subseteq \mu(A)$. Hence the result. $\square$

Theorem 2.4.3. *Let (X, μ) be a generalized topological space and A, B are subsets of X such that $A, B \notin \mu$. Then,*

1. *the simple expansion $\mu(B)$ is finer than the simple expansion $\mu(A)$ if and only if $A = G \cup B$ for some $G \in \mu$.*
2. *the simple expansion $\mu(B)$ is equal to the simple expansion $\mu(A)$ if and only if $A = B$.*

Proof.

1. First assume $\mu(A) \subseteq \mu(B)$, then $A \in \mu(B)$. Since $A \notin \mu$, $A = G \cup B$ for some $G \in \mu$. Conversely if $A = G \cup B$ for some $G \in \mu$, then $A \in \mu(B)$ thus getting $\mu(A) \subseteq \mu(B)$.
2. This result can be easily deduced from (1).

$\square$

Corollary 2.4.1. *Let (X, μ) be a generalized topological space. Then for every $A \subseteq X$ such that $A \notin \mu$, the simple expansion $\mu(A \backslash A^0)$ is always finer than $\mu(A)$.*

Proof. The set A can be written as $A = (A \backslash A^0) \cup A^0$. Then result follows from Theorem 2.4.3. $\square$

Corollary 2.4.2. *Let A, B are subsets of a set X such that $A, B \notin \mu$, where (X, μ) is a generalized topological space. If the simple expansion of μ by B is finer than the simple expansion of μ by A, then B is a subset of A.*

Proof. Assume $\mu(A) \subseteq \mu(B)$, then $A \in \mu(B)$. Since $A \notin \mu$, $A = G \cup B$ for some $G \in \mu$ which implies $B \subseteq A$. Hence the result. □

Remark 2.4.1. *Converse of Corollory 2.4.2 is not true.*
For example, let $X = \{a, b, c, d\}$. Consider the generalized topology $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Let $A = \{a, d\}$ and $B = \{a\}$. Here $B \subseteq A$, but $\mu(A) = \mu \cup \{\{a, d\}, \{a, b, d\}, \{a, c, d\}, X\}$ and $\mu(B) = \mu \cup \{\{a\}\}$. Here we see that $\mu(B)$ and $\mu(A)$ are not even comparable.

Theorem 2.4.4. *Let (X, μ) be a generalized topological space. Let $A \subseteq X$ and $A \notin \mu$ and $G \in \mu$. Then the following are equivalent.*

1. *The simple expansion $\mu(A)$ is an upper neighbor of μ.*
2. *$\mu(A) = \mu \cup \{A\}$.*
3. *$G \cup (A \backslash A^0) \in \mu(A) \backslash \mu \Rightarrow G = A^0$.*
4. *$G \cap A^c \neq \emptyset \Rightarrow G \cup A \in \mu$.*
5. *$\mu(A) = \mu(B)$ for every $B \in \mu(A) \backslash \mu$.*

Proof. (1) $\Leftrightarrow$ (5) by Theorem 2.4.1.
(1) $\Rightarrow$ (2): By Theorem 2.4.3, for A, B subsets of X and $A, B \notin \mu$, $\mu(A) = \mu(B)$ if and only if $A = B$. But since $\mu(A)$ is an upper neighbor and by (5), $\mu(A) \backslash \mu$ cannot have elements other than A. Hence $\mu(A) = \mu \cup \{A\}$.
(2) $\Rightarrow$ (1) is obvious.

(2) $\Leftrightarrow$ (3) is clear.

(2) $\Rightarrow$ (4): Assume $\mu(A) = \mu \cup \{A\}$. Now $G \cap A^c \neq \emptyset \Rightarrow G \nsubseteq A \Rightarrow G \cup A \neq A$ and $G \cup A \in \mu(A) = \mu \cup \{A\}$ proving $G \cup A \in \mu$.

(4) $\Rightarrow$ (2): Consider $\mu(A) = \mu \cup \{G \cup A : G \in \mu\}$. If $G \cap A^c = \emptyset$ then $G \subseteq A$ and thus $G \cup A = A$. Also if $G \cap A^c \neq \emptyset$, then $G \cup A \in \mu$. Thus in either case $G \cup A \in \mu \cup \{A\}$. Hence $\mu(A) = \mu \cup \{A\}$.

□

Norman Levine called a topology τ on a set X as a superset topology [31] if and only if for $\emptyset \neq O \subseteq A \subseteq X$ and $O \in \tau$, then $A \in \tau$. We define this concept in generalized topology also and discuss when will this possess an upper neighbor.

Definition 2.4.1. *A generalized topological space (X, μ) is said to be a superset generalized topological space if, whenever $\emptyset \neq G \in \mu$ and $G \subseteq H \subseteq X$, then $H \in \mu$.*

Theorem 2.4.5. *Let (X, μ) be a generalized topological space. Then μ is a superset generalized topology on X if and only if for every $A \subseteq X$ and $A \notin \mu$, $\mu(A)$ is an upper neighbor of μ.*

Proof. Suppose μ is a superset generalized topology on X. Let $A \subseteq X$ and $A \notin \mu$, $\mu(A) = \{G \cup A : G \in \mu\} \cup \mu$. Since $G \cup A$ is a superset of $G \in \mu$, $G \cup A \in \mu$. Then $G \cup A \in \mu$ for every $G \in \mu$ implying $\mu(A) = \mu \cup \{A\}$ proving $\mu(A)$ is an upper neighbor of μ.

Now assume $\mu(A)$ is an upper neighbor of μ for every $A \subseteq X$ and $A \notin \mu$. That is $\mu(A) = \mu \cup \{A\}$ for every $A \subseteq X$ and $A \notin \mu$. Let $G \in \mu$ and $G \subseteq H \subseteq X$. Suppose $H \notin \mu$. Take $A = H \backslash G$, then $A \notin \mu$, otherwise, if $A \in \mu$, then

$A \cup G = H \in \mu$, which is a contradiction to our assumption. Now consider the simple expansion of μ by $A = H \backslash G$. Then A and H are elements of $\mu(A) \backslash \mu$ implying that $\mu(A) \neq \mu \cup \{A\}$, a contradiction to our assumption that $\mu(A)$ is a cover for every $A \notin \mu$. Hence $H \in \mu$. Thus μ is a superset generalized topology on X. $\square$

We proved that every generalized topology has an upper neighbor when the underlying set is finite in the previous section. In general this is not true (see examples 2.4.1 and 2.4.2). Now, when does a generalized topology posses an upper neighbor if X is infinite? We couldn't find the answer in general. Here we attempt to solve the problem in particular cases.

The following theorem is used to prove our next result.

Theorem 2.4.6. *[46] A generalized topological space (X, μ) is μ-T_1 if and only if for each $x \in M_\mu$, $\{x\} \cup (X \backslash M_\mu)$ is a closed set, where M_μ is the union of all open sets in X.*

Theorem 2.4.7. *Every non μ-T_1 generalized topology has an upper neighbor.*

Proof. Let (X, μ) be a non μ-T_1 generalized topological space. If $X \notin \mu$, then $\mu(X)$ is an upper neighbor of μ. If $X \in \mu$, then since μ is non μ-T_1 by Theorem 2.4.6, there exists an $x \in X$ such that $\{x\}$ is not closed relative to μ showing that $\{x\}^c \notin \mu$.

Claim: $\mu(\{x\}^c)$ is an upper neighbor of μ.

Let $G \in \mu$, $G \cap (\{x\}^c)^c = G \cap \{x\}$. If $G \cap \{x\} \neq \emptyset$, then $x \in G$ and $G \cup \{x\} = G \in \mu$. Then by Theorem 2.4.4, $\mu(\{x\}^c)$ is an upper neighbor of μ. $\square$

We denote μ^c for the complement of a generalized topology μ on X, i.e., $\mu^c = P(X) \setminus \mu$ and we say μ is non trivial if $\mu \neq \{\emptyset\}$.

Theorem 2.4.8. *Let X be an infinite set and μ be a generalized topology on X. If μ or μ^c is finite, then μ has an upper neighbor.*

Proof. If $\mu = \{\emptyset\}$, then obviously μ has an upper neighbor. In fact each atom $\{\emptyset, A\}, \emptyset \neq A \subseteq X$ is an upper neighbor of μ. Assume the case when μ is finite and non trivial. If $X \notin \mu$, then $\mu \cup \{X\}$ is an upper neighbor of μ. Now let $X \in \mu$. Since X is infinite, there exists an $x \in X$ such that $X \setminus \{\{x\}\} \notin \mu$. Let $A = X \setminus \{\{x\}\}$, then $\mu(A) = \mu \cup \{A\}$ is easily seen to be an upper neighbor of μ.

Now let us assume μ^c is finite and μ is non trivial. Let $\mu^c = \{K_1, K_2, \ldots, K_p\}$. Consider K_1 and take one largest set, say D, in μ^c containing K_1, i.e., there exists no set $D^{'} \in \mu^c$ such that $D \subsetneq D^{'}$. Then for every $G \in \mu$, $G \cup D \in \mu$ or $G \cup D = D$. Otherwise, if $G \cup D \notin \mu$, then $G \cup D$ has to be an element in μ^c containing D, which is a contradiction to our assumption. Hence $\mu(D) = \mu \cup \{D\}$ is an upper neighbor of μ. $\square$

The following examples show that there are generalized topologies which do not have an upper neighbor.

Example 2.4.1. *Let X be any infinite set and $x \in X$. Define $\mu = \{G \subseteq X :$ either $x \notin G$ or $(x \in G$ and G^c is $finite)\}$. Then μ is a topology on X and hence a generalized topology. Let $A \subseteq X$ and $A \notin \mu$, then $x \in A$ and A^c is infinite. Also $\{y\} \in \mu$ for every $y \in A^c$, then $A \cup \{y\} \in \mu(A)$ for every $y \in A^c$ resulting $\mu(A) \neq \mu \cup \{A\}$. Thus (X, μ) does not have an upper neighbor.*

Example 2.4.2. *Consider the following generalized topology on the set of real numbers* $\mathbb{R}$ *.*
$\mu = \{\emptyset, \mathbb{R}\} \cup P(\mathbb{Q}) \cup \{X \cup Y : X = \mathbb{Q} \backslash F, F \subseteq \mathbb{Q}, F \; is \; finite, Y \subseteq \mathbb{R} \backslash \mathbb{Q}\}$ *where* $P(\mathbb{Q})$ *denotes power set of* $\mathbb{Q}$*, where* $\mathbb{Q}$ *denotes the set of all rational numbers.*
Let $A \subseteq \mathbb{R}$ *and* $A \notin \mu$*, then* A *can be* $A = G \cup H$*, where* $G \subseteq \mathbb{Q}, G^c \cap \mathbb{Q}$ *is infinite and* $H \subseteq \mathbb{R} \backslash \mathbb{Q}$*,* $H \neq \emptyset$
then $A \cup \{x\} \in \mu(A)$ *for every* $x \in G^c \cap \mathbb{Q}$ *implying* $\mu(A) \neq \mu \cup \{A\}$*.*
Thus μ *has no upper neighbor.*

In fact we can generalize these examples and we state this as a theorem.

Theorem 2.4.9. *Let* X *be any infinite set and* $A \subseteq X$*, is also infinite. Then the generalized topology* $\mu = \{\emptyset, X, P(A)\} \cup \{G \subseteq X : G \cap (A^c) \neq \emptyset, \text{ and } G^c \cap A \, is \, finite\}$ *does not have an upper neighbor.*

Proof. Consider any subset H of X such that $H \notin \mu$. Then the result follows easily from the fact that for all $x \in H^c \cap A$, $H \cup \{x\} \notin \mu \cup \{H\}$.

□

Remark 2.4.2. *Consider the lattice* $LT(X)$ *of topologies and the lattice* $LGT(X)$ *of generalized topologies on a set* X*. Let* τ *be a topology on* X*. Then* $\tau \in LT(X)$ *and* $\tau \in LGT(X)$*. Suppose* τ *has upper neighbors in* $LT(X)$ *and* $LGT(X)$*. Then upper neighbor of* τ *in* $LT(X)$ *and* $LGT(X)$ *are same if and only if there exists a subset* $A \subseteq X$ *and* $A \notin \tau$ *such that for every* $G \in \tau$*, If* $G \cap (A \backslash A^0) \neq \emptyset$ *then* $A \subseteq G$ *and If* $G \cap A^c \neq \emptyset$ *then* $G \cup A \in \tau$*. This is clear from the fact that immediate successor of* τ *in* $LT(X)$ *and* $LGT(X)$ *are same if and only if there exists* $A \subseteq X$ *and* $A \notin \tau$ *such that* $\tau(A) = \tau \cup \{A\}$ *and we use the following theorem.*

Theorem 2.4.10. *[31] Let (X, τ) be a topological space and A is a nonempty subset of X such that $A \notin \tau$. Then a necessary and sufficient condition for the simple expansion topology $\tau(A)$ is the union of the topology τ and the set A is that*

1. *$O \in \tau, O \cap (A \backslash A^o) \neq \emptyset \Rightarrow A \subseteq O$ and*
2. *$O \cap A^c \neq \emptyset \Rightarrow O \cup A \in \tau$.*

Remark 2.4.3. *Let μ be a generalized topology on X, which is not a topology. Then there exists an upper neighbor of μ, say $\mu^{'}$, which is a topology if and only if the set $\{G \cap H \notin \mu : G, H \in \mu\}$ is a singleton set and consequently $\mu^{'} = \mu(G \cap H) = \mu \cup \{G \cap H\}$ where $G, H \in \mu$ such that $G \cap H \notin \mu$.*

A similar concept here is the study of lower neighbors of a generalized topology. It is easy to show that if μ is a generalized topology on X, then every lower neighbor of μ is of the form $\mu \setminus \{A\}$ for some $A \in \mu$. This follows from Theorem 2.4.4 and the fact that $\mu^{'}$ is a lower neighbor of μ if and only if μ is an upper neighbor of $\mu^{'}$.

It is then natural to ask whether the existence of upper neighbors of a generalized topology implies the existence of a lower neighbor and vice versa. We have examples [Examples 2.4.3, 2.4.4] to show that neither of these implications are true. The existence of upper neighbors and lower neighbors of a generalized topology is left as an open problem.

Example 2.4.3. *Consider the generalized topology discussed in Example 2.4.2. We have shown that μ does not have an upper neighbor. But $\mu^{'} = \mu \setminus \{\{1\}\}$ is a lower neighbor of μ. In fact μ has infinitely many lower neighbors.*

Example 2.4.4. *Here we have an example for a generalized topology which has infinitely many upper neighbors but does not possess a lower neighbor.*

Define a generalized topology μ on $\mathbb{R}$ by $\mu = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\} \cup \{(-\infty, a) \cup (b, \infty) : a, b \in \mathbb{R}$ and $a \leq b\}$.
Note that $(-\infty, a) = \bigcup\limits_{n \in \mathbb{N}} (-\infty, a - \frac{1}{n})$ and $(b, \infty) = \bigcup\limits_{n \in \mathbb{N}} (b + \frac{1}{n}, \infty)$. Therefore we can not remove either $(-\infty, a)$ or (b, ∞) from μ inorder to get a lower neighbor. Similarly we can not remove $\mathbb{R}$ since $\mathbb{R} = (-\infty, 1) \cup (-1, \infty)$ and same is the case for elements of the form $(-\infty, a) \cup (b, \infty)$. Thus μ can not have a lower neighbor. But if we let $A = \mathbb{R} \setminus \{\{1, 2\}\}$, then $\mu(A)$ is an upper neighbor of μ.

2.5 Properties of the simple expansion

Here we discuss the general question, given a generalized topology μ on X with a property P when will a simple expansion of μ possess the same property P. We use the following notations here.
Consider the generalized topology μ on a set X. Let $A \subseteq X$. Then,

1. $\mu(A)$ denotes the simple expansion of μ by A.
2. A^o_μ denotes the interior of A with respect to μ.
3. $\overline{A_\mu}$ denotes the closure of A with respect to μ.
4. A^c denotes the set theoretic complement of A in X.
5. $\mu \cap A = \{G \cap A : G \in \mu\}$.

Theorem 2.5.1. *Let (X, μ) be a generalized topological space which is μ-T_o, μ-T_1 or μ-T_2. Let $A \subseteq X$ and $A \notin \mu$. Then $(X, \mu(A))$ is $\mu(A)$-T_o, $\mu(A)$-T_1 or $\mu(A)$-T_2 respectively.*

The above theorem can be easily verified.

Lemma 2.5.1. *Let (X, μ) be a generalized topological space and $A \subseteq X$ and $A \notin \mu$. Let $B \subseteq X$, then*

$$B^o_{\mu(A)} = \begin{cases} B^o_\mu \cup A, & \textit{If } A \subseteq B \\ B^o_\mu, & \textit{otherwise.} \end{cases}$$

Proof. The reader may easily supply the proof. □

Lemma 2.5.2. *Let (X, μ) be a generalized topological space and $A \subseteq X$ and $A \notin \mu$. Let $B \subseteq X$, then*

$$\overline{B}_{\mu(A)} = \begin{cases} \overline{B}_\mu \cap A^c, & If\ A \subseteq B^c \\ \overline{B}_\mu, & otherwise. \end{cases}$$

Proof. We have $\overline{B}_{\mu(A)} = [(B^c)^o_{\mu(A)}]^c$. By Lemma 2.5.1, $(B^c)^o_{\mu(A)} = (B^c)^o_\mu \cup A$ if $A \subseteq B^c$ and $(B^c)^o_{\mu(A)} = (B^c)^o_\mu$ otherwise. Thus $\overline{B}_{\mu(A)} = [(B^c)^o_\mu \cup A]^c = \overline{B}_\mu \cap A^c$ if $A \subseteq B^c$ and $\overline{B}_{\mu(A)} = \overline{B}_\mu$ otherwise. Hence the result is proved. □

Example 2.5.1.

1. *Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ be a generalized topology on X. Then μ is μ-regular and μ-normal generalized topology on X but the simple expansion of μ by the set $A = \{a, c\}$ is neither. (See Theorems 2.5.2 and 2.5.5).*

2. *Consider the generalized topology $\mu = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ on the set of all real numbers $\mathbb{R}$ and $\mathbb{Q}$ denote the set of all rational numbers. $(\mathbb{R}, \mu)$ is a connected*

generalized topological space. But the simple expansion of μ by $A = \mathbb{R} \setminus \mathbb{Q}$, $\mu(A) = \{\emptyset, \mathbb{R}, \mathbb{Q}, A\}$ is not connected. (See Theorem 2.5.7)

Theorem 2.5.2. *Let (X, μ) be a μ- regular generalized topological space and let A be a subset of X such that $A \notin \mu$ and $A^c \in \mu$. Then $(X, \mu(A))$ is $\mu(A)$-regular.*

Proof. Let $x \in X$ and $x \notin F$ where F is a closed set in $(X, \mu(A))$. If $F = O^c$ for some $O \in \mu$, then F is closed in (X, μ). Since (X, μ) is μ-regular there exist disjoint open sets $U, V \in \mu \subseteq \mu(A)$ such that $x \in \mu$ and $F \subseteq V$. Now let $F = (O \cup A)^c$ for some $O \in \mu$. Then $x \notin F$ implies $x \notin (O \cup A)^c = O^c \cap A^c$. Thus x does not belong to O^c or A^c.

Case 1: ($x \notin O^c$ and $x \notin A^c$) or ($x \in O^c$ and $x \notin A^c$)

Here $x \in A$ and $F = O^c \cap A^c \subseteq A^c$. Since A and A^c are open in $(X, \mu(A))$ and $A \cap A^c = \emptyset$, x and F can be separated by A and A^c in $(X, \mu(A))$.

Case 2: $x \notin O^c$ and $x \in A^c$

We have $F = O^c \cap A^c \subseteq O^c$ and since (X, μ) is μ-regular there exist disjoint open sets U, V in (X, μ) such that $x \in U$ and $O^c \subseteq V$. Thus $x \in U$ and $F = O^c \cap A^c \subseteq V$. Hence $(X, \mu(A))$ is $\mu(A)$-regular. □

Theorem 2.5.3. *Let (X, μ) be a strong generalized topological space and $A \notin \mu$. If A is dense in (X, μ), then $(X, \mu(A))$ is not $\mu(A)$-regular.*

Proof. Assume $(X, \mu(A))$ is $\mu(A)$-regular. Since $A \notin \mu$, $A \setminus A^o_\mu$ is non empty. Let $x \in A \setminus A^o_\mu$. Consider A^c, which is closed in $(X, \mu(A))$. Then by regularity, there exists disjoint open sets $G, G' \in \mu(A)$ such that $x \in G$ and $A^c \subseteq G'$. Then G must be contained in A, i.e., $x \in G \subseteq A$, then $x \in A^o_\mu$, which is a contradiction. Hence the result. □

Definition 2.5.1. *[43] Let (X, μ) be a generalized topological space. A collection $\mathcal{F}$ of subsets of X is said to be a μ-cover of X if the union of the elements of $\mathcal{F}$*

is equal to X. If every element in $\mathcal{F}$ are open in (X, μ), then $\mathcal{F}$ is called μ-open cover of X. A μ-sub cover of a μ-cover $\mathcal{F}$ is a sub collection $\mathcal{G}$ of $\mathcal{F}$ which itself is a μ-cover. The generalized topological space (X, μ) is said to be μ-compact space if each μ-open cover of X has a finite μ-open sub cover.

Lemma 2.5.3. *[43] Every μ-closed subset of a μ-compact generalized topological space (X, μ) is μ-compact.*

Theorem 2.5.4. *Let (X, μ) be a μ-compact generalized topological space and let $A \subseteq X$ and $A \notin \mu$. Then $(X, \mu(A))$ is $\mu(A)$-compact if and only if A^c is μ-compact in (X, μ).*

Proof. Necessity. Suppose $(X, \mu(A))$ is $\mu(A)$-compact. Since A^c is closed in $(X, \mu(A))$, by Lemma 2.5.3 A^c is $\mu(A)$-compact in $(X, \mu(A))$. Hence it is μ-compact in (X, μ), since $\mu \subseteq \mu(A)$.

Sufficiency. Assume that A^c is μ-compact in (X, μ). Consider a $\mu(A)$-open cover S for X, let $S = \{G_i \in \mu(A) : i \in I\}$ such that $X = \bigcup\limits_{i \in I} G_i$. Then since every $G_i \in \mu(A)$, either $G_i \in \mu$ or G_i is of the form $F_i \cup A$ for some open set F_i in μ. Since A^c is μ-compact, there exists a finite set $J \subseteq I$ and $A^c \subseteq \bigcup\limits_{j \in J} G_j$, where $G_j \in \mu$ for all $j \in J$. Now take any open set of the form $G_k \cup A$ from the collection S. Then it will form an $\mu(A)$-open cover for A and $X = \bigcup\limits_{j \in J} G_j \cup G_k \cup A$. Thus we get a finite $\mu(A)$-sub cover for X from the collection S. Hence $(X, \mu(A))$ is $\mu(A)$-compact. □

Lemma 2.5.4. *Let (X, μ) be a generalized topological space and A be a subset of X such that $A \notin \mu$. Then the generalized topological space $(A, \mu \cap A) = (A, \mu(A) \cap A)$ and $(A^c, \mu \cap A^c) = (A^c, \mu(A) \cap A^c)$.*

Lemma 2.5.5. *Every closed subspace $(A, \mu \cap A)$ of a μ-normal generalized topological space (X, μ) is $\mu \cap A$-normal.*

Proof. Proof is easy. $\square$

Theorem 2.5.5. *Let (X, μ) be a μ-normal generalized topological space. Let $A \subseteq X$ be such that $A \notin \mu$, $A^c \in \mu$ and $A \cap G \in \mu(A)$ for every $G \in \mu$. Then $(X, \mu(A))$ is $\mu(A)$-normal if and only if $(A^c, \mu \cap A^c)$ is $\mu \cap A^c$-normal.*

Proof. Assume $(X, \mu(A))$ is $\mu(A)$-normal. A^c is closed in $(X, \mu(A))$. We have by Lemma 2.5.5 that closed subspace of a normal space is normal. Thus $(A^c, \mu(A) \cap A^c)$ is normal. By Lemma 2.5.4, $(A^c, \mu(A) \cap A^c) = (A^c, \mu \cap A^c)$. Hence $(A^c, \mu \cap A^c)$ is $\mu \cap A^c$-normal.

Now assume the converse. Let F, G are closed and disjoint subsets of $(X, \mu(A))$. Then $F \cap A$ and $G \cap A$ are closed and disjoint in $(X, \mu(A) \cap A) = (X, \mu \cap A)$. Since A is closed in (X, μ), $F \cap A$ and $G \cap A$ are closed in (X, μ), which is μ-normal. Thus there exist disjoint open sets U and V in μ such that $F \cap A \subseteq U$ and $G \cap A \subseteq V$. Also $F \cap A^c$ and $G \cap A^c$ are disjoint and closed in $(A^c, \mu(A) \cap A^c) = (A^c, \mu \cap A^c)$, which is $\mu \cap A^c$-normal. Then there exist disjoint open sets U' and V' in $\mu \cap A^c$ such that $F \cap A^c \subseteq U'$ and $G \cap A^c \subseteq V'$. Now $F = (F \cap A) \cup (F \cap A^c) \subseteq (U \cap A) \cup U'$ which is open in $\mu(A)$ since $A \cap U$ and U' are open in $\mu(A)$ and hence the union. Similarly $G = (G \cap A) \cup (G \cap A^c) \subseteq (V \cap A) \cup V'$ which is also open by the same reason. Also $(U \cap A) \cup U'$ and $(V \cap A) \cup V'$ are disjoint since $(U \cap A)$ and $(V \cap A)$ are disjoint subsets of A and U' and V' are disjoint subsets of A^c. Hence $(X, \mu(A))$ is $\mu(A)$-normal. $\square$

Theorem 2.5.6. *Let (X, μ) be a generalized topological space and $A \subseteq X$ and $A \notin \mu$. Then*

1. *(X, μ) is μ-second countable if and only if $(X, \mu(A))$ is $\mu(A)$-second countable.*

2. *(X, μ) is μ-separable if and only if $(X, \mu(A))$ is $\mu(A)$-separable.*

Proof. *1.* Assume $(X, \mu(A))$ is $\mu(A)$-second countable. But $\mu(A) = \mu \cup \{G \cup A : G \in \mu\}$. If S is a countable base for $(X, \mu(A))$, then $S \cap \mu$ is a countable base for (X, μ). Now assume the converse and let $\{G_n\}_{n \in N}$, where N is the set of all Natural numbers, is a countable collection of open sets in (X, μ) which forms a base for (X, μ). Then clearly $\{G_n\}_{n \in N} \cup \{A\}$ forms a base for the generalized topological space $(X, \mu(A))$. Hence the result.
2. Assume the generalized topological space $(X, \mu(A))$ is $\mu(A)$-separable. Then (X, μ) is μ-separable since $\mu \subseteq \mu(A)$. Now if (X, μ) is μ-separable, then (X, μ) has a countable dense subset say H, implying $H \cup \{x\}$, where $x \in A$, is a countable dense subset of $(X, \mu(A))$ proving $(X, \mu(A))$ is $\mu(A)$-separable.

□

Theorem 2.5.7. *Let (X, μ) be a connected generalized topological space and if $A \notin \mu$, is a dense subset of (X, μ), then $(X, \mu(A))$ is a connected generalized topological space.*

Proof. If $(X, \mu(A))$ is not connected, let $U, V \in \mu(A)$ constitute a separation for $(X, \mu(A))$. Then U and V both cannot be open in (X, μ) since (X, μ) is connected and also both cannot belong to the set $\{O \cup A : O \in \mu\}$, since $U \cap V = \emptyset$. Therefore let $U \in \mu$ and $V = O \cup A$ where $O \in \mu$. $U \cap V = U \cap (O \cup A) = (U \cap O) \cup (U \cap A) \neq \emptyset$ because $U \cap A \neq \emptyset$ since A is dense in (X, μ), a contradiction. Hence the result. □

Chapter 3

Automorphism group of the lattice of fuzzy generalized topologies

3.1 Introduction

We have discussed the automorphism group of the lattice of generalized topologies in the previous chapter. Madhavan Namboothiri determined the automorphism group of lattice of fuzzy topologies when L is a finite chain and when L is the diamond-type lattice [37].

In this chapter we consider the similar problem in the lattice $LFGT(X, L)$, of fuzzy generalized topologies, on a set X and when L is a finite chain in the first part and when L is the diamond-type lattice in the second part of this chapter.

3.2 Preliminaries

First let us recall the definition of fuzzy generalized topology.

Definition 3.2.1. *[23] Let X be a nonempty ordinary set, L an F-lattice, $\mu \subseteq L^X$. Then μ is called an L-fuzzy generalized topology or fuzzy generalized topology on X, and (L^X, μ) is called an L-fuzzy generalized topological space or fuzzy generalized topological space, if μ satisfies the following conditions:*

1. $\underline{0} \in \mu$;
2. $\forall \mathcal{A} \subseteq \mu, \bigvee \mathcal{A} \in \mu$.

Consider the collection of all L-fuzzy generalized topologies on a nonempty set X, $LFGT(X, L)$ and let $\mu_1, \mu_2 \in LFGT(X, L)$, then μ_1 is said to be coarser than μ_2 (μ_2 is finer than μ_1) if $\mu_1 \subseteq \mu_2$. Let us denote the relation 'coarser than' by $\leq$ and with this partial order, $\leq$ on $LFGT(X, L)$, it form a complete lattice, where for a collection of L-fuzzy generalized topologies on X, say $\{\mu_i\}_{i\in I}$, $\bigvee\limits_{i\in I}\mu_i$ is the L-fuzzy generalized topology generated by $\underset{i\in I}{\cup}\mu_i$ and $\bigwedge\limits_{i\in I}\mu_i = \underset{i\in I}{\cap}\mu_i$. The smallest element of $LFGT(X, L)$ is $\{\underline{0}\}$ and the largest element is L^X. The atoms of $LFGT(X, L)$ are L-fuzzy generalized topologies of the form $\{\underline{0}, A\}$ where $A \in L^X$. Recall the following definition of lattice isomorphism and its equivalent form given in chapter 1.

Definition 3.2.2. *[21] The lattices $(L_0, \leq)$ and $(L_1, \leq')$ are isomorphic and the map $\phi : L_0 \to L_1$ is an isomorphism if and only if ϕ is one-to-one and onto and*

$$a \leq b \ in \ L_0 \ if \ and \ only \ if \ \phi(a) \leq' \phi(b) \ in \ L_1$$

Proposition 3.2.1. *[21] The lattices* $(L_0, \wedge, \vee)$ *and* $(L_1, \wedge, \vee)$ *are isomorphic and the map* $\phi : L_0 \to L_1$ *is an isomorphism if and only if* ϕ *is one-to-one and onto and*

$$\phi(a \vee b) = \phi(a) \vee \phi(b)$$
$$\phi(a \wedge b) = \phi(a) \wedge \phi(b)$$

An isomorphism of a lattice with itself is called an automorphism.

It can be shown that if the lattices are complete lattices, then an isomorphism between them preserves arbitrary join and arbitrary meet.

Remark 3.2.1. *Note that an automorphism of* $LFGT(X, L)$ *map an* L*-fuzzy generalized topology containing* n *elements onto an* L*-fuzzy generalized topology containing same number of elements if* n *is finite.*

3.3 Automorphism group of $LFGT(X, L)$ when L is a finite chain

Before proceeding to the main results, let us introduce some notations which will be using throughout this section. Let us denote the set $\{0, l_1, l_2, \ldots, l_n, 1\}$ by L and let the order in L be $0 < l_1 < l_2 < \ldots < l_n < 1$. We define an involution $'$ in L as $0' = 1, 1' = 0$ and $l_i' = l_{n-i+1}$ for every $i \in \{1, 2, \ldots, n\}$. Then L is an F-Lattice. Let us designate an atom of $LFGT(X, L)$ by $J_C = \{\underline{0}, C\}$ where $C \in L^X$ and $C \neq \underline{0}$.

For $l \in L, l \neq 0$ and $x \in X$,

$$x_l(t) = \begin{cases} l & \text{when } t = x \\ 0 & \text{otherwise} \end{cases}$$

and for $l \in L$, $l \neq 1$ and $x \in X$,

$$x^l(t) = \begin{cases} l & \text{when } t = x \\ 1 & \text{otherwise.} \end{cases}$$

For $i = 1, 2, \ldots, n$,

$$K_i = \{J_{x_{l_i}} : x \in X\},$$

$$M_i = \{J_{x^{l_i}} : x \in X\},$$

$$K_{n+1} = \{\{\underline{0}, x_1\} : x \in X\} \text{ and } M_{n+1} = \{\{\underline{0}, x^0\} : x \in X\}.$$

If A is an automorphism of $LFGT(X, L)$, let A^{-1} denote the inverse function of A which is again an automorphism of $LFGT(X, L)$. Let us first prove some preliminary results which will be using in our main theorem.

Lemma 3.3.1. *Let X be a set with more than one point. If A is an automorphism of $LFGT(X, L)$, then $A(\{\underline{0}, \underline{1}\}) = \{\underline{0}, \underline{1}\}$.*

Proof. Let J_C be an atom of $LFGT(X, L)$ and $C \neq \underline{1}, \underline{0}$.
Claim: There exists an L-fuzzy set $D \in L^X$ such that $J_C \vee J_D$ contain 4 elements.
Case 1: $C(y) = 0$ for some $y \in X$.
Since $C \neq \underline{0}$, there exists an $x \in X$ such that $C(x) \neq 0$. Now let $D = y_1$, then

$J_C \vee J_D = \{\underline{0}, C\} \vee \{\underline{0}, y_1\} = \{\underline{0}, C, y_1, C \vee y_1\}$. Since $(C \vee y_1)(y) = 1,\ C \vee y_1 \neq C$ and since $(C \vee y_1)(x) \neq 0, C \vee y_1 \neq y_1$. Hence $J_C \vee J_D$ contain exactly 4 elements.

Case 2: $C(y) \neq 0$ for every $y \in X$.

Since $C \neq \underline{1}$, there exists an element $x \in X$ such that $C(x) \neq 1$. Now considering $D = x_1$ we can prove as above that $J_C \vee J_D$ contain 4 elements.

Now join of $\{\underline{0}, \underline{1}\}$ with any atom of $LFGT(X, L)$ contain exactly 3 elements since $\underline{1}$ is comparable with every element of L^X. Thus if $A(\{\underline{0}, \underline{1}\}) = \{\underline{0}, C\}$ and $C \neq \underline{1}, \underline{0}$, then by claim, there exists an L-fuzzy set $D \in L^X$ such that $\{\underline{0}, C\} \vee \{\underline{0}, D\}$ contain 4 elements. Let $A^{-1}(J_D) = J_H$. We have $|\{\underline{0}, \underline{1}\} \vee \{\underline{0}, H\}| = 3$. By Remark 3.2.1, $|A(\{\underline{0}, \underline{1}\}) \vee A(\{\underline{0}, H\})| = |\{\underline{0}, C\} \vee \{\underline{0}, D\}| = 3$ which is a contradiction. Hence the proof. □

Lemma 3.3.2. *Let X be a set with more than one point. Then every automorphism of $LFGT(X, L)$ maps strong L-fuzzy generalized topologies onto strong L-fuzzy generalized topologies of $LFGT(X, L)$.*

Proof. Let A be an automorphism of $LFGT(X, L)$ and μ be a strong L-fuzzy generalized topology on X. Then $\mu = \bigvee_{C \in \mu} \{\underline{0}, C\}$ and $A(\mu) = A(\bigvee_{C \in \mu} \{\underline{0}, C\}) = \bigvee_{C \in \mu} A(\{\underline{0}, C\})$. Since $\underline{1} \in \mu$ and $A(\{\underline{0}, \underline{1}\}) = \{\underline{0}, \underline{1}\}$ by Lemma 3.3.1, $\bigvee_{C \in \mu} A(\{\underline{0}, C\})$ is a strong L-fuzzy generalized topology on X. Similarly the inverse image of a strong L-fuzzy generalized topology is a strong L-fuzzy generalized topology. □

Lemma 3.3.3. *Let X be a set with more than one point and let A be an automorphism of $LFGT(X, L)$. Then A maps M_n onto M_n.*

Proof. Consider the strong L-fuzzy generalized topologies of the form $\{\underline{0}, x^{l_n}, \underline{1}\}$ and let us denote this by $I_{x^{l_n}}$ for $x \in X$. Note that join of $I_{x^{l_n}}$ with any L-fuzzy generalized topology $I_C = \{\underline{0}, C, \underline{1}\}$, where $C \neq \underline{0}, \underline{1}$, contain exactly 4 elements.

Now we claim that for $C \in L^X$, $C \neq \underline{0}, \underline{1}$, such that $I_C \notin \{I_{x^{l_n}}\}_{x\in X}$, there exists an L-fuzzy set $D \in L^X$ such that $I_C \vee I_D$ contains 5 elements. Consider $I_C \notin \{I_{x^{l_n}}\}_{x\in X}$, $C \neq \underline{0}, \underline{1}$.

Case 1: Suppose for some $x \in X$, $C(x) = 0$. Since $C \neq \underline{0}$, there exists an element $y \in X$ such that $C(y) \neq 0$. Let us define $D \in L^X$ such that $D(x) = l_1$ and $D(y) = 0$. Since $(C \vee D)(x) = l_1, C \vee D \neq C$. Since $(C \vee D)(y) \neq 0$, we have $C \vee D \neq D$. Then $I_C \vee I_D = \{\underline{0}, C, D, C \vee D, \underline{1}\}$ contains 5 elements.

Case 2: Suppose $C(x) \neq 0$ for every $x \in X$. Note that $I_C \notin \{I_{x^{l_n}}\}_{x\in X}$. Then there exist elements $x, y \in X$ such that $C(x) = l_i$ where $i < n$ and $C(y) \neq 0$. Define $D \in L^X$ such that $D(x) = l_{i+1}$ and $D(y) = 0$. Then $C \vee D \neq C$ and $C \vee D \neq D$. Thus $I_C \vee I_D$ contains exactly 5 elements.
So the claim holds.

Now if $A(I_{x^{l_n}}) = I_C$ for some $I_C \notin \{I_{x^{l_n}}\}_{x\in X}$, then by above claim, there exists an L-fuzzy set $D \in L^X$ such that $I_C \vee I_D$ contains 5 elements. Since A is bijective, there exists an L-fuzzy set E such that $A(I_E) = I_D$. Thus $I_{x^{l_n}} \vee I_E$ contains 4 elements. But $A(I_{x^{l_n}} \vee I_E) = A(I_{x^{l_n}}) \vee A(I_E) = I_C \vee I_D$ contains 5 elements, which is not possible. Thus A map $\{I_{x^{l_n}}\}_{x\in X}$ onto itself. Now $I_{x^{l_n}} = \{\underline{0}, x^{l_n}, \underline{1}\} = \{\underline{0}, x^{l_n}\} \vee \{\underline{0}, \underline{1}\}$. Let $A(I_{x^{l_n}}) = I_{y^{l_n}}$ for some $y \in X$. Then

$$\begin{aligned} A(I_{x^{l_n}}) &= A(\{\underline{0}, x^{l_n}\} \vee \{\underline{0}, \underline{1}\}) = A(\{\underline{0}, x^{l_n}\}) \vee A(\{\underline{0}, \underline{1}\}) \\ &= I_{y^{l_n}} = \{\underline{0}, y^{l_n}\} \vee \{\underline{0}, \underline{1}\}. \end{aligned}$$

Thus $A(\{\underline{0}, x^{l_n}\}) = \{\underline{0}, y^{l_n}\}$, since $A(\{\underline{0}, \underline{1}\}) = \{\underline{0}, \underline{1}\}$ by Lemma 3.3.1. Since $x \in X$ is arbitrary, A maps M_n onto itself. $\square$

Lemma 3.3.4. *Let X be a set with more than one point. Then every automorphism of $LFGT(X,L)$ maps $\bigcup_{i=1}^{n+1} K_i$ onto itself.*

Proof. Let A be an automorphism of $LFGT(X, L)$ and $C \in L^X$. Then we can write C as $C = \bigvee\{x_l : x \in X \text{ and } l \in L \text{ such that } C(x) = l\}$, which implies

$$J_C \leq \bigvee\{J_{x_l} : x \in X \text{ and } l \in L \text{ such that } C(x) = l\}. \tag{3.1}$$

Now let $J_{y_m} \in \bigcup_{i=1}^{n+1} K_i$ for $y \in X$ and $m \in L$ and $m \neq 0$. Suppose $A(J_{y_m}) = J_D$ for some $D \in L^X$, $D \neq \underline{0}$.

Then by Equation 3.1,

$$J_D \leq \bigvee\{J_{x_l} : x \in X, l \in L \text{ such that } D(x) = l\}$$

Since A^{-1} preserve order and arbitrary join,

$$A^{-1}(J_D) \leq \bigvee\{A^{-1}(J_{x_l}) : x \in X, l \in L \text{ such that } D(x) = l\}$$

ie., $J_{y_m} \leq \bigvee\{A^{-1}(J_{x_l}) : x \in X, l \in L \text{ such that } D(x) = l\}$. This is true only when $A^{-1}(J_{x_l}) = J_{y_m}$ for some $x \in X$, $l \in L$, $l \neq 0$, such that $D(x) = l$ and hence $A(J_{y_m}) = J_{x_l}$. Thus we get $D = x_l$ and $J_D \in \bigcup_{i=1}^{n+1} K_i$. Since $J_{y_m} \in \bigcup_{i=1}^{n+1} K_i$ is arbitrary we have $A(J_{y_m}) \in \bigcup_{i=1}^{n+1} K_i$ for every $y \in X$ and $m \in L, m \neq 0$. Hence A map $\bigcup_{i=1}^{n+1} K_i$ into itself.

Now let $J_{z_t} \in \bigcup_{i=1}^{n+1} K_i$ for some $z \in X$ and $t \in L, t \neq 0$ and assume $A(J_E) = J_{z_t}$ for some $E \in L^X$, $E \neq \underline{0}$. If we replace C by E in equation 3.1 and since A preserve order and arbitrary join, we have,

$$A(J_E) \leq \bigvee\{A(J_{x_l}) : x \in X, l \in L \text{ such that } E(x) = l\}$$

ie., $J_{z_t} \leq \bigvee\{A(J_{x_l}) : x \in X, l \in L$ such that $E(x) = l\}$ implying $A(J_{x_l}) = J_{z_t}$ for some $x \in X$, $l \in L$ and $l \neq 0$ and $E(x) = l$. Thus we get $E = x_l$ and $J_E \in \bigcup_{i=1}^{n+1} K_i$. Thus given $J_{z_t} \in \bigcup_{i=1}^{n+1} K_i$ there exists $J_{x_l} \in \bigcup_{i=1}^{n+1} K_i$ such that $A(J_{x_l}) = J_{z_t}$, hence A is onto.

Thus A map $\bigcup_{i=1}^{n+1} K_i$ onto $\bigcup_{i=1}^{n+1} K_i$.

□

Lemma 3.3.5. *Let X be a set with more than one point and let A be an automorphism of the lattice $LFGT(X, L)$. If $C \in L^X$ and $A(J_C) = J_D$ for some $D \in L^X$, then for $x \in X$, $C(x) = 1$ if and only if there exists an element $y \in X$ such that $D(y) = 1$.*

Proof. Let $C(x) = 1$ for some $x \in X$, then $J_C \vee J_{x^{l_n}}$ is a strong L-fuzzy generalized topology. By Lemma 3.3.2, $A(J_C \vee J_{x^{l_n}}) = A(J_C) \vee A(J_{x^{l_n}})$ is a strong L-fuzzy generalized topology. Since A map M_n onto itself by Lemma 3.3.3, $A(J_{x^{l_n}}) = J_{y^{l_n}}$ for some $y \in X$ and let $A(J_C) = J_D$ for some $D \in L^X$. Then $J_D \vee J_{y^{l_n}}$ is a strong L-fuzzy generalized topology implying $D(y) = 1$. Similarly we can prove that if $A(J_C)(y) = 1$ for some $y \in X$, then $C(x) = 1$ for some $x \in X$. □

Lemma 3.3.6. *Let X be a set with more than one point. Then every automorphism of $LFGT(X, L)$ maps K_1 onto itself.*

Proof. Let A be an automorphism of $LFGT(X, L)$. By Lemma 3.3.4, A maps $\bigcup_{i=1}^{n+1} K_i$ onto itself. Let $x \in X$ and let $A(J_{x_{l_1}}) = J_{z_{l_i}}, i \geq 2$ and $z \in X$. Let $\mathfrak{C} = \{C \in L^X : C(t) \neq 0\ for\ every\ t \in X\}$ and $\mathfrak{D} = \{D \in L^X : A(J_C) = J_D, C \in \mathfrak{C}\}$. Note that $J_C \vee J_{x_{l_1}}$ contain 3 elements for every $C \in \mathfrak{C}$, then

$A(J_C \vee J_{x_{l_1}}) = A(J_C) \vee A(J_{x_{l_1}}) = J_D \vee J_{z_{l_i}}$ contains 3 elements for every $D \in \mathfrak{D}$. Hence for every $D \in \mathfrak{D}$, $\{\underline{0}, D\} \vee \{\underline{0}, z_{l_i}\} = \{\underline{0}, D, z_{l_i}\}$ and $D \vee z_{l_i} = D$ or z_{l_i}. If for some $D \vee z_{l_i} = z_{l_i}$, then $D = z_{l_j}$ for some $j < i$, then there exists an element $H \in \mathfrak{C}$ such that $A(J_H) = J_{z_{l_j}}$ which is not possible by Lemma 3.3.4. Thus $D \vee z_{l_i} = D$ for every $D \in \mathfrak{D}$ which implies that $D(z) \geq l_i$ for every $D \in \mathfrak{D}$ and $i \geq 2$. Then $z^{l_1} \notin \mathfrak{D}$ and hence $A^{-1}(J_{z^{l_1}}) \notin \{J_C\}_{C \in \mathfrak{C}}$. Let $A^{-1}(J_{z^{l_1}}) = J_H$ for some $H \in L^X$. Since $H \notin \mathfrak{C}$, there exists $t \in X$ such that $H(t) = 0$.
Define $F \in L^X$ such that

$$F(x) = \begin{cases} l_k & \text{whenever } H(x) = 0,\ k \in \{1, 2, \ldots n\} \\ H(x) & \text{otherwise.} \end{cases}$$

Then $F \in \mathfrak{C}$ and consequently $A(J_F) = J_E$ for some $E \in \mathfrak{D}$. In fact we can choose $k \in \{1, 2, \ldots, n\}$ such that $E \notin \{z^{l_2}, z^{l_3}, \ldots, z^{l_n}\}$. This is possible since A is a bijection and k has n choices and the set $\{z^{l_2}, z^{l_3}, \ldots, z^{l_n}\}$ has $n - 1$ elements. Now $|J_H \vee J_F| = 3$, since $H \leq f$, which implies $|A(J_H) \vee A(J_F)| = |J_{z^{l_1}} \vee J_E| = 3$. But $J_{z^{l_1}} \vee J_E = \{\underline{0}, z^{l_1}\} \vee \{\underline{0}, E\} = \{\underline{0}, z^{l_1}, E, z^{E(z)}\}$. Since $E(z) \geq l_i$ and $i \geq 2$ implying $z^{E(z)} \in \{z^{l_2}, z^{l_3}, \ldots, z^{l_n}\}$. But we have chosen F such that $E \notin \{z^{l_2}, z^{l_3}, \ldots, z^{l_n}\}$, thus $|J_{z^{l_1}} \vee J_E| = 4$, which is a contradiction. Hence $A(J_{x_{l_1}})$ cannot be $J_{z_{l_i}}$ for any $i \geq 2$ and by Lemma 3.3.4, A maps K_1 onto itself. □

Definition 3.3.1. *[6] Let X be a nonempty set and L be any F-Lattice. If $p : X \to X$ is a bijection, then $H_p : L^X \to L^X$ defined by $H_p(C)(x) = C(p^{-1}(x))$ for all $C \in L^X$ and $x \in X$ is an automorphism of L^X.*

Theorem 3.3.1. *Let X be a nonempty set and L be any F-Lattice. If μ is an L-fuzzy generalized topology on X, then the collection $H_p^*(\mu) = \{H_p(C) : C \in \mu\}$ is also an L-fuzzy generalized topology and H_p^* is an automorphism of $LFGT(X, L)$*

where H_p is as in the Definition 3.3.1.

Proof. Let μ be an L-fuzzy generalized topology on X. Then $\underline{0} \in H_p^*(\mu)$ because $H_p(\underline{0})(x) = \underline{0}(p^{-1}(x)) = \underline{0}$ for every $x \in X$. Now $\{C_i\}_{i \in I}$ be a collection of L-fuzzy sets in $H_p^*(\mu)$. Then for $i \in I$,

$$\begin{aligned} C_i &= H_p(K_i) \qquad\qquad \text{for some } K_i \in \mu \\ (\bigvee_{i \in I} C_i)(x) &= (\bigvee H_p(K_i))(x) \\ &= (\bigvee K_i)(p^{-1})(x) \\ &= H_p(\bigvee K_i)(x) \end{aligned}$$

Thus $H_p^*(\mu)$ is an L-fuzzy generalized topology on X and H_p^* map L-fuzzy generalized topologies into L-fuzzy generalized topologies. Also note that H_p^* is bijective. If $\mu, \tau \in LFGT(X)$ and $\mu \leq \tau$ if and only if $H_p^*(\mu) \leq H_p^*(\tau)$ by definition itself. Thus H_p^* is an automorphism of $LFGT(X)$. $\square$

Finally we are in a position to prove our main results. First we consider here the case when X is a singleton set.

Theorem 3.3.2. *Let X be a singleton set. Then the group of all automorphisms of the lattice $LFGT(X, L)$ is isomorphic to $S(L \setminus \{0\})$, the group of all permutations on $L \setminus \{0\}$.*

Proof. Let $X = \{x\}$ and L be as defined in the notation. Then the atoms of $LFGT(X, L)$ are $\{K_i\}_{i=1,2,\ldots,n,n+1}$ where $K_i = \{\underline{0}, x_{l_i}\}$ for $i = 1, 2, \ldots, n, n+1$ where $l_{n+1} = 1$. In fact these are the only elements of $LFGT(X, L)$ other than $\underline{0}$ since $X = \{x\}$. Let p be a permutation on $\{1, 2, \ldots, n+1\}$. Define a function

A_p on L^X, $A_p : L^X \to L^X$, for $i = 1, 2, \ldots, n, n + 1$

$$A_p(x_{l_i}) = x_{l_j} \text{ if and only if}, p(i) = j$$

and $A_p(\underline{0}) = \underline{0}$. For an L-fuzzy generalized topology $\mu \in LFGT(X, L)$, we define $A_p^*(\mu) = \{A_p(x_{l_i}) : x_{l_i} \in \mu\} \cup \{\underline{0}\}$. Then A_p^* is a bijection on $LFGT(X, L)$. Now for $\mu, \tau \in LFGT(X, L)$,

$$\mu \leq \tau \Leftrightarrow \mu \subseteq \tau \Leftrightarrow A_p^*(\mu) \subseteq A_p^*(\tau).$$

Hence A_p^* is an automorphism on $LFGT(X, L)$. Conversely if M is an automorphism on $LFGT(X, L)$, M must map atoms onto atoms of $LFGT(X, L)$. Then it will induce a bijection on $\{x_{l_i} : i = 1, 2, \ldots, n + 1\}$ and hence on $\{1, 2, \ldots, n + 1\}$. Thus it defines a bijection between the group of all automorphisms of $LFGT(X, L)$ and the group of all permutations on $\{1, 2, \ldots, n + 1\}$. Also if p and k are two permutations on $\{1, 2, \ldots, n + 1\}$, then $A_{p \circ k}^* = A_p^* \circ A_k^*$. This defines an isomorphism between the group of all automorphisms of $LFGT(X, L)$ and the group of all permutations on $L \setminus \{0\}$. $\square$

Theorem 3.3.3. *Let X be a set with more than one point. Then the group of all automorphisms of $LFGT(X, L)$ is precisely the collection $\{H_p^* : p$ is a bijection on $X\}$ where H_p^* is as in the Theorem 3.3.1.*

Proof. We have already proved in Theorem 3.3.1 that H_p^* is an automorphism on $LFGT(X, L)$.

Now let A be an automorphism on $LFGT(X, L)$. We need to prove that $A = H_p^*$ for some bijection p on X. By Lemma 3.3.6, A maps K_1 onto itself. Let $x \in X$, consider $J_{x_{l_1}}$ and let $A(J_{x_{l_1}}) = J_{y_{l_1}}$ for some $y \in X$. This y is unique.

Define $p : X \to X$ as $p(x) = y$ if and only if $A(J_{x_{l_1}}) = J_{y_{l_1}}$. For $t \in X$,

$$\begin{aligned} H_p(x_{l_1})(t) &= x_{l_1}(p^{-1}(t)) \\ &= \begin{cases} l_1 & if\ p^{-1}(t) = x, \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} l_1 & if\ t = y, \\ 0 & \text{otherwise.} \end{cases} \\ &= y_{l_1}(t). \end{aligned}$$

Also $H_p(\underline{0}) = \underline{0}$. Thus $H_p^*(J_{x_{l_1}}) = \{H_p(\underline{0}), H_p(x_{l_1})\} = \{\underline{0}, y_{l_1}\} = J_{y_{l_1}}$. Since $x \in X$ is arbitrary, $A = H_p^*$ on K_1.

Claim: If $A(J_{x_{l_1}}) = J_{y_{l_1}}$, then $A(J_{x_1}) = J_{y_1}$.

Suppose $A(J_{x_1}) = J_C = \{\underline{0}, C\}$, for some L-fuzzy set $C \in L^X$, $\underline{1} \neq C \neq \underline{0}$. Then,

$$|\{\underline{0}, C\} \vee \{\underline{0}, z_{l_1}\}| = 3, \text{ for every } z \in X \text{ such that } C(z) \neq 0.$$

Thus $|J_C \vee J_{z_{l_1}}| = 3$, for every $z \in X$ such that $C(z) \neq 0$.
So $|A^{-1}(J_C) \vee A^{-1}(J_{z_{l_1}})| = 3$, for every $z \in X$ such that $C(z) \neq 0$.
Hence $|J_{x_1} \vee A^{-1}(J_{z_{l_1}})| = 3$, for every $z \in X$ such that $C(z) \neq 0$.

But $A^{-1}(J_{z_{l_1}}) \in K_1$ and also $A^{-1}(J_{z_{l_1}})$ and J_{x_1} must be comparable for every $z \in X$ such that $C(z) \neq 0$. Then $A^{-1}(J_{z_{l_1}}) = J_{x_{l_1}}$ and thus $z = y$. But by Lemma 3.3.5, there exists an element $t \in X$ such that $C(t) = 1$. Hence $C = y_1$.

Claim: If $A(J_{x_{l_1}}) = J_{y_{l_1}}$ and $A(J_{x_1}) = J_{y_1}$, then

1. $A(J_{x_{l_i}}) = J_{y_{l_i}}$, where $i \in \{2, 3, \ldots, n\}$,
2. $A(J_{x^0}) = J_{y^0}$.

Proof of Claim (1): By Lemma 3.3.4, A maps $\bigcup_{i=1}^{n+1} K_i$ onto itself. Let $i \in \{2,3,\ldots,n\}$ and Suppose $A(J_{x_{l_i}}) = J_{z_{l_j}}$ for some $z \in X$ and $j \in \{2,3,\ldots,n\}$. We know that $|J_{x_{l_i}} \vee J_{x_{l_1}}| = 3$. Then $|A(J_{x_{l_i}} \vee J_{x_{l_1}})| = |A(J_{x_{l_i}}) \vee A(J_{x_{l_1}})| = |J_{z_{l_j}} \vee J_{y_{l_1}}| = 3$. This happens only if $z = y$. Thus $A(J_{x_{l_i}}) = J_{y_{l_j}}$ for some $j \in \{2,3,\ldots n\}$.

Now let $A(J_{x^{l_i}}) = \{\underline{0}, H\}$ for some $H \in L^X$, $H \neq \underline{0}$. Then we have $J_{x^{l_i}} \vee J_{x_1} = \{\underline{0}, x^{l_i}, x_1, \underline{1}\}$ is a strong fuzzy generalized topology. Thus $A(J_{x^{l_i}} \vee J_{x_1}) = A(J_{x^{l_i}}) \vee A(J_{x_1}) = J_H \vee J_{y_1}$ is a strong L-fuzzy generalized topology. So $H \vee y_1 = \underline{1}$. Hence $H(t) = 1$ for every $t \neq y$.

Let $H = y^{l_k}$ for some $k \in \{1,2,\ldots,n\}$. Now we have $A(J_{x_{l_i}}) = J_{y_{l_j}}$ and $A(J_{x^{l_i}}) = J_{y^{l_k}}$. Consider $J_{x_{l_i}} \vee J_{x^{l_i}} = \{\underline{0}, x_{l_i}, x^{l_i}\}$. Then

$$A(J_{x_{l_i}} \vee J_{x^{l_i}}) = A(J_{x_{l_i}}) \vee A(J_{x^{l_i}}) = J_{y_{l_j}} \vee J_{y^{l_k}} = \{\underline{0}, y_{l_j}, y^{l_k}\},$$

since $|J_{x_{l_i}} \vee J_{x^{l_i}}| = |A(J_{x_{l_i}} \vee J_{x^{l_i}})|$. But $\{\underline{0}, y_{l_j}, y^{l_k}\}$ is an L-fuzzy generalized topology. Thus $j \leq k$, otherwise $J_{y_{l_j}} \vee J_{y^{l_k}}$ contain 4 elements. Also we have $J_{x_{l_i}} \vee J_{x^{l_{i+1}}}$ contain 3 elements. So

$$A(J_{x_{l_i}} \vee J_{x^{l_{i+1}}}) = A(J_{x_{l_i}}) \vee A(J_{x^{l_{i+1}}}) = J_{y_{l_j}} \vee J_{y^{l_{k_1}}}$$

also contain 3 elements, where $A(J_{x^{l_{i+1}}}) = J_{y^{l_{k_1}}}$ for some $k_1 \in \{1,2,\ldots,n\}$. Hence k_1 must be greater than or equal to j. This is true for $J_{x^{l_{i+2}}}, J_{x^{l_{i+3}}}, \ldots, J_{x^{l_n}}$. Therefore for example,

we have $A(J_{x_{l_1}}) = J_{y_{l_1}}$, let $A(J_{x_{l_2}}) = J_{y_{l_j}}$, where $j \geq 2$. Then,

$$A(J_{x^{l_2}}) = J_{y^{l_{k_2}}},\ k_2 \geq j$$

$$A(J_{x^{l_3}}) = J_{y^{l_{k_3}}},\ k_3 \geq j$$

$$A(J_{x^{l_4}}) = J_{y^{l_{k_4}}},\ k_4 \geq j$$

$$\vdots$$

$$A(J_{x^{l_n}}) = J_{y^{l_{k_n}}},\ k_n \geq j.$$

Since A is a bijection j must be equal to 2. Then $A(J_{x_{l_2}}) = J_{y_{l_2}}$.

Similarly, $A(J_{x_{l_i}}) = J_{y_{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$.

Proof of Claim (2): Suppose $A(J_{x^0}) = \{\underline{0}, D\}$ for some L-fuzzy set $D \in L^X$, $D \neq \underline{0}$. We know that $J_{x^0} \vee J_{x_1} = \{\underline{0}, x^0, x_1, \underline{1}\}$ is a strong L-fuzzy generalized topology. Then by Lemma 3.3.2, $A(J_{x^0}) \vee A(J_{x_1})$ is a strong L-fuzzy generalized topology on X. Thus $\{\underline{0}, D\} \vee A(J_{y_1})$ is a strong L-fuzzy generalized topology implying $D(t) = 1$ for every $t \neq y$. If $D = y^{l_i}$ for some $i \in \{1, 2, \ldots, n\}$, then $J_D \vee J_{y_{l_i}} = J_{y^{l_i}} \vee J_{y_{l_i}}$ contain 3 elements, implying $J_{x^0} \vee J_{x_{l_i}}$ contain 3 elements, which is a contradiction. Hence D must be equal to y^0.

Claim: If $A(J_{x_{l_i}}) = J_{y_{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$, then $A(J_{x^{l_i}}) = J_{y^{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$.

Suppose $A(J_{x^{l_i}}) = \{\underline{0}, E\}$ for some L-fuzzy subset $E \in L^X$, $E \neq \underline{0}$. Then $J_{x^{l_i}} \vee J_{x_1}$ is a strong L-fuzzy generalized topology and thus $J_E \vee J_{y_1}$ is a strong L-fuzzy generalized topology, which implies that $E(t) = 1$ for every $t \neq y$. Let $E(y) = l_j$ for some $j = 1, 2, \ldots, n$. Then $E = y^{l_j}$. Also $|J_{x_{l_i}} \vee J_{x^{l_i}}| = 3$. Thus $|J_{y_{l_i}} \vee J_E| = |J_{y_{l_i}} \vee J_{y^{l_j}}| = 3$ implying $j \geq i$. So if $A(J_{x^{l_i}}) = J_{y^{l_j}}$, then $j \geq i$. But A map M_n onto itself. Thus $A(J_{x^{l_{n-1}}}) = J_{y^{l_{n-1}}}, A(J_{x^{l_{n-2}}}) = J_{y^{l_{n-2}}}$ and so on. Hence $A(J_{x^{l_i}}) = J_{y^{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$.

Now

$$\begin{aligned} H_p^*(J_{x_{l_i}}) &= \{H_p(\underline{0}), H_p(x_{l_i})\} \\ &= \{\underline{0}, y_{l_i}\} \\ &= J_{y_{l_i}} \\ &= A(J_{x_{l_i}}). \end{aligned}$$

Then $A = H_p^*$ on K_i, where $i \in \{1, 2, \ldots, n\}$, since x and y are arbitrary elements of X.

$$\begin{aligned} H_p^*(J_{x_1}) &= \{H_p(\underline{0}), H_p(x_1)\} \\ &= \{\underline{0}, y_1\} \\ &= J_{y_1} \\ &= A(J_{x_1}). \end{aligned}$$

Thus $A = H_p^*$ on K_{n+1}.

$$\begin{aligned} H_p^*(J_{x^0}) &= \{H_p(\underline{0}), H_p(x^0)\} \\ &= \{\underline{0}, y^0\} \\ &= J_{y^0} \\ &= A(J_{x^0}). \end{aligned}$$

So $A = H_p^*$ on M_{n+1}.

$$\begin{aligned} H_p^*(J_{x^{l_i}}) &= \{H_p(\underline{0}), H_p(x^{l_i})\} \\ &= \{\underline{0}, y^{l_i}\} \\ &= J_{y^{l_i}} \\ &= A(J_{x^{l_i}}). \end{aligned}$$

Hence $A = H_p^*$ on M_i, where $i \in \{1, 2, \ldots, n\}$. Therefore $A = H_p^*$ on $\bigcup_{i=1}^{n+1} (K_i \cup M_i)$.

Now let $V \notin \bigcup_{i=1}^{n+1} (K_i \cup M_i)$ be an L-fuzzy set. Suppose $A(\{\underline{0}, V\}) = \{\underline{0}, W\}$

for some L-fuzzy set $W \notin \bigcup_{i=1}^{n+1}(K_i \cup M_i)$ and $H_p^*(\{\underline{0}, V\}) = \{\underline{0}, H_p(V)\}$. To prove that $H_p(V) = W$, it is enough to prove the following results:

(a) $V(p^{-1}(y)) = 0$ if and only if $W(y) = 0$.

(b) $V(p^{-1}(y)) = l_i$ if and only if $W(y) = l_i$ where $i = 1, 2, \ldots, n$.

(c) $V(p^{-1}(y)) = 1$ if and only if $W(y) = 1$.

Proof of (a): We have $H_p(V)(t) = V(p^{-1})(t)$. Then $V(p^{-1}(y)) = 0 \Leftrightarrow V(x) = 0 \Leftrightarrow V \leq x^0 \Leftrightarrow |\{\underline{0}, V\} \vee \{\underline{0}, x^0\}| = 3 \Leftrightarrow |A(\{\underline{0}, V\}) \vee A(\{\underline{0}, x^0\})| = 3 \Leftrightarrow |\{\underline{0}, W\} \vee \{\underline{0}, y^0\}| = 3$, then $W(y)$ can not be greater than 0. Thus $W(y) = 0$.

Proof of (b): Assume $V(p^{-1}(y)) = l_i$ for some $i \in \{1, 2, \ldots, n\}$. But $p^{-1}(y) = x$, implying $V(x) = l_i$. Then $|\{\underline{0}, V\} \vee \{\underline{0}, x^{l_i}\}| = 3$. By Remark 3.2.1, $|\{\underline{0}, W\} \vee \{\underline{0}, y^{l_i}\}| = 3$ which implies $W(y) \leq l_i$, since $W \notin M_j$ for every $j \in \{1, 2, \ldots, n\}$.

Also if $V(x) = l_i$, then $|\{\underline{0}, V\} \vee \{\underline{0}, x_{l_i}\}| = 3$. By Remark 3.2.1, $|\{\underline{0}, W\} \vee \{\underline{0}, y_{l_i}\}| = 3$ implying $W(y) \geq l_i$, since $W \notin K_j$ for every $j \in \{1, 2, \ldots, n\}$. Thus we get $W(y) = l_i$. So if $V(p^{-1}(y)) = l_i$, then $W(y) = l_i$.

Similarly, it is also easy to show that, if $W(y) = l_i$, then $V(p^{-1}(y)) = l_i$ for every $i = 1, 2, \ldots, n$.

Proof of (c): Consider $V(p^{-1}(y)) = 1 \Leftrightarrow V(x) = 1 \Leftrightarrow |\{\underline{0}, V\} \vee \{\underline{0}, x_1\}| = 3 \Leftrightarrow |\{\underline{0}, W\} \vee \{\underline{0}, y_1\}| = 3 \Leftrightarrow W \geq y_1$(since $W \notin K_{n+1}$)$\Leftrightarrow W(y) = 1$.

Since x and y are arbitrary $A = H_p^*$ on all atoms in $LFGT(X, L)$. Also $LFGT(X, L)$ is an atomic lattice, hence $A = H_p^*$ on $LFGT(X, L)$. Thus the proof is complete. $\square$

3.4 Automorphism group of $LFGT(X, L)$ when L is the diamond-type lattice

Here we determine the automorphism group of lattice of fuzzy generalized topologies, $LFGT(X, L)$, when X is an arbitrary nonempty set and L is the diamond-type lattice.

First we look at the structure of diamond-type lattice $L = \{0, a, b, 1\}$ (see Figure 3.1). In L, 0 is the smallest element and 1 is the largest element, also $0 < a < 1$ and $0 < b < 1$ is the order relation in L. Here a and b are not comparable. Define order reversing involution on L as $0^{'} = 1, 1^{'} = 0, a^{'} = b$ and $b^{'} = a$. Then L is a complemented F-lattice.

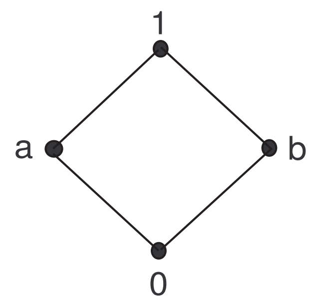

Figure 3.1: Diamond-type lattice

Throughout this section X will be an arbitrary non empty set and L will be the diamond-type lattice described above.

Definition 3.4.1. *Let X be a nonempty set and L be the diamond-type lattice*

$L = \{0, a, b, 1\}$. *Let* P *be a bijection on* $X \times \{a, b\}$ *defined as, for* $x \in X$ *and* $l \in \{a, b\}$, $P(x, l) = (P_1(x), P_2(l))$ *where* P_1 *and* P_2 *are bijections on* X *and* $\{a, b\}$ *respectively. Now let us define a bijection* P^* *on set of all* L*-fuzzy points of the lattice* L^X *by* $P^*(x_l) = y_m$ *if and only if* $P(x, l) = (y, m)$ *for every* $l, m \in \{a, b\}$ *and* $P^*(x_1) = P^*(x_a) \vee P^*(x_b)$ *where* $x, y \in X$.

By $P = (P_1, P_2)$ on $X \times \{a, b\}$, we simply mean $P(x, l) = (P_1(x), P_2(l))$ for $x \in X$ and $l \in \{a, b\}$.

Note that we are not considering all bijections on $X \times \{a, b\}$. The essence of this definition of bijection is that we have freedom in the choice of L-fuzzy points $\{x_a\}_{x \in X}$ only. For $x, y \in X$, if we map x_a onto y_a, then x_b has no other chance than y_b, in fact every z_l map onto $(h(z))_l$ where h is a bijection on X. On the other hand if x_a maps onto y_b, then every z_l maps onto $h(z)_{l'}$ where h is a bijection on X and l' is the pseudo-complement of l.

We explain the reason behind this in the following remark.

Remark 3.4.1. *Let* A *be an automorphism of* L^X. *We know that every automorphism of* L^X *map atoms onto atoms. Here atoms are* L*-fuzzy points. If we let* $A(x_a) = z_{l_1}$ *and* $A(x_b) = w_{l_2}$ *for* $x, z, w \in X$ *and* $l_1, l_2 \in \{a, b, 1\}$, *Then* $A(x_1) = A(x_a \vee x_b) = A(x_a) \vee A(x_b) = z_{l_1} \vee w_{l_2}$. *But since* $x_a < x_1$ *we have* $A(x_a) = z_{l_1} < A(x_1)$. *The value of* $A(x_1)$ *at the point* z *must be greater than* l_1, *thus getting* $(A(x_1))(z) = 1$. *Also* $z_{l_1} \vee w_{l_2}$ *must be an atom. So* z *must be equal to* w *and* $l_1 = l_2'$. *Hence* $A(x_1) = z_1$.

Before considering the main problem of this section we would like to find out the automorphism group of L^X. Before that let us prove the following Lemma.

Lemma 3.4.1. *For every bijection $P = (P_1, P_2)$ on $X \times \{a, b\}$ where $P_1 \in S(X)$ and $P_2 \in S(\{a, b\})$ and P^* on $Pt(L^X)$ defined as in Definition 3.4.1, $P^*(x_{(\bigvee_{i\in I} H_i)(x)}) = \bigvee_{i\in I} P^*(x_{H_i(x)})$ for $H_i \in L^X$ for all $i \in I$ and $x \in X$ with $H_i(x) \neq 0$ for every $i \in I$.*

Proof. Let $x \in X$ and let $\{H_i\}_{i\in I}$ be a collection of L-fuzzy sets of X with $H_i(x) \neq 0$ for all $i \in I$. If $H_i(x) = H_j(x)\ \forall i, j \in I$, then $(\bigvee_{i\in I} H_i)(x) = H_k(x)$ for some $k \in I$ and $P^*(x_{(\bigvee_{i\in I} H_i)(x)}) = P^*(x_{H_k(x)}) = \bigvee_{i\in I} P^*(x_{H_i(x)})$. If $H_i(x) \neq H_j(x)$ for some $i, j \in I$, then $(\bigvee_{i\in I} H_i)(x) = 1$ and $P^*(x_{(\bigvee_{i\in I} H_i)(x)}) = P^*(x_1) = \bigvee_{i\in I} P^*(x_{H_i(x)})$. Hence the result. $\square$

Theorem 3.4.1. *Let X be a nonempty set. Then automorphisms of L^X are $\{A_P : P = (P_1, P_2), P_1 \in S(X)$ and $P_2 \in S(\{a, b\})\}$ where $A_P(C) = \bigvee_{\substack{x\in X\\ C(x)\neq 0}} P^*(x_{C(x)})$, for $C \in L^X$.*

Proof. Let $C, D \in L^X$. If C or D is equal to $\underline{0}$, it follows at once that $A_P(C\vee D) = A_P(C) \vee A_P(D)$. So let us focus upon the case when $C \neq \underline{0}$ and $D \neq \underline{0}$.
By Lemma 3.4.1, we get

$$\begin{aligned}
A_P(C \vee D) &= \bigvee_{\substack{x\in X\\ (C\vee D)(x)\neq 0}} P^*(x_{(C\vee D)(x)}) \\
&= \bigvee_{\substack{x\in X\\ C(x)\neq 0\\ D(x)\neq 0}} [P^*(x_{C(x)}) \vee P^*(x_{D(x)})] \vee \bigvee_{\substack{x\in X\\ C(x)\neq 0\\ D(x)=0}} P^*(x_{C(x)}) \vee \bigvee_{\substack{x\in X\\ D(x)\neq 0\\ C(x)=0}} P^*(x_{D(x)}) \\
&= \bigvee_{\substack{x\in X\\ C(x)\neq 0}} P^*(x_{C(x)}) \vee \bigvee_{\substack{x\in X\\ D(x)\neq 0}} P^*(x_{D(x)}) \\
&= A_P(C) \vee A_P(D).
\end{aligned}$$

Thus $A_P(C \vee D) = A_P(C) \vee A_P(D)$.
Claim: A_P is injective.

Let $C, D \in L^X$ and assume $A_P(C) = A_P(D)$. Now if $x \in X$ and $C(x) \neq 0$, then there exists an element $y \in X$ and $D(y) \neq 0$ such that $P^*(x_{C(x)}) = P^*(y_{D(y)})$. Since P^* is bijective on $Pt(L^X)$, we have $x_{C(x)} = y_{D(y)}$, consequently $x = y$ and $C(x) = D(y)$. This is true for every $x \in X$ such that $C(x) \neq 0$. Thus for every $x \in X$, $C(x) \neq 0, C(x) = D(x)$. Similarly for every $x \in X$ such that $D(x) \neq 0$, we have $D(x) = C(x)$ and therefore $C = D$.

Claim: A_P is surjective

Let us start with an element $C \in L^X$. If $C = \underline{0}$, then $A_P(\underline{0}) = C$. Suppose $C \neq \underline{0}$. We define $D = \bigvee\limits_{\substack{x \in X \\ C(x) \neq 0}} (P^*)^{-1}(x_{C(x)})$. Then $A_P(D) = \bigvee\limits_{\substack{x \in X \\ C(x) \neq 0}} x_{C(x)} = C$. Hence A_P is surjective. Thus A_P is an automorphism of L^X.

Now let A be an automorphism of L^X. Then A map atoms of L^X onto itself and by Remark 3.4.1, for $x \in X$ A must map $\bigcup\limits_{x \in X} \{x_1\}$ onto itself. We define bijections P_1, P_2 on X and $\{a, b\}$ respectively as $P_1(x) = y$ and $P_2(l_1) = l_2$ if and only if $A(x_{l_1}) = y_{l_2}$, for $x, y \in X$ and $l_1, l_2 \in \{a, b\}$. This map is well defined by Remark 3.4.1. Now define P^* on $X \times \{a, b\}$ as $P^* = (P_1, P_2)$. Then $A = P^*$ on all atoms of L^X. Since L^X is an atomic lattice, $A = P^*$ on L^X. $\square$

Define A_P^* on $LFGT(X, L)$ as $A_P^*(\mu) = \{A_P(C) : C \in \mu\}$ where A_P as defined in Theorem 3.4.1. Then A_P^* is an automorphism of $LFGT(X, L)$ which we prove here as a theorem.

Theorem 3.4.2. *For every bijection $P = (P_1, P_2)$ on $X \times \{a, b\}$, where $P_1 \in S(X)$ and $P_2 \in S(\{a, b\})$, A_P^* is an automorphism of $LFGT(X, L)$.*

Proof. Let μ be an L-fuzzy generalized topology on X and $P = (P_1, P_2)$ be a bijection on $X \times \{a, b\}$ where $P_1 \in S(X)$ and $P_2 \in S(\{a, b\})$.

Claim 1: $A_P^*(\mu)$ is an L-fuzzy generalized topology on L^X.

By definition, $A_P^*(\mu) = \{A_P(C) : C \in \mu\}$. Since $\underline{0} \in \mu$ and $A_P(\underline{0}) = \underline{0}$, we have $\underline{0} \in A_p^*(\mu)$. Now let $\{C_i'\}_{i\in I} \subseteq A_P^*(\mu)$, then there exists a collection $\{C_i\}_{i\in I} \subseteq \mu$ such that $A_P(C_i) = C_i'$ for every $i \in I$. Consider $\bigvee\limits_{i\in I} C_i' = \bigvee\limits_{i\in I} A_P(C_i) = A_P(\bigvee\limits_{i\in I} C_i) \in \mu$, by Lemma 3.4.1. Thus A_P^* is closed under arbitrary join. Hence the claim.

Claim 2: A_P^* is a bijection on $LFGT(X, L)$.

Let $\mu_1, \mu_2 \in LFGT(X, L)$ and assume $A_P^*(\mu_1) = A_P^*(\mu_2)$. In other words $\{A_P(C) : C \in \mu_1\} = \{A_P(D) : D \in \mu_2\}$. But since A_P is a bijection, we have $\mu_1 = \mu_2$. Now to see the map A_P^* is onto, let $\tau' \in LFGT(X, L)$. Consider $\tau = \{A_P^{-1}(C) : C \in \tau'\}$. Then $A_P^*(\tau) = \{A_P(A_P^{-1}(C)) : C \in \tau'\} = \{C : C \in \mu'\}$, since A_P is a bijection. Thus $A_P^*(\tau) = \tau'$. Hence A_P^* is surjective and concluding that A_P^* is a bijection on $LFGT(X, L)$.

Now for $\mu_1, \mu_2 \in LFGT(X, L)$, if $\mu_1 \subseteq \mu_2$, then by definition of A_P^*, $A_P^*(\mu_1) \subseteq A_P^*(\mu_2)$. Thus A_P^* is order preserving. Hence the theorem is proved. □

Definition 3.4.2. *Let X be a nonempty set. Then the atoms of $LFGT(X, L)$ are sets of the form $\{\underline{0}, C\}$ where $C \in L^X$, $C \neq \underline{0}$. Let us designate this by J_C and also we use the following notations for special types of atoms.*

$$K = \{J_{x_l} : x \in X, l \in \{a, b\}\},\ K' = \{J_{x_1} : x \in X\}$$

$$M = \{J_{x^l} : x \in X, l \in \{a, b\}\} \text{ and } M' = \{J_{x^0} : x \in X\}.$$

where, for $l \in \{a, b, 1\}$,

$$x_l(t) = \begin{cases} l & \text{when } t = x \\ 0 & \text{otherwise} \end{cases}$$

and for $l \in \{0, a, b\}$,

$$x^l(t) = \begin{cases} l & \text{when } t = x \\ 1 & \text{otherwise.} \end{cases}$$

First we would like to see where these special types of atoms go under an arbitrary automorphism of LFGT(X,L). Once we know this, then since the lattice is atomic, result is easy to conclude.

Note 3.4.1. *Let X be a set with more than one point. Every automorphism A of $LFGT(X, L)$ map $\{\underline{0}, \underline{1}\}$ onto itself because join of any L-fuzzy generalized generalized topology of the type $\{\underline{0}, C\} \in LFGT(X, L)$ and $C \neq \underline{0}, \underline{1}$ with $\{\underline{0}, \underline{1}\}$ contain exactly 3 elements, otherwise we can find out an L-fuzzy set $D \in L^X$ and $D \neq \underline{0}, \underline{1}$ such that $\{\underline{0}, C\} \vee \{\underline{0}, D\}$ contain 4 elements. This characterizes $\{\underline{0}, \underline{1}\}$ and therefore A must map $\{\underline{0}, \underline{1}\}$ onto itself and as a result A map strong L-fuzzy generalized topologies onto strong L-fuzzy generalized topologies only.*

Lemma 3.4.2. *Let X be a set with more than one point and let A be an automorphism of $LFGT(X, L)$. Then A maps $M \cup M'$ onto itself.*

Proof. Let $I_{x^l} = \{\underline{0}, x^l, \underline{1}\}$ for $x \in X$ and $l \in \{0, a, b\}$ and consider the collection, $\eta = \{I_{x^l} : x \in X, l \in \{0, a, b\}\}$. Note that the join of any element in η with any L-fuzzy generalized topology of the form $I_C = \{\underline{0}, C, \underline{1}\}$, where $C \in L^X$ and $C \neq \underline{0}, \underline{1}$, contain exactly 4 elements. Also recall that every automorphism, A of $LFGT(X, L)$, maps an n element set onto an n element set and by Note 3.4.1, A map strong L-fuzzy generalized topologies onto strong L-fuzzy generalized topologies. So A must map $I_{x^l} \in \eta$ onto I_C for some $C \in L^X$,$C \neq \underline{0}$ and $C \neq \underline{1}$. If I_C doesn't belong to η, then we have a claim.

Claim: There exists an L-fuzzy set $D \in L^X$ such that $I_C \vee I_D$ contain 5 elements. If I_C doesn't belong to η, then either of the following cases arises

(1) there exist $x, y \in X$, such that $C(x) = 0$ and $C(y) \neq 0$

(2) there exist $x, y \in X$ such that $C(x) = m$ and $C(y) = k$ where $m, k \in \{a, b\}$.

If C is as in case(1), take $D = x_a$. Then, since $(C \vee D)(x) = a, C \vee D \neq C$ and since $(C \vee D)(y) \neq 0$, we have $C \vee D \neq D$. Thus $I_C \vee I_D = \{\underline{0}, C, D, C \vee D, \underline{1}\}$ contain 5 elements.

If C is as in case(2), then take $D = x_{m'}$, where m' is the pseudo-complement of m. Then $C \vee D \notin \{C, D, \underline{1}\}$ and hence $I_C \vee I_D$ contain exactly 5 elements.

This characterizes elements in η and A must map η onto itself. Since $A(\{\underline{0}, \underline{1}\}) = \{\underline{0}, \underline{1}\}$, A maps $M \cup M'$ onto itself. $\square$

Lemma 3.4.3. *Let X be a non empty set and let A be an automorphism of $LFGT(X, L)$, then A maps K onto itself.*

Proof. Let A be an automorphism of $LFGT(X, L)$ and $C \in L^X$. The proof is similar to the proof of Theorem 3.3.4. C can be written as $C = \bigvee\{x_l : x \in X \text{ and } C(x) = l \in \{a, b\}\} \bigvee \{x_a \vee x_b : x \in X, C(x) = 1\}$. Then as a consequence, we have $J_C \leq \bigvee\{J_{x_l} : x \in X \text{ and } C(x) = l \in \{a, b\}\} \bigvee \{J_{x_a} \vee J_{x_b} : x \in X, C(x) = 1\}$. Since A preserves order and arbitrary join, $A(J_C) \leq \bigvee\{A(J_{x_l}) : x \in X \text{ and } C(x) = l \in \{a, b\}\} \bigvee \{A(J_{x_a}) \vee A(J_{x_b}) : x \in X, C(x) = 1\}$. The last inequality implies that for $l \in \{a, b\}$, J_{x_l} is less than or equal to join of a collection of L-fuzzy generalized topologies if and only if J_{x_l} is already a member of that collection of L-fuzzy generalized topologies. This characterizes atoms of the form J_{x_l} for all $x \in X$ and $l \in \{a, b\}$. Thus A maps K onto itself. $\square$

Remark 3.4.2. *Let A be an automorphism of $LFGT(X, L)$. By Lemma 3.4.2, A maps $M \cup M'$ onto itself. Now let $x \in X$ and $J_{x^a} \in M$. Suppose $A(J_{x^a}) = J_{y^0}$ for some $y \in X$. Let $A(J_{x^b}) = J_{z^m}$ for some $z \in X$ and $m \in \{0, a, b\}$. Consider $J_{x^a} \vee J_{x^b}$, it is a strong L-fuzzy generalized topology, then $A(J_{x^a} \vee J_{x^b}) = A(J_{x^a}) \vee$*

$A(J_{x^b}) = J_{y^0} \vee J_{z^m}$ must be a strong L-fuzzy generalized topology. But this is true only if $y^0 \vee z^m = \underline{1}$, which is not possible. Therefore, $A(J_{x^a})$ can not be J_{y^0} for any $y \in X$. Similarly $A(J_{x^b})$ can not be J_{w^0} for any $w \in X$. Thus A map M onto itself and $M^{'}$ onto itself.

Theorem 3.4.3. *Let X be a non empty set. Then every automorphism maps $K^{'}$ onto itself.*

Proof. Let A be an automorphism of $LFGT(X, L)$. By Lemma 3.4.3, A map K onto itself. Let $x \in X$ and let $A(J_{x_a}) = J_{y_m}$ for some $y \in X$ and $m \in \{a, b\}$ and $A(J_{x_b}) = J_{z_t}$ for some $z \in X$ and $t \in \{a, b\}$. Let $A(J_{x_1}) = J_D$ for some $D \in L^X$, $D \neq \underline{0}$. We know that $\{\underline{0}, x_1\} \leq \{\underline{0}, x_a\} \vee \{\underline{0}, x_b\}$ implying $\{\underline{0}, D\} \leq \{\underline{0}, y_m\} \vee \{\underline{0}, z_t\}$. Since $D \notin M \cup M^{'} \cup K$, D must be equal to $y_m \vee z_t$ which implies that at atmost two points $y, z \in X$, D take non zero value.

Also note that $\{\underline{0}, x_1\} \vee \{\underline{0}, x^0\}$ is a strong L-fuzzy generalized topology. Let $A(\{\underline{0}, x^0\}) = \{\underline{0}, w^0\}$ for some $w \in X$. Hence $A(\{\underline{0}, x_1\}) \vee A(\{\underline{0}, x^0\}) = \{\underline{0}, D\} \vee \{\underline{0}, w^0\}$ is a strong L-fuzzy generalized topology implying that $D(w) = 1$. Hence D takes the value 1 at one point.

But we have $D = y_m \vee z_t$. If $y \neq z$, then D can not take the value 1 at any point of X because $m, t \in \{a, b\}$. Hence y must be equal to z. Now if $m = t$, then $D = y_m \vee z_t = y_m$, which is not possible. So $m \neq t$ and note that $m, t \in \{a, b\}$. Therefore $m = t^{'}$, where $t^{'}$ is the pseudo-complement of t. Hence $D = y_m \vee z_t = y_{t^{'}} \vee y_t = y_1$. Thus $A(\{\underline{0}, x_1\}) \in K^{'}$ for all $x \in X$.

We know that A^{-1} is also an automorphism on $LFGT(X, L)$, proceeding as above, we get $A^{-1}(\{\underline{0}, x_1\}) \in K^{'}$ for all $x \in X$. Hence A map $K^{'}$ onto itself. $\square$

Having proved all these preliminary results, now we prove the main result of this section.

Theorem 3.4.4. *Let X be a nonempty set and L be the diamond-type lattice. Then the automorphisms of $LFGT(X, L)$ are precisely $\{A_P^* : P = (P_1, P_2),$ where $P_1 \in S(X)$ and $P_2 \in S(\{a, b\})\}$.*

Proof. For any bijection $P = (P_1, P_2),$ *where* $P_1 \in S(X)$ *and* $P_2 \in S(\{a, b\})$, we have already proved in Theorem 3.4.2 that A_P^* is an automorphism of $LFGT(X, L)$. Now let A be any automorphism of $LFGT(X, L)$ and we need to prove that $A = A_P^*$ for some $P = (P_1, P_2),$ *where* $P_1 \in S(X)$ *and* $P_2 \in S(\{a, b\})$.

If X is a singleton set, say $\{x\}$, then the elements of L^X are $\underline{0}, x_a, x_b$, and x_1 and atoms of $LFGT(X, L)$ are J_{x_a}, J_{x_b} *and* J_{x_1}. By Lemma 3.4.3, A must map $\{J_{x_a}, J_{x_b}\}$ onto itself. If A maps J_{x_a} onto J_{x_b}, then A must map J_{x_b} onto J_{x_a} and J_{x_1} onto itself. Then $A = A_P^*$ for $P = (P_1, P_2)$ where P_1 is the identity function on X and P_2 on $\{a, b\}$ is defined as $P_2(a) = b$ and $P_2(b) = a$. Now if A is identity on $LFGT(X, L)$, then $A = A_P^*$, where $P = (P_1, P_2)$, and P_1, and P_2 are identity functions on X and $\{a, b\}$ respectively.

Now suppose X contain more than one point. Let $x \in X$ and $A(J_{x_a}) = J_{y_m}$ for some $y \in X$ and $m \in \{a, b\}$. Suppose $A(J_{x_b}) = J_{w_p}$ for some $w \in X$ and $p \in \{a, b\}$, $A(J_{x^a}) = J_D \in M$, where $\underline{0} \neq D \in L^X$ and $A(J_{x_1}) = J_{z_1}$ for some $z \in X$. Consider $J_{x^a} \vee J_{x_1} = \{\underline{0}, x^a, x_1, \underline{1}\}$. Then $A(\{\underline{0}, x^a, x_1, \underline{1}\}) = A(\{\underline{0}, x^a\}) \vee A(\{\underline{0}, x_1\}) = \{\underline{0}, D\} \vee \{\underline{0}, z_1\} = \{\underline{0}, D, z_1, D \vee z_1\}$. Since A map strong L-fuzzy generalized topologies onto strong L-fuzzy generalized topologies, $D \vee z_1$ must be $\underline{1}$ and by Remark 3.4.2, D must be z^l for some $l \in \{a, b\}$. Thus $A(J_{x^a}) = J_{z^l}$. Now consider $J_{x_a} \vee J_{x_b} \vee J_{x_1} = \{\underline{0}, x_a, x_b, x_1\}$ and hence $A(\{\underline{0}, x_a, x_b, x_1\}) = A(J_{x_a}) \vee A(J_{x_b}) \vee A(J_{x_1}) = J_{y_m} \vee J_{w_p} \vee J_{z_1}$. Since A map an n element set onto an n element set, the last term must be equal to $\{\underline{0}, y_m, w_p, z_1\}$.

So y_m, w_p and z_1 must be comparable and $y_m \vee w_p = z_1$, hence $y = w = z$ and $p = m^{'}$.
Thus If $A(J_{x_a}) = J_{y_m}$, then

1. $A(J_{x_b}) = J_{y_{m'}}$
2. $A(J_{x_1}) = J_{y_1}$ and
3. $A(J_{x^a}) = J_{y^l}$ for some $l \in \{a, b\}$.

Arguing similarly we can prove that $A(J_{x^b}) = J_{y^p}$ for some $p \in \{a, b\}$. Consider $J_{x^a} \vee J_{x^b} = \{\underline{0}, x^a, x^b, \underline{1}\}$ and $A(\{\underline{0}, x^a, x^b, \underline{1}\}) = A(J_{x^a}) \vee A(J_{x^b}) = J_{y^l} \vee J_{y^p} = \{\underline{0}, y^l, y^p, y^l \vee y^p\}$. Since A map strong L-fuzzy generalized topologies onto itself, $y^l \vee y^p$ must be $\underline{1}$ resulting $p = l^{'}$. But $A(J_{x_a}) = J_{y_m}$, so $J_{x_a} \vee J_{x^a} = \{\underline{0}, x_a, x^a\}$, and $A(\{\underline{0}, x_a, x^a\}) = A(J_{x_a}) \vee A(J_{x^a}) = J_{y_m} \vee J_{y^l}$. Since A is an automorphism and A map an n element set onto an n element set y_m and y^l must be comparable. But $l, m \in \{a, b\}$, since a and b are not comparable, l must be equal to m. Thus $A(J_{x^a}) = J_{y^m}$ and $p = l^{'} = m^{'}$ resulting $A(J_{x^b}) = J_{y^{m'}}$. Thus if $A(J_{x_a}) = J_{y_m}$, $m \in \{a, b\}$, then we have $A(J_{x_b}) = J_{y_{m'}}$, $A(J_{x^a}) = J_{y^m}$, $A(J_{x^b}) = J_{y^{m'}}$ and $A(J_{x_1}) = J_{y_1}$.

Let us define $P = (P_1, P_2)$ on $X \times \{a, b\}$ by $P_1(x) = y$ and $P_2(l) = m$ if and only if $A(J_{x_l}) = J_{y_m}$ where $x, y \in X$ and $l, m \in \{a, b\}$. The function P is a bijection on $X \times \{a, b\}$ since A map K onto itself. Also note that $P(x, l) = (y, m)$ for $x, y \in X$ and $l, m \in \{a, b\}$ if and only if $P^*(x_l) = y_m$. Now we need to prove that $A = A^*_P$. First our aim is to show that $A = A^*_P$ on all atoms of $LFGT(X, L)$

Let $A(J_{x_l}) = J_{y_m}$ for $x, y \in X$ and $l, m \in \{a, b\}$.

$$\begin{aligned} A_P^*(\{\underline{0}, x_l\}) &= \{A_P(\underline{0}), A_P(x_l)\} \\ &= \{\underline{0}, \bigvee_{\substack{y \in X \\ x_l(y) \neq 0}} P^*(y_{x_l(y)})\} \\ &= \{\underline{0}, P^*(x_l)\} \\ &= \{\underline{0}, y_m\} \\ &= A(\{\underline{0}, x_l\}) \end{aligned}$$

Thus $A = A_P^*$ on K.

Since $A(J_{x_l}) = J_{y_m}$, we have $A(J_{x^l}) = J_{y^m}$ and $A(J_{x_{l'}}) = J_{y_{m'}}$.

Now x^l can be written as $x^l = \bigvee_{\substack{w \in X \\ w \neq x}} (w_a \vee w_b) \vee x_l$, then

$$A_P(x^l) = \bigvee_{\substack{w \in X \\ w \neq x}} (P^*(w_a) \vee P^*(w_b)) \vee P^*(x_l).$$

Since $P^*(x_l) = y_m$, we have $y_m \leq A_P(x^l)$. Also $P^*(w_t) \leq A_P(x^l)$ for every $t \in \{a, b\}$ implying $z_n \leq A_P(x^l)$ for every $z \neq y$ and $n \in \{a, b\}$. Since $P^*(x_{l'}) = y_{m'}$ and $P^*(x_{l'}) \nleq A_P(x^l)$, we have $y_{m'} \nleq A_P(x^l)$. Therefore $A_P(x^l)$ must be y^m.

$$\begin{aligned} A(J_{x^l}) &= A(\{\underline{0}, x^l\}) = \{\underline{0}, y^m\} \\ &= \{A_P(\underline{0}), A_P(x^l)\} \\ &= A_P^*(J_{x^l}). \end{aligned}$$

Hence $A = A_P^*$ on M.

Now let $C \in L^X$, $C \notin K \cup M$ and let $A(J_C) = J_D$, then C can be written as

$$\begin{aligned} C &= \bigvee_{\substack{x\in X \\ C(x)=1}} (x_a \vee x_b) \vee \bigvee_{\substack{x\in X \\ C(x)\neq 0 \\ C(x)\neq 1}} x_{C(x)}. \\ A_P(C) &= \bigvee_{\substack{x\in X \\ C(x)=1}} (P^*(x_a) \vee P^*(x_b)) \vee \bigvee_{\substack{x\in X \\ C(x)\neq 0 \\ C(x)\neq 1}} P^*(x_{C(x)}). \\ &= \bigvee_{\substack{x\in X \\ C(x)\neq 0}} P^*(x_{C(x)}) = \bigvee_{\substack{x\in X \\ D(x)\neq 0}} x_{D(x)}, \ \textit{since } A(J_C) = J_D. \end{aligned}$$

Now to show that $A_P^*(J_C) = A(J_C)$,

$$\begin{aligned} A_P^*(J_C) &= A_P^*(\{\underline{0}, C\}) \\ &= \{A_P(\underline{0}), A_P(C)\} \\ &= \{\underline{0}, \bigvee_{\substack{x\in X \\ C(x)\neq 0}} P^*(x_{C(x)})\} \\ &= \{\underline{0}, \bigvee_{\substack{x\in X \\ D(x)\neq 0}} (x_{D(x)})\} \end{aligned}$$

Thus,

$$\begin{aligned} A_P^*(J_C) &= \{\underline{0}, \bigvee_{\substack{x\in X \\ D(x)\neq 0}} (x_{D(x)})\} \\ &= \{\underline{0}, D\} = J_D \\ &= A(J_C). \end{aligned}$$

Hence $A = A_P^*$ on $\{J_C\}_{C\in L^X}$, where C does not belong to $K \cup M$ and we proved that $A = A_P^*$ on all atoms of $LFGT(X, L)$. Since $LFGT(X, L)$ is an atomic lattice, it is clear that $A = A_P^*$ on $LFGT(X, L)$ and our proof is complete. $\square$

Chapter 4

Homogeneous generalized topological spaces

4.1 Introduction

Homogeneity in topological spaces is studied by many mathematicians. John Ginsburg in his paper [19] proved a simple representation theorem for finite topological spaces which are homogeneous. In the first section we characterize completely homogeneous generalized topological spaces. In the following sections we deal with homogeneous generalized topological spaces in a cyclic ordered set. We try to find out new homogeneous generalized topological spaces by considering the join of homogeneous generalized topologies and discuss the properties.

Let X be a nonempty set and μ be a generalized topology on X. We denote the union of all open sets in (X, μ) by M_μ. Let us recall the definition of homogeneous generalized topological space.

Definition 4.1.1. *[18] A generalized topological space (X, μ) is said to be homogeneous if for any two points x, $y \in M_\mu$ there exists a (μ, μ)-homeomorphism $f : (X, \mu) \to (X, \mu)$ such that $f(x) = y$ and (X, μ) is called completely homogeneous if every bijection on X is a homeomorphism on (X, μ).*

4.2 Completely homogeneous generalized topological spaces

In this section we try to characterize completely homogeneous generalized topologies and here we prove results without loss of generality for completely homogeneous strong generalized topologies only. If μ is a generalized topology on X which is not strong, then the results we prove here still hold if we replace X by M_μ.

We use some set theoretic results throughout this section. Consider a nonempty set X and A and B are subsets of X. Then there exists a bijection on X, which maps A onto B if and only if $|A| = |B|$ and $|X \setminus A| = |X \setminus B|$. If X is an infinite set, it is possible to choose subsets A and B of X such that $A \cup B = X, A \cap B = \emptyset$, and $|A| = |X| = |B|$ since $\alpha + \alpha = \alpha$ for any infinite cardinal α [27].

Throughout this chapter X will denote a nonempty ordinary set unless otherwise stated.

Examples of completely homogeneous generalized topologies.

1. $\{\emptyset\}$, $\{\emptyset, X\}$ and $P(X)$ on any set X.

2. $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$ on $X = \{a, b, c\}$.

3. $\tau = \{G \subseteq X : G \text{ is infinite}\} \cup \{\emptyset\}$ is a completely homogeneous generalized topology on an infinite set X.

Lemma 4.2.1. *Let (X, μ) be a completely homogeneous generalized topological space and C be a subset of X such that $|C| < |X|$. If C is open in (X, μ), then every subset B of X such that $|B| = |C|$ is also open in (X, μ).*

Proof. Let $B \subseteq X$ and $|B| = |C|$. Since $|C| < |X|$, we have $|X \setminus C| = |X \setminus B|$. Then there exists a bijection f on X, which map C onto B, consequently f is an open map since every bijection is a homeomorphism in a completely homogeneous generalized topological space and hence $f(C) = B$ is open in (X, μ). $\square$

Lemma 4.2.2. *Let (X, μ) be a completely homogeneous generalized topological space and let $C \subseteq X$, $C \neq \emptyset$, is open in (X, μ). Then supersets of C are also open in (X, μ).*

Proof. Let $C \subsetneq D \subseteq X$, then there exists an element $y \in D$ and $y \notin C$. Let $x \in C$. Consider the bijection f on X which map x onto y and y onto x and f is the identity map on all other elements. But every bijection is a homeomorphism on X and hence f is a homeomorphism on X. Since f is an open map, $f(C) = (C \setminus \{x\}) \cup \{y\}$ is open in (X, μ). Then $C \cup \{y\}$ is open since it is the union of two open sets, $C \cup \{y\} = C \cup (C \setminus \{x\} \cup \{y\})$. Thus D is open since D can be written as $D = \bigcup_{\substack{y \in D \\ y \notin C}} (C \cup \{y\})$. Hence the result. $\square$

Clearly the converse of previous lemma is not true. For example consider the generalized topology $\mu = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,b,c\}, X\}$ on the set $X = \{a,b,c,d\}$. It can be easily verified that the supersets of nonempty open sets are again open in (X,μ), but is not completely homogeneous generalized topological space.

Larson determined the completely homogeneous topologies in his paper [30]. He proved the following theorem.

Theorem 4.2.1. *[30] The only completely homogeneous topologies on a set X are:*

1. *The indiscrete topology*
2. *The discrete topology*
3. *Topologies of the form $\{G \subseteq X : |X \setminus G| < m\} \cup \{\emptyset\}$, where $\aleph_0 \leq m \leq |X|$.*

Next is a characterization theorem for completely homogeneous generalized topological spaces with a nonempty open subset of cardinality strictly less than that of X.

Theorem 4.2.2. *Let (X,μ) be a generalized topological space and C be a nonempty open subset of X such that $|C| < |X|$. Then μ is completely homogeneous generalized topology if and only if $\mu = \{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$ where $m < |X|$.*

Proof. Assume (X,μ) is completely homogeneous. If (X,μ) is a topological space, then we may use the preceding theorem by Larson. We observe that the only completely homogeneous topologies on a finite set are indiscrete and discrete topologies and if X is infinite, then μ is either discrete or every nonempty open set has cardinality the same as that of X. Therefore, since μ contain C and by Theorem 4.2.1, μ is completely homogeneous if and only if μ is $P(X) = \{G \subseteq X : |G| \geq 1\} \cup \{\emptyset\}$, if (X, μ) is a topological space.

Let (X, μ) be a completely homogeneous generalized topological space and not a topological space. Now consider the set $S = \{|G| : \emptyset \neq G \in \mu \textit{ and } |G| < |X|\}$. The set S is nonempty since $|C| \in S$. Let m be the smallest element in S. Then there exists a set $D \subsetneq X$ such that $|D| = m < |X|$ and D is open in (X, μ). By Lemma 4.2.1, if $B \subseteq X$ and $|B| = |D|$, then B is also open in (X, μ). Also by Lemma 4.2.2, supersets of B is also open for every $B \subseteq X$ such that $|B| = |D|$. On the other hand, nonempty subsets of cardinality less than m are not open. Thus μ is of the form $\{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$, where $m < |X|$. Conversely, if $\mu = \{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$ for some $m < |X|$, then it can be easily verified that μ is a completely homogeneous generalized topology on X. $\square$

Now consider the generalized topological space in which every non empty open set has cardinality same as that of whole set. Next we enquire when does this generalized topology completely homogeneous. First we prove some Lemmas.

Lemma 4.2.3. *Let μ be a completely homogeneous generalized topology on an infinite set X. Let G be an open subset of X with $|G| = |X|$ and $|G^c| = |X|$. Then every $H \subseteq X$ such that $|H| = |G|$ is open in (X, μ).*

Proof. Let $H \subseteq X$ and $|H| = |G|$. Since H is an infinite set, there exist disjoint subsets $A, B \subseteq H$ such that $|A| = |B| = |H|$ and $A \cup B = H$. Then $B \subseteq A^c$ and $|H| = |B| \leq |A^c| \leq |X| = |H|$. Hence $|A^c| = |H|$. But $|H| = |G| = |X| = |G^c|$ getting $|A^c| = |G^c|$. Also $|A| = |H| = |G|$ getting $|A| = |G|$. Then there exists a bijection f on X, which map A onto G. Since (X, μ) is a completely homogeneous generalized topological space, f is a homeomorphism. Consequently A is an open set since $A = f^{-1}(G)$ and G is open. But H is a superset of A. Hence by Lemma 4.2.2, H is open in (X, μ). $\square$

Lemma 4.2.4. *Let μ be a completely homogeneous generalized topology on an infinite set X. Let G be a subset of X with $|G| = |X|$ and $|G^c| < |X|$. If G is open in (X, μ), then for every $H \subseteq X$ such that $|H| = |G|$ and $|H^c| \leq |G^c|$ are open in (X, μ).*

Proof. Let H be a subset of X such that $|H| = |G|$ and $|H^c| \leq |G^c|$. If $|H^c| = |G^c|$, then there exists a bijection, say f, on X mapping G onto H. Since every bijection is a homeomorphism, $f(G) = H$ is open in (X, μ).

Now assume $|H^c| < |G^c|$. Consider a subset $A \subseteq H$ such that $|A \cup H^c| = |G^c|$. But $A \cup H^c = (H \setminus A)^c$. Therefore $|(H \setminus A)^c| = |G^c|$.

Case 1: G^c is a finite set.

Then H^c is finite and consequently A has to be finite and since $|H \setminus A| + |A| = |H|$, we have $|H \setminus A| = |H| = |G|$. Thus we obtain $|H \setminus A| = |G|$ and $|(H \setminus A)^c| = |G^c|$. Then there exists a bijection on X mapping G onto $H \setminus A$ and by proceeding as earlier we get $H \setminus A$ is open in (X, μ). But $H \setminus A \subseteq H$, therefore by Lemma 4.2.2, H is also open in (X, μ).

Case 2: G^c is an infinite set.

Note that $|(H \setminus A)^c| = |G^c|$, i.e., $|H^c \cup A| = |G^c|$ implying $|H^c| + |A| = |G^c|$. Since $|G^c|$ is infinite and $|H^c| < |G^c|$, we have $|A| = |G^c|$. But $|G^c| < |X| = |H|$, resulting $|A| < |H|$. Consider $|H \setminus A| + |A| = |H|$, consequently $|H \setminus A| = |H|$ since $|H|$ is infinite. Thus we have $|H \setminus A| = |G|$ and $|(H \setminus A)^c| = |G^c|$ and by similar arguments as in Case 1, we can prove that H is an open subset of X. Hence the proof is complete. $\square$

The previous lemmas enable us to prove the following characterization theorem.

The following definition is adopted from [14].

Definition 4.2.1. *The successor of a cardinal m is the least cardinal greater than m. A cardinal is said to be a limit cardinal if it is not the successor of a cardinal.*

Theorem 4.2.3. *Let μ be a generalized topology on an infinite set X and every $\emptyset \neq G \in \mu$ has cardinality as that of X. Then μ is a completely homogeneous generalized topology if and only if μ is of one of the following form.*

1. $\{G \subseteq X : |G| = |X|\} \cup \{\emptyset\}$.
2. $\{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$, *where* $m < |X|$.
3. $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$, *where* $m \leq |X|$ *and* m *is a limit cardinal,* $m \neq 0$.

Proof. Let (X, μ) be a completely homogeneous generalized topological space in which every $\emptyset \neq G \in \mu$ has $|G| = |X|$.

By Lemma 4.2.3, if for some $G \in \mu$ has $|G^c| = |X|$, then every $H \subseteq X$ such that $|H| = |G|$ is open in (X, μ). In other words, $\{H \subseteq X : |H| = |X|\} \subseteq \mu$. Moreover by the assumption every nonempty open set has cardinality the same as that of X. Therefore $\mu = \{G \subseteq X : |G| = |X|\} \cup \{\emptyset\}$.

Now suppose for every $\emptyset \neq G \in \mu$, $|G| = |X|$ and $|G^c| < |X|$. Consider the set $F = \{|G^c| : G \in \mu, G \neq \emptyset\}$. Since F is bounded by $|X|$, supremum of F exists and let $m = supF$.

Case 1: There exists $\emptyset \neq K \in \mu$ such that $|K^c| = m$

Then for every $\emptyset \neq G \in \mu$, $|G^c| \leq m$, i.e., $\mu \subseteq \{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$. Now by Lemma 4.2.4, every $G \subseteq X$ such that $|G^c| \leq |K^c|$, is also open in (X, μ). Hence $\{G \subseteq X : |G^c| \leq m\} \subseteq \mu$. Also note that here $m \neq |X|$. Hence $\mu = \{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$, where $m < |X|$.

Case 2: For every open set $\emptyset \neq G \in \mu$, $|G^c| \neq m$ and $m \neq 0$.

For every $\emptyset \neq G \in \mu$, $|G^c| < m$, i.e., $\mu \subseteq \{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$. Now since $m = supF$, given any $\alpha < m$, there exists $H \in \mu$ such that $|H^c| = \alpha$. Then every set $M \subseteq X$, with $|M^c| = \alpha$, is open in (X, μ). Moreover by Lemma 4.2.4, every set $U \subseteq X$ with $|U^c| < \alpha$ is also open in (X, μ). This is true for every cardinal number $\alpha < m$. Hence $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\} \subseteq \mu$ and thus we get $\mu = \{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$, where $m \leq |X|$.

If m is not a limit cardinal then there exists a cardinal n such that m is the successor of n. Therefore μ can be written as $\mu = \{G \subseteq X : |G^c| \leq n\} \cup \{\emptyset\}$. Hence if m is a limit cardinal and $m \neq 0$, then μ takes the form $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$.

Now the converse part of the theorem, we can easily verify that the generalized topologies listed in the theorem are completely homogeneous. Hence the proof. $\square$

To conclude this section, the competely homogeneous strong generalized topologies on an arbitrary nonempty set X are listed in the following theorem.

Theorem 4.2.4. *The competely homogeneous strong generalized topologies on an arbitrary nonempty set X are*

1. $\{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$, *where* $m \leq |X|$.

2. $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$, *where* $m \leq |X|$ *and* m *is a limit cardinal,* $m \neq 0$.

3. $\{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$, *where* $m < |X|$.

Note that $P(X)$ and $\{\emptyset, X\}$ can be obtained from (1) and (3) of the above list for $m = 1$ and $m = 0$ respectively. Also we may obtain generalized topologies of the form (3) from (2) if m is a limit cardinal and m has a successor.

4.3 On homogeneous generalized topological spaces

Here we consider a large collection of homogeneous generalized topologies on cyclically ordered set and we study the properties of the same. First let us go through the following examples.

Example 4.3.1. *Let* $X = \{a, b, c, d\}$. *Some homogeneous generalized topologies on* X *are,*

1. $\{\emptyset\}$

2. $\{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, \{a, d, c\}, \{a, b, d\}, X\}$

3. $\{\emptyset, \{a, b, c\}, \{b, c, d\}, \{c, d, a\}, \{d, a, b\}, X\}$

4. $\{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, d, c\}, \{a, b, d\}, X\}$

Note that in examples 2 and 3, there is a cyclicity in the sets and many homogeneous generalized topologies are obtained in this way. A cyclically ordered set is defined as follows.

Definition 4.3.1. *[44]A set X is said to be cyclically ordered if there is a ternary relation $[a < b < c]$ on X which satisfies*

1. *For any distinct $a, b, c \in X$ we have either $[a < b < c]$ or $[b < a < c]$, but not both.*
2. *$[a < b < c]$, if and only if $[b < c < a]$, if and only if $[c < a < b]$, for any $a, b, c \in X$.*
3. *If $[a < b < c]$ and $[a < c < d]$, then $[a < b < d]$.*

A cyclic interval of length n is an n-tuple $(x_1, x_2, \ldots, x_n)$ in which every three tuple (x_i, x_j, x_k) satisfies above three axioms, for every $i < j < k$ where $i, j, k \in \{1, 2, \ldots, n\}$ and also there exists no element $x \in X$ such that (x_{i-1}, x, x_i), $i = 1, 2, \ldots, n$, are in cyclic order. Let C denotes a cyclic interval of length n, $C = [x_1 < x_2 < \ldots < x_n]$, let us denote the set $\{x_1, x_2, \ldots, x_n\}$ also by C if there is no confusion. A cyclic subinterval C' is a subset of C which itself is a cyclic interval.

Let X be a cyclically ordered set. Then two intervals C_1 and C_2 of X are said to be k connected if $|C_1 \cap C_2| = k$ considering C_1 and C_2 as underlying sets and the intervals C_1 and C_2 are said to be disjoint if they are disjoint as subsets of X or if they are 0 connected.

Lemma 4.3.1. *[18]Let (X,μ) be a generalized topological space and let M_μ denotes the union of open sets in (X,μ). Then the following are equivalent.*

1. *(X,μ) is homogeneous.*
2. *(M_μ,μ) is homogeneous.*

Definition 4.3.2. *Consider a finite cyclically ordered set X and A be a cyclic interval of X. Let $A = [x_1 < x_2 < \ldots < x_n]$. For an integer k such that $1 \leq k \leq |A|$, we define cyclic subintervals C_i's of A, where $C_i = [x_i < x_{i\oplus 1} < \ldots < x_{i\oplus(k-1)}]$ for $i = 1,2,\ldots,n$ and $\oplus$ denotes the addition modulo n. Consider the generalized topology generated by the sets $C_1, C_2, \ldots, C_n$. Note that C_i's are subintervals of length k, with $k \geq 1$ and C_i and $C_{i\oplus 1}$ are $k-1$ connected for every $i = 1,2,\ldots,n$ and let us denote this generalized topology by $\mu_k(A)$.*

We discuss the properties of the generalized topology $(X,\mu_k(A))$ in this section. We use the cycles in group theory in the proofs of some of the theorems and these cycles are different from the cyclic interval we discussed in this section.

Theorem 4.3.1. *Let X be a finite cyclically ordered set and $A \subseteq X$ be a cyclic interval of X. Then $\mu_k(A)$, for $1 \leq k \leq |A|$, is a homogeneous generalized topology on X.*

Proof. Let $A = \{x_1, x_2, \ldots, x_n\}$ and by the definition of $\mu_k(A)$, there exist intervals $C_1, C_2 \ldots, C_n$, where each C_i, for $i = 1,2,\ldots,n$, is of length k and each C_i and $C_{i\oplus 1}$ are $k-1$ connected, such that the sets $B = \{C_1, C_2, \ldots, C_n\}$ generate the generalized topology $\mu_k(A)$. Now let $x_i, x_j \in A$ and we need a homeoemorphism h on $(A,\mu_k(A))$ which map x_i onto x_j. Let $S(X)$ denotes the group of all permutations on X. Consider the subgroup G of $S(X)$ generated by the cycle

$g = (x_1\ x_2\ \dots\ x_n) \in S(X)$. Then it is easy to verify that g and all of its powers are homeomorphisms on A. Define h as $h = g^{j-i}$ if $i < j$ and $g^{n-(i-j)}$ if $i > j$. Then $h \in G$ and h is a homeomorphism on A which map x_i onto x_j. Thus $\mu_k(A)$ is a homogeneous generalized topology on X. □

Given a nonempty set X, we can give several cyclic order for X to obtain homogeneous generalized topologies. Considering the collection of all generalized topologies on X, we saw that it form a complete lattice. Thus we can talk about the join of two generalized topologies. See the following examples.

Example 4.3.2. *Let* $X = \{1, 2, 3, 4, 5\}$ *with cyclic order* $[1 < 2 < 3]$ *and* $[4 < 5]$. *Then* $A = \{1, 2, 3\} \subseteq X$ *is a cyclic interval of* X. *Let* $k = 2$. *Then* $\mu_2(A) = \{\emptyset, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$ *is a homogeneous generalized topology on* A*, in fact it is completely homogeneous.*

Example 4.3.3. *Let* $X = \{1, 2, 3, 4, 5, 6, 7\}$ *and* $A = [1 < 2 < 3]$ *and* $B = [3 < 4 < 5]$ *are cyclic intervals with respect to two different cyclic orders on* X. *Then* $\mu_2(A) = \{\emptyset, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$ *and* $\mu_2(B) = \{\emptyset, \{3, 4\}, \{4, 5\}, \{5, 3\}, \{3, 4, 5\}\}$. *Even though* $\mu_2(A)$ *and* $\mu_2(B)$ *are homogeneous it can be easily seen that* $\mu_2(A) \vee \mu_2(B)$ *is not a homogenous generalized topology on* X. *See Theorem 4.3.2.*

Example 4.3.4. *Let* $X = \{1, 2, 3, 4, 5\}$ *and* $A = [1 < 2 < 3 < 4 < 5]$ *and* $B = [3 < 2 < 4 < 1 < 5]$ *are cyclic intervals with respect to two different cyclic orders on* X. *Then* $\mu_k(A)$ *and* $\mu_k(B)$ *are homogeneous for any* k *such that* $1 \leq k \leq 5$. *See Note 4.3.1.*

Remark 4.3.1. *Minimal open sets of* $\mu_k(A)$ *are the cyclic intervals in the base, namely* $C_1, C_2, \dots, C_n$. *Also if* $k = |A| - 1$. *then* $\mu_k(A)$ *is a completely homogeneous generalized topology on* A.

Note that given any finite set X, we obtain several homogeneous generalized topologies on X by giving some cyclic order to elements of X.

Remark 4.3.2. *Homeomorphism preserves cyclic order.*
Let X be a cyclically ordered set and $A = [x_1 < x_2 < \ldots < x_n]$ is a cyclic interval of X with generalized topology $\mu_k(A)$. Let h be a homeomorphism on $(A, \mu_k(A))$. Let $C_1, C_2, \ldots, C_n$ are cyclic intervals which generate $\mu_k(A)$. By Remark 4.3.1, $\{C_1, C_2, \ldots, C_n\}$ is a collection of minimal open sets [18]. Then $\{h(C_1), h(C_2), \ldots, h(C_n)\}$ is again a collection of minimal open sets. Also C_i and C_j are $k-1$ connected implies $h(C_i)$ and $h(C_j)$ are $k-1$ connected. Hence $[h(x_1) < h(x_2) < \ldots < h(x_n)]$ is a cyclic interval, in fact h map a cyclic interval of length n onto a cyclic interval of same length.

Note 4.3.1. *Thus for each cyclic interval $A \subseteq X$ and each integer k such that $1 \leq k \leq |A|$, $\mu_k(A)$ is a homogeneous generalized topology on A or X. Now fix k and change the cyclic order on A. Let A and B are cyclic intervals of X such that $|A| = |B|$. Then it is easy to verify that $(X, \mu_k(A))$ is homeomorphic to $(X, \mu_k(B))$. Thus varying cyclic order on A can no longer make non homeomorphic homogeneous generalized topologies. But varying k in $\mu_k(A)$ gives non homeomorphic homogeneous generalized topologies on A or X. Here we try to find out new homogeneous generalized topologies.*

Theorem 4.3.2. *Let F and G be two disjoint cyclic intervals of a finite cyclically ordered set X and consider the generalized topologies $\mu_k(F)$ and $\mu_{k'}(G)$ on X where k and $k^{'}$ are integers vary in the range $1 \leq k \leq |F|$ and $1 \leq k^{'} \leq |G|$. Then the join of $\mu_k(F)$ and $\mu_{k'}(G)$ is a homogeneous generalized topology on X if and only if*

1. *Cardinality of F and G are same and*

2. $k = k^{'}$.

Proof. Let $F = \{x_1, x_2, \ldots, x_m\}$ and $G = \{y_1, y_2, \ldots, y_n\}$. Let $B = \{C_1, C_2 \ldots, C_m\}$ and $B^{'} = \{D_1, D_2 \ldots, D_n\}$ are collections of cyclic sub intervals of F and G respectively which satisfy properties in Definition 4.3.2, where $C_i^{'}s$ are of length k and $D_j^{'}s$ are of length $k^{'}$ and each C_i and $C_{i\oplus 1}$ are $k-1$ connected for $i = 1, 2, \ldots, m$. Also each D_j and $D_{j\oplus 1}$ are $k^{'}-1$ connected for $j = 1, 2, \ldots, n$.

Assume $\mu = \mu_k(F) \vee \mu_{k'}(G)$ is a homogeneous generalized topology on X. Let $x \in F$ and $y \in G$. Then there exist a homeomorphism h on (X, μ) such that $h(x) = y$. Since $x \in F$, $x \in C_i$ for some $i \in \{1, 2, \ldots, m\}$. Note that here $min(X, \mu) = B \cup B^{'}$. Then $h(C_i)$ is a minimal open set containing y, since homeomorphism maps minimal open sets onto minimal open sets. Thus $h(C_i) \in B^{'}$ implies $h(C_i) = D_l$ for some $l \in \{1, 2, \ldots, n\}$. Since h is a bijection $|C_i| = |h(C_i)| = |D_l|$. But $|D_l| = |D_j|$ for every $j \in \{1, 2, \ldots, n\}$. Hence $k = k^{'}$ and also $h(\bigcup\limits_{i=1}^{m} C_i) = \bigcup\limits_{j=1}^{n} D_j$. Also by Remark 4.3.2, h preserves cyclic order. Thus $h(F)$ is a cyclic interval and $h(F) = G$, since $F \cap G = \emptyset$, we have $|F| = |G|$.
Next assume the converse. Then $n = m$ and $k = k^{'}$. Let $a, b \in F \cup G$.
Case 1: $a, b \in F$. Let $a = x_i$ and $b = x_j$ for some $i, j \in \{1, 2 \ldots m\}$. Define $h = g^{j-i}\, if\, i < j$ and $h = g^{m-(i-j)}\, if\, i > j$ where $g = (x_1\ x_2\ \ldots\ x_n) \in S(X)$ where $S(X)$ is the group of all permutations on X. Then h is a homeomorphism on $F \cup G$ and hence h is a homeomorphism on X.
Case 2: $a, b \in G$. Similar to Case 1.
Case 3: $a \in F$ and $b \in G$. Let $a = x_i$ and $b = y_j$ for some $i, j \in \{1, 2, \ldots, m\}$. Define h by $h(x_{i\oplus n}) = y_{j\oplus n}$ where n is a natural number. Then it is easy to check that h is a homeomorphism on X. Hence the proof is complete. □

Remark 4.3.3. *Above theorem can be extended to a finite number of disjoint* *subintervals of X.*

Lemma 4.3.2. *Let (X, μ) be a homogeneous generalized topological space. Then the number of minimal open sets containing x is same for every $x \in X$.*

Proof. Let $\{U_i\}_{i \in I}$, where I is an indexing set, be the collection of all minimal open sets containing x. Let $y \in X$ be arbitrary and $y \neq x$. Then there exists a homeomorphism h on X such that $h(x) = y$. Then $h(U_i)$ is a minimal open set containing y for every $i \in I$. Also these are the only minimal open set containing y. If not, suppose G is a minimal open set containing y such that G is not of the form $h(U_i)$. Now consider $h^{-1}(G)$, this is a minimal open set containing x so $h^{-1}(G) = U_i$ for some i implies $G = h(U_i)$, a contradiction to our assumption. □

Thus if elements of $min(X, \mu)$ has finite cardinality, say m, and the number of minimal open sets containing x is k, then we obtain the following result.

Proposition 4.3.1. *Let μ be a homogeneous generalized topology on a finite set X and for each $U \in min(X, \mu)$ has cardinality m. Let k denote the number of minimal open sets cotainining x. Then $m.|min(X, \mu)| = n.k$, where $n = |X|$.*

Remark 4.3.4. *Let A be a cyclic interval of a finite set X and consider the generalized topology $\mu_k(A)$ where $1 \leq k \leq |A|$. Then the number of minimal open set containing $x \in A$ is k.*

Theorem 4.3.3. *Let F and G be cyclic subintervals of a finite cyclically ordered set X such that $F \cap G$ is nonempty. Consider the generalized topologies $\mu_k(F)$ and $\mu_{k'}(G)$ on X where $1 \leq k \leq |F|$ and $1 \leq k' \leq |G|$. If the join of $\mu_k(F)$ and $\mu_{k'}(G)$ is a homogeneous generalized topology on X then $F = G$.*

Proof. Let $B = \{C_1, C_2 \ldots, C_m\}$ *and* $B^{'} = \{D_1, D_2 \ldots, D_n\}$ are collections of cyclic intervals of F and G respectively which satisfy properties in 4.3.2, where $C_i^{'}s$ are of length k and $D_j^{'}s$ are of length $k^{'}$ and each C_i and $C_{i\oplus 1}$ are $k-1$ connected for $i = 1, 2, \ldots, m$. And each D_j and $D_{j\oplus 1}$ are $k^{'}-1$ connected for $j = 1, 2, \ldots, n$.

Assume that the join of $\mu_k(F)$ and $\mu_{k'}(G)$, say μ, is a homogeneous generalized topology on X. Then $B \cup B^{'} \cup \{\emptyset\}$ form a base for a homogeneous generalized topology μ on X. Suppose $F \neq G$. Let $a \in F$. Without loss of generality let us assume that there exist an element $b \in G$ such that $b \notin F$. ie., $G \nsubseteq F$.

Case 1: $k \neq k^{'}$

Let h be a homeomorphism on (X, μ) mapping a onto b. If C_i is a minimal open set containing a for some $i \in \{1, 2, \ldots, m\}$, then $h(C_i) = D_j$ for some $j = \{1, 2, \ldots, n\}$, is a minimal open set containing b. Since h is a bijection $|C_i| = |D_j|$ and hence $k = k^{'}$, which is a contradiction. Similar is the case if we assume $F \nsubseteq G$. Thus $F = G$.

Case 2: $k = k^{'}$

Given $F \cap G \neq \emptyset$, choose $c \in F \cap G$. Then By Lemma 4.3.2, number of minimal open set containing every $x \in F \cup G$ is constant. Then the minimal open set containing c in B and $B^{'}$ are same, otherwise the number of minimal open set containing c is strictly greater than the number of minimal open set containing $b \in G$, since $b \notin F$. Thus for each element $x \in F \cap G$, minimal open set containing x in the collection B is same as that in $B^{'}$. Since $c \in F$ there exists a $p \in \{1, 2, \ldots, m\}$ such that $c \in C_p \in B \Rightarrow C_p \in B^{'}$. But by Remark 4.3.4, there are exactly k minimal open sets containing c in the collection B, without loss of generality let $C_1, C_2, \ldots, C_k$ are the minimal open sets containing c which implies $C_1, C_2, \ldots, C_k \in B^{'}$ consequently, $C_i \subseteq F \cap G$ for every $i \in \{1, 2, \ldots, k\}$.

But C_k and C_{k+1} are $k-1$ connected implies C_{k+1} is a minimal open set for all elements in $C_k \cap C_{k+1} \subseteq C_k \subseteq F \cap G$, consequently, $C_{k+1} \in B^{'}$. Proceeding like this we get $C_i \in B^{'}$ for every $i \in \{1, 2, \ldots, m\}$. That is $B \subseteq B^{'}$ and hence $F \subseteq G$. Similarly we can prove that $B^{'} \subseteq B$ implying $G \subseteq F$. Hence $F = G$.

□

Remark 4.3.5. *Converse of above theorem is not true. For example, let* $X = \{a, b, c, d, e, f\}$, $F = \{a, b, c, d, e\}$ *with order* $[a < b < c < d < e]$ *and* $k = 2$ *and* $G = \{a, b, c, d, e\}$ *with order* $[a < c < b < d < e]$ *and* $k^{'} = 2$. *Consider* $\mu_2(F)$ *and* $\mu_2(G)$ *with base* $B = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, a\}, \emptyset\}$ *and* $B^{'} = \{\emptyset, \{a, c\}, \{c, b\}, \{b, d\}, \{d, e\}, \{e, a\}\}$ *respectively. Consider the generalized topology* $\mu_2(F) \vee \mu_2(G)$, *then we can not find a homeomorphism mapping* a *onto* b. *Therefore* $\mu_2(F) \vee \mu_2(G)$ *is not homogeneous.*

Remark 4.3.6. *In Theorem 4.3.3,* $F = G$ *does not imply that cyclic orders on* F *and* G *are same. That is there are generalized topologies* $\mu_k(F)$ *and* $\mu_{k'}(G)$, *with* $F = G$, *cyclic orders on* F *and* G *are different and generalized topology* $\mu_k(F) \vee \mu_{k'}(G)$ *is homogeneous. Let* $X = \{a, b, c, d, e\}$ *and* $F = G = \{a, b, c, d\}$. $[a < b < c < d]$ *is the cyclic order in* F *and* $[a < c < b < d]$ *is the cyclic order in* G. *Let* $k = 2$ *and* $k^{'} = 3$. *Then* $\mu_k(F) = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{a, b, c\}, \{b, c, d\}, \{c, d, a\}, \{a, d, b\}, F\}$ *and* $\mu_{k'}(G) = \{\emptyset, \{a, c, b\}, \{c, b, d\}, \{b, d, a\}, \{a, d, c\}, G\}$. *Then* $\mu_k(F) \vee \mu_{k'}(G)$ *is homogeneous.*

4.4 Completely homogeneous fuzzy generalized topologies

In this section we introduce the concept of homogeneous spaces and completely homogeneous spaces in fuzzy generalized topologies and discuss few properties of level generalized topologies.

Throughout this section L will denote an F-lattice.

Definition 4.4.1. *[23] Let μ_1, μ_2 be L-fuzzy generalized topologies on X and Y respectively. Let $f : X \to Y$. Then the function f is called continuous if for every $A \in \mu_2$, $f^{-1}(A) \in \mu_1$, where f^{-1} is the L-fuzzy reverse mapping from L^Y to L^X induced from $f : X \to Y$. Also f is called homeomorphism if it is bijective and the induced L-fuzzy map, f and L-fuzzy reverse map, f^{-1} are continuous.*

We introduce the concept of homogeneity in L-fuzzy generalized topological spaces.

Definition 4.4.2. *Let X be a nonempty set and L be an F-lattice. Then the L-fuzzy generalized topological space (L^X, μ) is called homogeneous if for every pair $x, y \in X$, there exists a bijection on X mapping x onto y, which induces a homeomorphism on (L^X, μ) and (L^X, μ) is called completely homogeneous if every bijection on X induces a homeomorphism on (L^X, μ).*

Note that a necessary and sufficient condition for a permutation h of a set X to be an L-fuzzy homeomorphism of (L^X, μ) onto itself is that $f \in \mu$ if and only if $f \circ h \in \mu$.

Definition 4.4.3. *[47] Let X be a nonempty set and L be a complete lattice. Consider the L-fuzzy space L^X, for $A \in L^X$ and $a \in L$, we define $a-$level(or $a-$stratification) of A as the ordinary set $\{x \in X : A(x) \geq a\}$ denoted by $A_{[a]}$.*

Proposition 4.4.1. *Let L^X be an L-fuzzy space and μ be an L-fuzzy generalized topology on X. Then the set $G_{[a]}(\mu) = \{f_{[a]} : f \in \mu\}$, where $a \in L$ and $a \neq 0$, is a generalized topology on X.*

Proof. The level set corresponds to $\underline{0} \in \mu$ is $\emptyset$, therefore $\emptyset \in G_{[a]}(\mu)$. Let $\{f_{i_{[a]}}\}_{i \in I}$ be an arbitrary collection of elements in $G_{[a]}(\mu)$. Then, $\bigcup_{i \in I} f_{i_{[a]}} = \bigcup_{i \in I}\{x \in X : f_i(x) \geq a\} = \{x \in X : \bigvee_{i \in I} f_i(x) \geq a\}$. Since $\bigvee_{i \in I} f_i \in \mu$, we have $\bigcup_{i \in I} f_{i_{[a]}} \in G_{[a]}(\mu)$. Thus $G_{[a]}(\mu)$ is a generalized topology on X. □

Let μ be an L-fuzzy generalized topology on X. Then the collection $G_{[a]}(\mu) = \{f_{[a]} : f \in \mu\}$ for $a \in L$ and $a \neq 0$, is called level generalized topology with respect to a.

Theorem 4.4.1. *Let L^X be an L-fuzzy space and μ be a completely homogeneous L-fuzzy generalized topology on X. Then all the level generalized topologies are also completely homogeneous.*

Proof. Let h be a bijection on X, since (L^X, μ) is completely homogeneous, h will induce a homeomorphism on (L^X, μ). Note that h is a homeomorphism of (L^X, μ), for $f \in L^X$, $h(f)$ and $h^{-1}(f)$ are in μ, where $h(f)(y) = \bigvee\{f(x) : x \in X, h(x) = y\}$ for all $y \in X$ and $h^{-1}(f)(x) = f(h(x))$. Let $a \in L$ and $G_{[a]}(\mu)$ be a level generalized topology on X and let $U \in G_{[a]}(\mu)$. Then $U = f_{[a]}$ for some $f \in \mu$. It is enough to show that $h(U) \in G_{[a]}(\mu)$ and $h^{-1}(U) \in G_{[a]}(\mu)$. Consider $h(U) = h(f_{[a]}) = \{h(x) : f(x) \geq a\} = \{x \in X : f(h^{-1}(x)) \geq a\} = \{x \in X :$

$f \circ h^{-1}(x) \geq a\} = \{x \in X : h(f)(x) \geq a\} = h(f)_{[a]}$. But $h(f) \in \mu$ and thus $h(U) \in G_{[a]}(\mu)$.

Similarly $h^{-1}(U) = h^{-1}(f_{[a]}) = \{h^{-1}(x) : f(x) \geq a\} = \{x \in X : f(h(x)) \geq a\} = \{x \in X : h^{-1}(f)(x) \geq a\} = (h^{-1}(f))_{[a]}(\mu)$. Since $h^{-1}(f) \in \mu$, we have $h^{-1}(U) \in G_a(\mu)$. Thus h is a homeomorphism on $(X, G_{[a]}(\mu))$. Since h and $G_{[a]}$ are arbitrary, all level generalized topologies are completely homogeneous. $\square$

Remark 4.4.1. *Converse of Theorem 4.4.1 is not true. For example, consider the set* $X = \{a, b, c\}$ *and* $L = \{0, \frac{1}{2}, 1\}$ *with usual order and* $0^{'} = 1, 1^{'} = 0$ *and* $(\frac{1}{2})^{'} = \frac{1}{2}$. *Then* L^X *is an L-fuzzy space and consider the L-fuzzy generalized topology* μ *having base* $\mathcal{B} = \{\underline{0}, a_1, b_1, c_1, f\}$ *where* $f(a) = \frac{1}{2}, f(b) = \frac{1}{2}$ *and* $f(c) = 1$. *Then* $G_{[\frac{1}{2}]}(\mu)$ *and* $G_{[1]}(\mu)$ *are* $P(X)$ *which is obviously completely homogeneous. But as you see here* μ *is not completely homogeneous.*

Chapter 5

Conclusion

In this thesis we had investigated some properties of the lattice of generalized topological spaces and introduced simple expansion of a generalized topology and characterized the same. Using simple expansion we studied the properties of adjacent topologies and compared the lattice of generalized topologies and lattice of topologies on same set.

We determined the automorphism group of lattice of generalized topologies, so that we could obtain the generalized topologies possessing a property simply from the lattice structure of $LGT(X)$. Also, we determined the automorphism group of the lattice $LFGT(X, L)$ of L-fuzzy generalized topologies on X when L is a finite chain and when L is the diamond-type lattice. Homogeneity in generalized topological spaces and L-fuzzy generalized topological spaces have been discussed and characterized completely homogeneous generalized topological spaces.

In this dissertation we had investigated the properties of generalized topologies and fuzzy generalized topologies with special reference to the lattice theoretic properties.

Scope for further research

Many results in this thesis open up new areas of research.

We had given several equivalent conditions for the simple expansion of a generalized topology to be an upper neighbor of the same but the characterization of simple expansion of T_1 generalized topologies is yet to be obtained. Also we had examples that some generalized topologies do not possess an upper neighbor in the lattice of generalized topologies on an infinite set. We proved that when the generalized topology μ is non μ-T_1 or when μ or μ^c is finite then μ has an upper neighbor in $LGT(X)$. But we couldn't tackle the problem in general and is left open.

Similarly the study of lower neighbors of generalized topologies can also be attempted and existence of lower neighbors in $LGT(X)$ is also an open problem.

In Chapter 3, we determined the automorphism group of $LFGT(X, L)$ when L is a finite chain and when L is the diamond-type lattice only. For the general lattice L or for an infinite chain, the determination of automorphism group of $LFGT(X, L)$ is still open.

In the last chapter we couldn't completely characterize homogeneous generalized topologies and homogeneous fuzzy generalized topologies on an arbitrary set. Also the related problems are characterization of hereditarily homogeneous generalized topologies, rigid and antirigid properties of generalized topologies.

Bibliography

[1] Abd El-Monsef M. E., El-Deep S. N. and Mahmoud R. A. : *β-Open Sets and β-Continuous Mappings*, Bull. Fac. Sci. Assiut Univ., **12**, 77-90(1966).

[2] Agashe P. and Levine N. : *Adjacent Topologies*, J. Math. Tokushima Univ., **7**, 21-35(1973).

[3] Ayawan J. B. T. and Canoy S. R. : *Axioms of Countability in Generalized Topological Spaces*, International Mathematical Forum, **8**, No. 31, 1523 - 1530(2013).

[4] Baskaran R., Murugalingam M. and Sivaraj D. : *Lattice of Generalized Topologies*, Acta Math. Hung., **133**, No.4(2011).

[5] Birkhoff G. : *Lattice Theory*, American Mathematical Soc., Second Edition (1984).

[6] Chacko B. : *Some Lattice Theoretic Problems Related to General and Fuzzy Topology*, Thesis for Ph.D Degree, University of Calicut, India (2008).

[7] Chang C. L. : *Fuzzy Topological Spaces*, J. Math. Anal. Appl. **24**, 182-190(1968).

[8] Császár Á. : *Generalized Open Sets*, Acta Math. Hung., **75**, 65-87 (1997).

[9] Császár Á. : *Generalized Topology, Generalized Continuity*, Acta Math. Hung., **96**, 351-357(2002).

[10] Császár Á. : *Products of Generalized Topologies*, Acta Math. Hung. **123** 127-132(2009).

[11] Császár Á. : γ-*Connected Sets*, Acta Math. Hung., **101**, 273-279(2003).

[12] Davey B. A. and Priestley H. A. : *Introduction to Lattices and Order*, Cambridge University Press, Second Edition (1990).

[13] de Arruda Saraiva L. E. : *Generalized Quotient Topologies*, Acta Math. Hung., **132**, 168-173(2011).

[14] Devlin K. : *The Joy of Sets: Fundamentals of Contemporary Set Theory*, Springer-Verlag Newyork Inc., Second Edition (1993).

[15] Dhanya P. M. : *On the Lattice of Generalized Topologies*, Int. J. Pure Appl. Math., **102**, No. 1, 85-95(2015).

[16] Fraleigh J. B. : *A First Course in Abstract Algebra*, Pearson Education, Inc., Seventh Edition(2003).

[17] Fröhlich O. : *Das Halbordnungssystem der Topologischen Räume auf einer Menge*, Math. Ann. **156**, 79-95(1964).

[18] Ghour S. A, Omari A. A. and Noiri T. : *On Homogeneity and Homogeneity Components in Generalized Topological Spaces*, Faculty of Sciences and Mathematics, University of Niś, Serbia, 1097-1105(2013).

[19] Ginsburg J. : *A Structure Theorem in Finite Topology*, Canad. Math. Bull., **26**, 121-122(1983).

[20] Goguen J. A. : *L–Fuzzy Sets*, J. Math. Anal. Appl. **18**, 145-174(1967).

[21] Grätzer G. : *General Lattice Theory*, Birkhäuser Verlag, Second Edition, 1998.

[22] Hartmanis J. : *On the Lattice of Topologies*, Canad. J. Math., **10**, 547-553(1958).

[23] Heba I.M. : *On L-fuzzy Generalized Topology*, Int. J. Comput. Appl. **69**, No. 28, 4-7(2013).

[24] Jafari S., Latif R. M. and Moshokoa S. P. : *A Note on Generalized Topological Spaces and Preorder*, CUBO A Mathematical Journal, **12**, No.02, 123-126, June (2010).

[25] Kannan V., Ramachandran P. T. : *Hereditarily Homogeneous Spaces*, Proceedings, International Seminar on Recent Trends in Topology & its Applications., St.Joseph College, Irinjalakuda (2009).

[26] Khayyeri R. and Mohamadian R. : *On Base for Generalized Topological Spaces*, Int. J. Contemp. Math. Sciences, **6**, No. 48, 2377 - 2383(2011).

[27] Kuratowski K. : *Introduction to Set Theory and Topology*, *Pergamon Press*, Oxford (1961).

[28] Larson R. E. and Andima S. J. : *The Lattice of Topologies: A Survey*, Rocky Mountain J. Math., **5** , No.2(1975).

[29] Larson R. E. : *On the Lattice of Topologies*, Ph.D Thesis submitted to the University of Colorado (1970).

[30] Larson R. E. : *Minimum and Maximum Topological Spaces*, Bulletin De L'Academie, **XVIII**, No.12(1970).

[31] Levine N. : *Minimal Simple Extensions of Topologies*, Kyungpook Math J., **19**, No. 1, 43-55(1979).

[32] Levine N. : *Semi-open Sets and Semi-continuity in Topological Spaces*, Amer. Math. Monthly, **70**, 36-41(1963).

[33] Maheswary S. N. and Tapi U. : *Note on Some Applications of Feebly Open Sets*, M. B. J. Univ. of Saugar (1979).

[34] Mashhour A. S., Abd El-Monsef M. E. and El Deeb S. N. : *On Precontinuous and Weak Precontinuous Mappings*, Proc. Math. Phys. Soc. Egpt, **53**, 47-53(1982).

[35] Min W. K. : *Remarks on Separation Axioms on Generalized Topological Spaces*, J. Chungcheong Math. Soc., **23**, No. 2, June (2010).

[36] Ming L. Y. and Kang L. M. : *Fuzzy Topology*, World Scientific Publishing Co. Pte. Ltd., Advances in Fuzzy Systems- Applications and Theory, **9**(1997).

[37] Namboothiri M. N. M. : *Some Problems Relating Order and Fuzzy Topologies*, Thesis for Ph.D Degree, University of Calicut, India (2012).

[38] Njástad O. : *On Some Classes of Nearly Open Sets*, Pacific J. Math., **15**, 961-970(1965).

[39] Rajagopalan M. and Wilansky A. : *Reversible Topological Spaces*, J. Austral. Math. Soc., **6**, 129-138(1966).

[40] Sarma R. D. : *On Extremally Disconnected Generalized Topologies*, Acta Math. Hung., **134**, No. 4, 583-588(2012).

[41] Stadler B. M.R. and Stadler P. F. : *Generalized Topological Spaces in Evolutionary Theory and Combinatorial Chemistry*, J. Chem. Inf. Comput. Sci., **42**, 577-585(2002).

[42] Száz A. : *Minimal Structures, Generalized Topologies, and Ascending Systems should not be studied without Generalized Uniformities*, FILOMAT, **27**, No.6, 87-97(2007).

[43] Thomas J. and John S. J. : *$\mu-$Compactness in Generalized Topological Spaces*, J. Adv. Stud. Topol., **3**, No. 3, 18-22(2012).

[44] Viro O. Ya., Ivanov O. A., Netsvetaev N. Yu. and Kharlamov V. M. : *Elementary Topology, Problem Textbook*, American Math. Soc. (2012).

[45] Weston J. D. : *On the Comparison of Topologies*, J. London Math. Soc., **32**, 342-354(1957).

[46] Xun G. E. and Ying G. E. : *μ-Separations in Generalized Topological Spaces*, Appl. Math. J. Chin. Univ., **25**, No.2, 243-252(2010).

[47] Zadeh L. A. : *Fuzzy Sets*, Inform. and Control, **8** , 338-353(1965).

Papers published

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2 Dhanya P. M., Ramachandran P. T.: *On the Simple Expansion of Generalized Topologies*, International Journal of Pure and Applied Mathematics, **104**, No. 4, 621-630 (2015) ISSN: 1311-8080 (Print version); ISSN: 1314-3395 (Online version) .

3 Dhanya P. M.: *On Completely Comogeneous Generalized Topological Space*, Mathematical Sciences International Research Journal, 357-359 (2015) ISSN: 2278-8697.

4 Dhanya P. M.: *Automorphism Group of Lattice of Fuzzy Generalized Topologies*, Annals of Fuzzy Mathematics and Informatics (Accepted) ISSN: 2093-9310(Print version), ISSN: 2287-6235(Online version).

5 Sini P., Dhanya P. M., Ramachandran P. T.: *On Completely Homogeneous L-topological Space*, Annals of Fuzzy Mathematics and Informatics (Accepted) ISSN: 2093-9310(Print version), ISSN: 2287-6235(Online version).