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CERTIFICATE

I hereby certify that the thesis entitled "A STUDY ON COMMON NEIGHBOR POLYNOMIAL OF GRAPHS" is a bonafide work carried out by **Smt. Shikhi M,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**A STUDY ON COMMON NEIGHBOR POLYNOMIAL OF GRAPHS**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

List of symbols

G	A simple finite graph
$E(G)$	Edge set of G
$V(G)$	Vertex set of G
$N(G, i)$	i-common neighbor set of G
$\|S\|$	Cardinality of the set S
$N[G; x]$	Common neighbor polynomial of G
$N^m[G; x]$	m^{th} derivative of $N[G; x]$
K_n	Complete graph on n vertices
P_n	Path on n vertices
C_n	Cycle on n vertices
$K_{m,n}$	Complete bipartite graph with $m + n$ vertices
$B_{n,n}$	Bistar graph on $2n + 2$ vertices
$K_{n_1,n_2,\ldots,n_m}$	Complete m-partite graph
$L_{m,n}$	Lollipop graph
W_n	Wheel graph
H_n	Helm

WB_n	Web graph
S_n	Shell graph
B_N	Bow graph
BF	Butterfly graph
F_n	Friendship graph
$d_u(G)$	Degree of the vertex u in G
$B_{n,1}$	n-Barbell graph
B_n	Bipartite cocktail party graph
$W_n^{(m)}$	Windmill graph
$D_n^{(m)}$	Dutch windmill graph
$C_n \odot P_m$	Armed crown of C_n and P_m
$f_{n\times m}$	Flower graph
$C_p \odot C_q^t$	Chaplet graph
$S_{n,m}$	Snake graph
$RK_{n\times n}$	$n \times n$ square rook's graph
$P_n(m_1, m_2, \ldots, m_n)$	Caterpillar tree
$H \vee K$	Join of the graphs H and K
$H \circ K$	Corona of the graphs H and K
$H \square K$	Cartesian product of the graphs H and K
L_n	Ladder graph
CL_n	Circular ladder graph
B_m	m-book graph
$H \times K$	Tensor product of the graphs H and K
$S(G)$	Splitting graph of G
$Sh(G)$	Shadow graph of G

$\mu(G)$	Mycielski graph of G
$\overset{\mathcal{N}}{\sim}$	CNP-equivalent
$[G]_{\mathcal{N}}$	CNP-equivalent class of G
$p(G)$	Disjoint union of p copies of G
$\overline{G}$	Complement of G
$G+H$	Disjoint union of the graphs G and H
$\mathcal{N}$	Number of real common neighbor roots of G
$N_r(G,i)$	Generalized i-common neighbor set of G
$N_r[G;x]$	Generalized common neighbor polynomial of G
$[G]_{\mathcal{N}_r}$	CNP_r-equivalent class of G
$clr(v)$	Cluster of the vertex v
$W(G)$	Wiener index of G
$WW(G)$	Hyper wiener index of G
$H(G,x)$	Hosoya polynomial of G
$D_3[n]$	Nanostar dendrimer of third generation
D_k	PAMAM dendrimer of $k^(th)$ generation
$\mathcal{K}(G)$	Kronecker double cover of G

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Introduction

Graph theory is one of the well flourished branches of Mathematics. Originating from the modeling and negative resolution of famous Konigsberg bridge problem by Leonard Euler[23], graph theory has entrenched as one of the best tool to model network systems involved in complex real life problems. Beauty of graph theory lies in its wide scope of applications in the fields ranging from network theory, chemistry and operational research to architecture and linguistics. Performing as a translator of real life problems to mathematical models, graph theory has an astounding position amidst various branches of applied mathematics.

Among various branches of graph theory, graph polynomials is one of the well studied concepts as they are used to unveil the structural properties of graphs. Roughly speaking, a graph polynomial is a polynomial assigned to a graph whose coefficients are the indicators of some graph theoretic parameters. It can be defined as a function from the set of all finite graphs to the polynomial ring over the set of real numbers such that isomorphic graphs are assigned to the same polynomial.

In the present work, emphasizing on the structural similarity of pairs of nodes in a network system, a new graph polynomial is introduced named as 'Common neighbor polynomial of graphs'. While modeling the structure of a social network system, usually pairs of individuals with shared interests are represented by pairs of vertices with common neighbors. The number of such common neighbors serves as a measure of consensus and proclivities between the corresponding pair of individuals. Moreover, it is conjectured that two persons having one or more common acquaintances are more likely to be acquainted in future[7]. Hence the study of common neighbors of pairs of nodes in a network system is significant in predicting the possibility of future links as well as in clustering analysis.

An overview of the thesis

The thesis comprises an introductory chapter together with nine chapters in which a new graph polynomial called "Common Neighbor Polynomial of graphs" is introduced and studied in a detailed manner. In the introductory chapter, a concise description is given detailing the motivational facts behind the introduction of the new graph polynomial. Moreover, a blueprint of the upcoming chapters is also provided.

In **chapter 1**, the terminology and notations that will appear in the subsequent chapters are detailed. Basic graph theoretic definitions are explained in the first section. Second section of the chapter describes some important graph operations. Section 1.3 includes an introduction to the theory of graph polynomials along with some theorems on polynomials which are beneficial in the study of roots of polynomials.

In **chapter 2**, a new graph polynomial called 'Common neighbor polynomial' is introduced whose coefficients are the cardinalities of i-common neighbor sets which are defined as subsets of $V(G) \times V(G)$. The definition of i-common neighbor set and common neighbor polynomial of graphs is introduced in section 2.2. Let $G(V, E)$ be a graph of order n. Then for $0 \leq i \leq n-2$, the i-common-neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$. The common-neighbor polynomial of G denoted by $N[G; x]$ is defined as $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)| x^i$. In section 2.3, the common neighbor polynomial of many well known graph classes are identified. The common neighbor polynomial of strongly regular graphs and trees are studied in section 2.4 and 2.5 respectively. The common neighbor polynomial of some special graph constructions are discussed in section 2.6.

The common neighbor polynomial of graphs obtained by the unary graph operations such as splitting graph, shadow graph or mycielsky graph of a given graph are discussed in **chapter 3**.

Binary graph operations are used to model the action between two network systems. Binary graph operations are usually known as graph products in which two initial graphs are acted together according to some specific rules to produce a new graph. **Chapter 4** provides explicit formulae to find common neighbor polynomial of some well known graph products such as join, corona, cartesian product, rooted product and tensor product of graphs in terms of the common neighbor polynomial of the parent graphs.

Structural equivalence of network systems is one of the prime concerns of network analysis. Usually in graph theory, isomorphic graphs are referred to as equal graphs. But, the existence of isomorphism may not be a criteria for identifying two graphs as equivalent as far as structural equivalence is concerned. From this point of view, CNP-equivalent classes of graphs are defined and studied in **chapter 5**. Two graphs G and H are said to be CNP-equivalent ($G \overset{\mathcal{N}}{\sim} H$) if and only if $N[G;x] = N[H;x]$. Obviously, the relation $\overset{\mathcal{N}}{\sim}$ is an equivalence relation on the class $\mathcal{G}$ of all simple finite graphs. The set of all graphs CNP-equivalent to a graph G is denoted as $[G]_{\mathcal{N}}$ and is defined as $[G]_{\mathcal{N}} = \{H \in \mathcal{G} : N[H;x] = N[G;x]\}$. A graph H is said to be $CNP-$unique if $[H]_{\mathcal{N}} = \{H\}$. Some CNP-equivalent classes of graphs are identified in section 5.2. In section 5.3, it is showed that graph classes like complete graphs and complete bipartite graphs are CNP-unique graphs.

While introducing a new graph polynomial, it is customary to verify whether it can be the graphical model of a stable physical system. A polynomial all of whose non zero roots lie in the open left half plane is said to be stable with respect to the closed right half plane and such a polynomial is called a Hurwitz polynomial. Identification of Hurwitz polynomials are beneficial in control systems theory as they represent the characteristic equations of stable linear systems. In **chapter 6**, we identify the conditions under which the common neighbor polynomial of some graph classes becomes a Hurwitz polynomial.

Chapter 7 focuses on the real roots of common neighbor polynomial of graphs. The roots of common neighbor polynomial of a graph G are called the common neighbor roots of G. The number of real common neighbor roots of a graph G where the multiplicities counted, is denoted by $\mathcal{N}(G)$. In chapter 7, we study the number of real common neighbor roots of some well known graph classes.

In **chapter 8** we generalize the concepts of i-common neighbor sets and common neighbor polynomial of graphs and define generalized i-common neighbor sets and generalized common neighbor polynomial of graphs. In section 8.2. The generalized common neighbor polynomial of some well known graph classes are identified. Moreover, some characterizations on graphs in terms of generalized common neighbor polynomial of graphs are also discussed. In section 8.3, we define the simplicial complexes of graphs and introduce the concept of cluster of a vertex in a graphs. In the light of these concepts, generalized i-common neighbor sets of graphs is studied.

Chapter 9 spot lighted on the significance of common neighbor polynomial of graphs in some applied areas. In section 9.1, we study common neighbor polynomial of graphs incorporated with chemical graph theory. Structural analysis of chemical molecules is a prime concern of mathematical chemistry. The common neighbor polynomial of nanostar dendrimers and PAMAM dendrimers are studied in subsections 9.1.1 and 9.1.2 respectively. In 9.1.3, the Hosoya polynomial of graphs with diameter not more than three is derived using the common neighbor polynomial of corresponding graphs. Section 9.2 deals with the significance of common neighbor polynomial of graphs in network data clustering. In 9.2.1, we discuss the Shared Nearest Neighbor(SNN) clustering and explains the way in which the common neighbor polynomial of graphs is useful in the formation of meaningful clusters. In section 9.3, we establish a relation which connects common neighbor polynomial of a graph with the adjacency matrix of the graphs. Making use of this relation, a C^{++} program is developed for generating coefficients of common neighbor polynomial of a graph and is provided as an Appendix.

In the concluding chapter of the thesis, some directions for further research are included. Also this chapter includes a list of publications and bibliography.

Chapter 1

Preliminaries

The chapter explores the graph theoretic terminology and notations that will appear in the subsequent chapters. We adopt the basic definitions and notations as in Graph Theory [20], written by J.A. Bondy and U.S.R. Murty. This chapter includes three sections. The first section deals with basic definitions and notations that may appear in the forthcoming chapters. In the second section various graph theoretic operations are discussed. Third section incorporates some basic results and theorems which are used in the forthcoming chapter to study the roots of polynomials.

1.1 Basic terminology

A **graph** G is an ordered pair (V, E) consisting of the disjoint sets V of vertices and E of edges, together with an incidence function $\psi : E \to V \times V$ which associates each edge of G with an unordered pair of vertices of G. A graph having finite number of vertices and edges is called a **finite graph**. The number of vertices and number of edges of a finite graph G are called the **order** and **size** of G respectively. Two or more edges having same end vertices are called **multiple edges** and an edge with identical end vertices is called a **loop**. A graph is **simple** if it has no multiple edges or loops.

The end vertices of an edge are said to the **incident** with the edge. Two **vertices are adjacent** if they are incident with a common edge and two **edges are adjacent** if they are incident to a common end vertex. Two adjacent vertices are said to be neighbors of each other. The set of all neighbors of a vertex $v \in V$ is called the **neighbor set of** v and is denoted by $N(v)$. The number of vertices in $N(v)$ is called the **degree of** v. Vertices of degree 1 are called **pendent vertices**. A graph having all the vertices with same degree is called a **regular graph**. A subset S of the set of vertices of a graph G in which any two distinct vertices are adjacent is called a **clique** in G.

Let G be a graph of order n. Then the **adjacency matrix** of G is a $n \times n$ matrix in which the ij^{th} entry becomes 1 or 0 according as the pair of vertices v_i and v_j are adjacent or not in G.

A **complete graph** is a simple graph in which all the pairs of vertices are adjacent. A graph is **bipartite** if its vertex set can be partitioned into two subsets X and Y so that any edge of G has one end vertex in X and the other in Y. If each vertex of X is joined to every vertex of Y in a bipartite graph, it is called a **complete bipartite graph**.

A **complete m-partite graph** $K_{n_1,n_2,\ldots,n_m}$ is a graph whose vertex set can be partitioned into m non empty sets $V_i, i = 1, 2, \ldots, m$ such that every vertex in V_i is adjacent to every vertex in V_j for every $i \neq j$ and $i, j \in \{1, 2, \ldots, m\}$.

A **walk** is an alternating sequence $v_0e_1v_1e_2 \ldots v_{i-1}e_iv_i \ldots v_n$ of vertices and edges in which the vertices v_{i-1} and v_i are the end points of the edge e_i. The length of a walk is the number of edges in the walk. A **path** is a walk having all the vertices distinct. A path on n vertices is denoted by P_n. A **trail** is a walk where all the edges are distinct. A closed trail in which all the vertices are distinct is called a **cycle**. A cycle of length n is denoted by C_n. A graph G is **connected** if for each pair of vertices u and v in $V(G)$, there is a u-v path in G. A **disconnected graph** is a graph which is not connected. A graph is **acyclic** if it contains no cycles. A connected acyclic graph is called a **tree**.

The **distance** between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest u-v path in G. The maximum distance between any pair of vertices of G is called the diameter of G. The **Hosoya polynomial**[26] of G is defined as $H(G, x) = \sum_{j=1}^{l} d(G, j)x^j$ where $d(G, j)$ denote the number of pairs of vertices in G having distance j apart and l denote the diameter of the graph.

A **Wheel graph** $W_n, n > 3$ is obtained by taking the join of the cycle C_{n-1} and K_1. A **helm,** $H_n, n > 3$ is obtained from a wheel graph W_n by adding pendent edges to every vertices on the wheel rim. A **web graph** $WB_n, n > 3$ is obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. WB_n has $3n - 2$ vertices and $3(n - 1)$ edges. A **shell graph** S_n where $n \geq 3$ is obtained from the cycle graph C_n by adding the edges corresponding to the $(n - 3)$ concurrent chords of the cycle. The vertex at which all the chords are concurrent is called the apex of the shell. A **bow graph** is a double shell with same apex in which each shell has any order.

A **butterfly graph** is a bow graph along with exactly two pendent edges at the apex. A **friendship graph** F_n is the one point union of n copies of the cycle C_3. A **Tadpole** $T_{(n,l)}$ is a graph obtained by attaching a path P_l to one of the vertices of the cycle C_n by a bridge. The n**- barbell graph** $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge. The **Lollipop graph** $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path P_n with a bridge.

A **bistar graph** $B_{m,n}$ is obtained by connecting the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by a bridge. The **bipartite Cocktail party graph** B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$. The **Windmill graph** $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common. An **armed crown** $C_n \bigodot P_m$ is a graph obtained by attaching a path P_m to every vertex of the cycle C_n.

A simple k-regular graph G on n vertices is said to be **strongly regular of type** (n, k, λ, l) if there exists integers λ, l such that any adjacent pair of vertices of G have exactly λ common neighbors and any non-adjacent pair of vertices of G have exactly l common neighbors.

A **rooted tree**[13] is a tree in which one of the vertices is distinguished as the root. According to the distance of other vertices from the root vertex, there is a hierarchy on the vertices of a rooted tree. The distance of a vertex v from the root is called the depth or level of the vertex. The height of a rooted tree is the greatest depth of a vertex of the tree. Considering a path from the root to a vertex w, if a vertex v immediately precedes w , then v is called the parent of w and w is called the child of v. Vertices having same parent are called siblings.

An **m-ary tree** ($m \geq 2$) is a rooted tree in which every vertex has m or fewer number of children. A **complete m-ary tree** is an m-ary tree in which every internal vertices has exactly m children and all leaves are of same distance from the root.

The **derivative of a graph** G is a graph obtained from G by deleting all the pendent vertices of G. A **caterpillar**[39] is a tree graph whose derivative is a path graph . Consequently, a caterpillar $P_n(m_1, m_2, \ldots, m_n)$ is obtained by attaching m_i pendent edges to the vertex v_i of a path P_n where $i \in \{1, 2, \ldots, n\}$. A **star like tree graph** $S(n_1, n_2, \ldots, n_k)$[24] is a graph having only one vertex w of degree greater than 2 such that deletion of w results in a disjoint union of the path graphs $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$. The star like tree graphs are used to represent proteins which will have generally 20 branches where each branch indicates the presence of one of the 20 natural amino acids.

Let G and H be two graphs with incidence functions ψ_G and ψ_H respectively. Then G and H are **isomorphic**[33] if there exists bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$ where $u, v \in V(G)$ and $e \in E(G)$.

1.2 Graph operations

The **splitting graph** [12] $S(G)$ of a graph G is obtained by adding new vertices v' to G corresponding to each vertex v of G and then joining the vertex v' to all vertices of G adjacent to v in G. The **shadow graph** $Sh(G)$ of a graph G is obtained by taking two copies of G, say G_1 and G_2 and joining each vertex of

G_1 to the neighbors of the corresponding vertex of G_2. The **Mycielski graph,** $\mu(G)$ [22] of a graph G contains G itself as an isomorphic subgraph together with $n+1$ additional vertices; a vertex v_i corresponding to each vertex u_i of G and another vertex w. Each v_i is connected by an edge to w and for each edge u_iu_j of G, $\mu(G)$ includes two additional edges u_iv_j and v_iu_j.

Consider the graph $G(V,E)$ and let $w \notin V$. Then the graph $G' = G + w$ is a graph obtained from G by including the vertex w in G and joining it to all other vertices of G. If H and K are two graphs, then the **join,** $H \vee K$ is the graph with vertex set $V(H) \cup V(K)$ and edge set $E(H) \cup E(K) \cup \{uv : u \in V(H), v \in V(K)\}$.

The **corona of two graphs**[13] K and H is formed from one copy of K and $|V(K)|$ copies of H where the i^{th} vertex of K is adjacent to every vertex in the i^{th} copy of H[35]. It is denoted by $K \circ H$. The **Cartesian product**[13] of two graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and the vertices (u,v) and (x,y) are adjacent if and only if $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$.

A **rooted graph** is a graph in which one vertex is distinguished as a root. The **rooted product**[3] of a graph G and a rooted graph H is obtained as follows: Take $|V(G)|$ copies of H and for each vertex v_i of G, identify v_i with the root vertex of the i^{th} copy of H. The **tensor product**[13] of two graphs K and H is the graph $K \times H$ with vertex set $V(K) \times V(H)$ and the vertices (u,v) and (x,y) are adjacent if and only if $ux \in E(K)$ and $vy \in E(H)$.

1.3 Polynomials

The following theorems can be used to study the number of real roots of polynomials.

Theorem 1.3.1 (de Gua's Theorem [42])**.** *If the polynomial $f(x)$ lacks $2m$ consecutive terms then it has no less than $2m$ imaginary roots. If $2m+1$ consecutive terms are missing then, if they are between terms of different signs, the polynomial has no less than $2m$ imaginary roots, whereas, if the missing terms are between terms of same sign, the polynomial has no less than $2m+2$ imaginary roots.*

Theorem 1.3.2 (S. Kakeya [40])**.** *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with real coefficients satisfying $a_0 < a_1 \leq a_2 \leq \ldots \leq a_n$, then all the zeros of $p(z)$ lie in $|z| \leq 1$.*

Theorem 1.3.3. *[37] Consider the cubic equation $ax^3 + bx^2 + cx + d = 0$. Then the discriminant of the cubic equation is given by $\Delta = b^2c^2 - 4ac^3 - 4b^d + 18abcd - 27a^2d^2$. If $\Delta > 0$, the equation has three real distinct roots; if $\Delta = 0$, the equation has three real roots in which one of them is a multiple root; if $\Delta < 0$, the equation has one real root and two imaginary roots.*

A polynomial $f(x_1, \ldots, x_n)$ is said to be **stable** [28] with respect to a region $\Omega \in \mathbb{C}^n$ if no root of f lies in Ω. Polynomials which are stable with respect to the closed right half plane and with respect to the open unit disk are called Hurwitz polynomial and Schur polynomial respectively. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems[15].

Let $\mathcal{G}$ be the set of finite graphs on n vertices and $R[x]$ the polynomial ring over the real numbers. Then a graph polynomial is a function $P : \mathcal{G} \to R[x]$ such that for any two graphs $G, H \in \mathcal{G}$, if G is isomorphic to H, then $P(G) = P(H)$. A graph polynomial encodes information about the graph and enables algebraic methods for extracting this information.

With the introduction of edge difference polynomial[21] in 1878, J.J. Sylvester initiated the study of graph polynomials which was further studied by J. Petersen in 1891. Since then many graph polynomials were introduced among which matching polynomial[9], chromatic polynomial[16], Hosoya polynomial[26] and domination polynomial[36] are well popularized.

Chapter 2

Common neighbor polynomial of graphs

In social network systems, pair of nodes having same common neighbors must have some similarity in the social sense. Also the number of common neighbors of two nodes serves as a measure of their similarity. Similarities of nodes in network systems were discussed in [8]. In this chapter, emphasizing on the structural equivalence of pairs of nodes in a network system, we introduce the i-common neighbor set and the common neighbor polynomial of graphs. Throughout this work G denotes a simple and finite graph with vertex set $V(G)$ and edge set $E(G)$ and (u, v) denotes an unordered vertex pair of distinct vertices of G.

2.1 Common neighbor polynomial of graphs

In this section we first introduce the concept of *i-common neighbor set* and then define the *common neighbor polynomial* of a graph. Moreover, we derive the common neighbor polynomial of some well known graph classes.

Definition 2.1.1. *Let $G = (V, E)$ be a graph of order n. Then for $0 \leq i \leq n-2$, the i-common neighbor set of G is defined as:*

$$N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}.$$

Definition 2.1.2. *Let G be a graph of order n. Then the common neighbor polynomial of G denoted by $N[G; x]$ is defined as*

$$N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)| x^i.$$

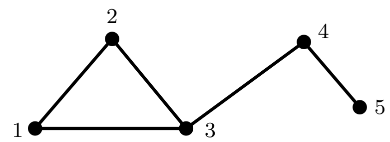

Figure 2.1: The graph G

Example 2.1.3. *For $i = 0, 1, 2$ and 3, the i-common neighbor sets of the graph G shown in 2.1 are*

$N(G, 0) = \{(1, 5), (2, 5), (3, 4), (4, 5)\}$,

$N(G, 1) = \{(1, 2), (2, 3), (1, 3), (1, 4), (2, 4), (3, 5)\}$,

$N(G, i) = \phi$ *for* $i = 2, 3$.

Hence the common neighbor polynomial of G is $N[G; x] = 6x + 4$.

We observe the following simple properties of $N[G; x]$:

(i) $N[G;x]$ is a polynomial of degree at most $(n-2)$.

(ii) Isomorphic graphs have same common neighbor polynomials.

(iii) $N(G,n)=\phi$ and $N(G,n-1)=\phi$.

(iv) $N[G;1]=\sum_{i=1}^{n-2}|N(G,i)|=\binom{n}{2}$.

(v) $N[G;0]$ gives the number of vertex pairs of G having no common neighbors.

(vi) $N^{(m)}[G;0]=m!|N(G,m)|$, $m=1,2,\ldots,(n-2)$ where $N^{(m)}[G;x]$ denotes the m^{th} derivative of $N[G;x]$ with respect to x.

Theorem 2.1.4. *Let G be a simple graph of order n. Then $N[G;x]$ is a non constant polynomial if and only if there exists a path of length 2 in G.*

Proof. Suppose $N[G;x]$ is a non constant polynomial of degree m. Then there exists a pair of vertices (u,v) which have m common neighbors. Let w be one such neighbor. Then uwv is a path of length 2 in G. Conversely, suppose there exists a path uwv of length 2 in G. Then the pair (u,v) has at least one common neighbor w. Let the number of common neighbors of $(u,v)=l\geq 1$. Then $N(G,l)\neq\phi$. Then $N[G;x]$ is a non constant polynomial. □

Theorem 2.1.5. *If $|N(G,i)|=m$ where $i>1$, then G contains at least $\frac{1}{2}m\binom{i}{2}$ cycles of length 4.*

Proof. Since $|N(G,i)|=m$, there exist m pairs of vertices which share i common neighbors for $i>1$. Let (u,v) be one such pair. If w_1 and w_2 are two common neighbors of u and v, then uw_1vw_2u is a cycle of length 4. Therefore, if there are i common neighbors, there exist iC_2 cycles of length 4 containing u and v. Since each such cycle corresponds to a maximum of 2 vertex pairs, if there are m such pairs (u, v), there exist at least $\frac{1}{2}m\binom{i}{2}$ cycles of length 4. $\square$

2.2 Common neighbor polynomial of some well known graph classes

Theorem 2.2.1. *For a complete graph K_n, we have*

$$N[K_n; x] = \binom{n}{2}x^{n-2}, n \geq 2.$$

Proof. In K_n, any pair of vertices have $(n-2)$ common neighbors and there are $\binom{n}{2}$ such pairs of vertices. $\square$

Theorem 2.2.2. *For a path graph P_n where $n \geq 2$, we have*

$$N[P_n; x] = (n-2)x + \binom{n-1}{2} + 1.$$

Proof. Any pair (u_i, u_{i+2}) of vertices of P_n has one common neighbor for $i = 1, 2, \ldots, (n-2)$. All other pairs of vertices have no common neighbors and there are $\binom{n}{2} - (n-2) = \binom{n-1}{2} + 1$ such pairs. It follows that $N(P_n, 1) = n - 2$ and $N(P_n, 0) = \binom{n-1}{2} + 1$. Hence the result follows. $\square$

Lemma 2.2.3. *Let the vertices of a cycle be $u_1, u_2, \ldots, u_n$. Then the unordered vertex pairs (u_i, u_{i+2}), $i = 1, 2, \ldots, n$ (where the indices $i > n$ are taken modulo n) are all distinct unless $n = 4$.*

Proof. For $i = 1, 2, \ldots, n$, the vertex pairs under consideration are (u_1, u_3), (u_2, u_4), $\ldots, (u_{n-2}, u_n), (u_{n-1}, u_1), (u_n, u_2)$. Clearly, the first $n-2$ pairs are all distinct and the equality of pairs occur in the cases when $n-1=3$ or $n=4$. In both these cases, $n=4$. This completes the proof. $\square$

Theorem 2.2.4. *For a cycle graph C_n, we have the following:*

$$N[C_n;x] = \begin{cases} nx + \frac{n(n-3)}{2}, & n > 2, n \neq 4, \\ 2x^2+4, & n = 4. \end{cases}$$

Proof. By above lemma, if $n \neq 4$, the vertex pairs $(u_i, u_{i+2}), i = 1, 2, \ldots, n$ of C_n are all distinct and each has one common neighbor u_{i+1} where the indices $i > n$ are taken over modulo n. All other pairs of vertices have no common neighbors. Thus we have $N(C_n, 1) = n$ and $N(C_n, 0) = \binom{n}{2} - n = \frac{n(n-3)}{2}$. If $n = 4$, clearly $N[C_4; x] = 2x^2 + 4$. $\square$

Theorem 2.2.5. *For a complete bipartite graph $K_{m,n}$ where $m, n \geq 2$, we have*

$$N[K_{m,n};x] = \binom{m}{2}x^n + \binom{n}{2}x^m + mn.$$

Proof. Let M, N be the bipartite sets of vertices of $K_{m,n}$ and let $|V_M| = m$ and $|V_N| = n$. Any pair of vertices of M have n common neighbors and there are $\binom{m}{2}$ such pairs. Any pair of vertices of N have m common neighbors and there are $\binom{n}{2}$ such pairs. The pairs (u, v) where $u \in M$and $v \in N$ have no common neighbors and there are mn such pairs. Hence the result follows. $\square$

Corollary 2.2.6. *For a star graph $K_{1,n}$, $N[K_{1,n}; x] = \binom{n}{2}x + n$.*

Theorem 2.2.7. *The common neighbor polynomial of bistar graph $B_{n,n}$ is given by the relation*

$$N[B_{n,n};x] = n(n+1)x + (n+1)^2.$$

Proof. Let $B_{n,n}$ be a bistar graph which is the union of two star graphs $K_{1,n}$ with centres u and v together with a new edge uv. For $I = \{1, 2, \dots, n\}$, let $\{u_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ be the set of vertices of the star graphs with centers u and v respectively. The pairs (u_i, v) have one common neighbor u and the pairs (v_i, u) have one common neighbor v for $i = 1, 2, \dots, n$. And there are $2n$ such pairs. For $i, j \in I$ and $i \neq j$, the pairs (u_i, u_j) have one common neighbor u and the pairs (v_i, v_j) have one common neighbor v where there are $2\binom{n}{2}$ such pairs. The pairs $(u, v), (u_i, v_j), (u_i, u)$ and (v_i, v) have no common neighbors and there are $1 + n^2 + 2n$ such pairs. It follows that

$$\begin{aligned} N[B_{n,n}; x] &= \left(2n + 2\binom{n}{2}\right)x + 1 + n^2 + 2n \\ &= n(n+1)x + (n+1)^2. \end{aligned}$$

This completes the proof. □

Theorem 2.2.8. *For $n_i > 1, i = 1, 2, \dots, m,$ and $\sum_{i=1}^{m} n_i = N$, the common neighbor polynomial of complete m-partite graph $K_{n_1,n_2,\dots,n_m}$ is given by*

$$N[K_{n_1,n_2,\dots,n_m}; x] = \sum_{i=1}^{m} \binom{n_i}{2} x^{N-n_i} + \sum_{i\neq j; i,j\in\{1,2,\dots,m\}} n_i n_j x^{N-(n_i+n_j)}.$$

Proof. Let $V_1, V_2, \dots, V_m$ be the m-partite sets of vertices of $G = K_{n_1,n_2,\dots,n_m}$ with $|V_i| = n_i$ and $\sum_{i=1}^{m} n_i = N$. Let (u, v) be any pair of vertices of G. We consider the following two cases:

Case(i) Let $u, v \in V_i$; $i \in \{1, 2, \dots, m\}$. Then (u, v) has $N - n_i$ common neighbors and for each i, there are $\binom{n_i}{2}$ such pairs.

Case(ii) Let $u \in V_i$ and $v \in V_j$ where $i, j \in \{1, 2, \dots, m\}$ and $i \neq j$.

Then (u,v) has $N - n_i - n_j$ common neighbors and there are $n_i n_j$ such pairs. Hence the result follows. □

Corollary 2.2.9. *For a complete m partite graph $K_{n,n,\ldots,n}$ where n repeats m times,*

$$N[K_{n,n,\ldots,n};x] = m\binom{n}{2}x^{n(m-1)} + n^2\binom{m}{2}x^{n(m-2)}.$$

Proof. In the above theorem, put $n_i = n$ for each $i = 1, 2, \ldots, m$. □

Theorem 2.2.10. *For a lollipop graph $L_{n,1}$, we have*

$$N[L_{n,1};x] = \binom{n}{2}x^{n-2} + (n-1)x + 1.$$

Proof. The lollipop graph $L_{n,1}$ can be viewed as a complete graph K_n with a pendent vertex attached to one of its vertices through a bridge. Let $u_1, u_2, \ldots, u_n, v$ be the vertices of $L_{n,1}$ with the pendent vertex v joined to u_n with a bridge. Then any pair of vertices (u_i, u_j) share $(n-2)$ common neighbors and there are $\binom{n}{2}$ such pairs. The pair of vertices $(u_i, v), i = 1, 2, \ldots, (n-1)$ share one common neighbor u_n. The pair (u_n, v) has no common neighbor. Hence the result follows. □

Theorem 2.2.11. *For a wheel graph W_n, we have the following.*

$$N[W_n;x] = \begin{cases} \frac{(n-1)(n-4)}{2}x + 2(n-1)x^2, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 4x, & \text{if } n = 5. \end{cases}$$

Proof. Note that $W_n \cong C_{n-1} \vee K_1$. Let $u_1, u_2, \ldots, u_{n-1}$ be the vertices of C_{n-1} and let u_n be the vertex of K_1. Let (u,v) be any pair of vertices of W_n.

Case(i) Let $u, v \in \{u_1, u_2, \ldots, u_{n-1}\}$. Then the common neighbors of (u, v) in W_n are the common neighbors of (u, v) in C_{n-1} and the vertex u_n. Hence the number of common neighbors of vertex pairs (u, v) under this case equals one more than the number of common neighbors of (u, v) in C_{n-1}.

Case(ii) Let $u \in \{u_1, u_2, \ldots, u_{n-1}\}$ and $v = u_n$. Then the pairs (u, v) have 2 common neighbors u_{i+1} and u_{i-1} where all indices i are taken over modulo $(n-1)$. So we have,

$$\begin{aligned} N[W_n; x] =& xN(C_{n-1}; x) + (n-1)x^2 \\ =& \begin{cases} \frac{(n-1)(n-4)}{2}x + 2(n-1)x^2, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 4x, & \text{if } n = 5. \end{cases} \end{aligned}$$

This completes the proof. □

Theorem 2.2.12. *For a Helm H_n, we have the following*

$$N[H_n; x] = \begin{cases} 2(n-1)x^2 + \frac{(n-1)(n+2)}{2}x + \frac{(n-1)(3n-8)}{2}, & \textit{if } n \neq 5, \\ 2x^3 + 4x^2 + 16x + 14, & \textit{if } n = 5. \end{cases}$$

Proof. Let w be the centre vertex, $u_1, u_2, \ldots, u_{n-1}$ be the vertices on the wheel rim and $v_1, v_2, \ldots, v_{n-1}$ be the pendent vertices. Let (u, v) be any pair of vertices of H_n. We consider the following four cases:

Case(i) Let $u, v \in \{u_1, u_2, \ldots, u_{n-1}, w\}$. Then the number of vertex pairs (u, v) with i common neighbors in H_n equals number of vertex pairs (u, v) with i common neighbors in W_n.

Case(ii) Let $u \in \{u_1, u_2, \ldots, u_{n-1}\}$ and $v \in \{v_1, v_2, \ldots, v_{n-1}\}$. Then the pairs (u_i, v_{i+1}) and (u_i, v_{i-1}) have common neighbors u_{i+1} and u_{i-1} respectively where the indices i are taken modulo $(n-1)$. There are $2(n-1)$ such pairs. The remaining $(n-1)^2 - 2(n-1)$ pairs under this case have no common neighbors.

Case(iii) Let $u \in \{v_1, v_2, \ldots, v_{n-1}\}$ and $v = w$. Each such $(n-1)$ pairs (v_i, w) has exactly one common neighbor u_i.

Case(iv) $u, v \in \{v_1, v_2, \ldots, v_{n-1}\}$. There are no common neighbors for any of the $\binom{n-1}{2}$ pairs under this case.

So we obtain,

$$N[H_n; x] = N[W_n; x] + 2(n-1)x + (n-1)^2 - 2(n-1) + (n-1)x + \binom{n-1}{2}$$

$$= \begin{cases} 2(n-1)x^2 + \frac{(n-1)(n+2)}{2}x + \frac{(n-1)(3n-8)}{2}, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 16x + 14, & \text{if } n = 5. \end{cases}$$

This completes the proof. □

Theorem 2.2.13. *For a web graph WB_n where $n > 3$, we have*

$$N[WB_n; x] = \begin{cases} 4(n-1)x^2 + \frac{(n-1)(n+6)}{2}x + (n-1)(4n-10), & \text{if } n \neq 5, \\ 2x^3 + 14x^2 + 20x + 42, & \text{if } n = 5. \end{cases}$$

Proof. Let w be the center vertex, $u_1, u_2, \ldots, u_{n-1}$ be the vertices on the inner wheel rim, $v_1, v_2, \ldots, v_{n-1}$ be the vertices on the outer wheel rim and let $w_1, w_2, \ldots, w_{n-1}$ be the pendent vertices of the web graph. Let (u, v) be any pair of vertices of the web graph WB_n. We will consider the following 8 cases:

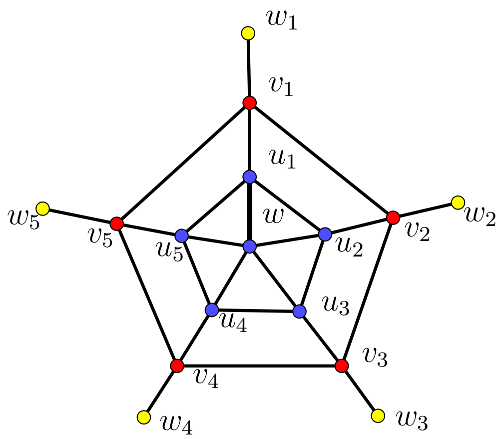

Figure 2.2: The web graph W_6

Case(i) Let $u, v \in \{u_1, u_2, \ldots, u_{n-1}, w\}$. Then number of vertex pairs (u, v) with i common neighbors in WB_n equals the number of vertex pairs with i common neighbors in W_n.

Case(ii) Let $u, v \in \{v_1, v_2, \ldots, v_{n-1}\}$. Then number of vertex pairs (u, v) with i common neighbors in WB_n equals the number of vertex pairs with i common neighbors in the cycle C_{n-1}.

Case(iii) Let $u \in \{u_1, u_2, \ldots, u_{n-1}\}$ and $v \in \{v_1, v_2, \ldots, v_{n-1}\}$. All the pairs (u_i, v_{i-1}) and (u_i, v_{i+1}) have two common neighbors u_{i+1} and v_i where the indices $i > 1$ are taken modulo n and there are $2(n-1)$ such pairs. All the remaining $(n-1)^2 - 2(n-1)$ pairs have no common neighbors.

Case(iv) Let $u \in \{v_1, v_2, \ldots, v_{n-1}\}$ and $v = w$. All the $n-1$ pairs $(v_i, w), i = 1, 2, \ldots, n-1$ have one common neighbor u_i.

Case(v) Let $u, v \in \{w_1, w_2, \ldots, w_{n-1}\}$. Then (u, v) has no common neighbors and there are $\binom{n-1}{2}$ such pairs.

Case(vi) Let $u \in \{w_1, w_2, \ldots, w_{n-1}\}$ and $v \in \{v_1, v_2, \ldots, v_{n-1}\}$. The vertex pairs (w_i, v_{i+1}) and (w_i, v_{i-1}) have one common neighbor v_i where the indices $i > 1$ are taken modulo n and there are $2(n-1)$ such pairs. All the remaining $(n-1)^2 - 2(n-1)$ pairs have no common neighbors.

Case(vii) Let $u \in \{w_1, w_2, \ldots, w_{n-1}\}$ and $v \in \{v_1, v_2, \ldots, v_{n-1}\}$. The pairs (w_i, u_i) have one common neighbor v_i and there are $n-1$ such pairs. All the remaining $(n-1)^2 - (n-1)$ pairs have no common neighbors.

Case(viii) $u \in \{w_1, w_2, \ldots, w_{n-1}\}$ and $v = w$.

The pairs (w_i, w) have no common neighbors and there are $n-1$ such pairs. So we obtain

$$\begin{aligned}
N[WB_n; x] &= N[W_n; x] + N[C_{n-1}; x] + 2(n-1)x^2 + \\
&\quad \big[(n-1)^2 - 2(n-1)\big] + (n-1)x + \binom{n-1}{2} + \\
&\quad 2(n-1)x + \big[(n-1)^2 - 2(n-1)\big] + (n-1)x + \\
&\quad \big[(n-1)^2 - (n-1)\big] + (n-1). \\
&= 4(n-1)x^2 + \frac{(n-1)(n+6)}{2}x + (n-1)(4n-10), n \neq 5. \\
N[WB_5; x] &= N[W_5; x] + N[C_4; x] + 8x^2 + 16x + 38. \\
&= (2x^3 + 4x^2 + 4x) + (2x^2 + 4) + 8x^2 + 16x + 38. \\
&= 2x^3 + 14x^2 + 20x + 42.
\end{aligned}$$

This completes the proof. $\square$

Theorem 2.2.14. *For a shell graph S_n, we have*

$$N[S_n; x] = 2(n-3)x^2 + \left[\binom{n-2}{2} + 3\right]x.$$

Proof. S_n can be considered as the join of P_{n-1} and K_1. Let $u_1, u_2, \ldots, u_{n-1}$ be the vertices of P_{n-1} and let u be the vertex of K_1.

Case(i) The vertex pairs $(u, u_i), i = 2, 3, \ldots, (n-2)$ have 2 common neighbors u_{i-1} and u_{i+1}. The pair (u, u_1) has only one common neighbor u_2 and the pair (u, u_{n-1}) has only one common neighbor u_{n-2}.

Case(ii) The vertex pairs $(u_i, u_{i+2}), i = 1, 2, \ldots, (n-3)$ have two common neighbors u_{i+1} and u. All other pairs (u_i, u_j) where $i, j \in \{1, 2, \ldots, (n-1)\}$ and $i \neq j, j-2$ have only one common neighbor u. And there are $\binom{n-1}{2} - (n-3)$ such pairs.

Thus we have

$$\begin{aligned} N[S_n; x] &= (n-3)x^2 + 2x + (n-3)x^2 + \left[\binom{n-1}{2} - (n-3)\right]x \\ &= 2(n-3)x^2 + \left[\binom{n-2}{2} + 3\right]x. \end{aligned}$$

This completes the proof. □

Theorem 2.2.15. *If B_N is a bow graph with $N \geq 5$ vertices, then*

$$N[B_N; x] = 2(N-5)x^2 + \left[\frac{N(N-5)}{2} + 10\right]x.$$

Proof. Let the bow graph B_N includes the shells S_n and S_m with the unique apex w. Then, $N = n + m - 1$. Any pair of vertices of S_n has as many common neighbors in B_N as in S_n and any pair of vertices of S_m has as many common neighbors in B_N as in S_m. Any vertex pair (u, v) where $u \in S_n$, $v \in S_m$ and $u, v \neq w$ has only one common neighbor w and there are $(n-1)(m-1)$ such pairs.

$$N[B_N; x] = N[S_n; x] + N[S_m; x] + (n-1)(m-1)x.$$

$$= 2(n-3)x^2 + \Big[\binom{n-2}{2} + 3\Big]x + 2(m-3)x^2 +$$

$$\Big[\binom{m-2}{2} + 3\Big]x + (n-1)(m-1)x.$$

$$= 2(n+m-1-5)x^2 + \left[\frac{(n+m-1)^2 - 5(n+m-1) + 20}{2}\right]x.$$

$$= 2(N-5)x^2 + \Big[\frac{N^2-5N+20}{2}\Big]x.$$

$$= 2(N-5)x^2 + \left[\frac{N(N-5)}{2} + 10\right]x.$$

This completes the proof. □

Theorem 2.2.16. *If BF is a butterfly graph with $N \geq 7$ vertices, then*

$$N[BF;x] = 2(N-7)x^2 + \left[\frac{N(N-5)}{2} + 12\right]x + 2.$$

Proof. A butterfly graph BF with N vertices includes a bow graph B_{N-2} with $N-2$ vertices and two pendent vertices at the apex w. Any pair of vertices of B_{N-2} has as many common neighbors in BF as in B_{N-2}. If u, v are pendent vertices, (u, v) has only one common neighbor w. If u is a pendent vertex and v the apex vertex, there are no common neighbors for the vertex pair (u, v) and there are 2 such pairs. If u is a pendent vertex and v is any of the vertices of B_{N-2} other than the apex w, there is only one common neighbor for the vertex pair (u, v) where there are $2(N-3)$ such pairs. Thus we have

$$N[BF;x] = N[B_{N-2};x] + x + 2 + 2(N-3)x.$$

$$= 2(N-7)x^2 + \left[\frac{(N-2)(N-7)}{2} + 10\right]x + x + 2 + 2(N-3)x.$$

$$= 2(N-7)x^2 + \Big[\frac{N^2-9N+14+20+2+4N-12}{2}\Big]x + 2.$$

$$= 2(N-7)x^2 + \Big[\frac{N^2-5N+24}{2}\Big]x + 2.$$

$$= 2(N-7)x^2 + \left[\frac{N(N-5)}{2} + 12\right]x + 2.$$

This completes the proof. □

Theorem 2.2.17. *For the friendship graph F_n, we have*

$$N[F_n; x] = n(2n+1)x.$$

Proof. A friendship graph F_n is the one point union of n copies of the cycle C_3. Let (u, v) be any pair of vertices of F_n. We consider the following 2 cases.

Case(i) Let u and v be vertices of the i^{th} copy of C_3 where $i = 1, 2, \ldots, n$. Then there is one common neighbor for each pair (u, v) and there are 3 such pairs corresponding to each i.

Case(ii) Let u and v be vertices of i^{th} copy of C_3 and j^{th} copy of C_3 respectively, other than the centre vertex where $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. Then every pair (u, v) has one common neighbor and there are $4\binom{n}{2}$ such pairs.

It follows that

$$\begin{aligned} N[F_n; x] &= 3nx + 4\binom{n}{2}x \\ &= n(2n+1)x. \end{aligned}$$

This completes the proof. □

Theorem 2.2.18. *If G is a graph having 2 components G_1 and G_2 with n and m vertices respectively, then $N[G; x] = N[G_1; x] + N[G_2; x] + nm$.*

Proof. For $k = 1, 2$ the number of vertex pairs of G_k with i common neighbors in G equals the number of vertex pairs of G_k with i common neighbors in G_k. Now, for a pair of vertices (u, v) in G with $u \in G_1$ and $v \in G_2$, there is no common neighbor; and there are nm such pairs. □

Corollary 2.2.19. *Let G be a graph having m components $G_1, G_2, \ldots, G_m$ where $|V(G_i)| = n_i$ for $i = 1, 2, \ldots, m$. Then for $I = \{1, 2, \ldots, m\}$, we have,*

$$N[G; x] = \sum_{i \in I} N[G_i; x] + \sum_{\substack{i,j \in I \\ i \neq j}} n_i n_j.$$

Proof. The proof follows from Theorem 2.2.18, using mathematical induction on the number of components m of G. □

Theorem 2.2.20. *If G is a connected graph and $e = uv$ a cutedge of G, then $N[G-e; x] = N[G; x] - [d_u(G-e) + d_v(G-e)]x + [d_u(G-e) + d_v(G-e)]$ where $d_u(G)$ denote the degree of the vertex u in G.*

Proof. Let $G-e$ has two components G_1 and G_2, where $u \in G_1$ and $v \in G_2$. If u_i is a neighbor of u in G_1, the pair (u_i, v) has exactly one common neighbor u in G for $i = 1, 2, \ldots, d_u(G_1)$. If v_i is a neighbor of v in G_2, the pair (v_i, u) has exactly one common neighbor v in G for $i = 1, 2, \ldots, d_v(G_2)$. Therefore, deletion of the cut edge $e = uv$ reduces the number of vertex pairs with 1 common neighbor by $d_u(G-e) + d_v(G-e)$ and increases the number of vertex pairs with no common neighbors by $d_u(G-e) + d_v(G-e)$. And the deletion of a cutedge produces no change in the number of vertex pairs with more than one common neighbors. □

Corollary 2.2.21. *$N[P_n - e; x] = N[P_n; x] - 2x + 2$ if e is not a pendent edge of P_n and $N[P_n - e; x] = N[P_n; x] - x + 1$ if e is a pendent edge of P_n.*

Proof. The result follows from Theorem 2.2.20 using the fact that all the edges of P_n are cutedges of P_n. □

Corollary 2.2.22. *For a tadpole graph $T_{(n,l)}$ with $n > 2$, we have the following:*

$$N[T_{(n,l)}; x] = \begin{cases} N[C_n; x] + N[P_l; x] + 3x + nl - 3, & \text{if } l > 1, \\ N[C_n; x] + N[P_l; x] + 2x + nl - 2, & \text{if } l = 1. \end{cases}$$

Proof. A Tadpole $T_{(n,l)}$ is a graph obtained by attaching a path P_l to one of the vertices of the cycle C_n by a bridge. Let the vertex u of P_l be attached to the vertex v of C_n through the bridge uv. Removing the bridge uv from $T_{n,l}$, the resulting graph is the union of the path P_l and the cycle C_n. So the result follows from Theorems 2.2.18 and 2.2.20. □

Corollary 2.2.23. *For a n-barbell graph $B_{n,1}$, we have*

$$N[B_{n,1}; x] = 2\binom{n}{2}x^{n-2} + 2(n-1)x + (n-1)^2 + 1.$$

Proof. The n- barbell graph $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge. Let two copies of K_n be connected by the bridge e.

Note that $B_{n,1} - e = K_n + K_n$, the disjoint union of two copies of K_n.

$$\begin{aligned} N[B_{n,1}; x] &= N[B_{n,1} - e; x] + 2(n-1)x - 2(n-1). \\ &= N[K_n + K_n; x] + 2(n-1)x - 2(n-1) \end{aligned}$$

By Theorem 2.2.18 it follows that

$$\begin{aligned} N[B_{n,1}; x] &= 2\binom{n}{2}x^{n-2} + n^2 + 2(n-1)x - 2(n-1) \\ &= 2\binom{n}{2}x^{n-2} + 2(n-1)x + (n-1)^2 + 1. \end{aligned}$$

This completes the proof. □

Corollary 2.2.24. *For a lollipop graph $L_{m,n}$,*

$$N[L_{m,n}; x] = \binom{m}{2} x^{m-2} + (m + n - 2)x + \binom{n-1}{2} + m(n - 1) + 1.$$

Proof. The lollipop graph $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path P_n with a bridge e. Note that $L_{m,n} - e = K_m + P_n$, the disjoint union of complete graph K_n and path graph P_n. Therefore using theorems 2.2.18 and 2.2.20, it follows that,

$$\begin{aligned}
N[L_{m,n}; x] &= N[L_{m,n} - e; x] + mx - m \\
&= N[K_m + P_n; x] + mx - m \\
&= \binom{m}{2} x^{m-2} + (n - 2)x + \binom{n-1}{2} + 1 + mn + mx - m \\
&= \binom{m}{2} x^{m-2} + (m + n - 2)x + \binom{n-1}{2} + m(n - 1) + 1.
\end{aligned}$$

This completes the proof. □

Corollary 2.2.25. *For a bistar graph $B_{m,n}$,*

$$N[B_{m,n}; x] = \left[\binom{m}{2} + \binom{n}{2} + m + n\right] x + m + n + mn + 1.$$

Proof. A bistar graph $B_{m,n}$ is obtained by connecting the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by a bridge e. Let $K_{1,m}$ and $K_{1,n}$ be joined by the bridge e to form $B_{m,n}$. Note that $B_{m,n} - e = K_{1,m} + K_{1,n}$. Applying Theorems 2.2.18 and 2.2.20 we have,

$$\begin{aligned}
N[B_{m,n}; x] &= N[B_{m,n} - e; x] + (m + n)x - (m + n). \\
&= N[K_{1,m} + K_{1,n}; x] + (m + n)x - (m + n). \\
&= \binom{m}{2} x + m + \binom{n}{2} x + n + (m + 1)(n + 1) + (m + n)x - (m + n).
\end{aligned}$$

$$= \left[\binom{m}{2} + \binom{n}{2} + m + n\right] x + m + n + mn + 1.$$

This completes the prof. □

Theorem 2.2.26. *For a bipartite Cocktail party graph B_n,*

$$N[B_n; x] = 2\binom{n}{2} x^{n-2} + n^2.$$

Proof. The bipartite Cocktail party graph B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$. Let U, V be the bipartite sets of vertices of B_n. For $i, j \in \{1, 2, \ldots, n\}$, every pair (u_i, u_j) where $u_i, u_j \in U$ and every pair (v_i, v_j) where $v_i, v_j \in V$ have $n - 2$ common neighbors each. There are $2\binom{n}{2}$ such pairs of vertices in B_n. A vertex pair of the form (u_i, v_j) where $u_i \in U$ and $v_j \in V$ has no common neighbors and there are n^2 pairs of vertices under this case. Hence the result follows. □

Theorem 2.2.27. *For a windmill graph $W_n^{(m)}$,*

$$N[W_n^{(m)}; x] = m\binom{n}{2} x^{n-2} + \binom{m}{2}(n-1)^2 x.$$

Proof. The Windmill graph $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common. A vertex pair with vertices of same K_n has as many common neighbors in $W_n^{(m)}$ as in K_n. A vertex pair with vertices other than the common vertex taken from two distinct copies of K_n has exactly one common neighbor which is the common vertex. There are $\binom{m}{2}(n-1)^2$ such vertex pairs. It follows that

$$\begin{aligned} N[W_n^{(m)}; x] &= mN[K_n; x] + \binom{m}{2}(n-1)^2 x \\ &= m\binom{n}{2} x^{n-2} + \binom{m}{2}(n-1)^2 x. \end{aligned}$$

This completes the proof. □

Dutch Windmill graph $D_n^{(m)}$ is a Windmill graph $W_n^{(m)}$ with $n = 3$.

Corollary 2.2.28. $N[D_n^{(m)}; x] = m(2m+1)x$.

Theorem 2.2.29. *For an armed crown $C_n \odot P_m$, we have the following:*

$$N[C_n \odot P_m; x] = N[C_n; x] + n\ N[P_{m+1}; x] + 2nx + \binom{n}{2} m(m+2) - 2n.$$

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n and let $w_{i1}, w_{i2}, \ldots, w_{im}$ be the vertices of the i^{th} copy of P_m attached to the i^{th} vertex of C_n. Let (u, v) be a pair of vertices of $C_n \odot P_m$. Here we consider the following 4 cases.

Case(i) Let $u, v \in \{u_1, u_2, \ldots, u_n\}$. Number of vertex pairs (u, v) with i common neighbors in $C_n \odot P_m$ equals number of vertex pairs with i common neighbors in C_n.

Case(ii) Let $u \in \{w_{i1}, w_{i2}, \ldots, w_{im}\}$ and $v \in \{w_{j1}, w_{j2}, \ldots, w_{jm}\}$; $i \neq j$; $i, j = 1, 2, \ldots, n$.

Then (u, v) have no common neighbors and there are $\binom{n}{2}\ m^2$ such pairs.

Case(iii) Let $u, v \in \{w_{i1}, w_{i2}, \ldots, w_{im}, u_i\}, i = 1, 2, \ldots, n$. Then number of vertex pairs (u, v) with i common neighbors in $C_n \odot P_m$ equals number of vertex pairs with i common neighbors in P_{m+1}. And there are $n\ |N(P_{m+1}, i)|$ such pairs.

Case(iv) Let $u = u_i$ and $v \in \{w_{j1}, w_{j2}, \ldots, w_{jm}\}, i \neq j; i, j = 1, 2, \ldots, n$.

Then the pairs $(u_i, w_{(i-1)1})$ and $(u_i, w_{(i+1)1})$ have common neighbors u_{i-1} and u_{i+1} respectively. There are $2n$ such vertex pairs with 1 common neighbor. The remaining $m(n-1)n - 2n$ vertex pairs under case(iv) have no common neighbors.

It follows that

$$N[C_n \odot P_m;x] = N[C_n;x] + n\ N[P_{m+1};x] + \binom{n}{2}\ m^2 + 2nx + mn(n-1) - 2n$$
$$= N[C_n;x] + n\ N[P_{m+1};x] + 2nx + \binom{n}{2} m(m+2) - 2n.$$

This completes the proof. $\square$

A flower graph[10] $f_{n\times m}$ is a graph with a n-cycle and n number of m-cycles each intersects with the n-cycle on a unique single edge .

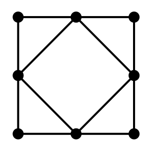

Figure 2.3: The flower graph $f_{4\times 3}$

Theorem 2.2.30. *If $f_{n\times m}$ is a flower graph, then, the following results hold:*

1. *If $m \neq 4$, $N[f_{n\times m};x] = N[C_n;x] + n\ N[P_{m-2};x] + 5nx + (m-2)n^2 + \binom{n}{2}(m-2)^2 - 5n$.*

2. *If $m = 4$, $N[f_{n\times m};x] = N[C_n;x] + 2nx^2 + 3nx + 4n^2 - 6n$.*

Proof. Let C_n be the inner cycle and $C_m^1, C_m^2, \ldots, C_m^n$ be the m-cycles having one of the edges common to C_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n and for each $j \in \{1, 2, \ldots, n\}$, let $U_j = \{u_1^j, u_2^j, \ldots, u_{m-2}^j\}$ be the set of $m-2$ vertices which form the m-cycle C_m^j together with the edge $v_j v_{j+1}$ of C_n. Let (u, v) be any pair of vertices of $f_{n,m}$. We consider the following 3 cases.

Case(i) Let $u, v \in V(C_n)$.

Then the number of pairs (u, v) with i common neighbors in $f_{n,m}$ equals $|N(C_n, i)|$.

Case(ii) Let $u, v \in U_j$ where $j \in \{1, 2, \ldots, n\}$.

Then for each $j \in \{1, 2, \ldots, n\}$ the number of pairs (u, v) with i common neighbors in $f_{n,m}$ equals $|N(P_{m-2}, i)|$.

Case(iii) Let $u \in U_j$ and $v \in U_k$ where $j, k \in \{1, 2, \ldots, n\}$ and $j \neq k$. Then the n pairs (u^{j-1}_{m-2}, u^j_1) has exactly one common neighbor v_j where the index j is taken modulo m. All other $\binom{n}{2}(m-2)^2 - n$ pairs of vertices under this case have no common neighbors.

Case(iv) Let $u \in V(C_n)$ and $v \in U_j$ where $j \in \{1, 2, \ldots, n\}$.

Then the pairs of the form (u^j_1, v_{j-1}) and (u^{j-1}_{m-2}, v_{j+1}) has exactly one common neighbor v_j . Also the pairs (u^j_1, v_{j+1}) has exactly one common neighbor v_j if $m \neq 4$ and has two common neighbors u^j_{m-2} and v_j if $m = 4$. Similarly, the pairs (u^j_{m-2}, v_j) has one common neighbor v_{j+1} if $m \neq 4$ and has two common neighbors u^j_1 and v_{j+1} if $m = 4$. All other $(m-2)n^2 - 4n$ pairs under this case has no common neighbors.

It follows that

1. If $m \neq 4$, $N[f_{n\times m}; x] = N[C_n; x] + n\ N[P_{m-2}; x] + 4nx + (m-2)n^2 - 4n + nx + \binom{n}{2}(m-2)^2 - n.$
   $= N[C_n; x] + n\ N[P_{m-2}; x] + 5nx + (m-2)n^2 + \binom{n}{2}(m-2)^2 - 5n.$

2. If $m = 4$, $N[f_{n\times m}; x] = N[C_n; x] + n\ N[P_2; x] + 2nx^2 + 2nx + 2n^2 - 4n +$

$nx + 4\binom{n}{2} - n$.

$=N[C_n; x] + 2nx^2 + 3nx + 4n^2 - 6n$.

This completes the proof. □

A chaplet graph[38] $C_p \odot C_q^t$ where $p, q, t \geq 3$ is obtained by taking one point union of t-copies of the cycle C_q and attaching the same to each vertex of the cycle C_p.

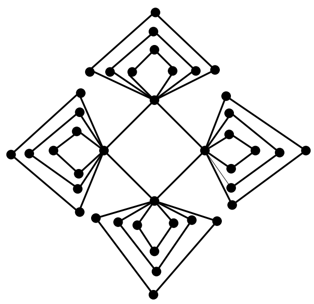

Figure 2.4: The chaplet graph $C_4 \odot C_4^3$

Theorem 2.2.31. $N[C_p \odot C_q^t; x] = N[C_p; x] + tpN[C_q; x] + [4tp + 3pt(t-1)]x + \binom{p}{2}t^2(q-1)^2 + p(q^2 - 2q - 5)\binom{t}{2} + [(p-1)(q-1) - 4]tp$.

Proof. Let $u_1, u_2, \ldots, u_p$ be the vertices of the cycle C_p. For $j \in \{1, 2, \ldots, t\}$ and $k \in \{1, 2, \ldots, p\}$, let $u_k, u_{k1}^j, u_{k2}^j, \ldots, u_{k(q-1)}^j$ be the vertices of j^{th} copy of the cycle C_q attached to the vertex u_k of C_p. Let (u, v) be any pair of vertices of $C_p \odot C_q^t$. We consider the following cases:

Case(i) Let $u, v \in \{u_1, u_2, \ldots, u_p\}$.

In this case, the number of vertex pairs (u, v) with i common neighbors equals $|N(C_p, i)|$.

Case(ii) Let $u, v \in \{u_k, u^j_{k1}, u^j_{k2}, \ldots, u^j_{k(q-1)}\}$ where $j \in \{1, 2, \ldots, t\}$ and $k \in \{1, 2, \ldots, p\}$.

Fixing the variables j, k, the number of vertex pairs (u, v) with i common neighbors equals $|N(C_q, i)|$ and there are tp choices for fixing j and k.

Case(iii) Let $u \in \{u^j_{k1}, u^j_{k2}, \ldots, u^j_{k(q-1)}\}$ and $v \in \{u_1, u_2, \ldots, u_{k-1}, u_{k+1}, \ldots, u_p\}$ where $j \in \{1, 2, \ldots, t\}$ and $k \in \{1, 2, \ldots, p\}$.

In this case, pairs of vertices of the form (u^j_{k1}, u_{k+1}), (u^j_{k1}, u_{k-1}), $(u^j_{k(q-1)}, u_{k+1})$ and $(u^j_{k(q-1)}, u_{k-1})$ have exactly one common neighbor u_k and there are $4tp$ pairs of vertices of this form. All other vertices under this case have no common neighbors and there are $(p-1)(q-1)tp - 4tp$ such pairs.

Case(iv) Let $u \in \{u^j_{k1}, u^j_{k2}, \ldots, u^j_{k(q-1)}\}$, $v \in \{u^l_{k1}, u^l_{k2}, \ldots, u^l_{k(q-1)}\}$ where $j, l \in \{1, 2, \ldots, t\}$, $k \in \{1, 2, \ldots, p\}$ and $j \neq l$.

In this case, pairs of vertices of the form (u^j_{k1}, u^l_{k1}),$(u^j_{k(q-1)}, u^l_{k(q-1)})$ and $(u^j_{k1}, u^l_{k(q-1)})$ have exactly one common neighbor u_k and there are $2p\binom{t}{2} + pt(t-1) = 4p\binom{t}{2}$ pairs of vertices of this form. All the remaining vertices under this case have no common neighbors and the number of such vertices are given by $\binom{t}{2}p(q-1)^2 - 4p\binom{t}{2}$ which equals $p(q^2 - 2q - 3)\binom{t}{2}$.

Case(v) Let $u \in \{u^j_{k1}, u^j_{k2}, \ldots, u^j_{k(q-1)}\}$, $v \in \{u^l_{s1}, u^l_{s2}, \ldots, u^l_{s(q-1)}\}$ where $j, l \in \{1, 2, \ldots, t\}$ and $k, s \in \{1, 2, \ldots, p\}$ and $k \neq s$.

In this case the pairs of vertices (u, v) have no common neighbors and there are $\binom{p}{2}t^2(q-1)^2$ such vertex pairs.

Hence it follows that

$$N[C_p \odot C^t_q; x] = N[C_p; x] + tpN[C_q; x] + 4tp\ x + [(p-1)(q-1) - 4]tp+$$

$$
\begin{aligned}
&4p\binom{t}{2}x + p(q^2-2q-3)\binom{t}{2} + \binom{p}{2}t^2(q-1)^2 \\
&= N[C_p;x] + tpN[C_q;x] + [4tp + 2pt(t-1)]x + \binom{p}{2}t^2(q-1)^2 \\
&\quad + p(q^2-2q-3)\binom{t}{2} + [(p-1)(q-1)-4]tp.
\end{aligned}
$$

This completes the proof. □

A snake graph[19] $S_{n,m}$ is obtained from a path graph P_n by replacing each edge of P_n by the cycle graph C_m . $S_{n,3}$ is known as the triangular snake graph and $S_{n,4}$ the rectangular snake graph.

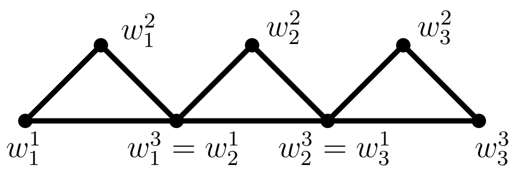

Figure 2.5: The snake graph $S_{3,3}$

Theorem 2.2.32. *For a snake graph $S_{n,m}$ we have,*

$$N[S_{n,m};x] = nN[C_m;x] + 4(n-1)x + [(m-1)^2-4](n-1) + (m-1)^2\binom{n-1}{2}.$$

Proof. Let the vertices of the i^{th} cycle of $S_{n,m}$ be represented by $w_i^1, w_i^2, \ldots, w_i^m$ respectively. Let (u,v) be any pair of vertices of $S_{n,m}$. We will consider 3 cases:

Case(i) Let $u, v \in \{w_i^1, w_i^2, \ldots, w_i^m\}$; $i \in \{1, 2, \ldots, n\}$.

Then for each i, the number of vertex pairs (u,v) with k common neighbors equals $|N(C_m,k)|$.

Case(ii) Let $u \in \{w_i^1, w_i^2, \ldots, w_i^{m-1}\}$ and $v \in \{w_{i+1}^2, w_{i+1}^3, \ldots, w_{i+1}^m\}$ where $i \in \{1, 2, \ldots, n-1\}$.

Then the pairs (w_i^1, w_{i+1}^2), (w_i^1, w_{i+1}^m), (w_i^{m-1}, w_{i+1}^2), (w_i^{m-1}, w_{i+1}^m) have exactly one common neighbor w_i^m and there are $4(n-1)$ such pairs. The remaining $[(m-1)^2 - 4](n-1)$ pairs under this case have no common neighbors.

Case(iii) Let $u \in \{w_i^1, w_i^2, \ldots, w_i^{m-1}\}$ and $v \in \{w_j^2, w_j^3, \ldots, w_j^m\}$; $i \in \{1, 2, \ldots, n-2\}$ and $j \in \{i+2, i+3, \ldots, n\}$.

The vertex pairs under this case have no common neighbors and there are $(m-1)^2 \sum_{i=1}^{n-2}(n-i-1) = (m-1)^2\binom{n-1}{2}$ such pairs.

It follows that $N[S_{n,m}; x] = nN[C_m; x] + 4(n-1)x + [(m-1)^2 - 4](n-1) + (m-1)^2\binom{n-1}{2}$. □

Corollary 2.2.33. *For a triangular snake graph $S_{n,3}$, we have the following:*

$$N[S_{n,3}; x] = nN[C_3; x] + 4(n-1)x + 4\binom{n-1}{2}.$$

2.3 Strongly regular graphs

The concept of strongly regular graphs was introduced by R C Bose[34] as follows: A simple k-regular graph G on n vertices is said to be strongly regular of type (n, k, λ, l) if there exists integers λ, l such that any adjacent pair of vertices of G have exactly λ common neighbors and any non-adjacent pair of vertices of G have exactly l common neighbors. From the definition itself, it follows that if G

is a strongly regular graph of type (n, k, λ, l),

$$N[G; x] = mx^{\lambda} + \left[\binom{n}{2} - m\right]x^{l}$$

where m is the number of edges of G.

Many interesting graphs like Petersen graph, Clebsch graph, Shrikhande graph etc. are known to be strongly regular. Hence their common neighbor polynomial can be easily evaluated. Some of the results are as follows:

1. For a cycle graph C_5 which is strongly regular of type $(5, 2, 0, 1)$, we have $N[C_5; x] = 5 + [\binom{5}{2} - 5]x = 5x + 5$.

2. The Petersen graph P which is strongly regular of type $(10, 3, 0, 1)$ contains 15 edges and hence $N[P; x] = 30x + 15$.

3. Srikhande graph S is a named graph with 48 edges which is discovered by renowned Indian Mathematician S.S. Srikhande. It has many interesting properties including the one that it is strongly regular of type $(16, 6, 2, 2)$. So its common neighbor polynomial is given by $N[S; x] = 120x^2$.

4. A $n \times n$ square rook's graph $RK_{n\times n}$ which is the line graph of complete bipartite graph $K_{n,n}$ represents all legal moves of 'rook' on a chessboard. It is known to be strongly regular of type $(n^2, 2n - 2, n - 2, 2)$. $RK_{n\times n}$ contains $n^3 - n^2$ edges. It follows that

$$\begin{aligned} N[RK_{n\times n}; x] &= (n^3 - n^2)x^{n-2} + [\binom{n^2}{2} - (n^3 - n^2)]x^2 \\ &= n^2(n-1)x^{n-2} + 2\binom{n}{2}^2 x^2. \end{aligned}$$

5. Chang graphs are named after Chang Li-Chien who revealed[5] some interesting properties of the graphs. These are a set of three 12-regular graphs

with 28 vertices and 168 edges which are obtained by graph switching of the line graph of K_8. Chang graphs are strongly regular of type $(28, 12, 6, 4)$ and so has the common neighbor polynomial $168x^6 + 210x^4$.

6. Paley graphs have vertices from a finite field and two vertices are connected if their difference is a square in the field. Godsil and Royle[4] proved that paley graphs are strongly regular of type $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$. If m is the number of edges of a paley graph $P(q)$ on q vertices, then $N[P(q); x] = mx^{\frac{1}{4}(q-5)} + [\binom{q}{2} - m]x^{\frac{1}{4}(q-1)}$.

2.4 Common neighbor polynomial of trees

In this section we study common neighbor polynomial of tree graphs, in particular the rooted trees and caterpillar trees.

Theorem 2.4.1. *Let T be a tree on n vertices. Let v be a vertex of T with degree k. If T' is a tree obtained from T by attaching p pendent edges at the vertex v, we have the following:*

$$N[T'; x] = N[T; x] + \frac{p}{2}(2k + p - 1)x + p(n - k).$$

Proof. Let $\{v_1, v_2, \ldots, v_k\}$ be the neighbors of v in T. When we attach a pendent edge vw to v, n number of new pairs of vertices are introduced in which the pairs (v_i, w) where $i \in \{1, 2, \ldots, k\}$ have one common neighbor v and the remaining $n - k$ new pairs have no common neighbors. There will be no change in the number of common neighbors of pairs of vertices of T by the introduction of the pendent edge vw. Hence the common neighbor polynomial becomes $N[T; x] +$

$kx + (n-k)$. Repeating the process p times, after attaching the p-th pendent edge to v, the common neighbor polynomial of resulting graph becomes,

$$\begin{aligned} N[T';x] &= N[T;x] + kx + (n-k) + (k+1)x + (n-k) + \dots \\ &\qquad + (k+p-1)x + (n-k) \\ &= N[T;x] + [k + (k+1) + (k+2) + \dots + (k+p-1)]x + p(n-k) \\ &= N[T;x] + \frac{p}{2}(2k+p-1)x + p(n-k). \end{aligned}$$

This completes the proof. □

Theorem 2.4.2. *Let T be a complete m-ary tree with p levels where the root vertex is considered to be in the 0-th level. Then we have the following:*

$$\begin{aligned} N[T;x] = \frac{m^2(m^{p-1}-1)}{m-1}x + \binom{m}{2}\frac{m^p-1}{m-1}x + \frac{m[m^{2p-2}-m^p+m-1]}{m-1} + m^{2p-1} \\ + \sum_{i=0}^{p-3}\left[\frac{m^{2i+3}-m^{p+i+1}}{1-m}\right] + \frac{m^2[m^{2p}-m^{p+1}-m^p+m]}{2(m^2-1)}. \end{aligned}$$

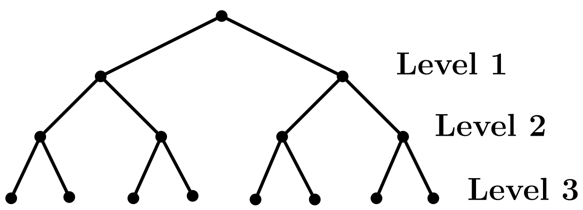

Figure 2.6: Complete binary tree of level 3

Proof. Let (u,v) be any pair of vertices of T. Here we consider 4 different cases according to the levels in which the vertices u and v lie in T.

Case(i) For $i \in \{0,1,2,\dots,p-2\}$ let u be a vertex in the i-th level and v a vertex in the $(i+1)^{th}$ or $(i+2)^{th}$ level.

If v is an $(i+1)^{th}$ level vertex, the vertex pair (u,v) have no common neighbors and there are $m^i m^{i+1}$ such pairs of vertices in T. If v is in the $(i+2)^{th}$ level, then there are $m^i m^{i+2}$ pairs of vertices (u, v) in which $m^i[m^{i+2} - m^2]$ pairs of vertices have no common neighbors and $m^i m^2$ pairs have exactly one common neighbor.

Case(ii) Let u be a vertex in the $(p-1)$-th level and v a vertex in the p-th level. In this case the pairs of vertices (u, v) have no common neighbors and there are $m^{p-1}m^p$ such pairs of vertices.

Case(iii) For $i \in \{0, 1, 2, \ldots, p-3\}$ let u be a vertex in the i-th level and v a vertex in the j-th level where $j = i+3, i+4, \ldots, p$.
All the pairs of vertices under this case have no common neighbors and there are $m^i[m^{i+3} + m^{i+4} + \ldots + m^p]$ such pairs of vertices.

Case(iv) For $i \in \{1, 2, \ldots, p\}$ let u and v be vertices of same level.
In this case $\frac{1}{2}m^i[m^i - m]$ distinct pairs of vertices which are not siblings have no common neighbors and $\binom{m}{2}m^{i-1}$ pairs of vertices which are siblings have exactly one common neighbor.

From the above cases, it follows that

$$
\begin{aligned}
N[T; x] = &\sum_{i=0}^{p-2} m^i m^2 x + \binom{m}{2} \sum_{i=1}^{p} m^{i-1} x + \sum_{i=0}^{p-2} m^i[m^{i+1} + m^{i+2} - m^2] + m^{2p-1} \\
&+ \sum_{i=0}^{p-3} m^i[m^{i+3} + m^{i+4} + \ldots + m^p] + \sum_{i=1}^{p} \frac{1}{2} m^i(m^i - m). \\
= &\frac{m^2(m^{p-1} - 1)}{m-1} x + \binom{m}{2} \frac{m^p - 1}{m-1} x + \frac{m[m^{2p-2} - m^p + m - 1]}{m-1} + m^{2p-1} \\
&+ \sum_{i=0}^{p-3} \left[\frac{m^{p+i+1} - m^{2i+3}}{m-1}\right] + \frac{m^2[m^{2p} - m^{p+1} - m^p + m]}{2(m^2-1)}.
\end{aligned}
$$

This completes the proof. $\square$

Theorem 2.4.3. *The common neighbor polynomial of a caterpillar tree* $P_n(m_1, m_2, \ldots, m_n)$ *is given by the following:*

$$N[P_n(m_1, m_2, \ldots, m_n); x] = N[P_n; x] + \sum_{j=1}^{n} N[K_{1,mj}; x] + \sum_{\substack{l,k \\ l \neq k}} m_l m_k$$

$$+ \left[m_1 + m_n + 2\sum_{j=2}^{n-1} m_j\right] x + (n-2)(m_1 + m_n) + (n-3)\sum_{j=2}^{n-1} m_j.$$

Proof. Let $P_n(m_1, m_2, \ldots, m_n)$ be a caterpillar tree and let $v_1, v_2, \ldots, v_n$ be the vertices of its derived graph which is a path. Also let $v_1^{(j)}, v_2^{(j)}, \ldots, v_{m_j}^{(j)}$ be the pendent vertices of the caterpillar tree attached to the vertex v_j where $j \in \{1, 2, \ldots, n\}$.

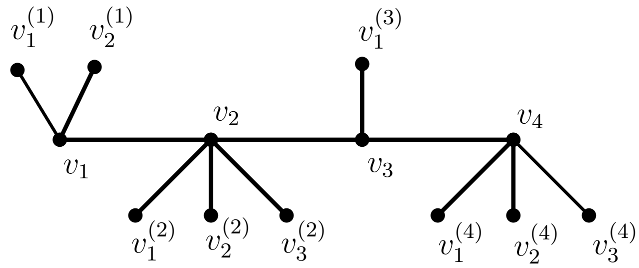

Figure 2.7: The caterpillar $P_4(2, 3, 1, 3)$

Let (u, v) be any pair of vertices of the caterpillar. We consider the following cases to build up its common neighbor polynomial.

Case(i) Let $u, v \in \{v_1, v_2, \ldots, v_n\}$. Here the number of pairs of vertices (u, v) with i common neighbors equals $|N(P_n, i)|$. So the pairs of vertices under this case contribute the term $N[P_n; x]$ to the common neighbor polynomial of the caterpillar.

Case(ii) Let $u, v \in \{v_j, v_1^{(j)}, v_2^{(j)}, \ldots, v_{mj}^{(j)}\}$. Here the vertices of the set under consideration spans a star graph $K_{1,mj}$ and hence the number of pairs of vertices (u, v) with i common neighbors equals $|N(K_{1,mj}, i)|$.

Case(iii) Let $u \in \{v_1^{(l)}, v_2^{(l)}, \ldots, v_{ml}^{(l)}\}$ and $v \in \{v_1^{(k)}, v_2^{(k)}, \ldots, v_{mk}^{(k)}\}$ where $l \neq k$ and $l, k \in \{1, 2, \ldots, n\}$.

Here u and v are the pendent vertices attached to the vertices v_l and v_k respectively where $l \neq k$. No pair of vertices under this case have common neighbors and there are $\sum_{\substack{l,k \\ l \neq k}} m_l m_k$ such pairs.

Case(iv) Let $u \in \{v_1, v_n\}$, a pendent vertex of the derived graph P_n and let v be any vertex attached to the vertices of P_n such that uv is not an edge of the caterpillar.

In this case pairs of vertices of the form $(v_1, v_l^{(2)})$ and $(v_n, v_k^{(n-1)})$ where $l \in \{1, 2, \ldots, m_2\}$ and $k \in \{1, 2, \ldots, m_{n-1}\}$ have exactly one common neighbor each and there are $m_2 + m_{n-1}$ such pairs of vertices. There remains $m_1 + m_2 + m_{n-1} + m_n + 2\sum_{j=3}^{n-2} m_j$ pairs of vertices under this case which have no common neighbors.

Case(v) Let $u \in \{v_2, v_3 \ldots, v_{n-1}\}$ and let v be any vertex selected in a way same as in Case(iv).

For $i \in \{2, 3, \ldots, n-1\}$, the pairs of vertices of the form $(v_i, v_l^{(i-1)})$ where $l \in \{1, 2, \ldots, m_{i-1}\}$ have exactly one common neighbor v_{i-1} and pairs of vertices of the form $(v_i, v_k^{(i+1)})$ where $k \in \{1, 2, \ldots, m_{i+1}\}$ have exactly one common neighbor v_{i+1}. There are $m_1 + m_2 + m_{n-1} + m_n + 2\sum_{j=3}^{n-2} m_j$ such pairs of vertices.

The remaining pairs of vertices under this case have no common neighbors and there are $(n-3)(m_1 + m_n) + (n-4)(m_2 + m_{n-1}) + (n-5)\sum_{j=3}^{n-2} m_j$

such pairs of vertices.

From the above cases, it follows that

$$\begin{aligned}
N[P_n(m_1, m_2, \ldots, m_n); x] &= N[P_n; x] + \sum_{j=1}^{n} N[K_{1,mj}; x] + \sum_{\substack{l,k \\ l \neq k}} m_l m_k \\
&\quad + [m_2 + m_{n-1}]x + m_1 + m_2 + m_{n-1} + m_n + 2\sum_{j=3}^{n-2} m_j \\
&\quad + \left[m_1 + m_2 + m_{n-1} + m_n + 2\sum_{j=3}^{n-2} m_j\right]x + (n-3)(m_1 + m_n) \\
&\quad + (n-4)(m_2 + m_{n-1}) + (n-5)\sum_{j=3}^{n-2} m_j.
\end{aligned}$$

Now the result follows after some rearrangement of the terms. $\square$

Corollary 2.4.4. *For a caterpillar tree $P_n(m, m, \ldots, m)$ where same number of vertices are attached to each vertex of the derived graph P_n, we have,*

$$\begin{aligned}
N[P_n(m, m, \ldots, m); x] = N[P_n; x] + nN[K_{1,m}; x] + \binom{n}{2} m^2 + 2m(n-1)x \\
+ m(n-1)(n-2).
\end{aligned}$$

Theorem 2.4.5. *The common neighbor polynomial of a star like tree graph $S(n_1, n_2, \ldots, n_k)$ with $N+1$ vertices is given by*

$$N[S(n_1, n_2, \ldots, n_k); x] = \sum_{r=1}^{k} N[P_{n_r+1}; x] + \binom{k}{2}x + \binom{N}{2} - \sum_{r=1}^{k}\binom{n_r}{2} - \binom{k}{2}$$

where $N = n_1 + n_2 + \ldots + n_k$.

Proof. Let $S(n_1, n_2, \ldots, n_k)$ be a star like tree graph with a vertex w such that $S(n_1, n_2, \ldots, n_k) - w = P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_k}$. Any pair of vertices $(u, v) \in P_{n_r} \cup \{w\}$ where $r \in \{1, 2, \ldots, k\}$, has as many common neighbors in $S(n_1, n_2, \ldots, n_k)$ as it has in P_{n_r+1}.

Let (u, v) be a pair of vertices in $S(n_1, n_2, \ldots, n_k)$ such that $u \in P_{n_r}$ and $v \in P_{n_s}$ where $r \neq s$, $r, s \in \{1, 2, \ldots, k\}$. Then the vertex pair (u, v) has a single common neighbor w if both u and v are adjacent to w and there are no common neighbors otherwise. Hence there are $\binom{k}{2}$ pairs of vertices with one common neighbor and $\binom{N}{2} - \sum_{r=1}^{k} \binom{n_r}{2} - \binom{k}{2}$ pairs of vertices with no common neighbors. Hence the result follows. $\square$

The strand A of human insulin has 21 amino acids of 11 kinds is usually represented by a star like tree graph with 11 branches as shown in Figure 2.8.

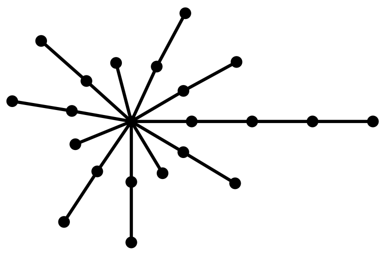

Figure 2.8: Strand A of human insulin

Corollary 2.4.6. *The common neighbor polynomial of the graphical representation of Strand A of human insulin is given by*

$$N[G; x] = 65x + 166.$$

Proof. The proof follows from the fact that Strand A of human insulin can be graphically represented as a star like tree graph $S(1, 2, 4, 2, 2, 1, 2, 2, 1, 2, 2)$. $\square$

The (n, k) firecracker graph[11] is obtained by identifying each vertex of a path P_n with one of the pendent vertices of the star graph $K_{1,k}$. In particular, the $(n, 2)$ firecracker graph is known as the centipede graph.

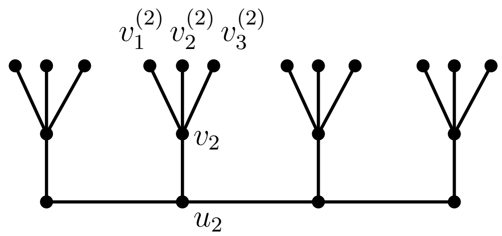

Figure 2.9: The $(4,4)$- firecracker graph

Theorem 2.4.7. *The common neighbor polynomial of (n,k) firecracker graph G is given by the following:*
$N[G;x] = \Big[n\binom{k}{2}+3n-4\Big]x+nk+\binom{n-1}{2}+1+\binom{n}{2}k^2+(n-1)(n-2)+n(n-1)(k-1)$.

Proof. For $j \in \{1,2,\ldots,n\}$, let $v_1^j, v_2^j, \ldots, v_k^j$ be the pendent vertices of the j^{th} star where the vertex v_k^j is identified with the vertex u_j of the path P_n. Let v_j be the center vertex of the j^{th} star attached to the vertex u_j of P_n. Let (u,v) be any pair of vertices of G. Here we consider 5 cases:

Case(i) Let $u, v \in \{v_j, v_1^j, v_2^j, \ldots, v_k^j\}$ where $j \in \{1,2,\ldots,n\}$.

In this case, the pair (u,v) has as many common neighbors in G as in $K_{1,k}$.

Case(ii) Let $u, v \in \{u_1, u_2, \ldots, u_n\}$.

Here the vertex pair (u,v) has as many common neighbors in G as in P_n.

Case(iii) For $j \in \{1,2,\ldots,n\}$, let $u = v_j$ and $v \in \{u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n\}$.

In this case, for each j, the pairs (v_j, u_{j-1}) and (v_j, u_{j+1}) has exactly one common neighbor. Also the pairs (v_1, u_2) and (v_n, u_{n-1}) have one common neighbor each. Hence there are $2(n-1)$ pairs of vertices (u,v) with one common neighbor and all other $(n-1)(n-2)$ pairs of vertices have no common neighbors.

Case(iv) Let $u \in \{v_r, v_1^r, v_2^r, \ldots, v_{k-1}^r\}$ and $v \in \{v_s, v_1^s, v_2^s, \ldots, v_{k-1}^s\}$ where $r, s \in \{1, 2, \ldots, n\}$.

In this case, there are $\binom{n}{2}k^2$ pairs of vertices which have no common neighbors.

Case(v) For $j \in \{1, 2, \ldots, n\}$, let $u \in \{v_1^j, v_2^j, \ldots, v_{k-1}^j\}$ and $v \in \{u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n\}$.

In this case, there are $n(n-1)(k-1)$ pairs of vertices which have no common neighbors.

Using theorems 2.2.2 and 2.2.5, it follows that,

$$\begin{aligned} N[G;x] &= nN[K_{1,k};x] + N[P_n;x] + 2(n-1)x + (n-1)(n-2) + \binom{n}{2}k^2 \\ &\qquad + n(n-1)(k-1) \\ &= n\left[\binom{k}{2}x + k\right] + (n-2)x + \binom{n-1}{2} + 1 + 2(n-1)x + \binom{n}{2}k^2 \\ &\qquad + (n-1)(n-2) + n(n-1)(k-1) \\ &= \left[n\binom{k}{2} + 3n - 4\right]x + nk + \binom{n-1}{2} + 1 + \binom{n}{2}k^2 \\ &\qquad + (n-1)(n-2) + n(n-1)(k-1). \end{aligned}$$

This completes the proof. □

Corollary 2.4.8. *The common neighbor polynomial of the centipede graph G is given by*

$$N[G;x] = 4(n-1)x + 2n + \frac{3(n-1)(3n-2)}{2} + 1.$$

Proof. The proof follows from the fact that the centipede graph is a special case of (n,k)- firecracker graph when $k = 2$. □

2.5 Common neighbor polynomial of some graph constructions

In this section we study common neighbor polynomial of some graph constructions.

Let v_0 be a specific vertex of a graph G. Let $G_{v_0}(m)$ be a graph obtained from G by identifying the vertex V_0 of G with an end vertex of the path P_{m+1} with $m+1$ vertices [36].

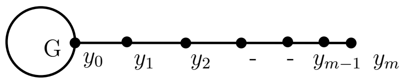

Figure 2.10: The graph $G_{v_0}(m)$

Theorem 2.5.1. *Let G be a graph with n vertices and let $v_0 \in V(G)$. If $deg(v_0) = d$, we have* $N[G_{v_0}(m); x] = N[G; x] + (m + d - 1)x + mn - d + \binom{m-1}{2}$.

Proof. Let $y_0, y_1, \ldots, y_m$ be the vertices of the path P_{m+1}. Let the vertex v_0 of G be identified with the end vertex y_0 of P_{m+1}. Let (u, v) be any pair of vertices of $G_{v_0}(m)$. We consider 3 cases:

Case(i) Let $u, v \in V(G)$.

Then the number of vertex pairs (u, v) with i common neighbors in $G_{v_0}(m)$ equals $|N(G, i)|$.

Case(ii) Let $u, v \in \{y_1, y_2, \ldots, y_m\}$.

Then the number of vertex pairs (u, v) with i common neighbors in $G_{v_0}(m)$ equals $|N(P_m, i)|$.

Case(iii) Let $u \in V(G)$ and $v \in \{y_1, y_2, \ldots, y_m\}$.

If $u = y_0$, then (u, y_2) has one common neighbor and if u is a neighbor of y_0, then (u, y_1) has one common neighbor. Thus $d+1$ pairs of vertices under this case have exactly one common neighbor. All other $(mn-d-1)$ vertices under this case have no common neighbors.

It follows that

$$\begin{aligned} N[G_{v_0}(m);x] &= N[G;x] + N[P_m;x] + (d+1)x + (mn-d-1) \\ &= N[G;x] + (m-2)x + \binom{m-1}{2} + 1 + (d+1)x + (mn-d-1) \\ &= N[G;x] + (m+d-1)x + mn - d + \binom{m-1}{2}. \end{aligned}$$

This completes the proof. $\square$

Let a and b be two specific vertices of a graph G. Let $G'_{a,b}(m)$ or simply, $G'(m)$ be a graph obtained from G by identifying the vertices a and b of G with the two end vertices of a path P_m [36].

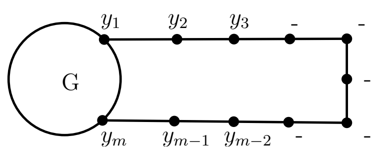

Figure 2.11: The graph $G'(m)$

Theorem 2.5.2. *Let G be a graph with n vertices. Let a, b be two specific vertices of G. Then for $m > 2$, we have $N[G'(m);x] = N[G;x] + (m+d-2)x + \binom{m-3}{2} + n(m-2) - (d+1)$ where d denotes the sum of degrees of vertices a and b in G.*

Proof. Let $y_1, y_2, \ldots, y_m$ be the vertices of a path P_m. Let the vertices a, b of G be identified with the end vertices y_1 and y_m of P_m respectively. Let (u, v) be any pair of vertices of $G'(m)$.

Here we consider the following 3 cases:

Case (i) Let $u, v \in V(G)$.

Then the number of vertex pairs (u, v) with i common neighbors in $G'(m)$ equals $|N(G, i)|$.

Case(ii) Let $u, v \in \{y_2, y_3, \ldots, y_{m-1}\}$.

Then the number of vertex pairs (u, v) with i common neighbors in $G'(m)$ equals $|N(P_{m-2}, i)|$.

Case(iii) Let $u \in V(G)$ and $v \in \{y_2, y_3, \ldots, y_{m-1}\}$.

If $u = y_1$, then (u, y_3) has one common neighbor and if $u = y_m$, then (u, y_{m-2}) has one common neighbor. If $uy_1 \in E(G)$ then (u, y_2) has one common neighbor in $G'(m)$ and if $uy_m \in E(G)$ then (u, y_{m-1}) has one common neighbor in $G'(m)$. Thus $d + 2$ pairs of vertices (u, v) have 1 common neighbor in $G'(m)$. All other $n(m-2)-(d+2)$ vertex pairs under this case have no common neighbors. It follows that

$$\begin{aligned} N[G'(m); x] &= N[G; x] + N[P_{m-2}; x] + (d+2)x + n(m-2) - (d+2) \\ &= N[G; x] + (m+d-2)x + \binom{m-3}{2} + n(m-2) - (d+1). \end{aligned}$$

This completes the proof. $\square$

Let G_1 and G_2 be two disjoint graphs. Let $(G_1, G_2)_{u,v}(m)$ be a graph obtained by identifying the vertices u of G_1 and v of G_2 with the end vertices y_1 and y_m respectively, of a path P_m.

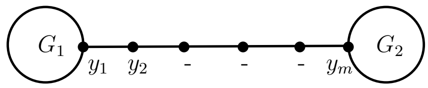

Figure 2.12: The graph $(G_1, G_2)_{u,v}(m)$

Theorem 2.5.3. *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices respectively. Let $u \in V(G_1)$ is of degree d_1 and $v \in V(G_2)$ is of degree d_2. Then $N[(G_1, G_2)_{u,v}(m); x] = N[G_1; x] + N[G_2; x] + N[P_{m-2}; x] + (d_1 + d_2 + 2)x + (n_1 + n_2)(m-2) - (d_1 + d_2) + n_1 n_2 - 2$ where $m > 3$.*

Proof. Let $y_1, y_2, \ldots, y_m$ be the vertices of the path P_m. Let the vertex u of G_1 be identified with the end vertex y_1 of P_m and let the vertex v of G_2 be identified with the vertex y_m. Let (x, y) be any pair of vertices of $(G_1, G_2)_{u,v}(m)$. We consider the following 6 cases:

Case(i) Let $x, y \in V(G_1)$.

Then the number of vertex pairs (x, y) with i common neighbors in $(G_1, G_2)_{u,v}(m)$ equals $|N(G_1, i)|$.

Case(ii) Let $x, y \in V(G_2)$.

Then the number of vertex pairs (x, y) with i common neighbors in $(G_1, G_2)_{u,v}(m)$ equals $|N(G_2, i)|$.

Case(iii) Let $x \in V(G_1)$ and $y \in \{y_2, y_3, \ldots, y_{m-1}\}$.

In this case, in $x = u$, the vertex pair (u, y_3) has exactly one common neighbor y_2 and if x is a neighbor of u in G_1, then there are d_1 pairs of vertices of the form (x, y_2) which have exactly one common neighbor y_1. The remaining $n_1(m-2)-(1+d_1)$ vertex pairs have no common neighbors.

Case(iv) Let $x \in V(G_2)$ and $y \in \{y_2, y_3, \ldots, y_{m-1}\}$.

As in Case(iii), the vertex pair (y_m, y_{m-2}) has exactly one common neighbor y_{m-1} and d_2 pairs of vertices has exactly one common neighbor y_m. The remaining $n_2(m-2)-(1+d_2)$ vertex pairs have no common neighbors.

Case(v) Let $x, y \in \{y_2, y_3, \ldots, y_{m-2}\}$.

Then the number of pairs of vertices having i common neighbors equals $|N(P_{m-2}, i)|$.

Case(vi) Let $x \in V(G_1)$ and $y \in V(G_2)$.

Since $m > 3$, all the $n_1 n_2$ pairs of vertices (x, y) under this case have no common neighbors.

$$\begin{aligned} N[(G_1, G_2)(m); x] &= N[G_1; x] + N[G_2; x] + N[P_{m-2}; x] + (1 + d_1)x \\ &\quad + n_1(m-2) - (d_1 + 1) + (1 + d_2)x + n_2(m-2) \\ &\quad - (d_2 + 1) + n_1 n_2 \\ &= N[G_1; x] + N[G_2; x] + N[P_{m-2}; x] + (d_1 + d_2 + 2)x + \\ &\quad (n_1 + n_2)(m-2) - (d_1 + d_2) + n_1 n_2 - 2. \end{aligned}$$

This completes the proof. $\square$

Chapter 3

Common neighbor polynomial of some unary graph operations

The vertex and edge modification problems are very common in graph theory in the context of constructing graphs with some intended properties. For example, when graphs are used to represent an experimental data, vertex or edge modifications are useful for correcting errors in the data. Usually the positive and negative errors in the data are corrected by deleting an edge(or vertex) and adding an edge(or vertex) respectively in the modelled graph. In this chapter we study the common neighbor polynomial of some modifications of graphs.

3.1 Splitting graph of a given graph

The *splitting graph* $S(G)$ of a graph G is obtained by adding new vertices v' to G corresponding to each vertex v of G and then joining the vertex v' to all vertices of G adjacent to v in G.The vertex v' corresponding to v is called the tag vertex of v [12].

Figure 3.1: The path graph P_4 and its splitting graph

Theorem 3.1.1. *If G is a graph with n vertices, then the common neighbor polynomial of splitting graph of G is given by,*

$$N[S(G);x] = N[G;x^2] + 3N[G;x] + \sum_{i=0}^{n} n(G,i)x^i$$

where $n(G,i)$ represents the number of vertices of G with degree i.

Proof. Let $V'(G)$ be the set of tag vertices of G and let (u,v) be any pair of vertices of $S(G)$.

Case(i) Let $u,v \in V(G)$.

If w is a common neighbor of (u,v) in G, the common neighbors of (u,v) in $S(G)$ are exactly w and its tag vertex w'. Therefore, number of pairs (u,v) with i common neighbors in $S(G)$ equals the number of pairs (u,v) with $\frac{i}{2}$ common neighbors in G which equals $|N(G,\frac{i}{2})|$. Since the number of common neighbors of any pair of vertices of $S(G)$ under this case is even, it is enough to consider the cases when i is even.

Case(ii) Let $u,v \in V'(G)$.

Then the common neighbors of (u,v) in $S(G)$ are exactly the common neighbors of corresponding vertices in G. Hence number of vertex pairs (u, v) with i common neighbors in $S(G)$ equals $|N(G, i)|$.

Case(iii) Let $u \in V(G)$ and $v = u'$, the tag vertex of $u \in G$.

Then the common neighbors of (u, v) in $S(G)$ are exactly the neighbors of u in G. Hence number of vertex pairs (u, v) with i common neighbors in $S(G)$ equals $n(G, i)$.

Case(iv) Let $u \in V(G)$ and $v = w'$, the tag vertex of $w \in G$ where $w \in V(G)$ and $w \neq u$.

In this case, the common neighbors of (u, w') in $S(G)$ are exactly the common neighbors of (u, w) in G. Note that, each common neighbor z of the vertex pair (x, y) of G is a common neighbor for the pairs (x', y) and (x, y') where x', y' are the tag vertices of x, y respectively. Using this fact we can conclude that the number of vertex pairs (u, v) under this case with i common neighbors in $S(G)$ equals $2|N(G, i)|$.

From the above cases, it follows that

$$N(S(G); x) = \sum_{\substack{i=0 \\ i\ even}}^{n} |N(G, \frac{i}{2})| x^i + 3\sum_{i=0}^{n} |N(G, i)|\, x^i + \sum_{i=0}^{n} n(G, i) x^i$$

$$N[S(G); x] = N[G; x^2] + 3N[G; x] + \sum_{i=0}^{n} n(G, i) x^i.$$

This completes the proof. $\square$

3.2 Shadow graph of a given graph

The *shadow graph* $Sh(G)$ of a graph G is obtained by taking two copies of G, say G_1 and G_2 and joining each vertex of G_1 to the neighbors of the corresponding vertex of G_2.

Figure 3.2: The path graph P_4 and its shadow graph

Theorem 3.2.1. *If G is a graph with n vertices, the common neighbor polynomial of the shadow graph of G is given by*

$$N[Sh(G);x] = 4N[G;x^2] + \sum_{\substack{i=1 \\ i\ even}}^{|V(G)|} n(G, \frac{i}{2})x^i$$

where $n(G,i)$ represents the number of vertices of the graph G with degree i.

Proof. Let (u,v) be any pair of vertices of $Sh(G)$. Here we consider 4 cases.

Case(i) Let $u, v \in V(G_1)$.

If the vertex pair (u,v) has i common neighbors in G_1, it has $2i$ common neighbors in $Sh(G)$ which are the neighbors in G_1 and the vertices corresponding to these neighbors in G_2. Hence number of pairs (u,v) with i common neighbors in $Sh(G)$ equals $|N(G, \frac{i}{2})|$.

Case(ii) Let $u, v \in V(G_2)$.

As in Case(i), number of pairs (u,v) with i common neighbors in $Sh(G)$ equals $|N(G, \frac{i}{2})|$.

Case(iii) Let $u \in V(G_1)$ and $v \in V(G_2)$ such that $v = u'$, the vertex of G_2 corresponding to the vertex u of G_1.

The common neighbors of (u, u') are the neighbors of u in G_1 and the neighbors of u' in G_2. Since u and u' are corresponding vertices, if u has i neighbors in G, then (u, u') has $2i$ common neighbors in $Sh(G)$. Thus, number of pairs (u, u') with i common neighbors in $Sh(G)$ equals the number of vertices in G with degree $\frac{i}{2}$ where it is enough to consider only even integers i.

Case(iv) Let $u \in V(G_1)$ and $v \in V(G_2)$ where $v = w'$, the vertex of G_2 corresponding to the vertex w of G_1 such that $w \neq u$.

Note that, if the vertex pair (x, y) of G has i common neighbors in G, then the pairs (x, y') and (y, x') of $Sh(G)$ have $2i$ common neighbors; viz,the neighbors of (x, y) in G and the vertices corresponding to that neighbors in $Sh(G)$. Using this fact, we can conclude that the number of pairs (u, w') under this case with i common neighbors in $Sh(G)$ equals $2|N(G, \frac{i}{2})|$ where it is enough to consider only even integers i.

It follows that

$$|N(Sh(G), i)| = 4|N(G, \frac{i}{2})| + n(G, \frac{i}{2})$$

$$N[Sh(G); x] = 4N[G; x^2] + \sum_{\substack{i=1 \\ i \text{ even}}}^{n} n(G, \frac{i}{2})x^i.$$

This completes the proof. $\square$

3.3 Mycielski graph of a given graph

The *Mycielski graph [22],* $\mu(G)$ of a graph G contains G itself as an isomorphic subgraph together with $n+1$ additional vertices; a vertex v_i corresponding to each vertex u_i of G and another vertex w. Each v_i is connected by an edge to w and for each edge u_iu_j of G, $\mu(G)$ includes two additional edges u_iv_j and v_iu_j.

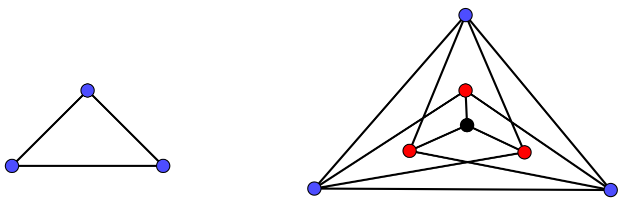

Figure 3.3: The cycle graph C_3 and its mycielski graph

Theorem 3.3.1. *If G is a graph with n vertices and if $n(G,i)$ denote the number of vertices of G with degree i, then we have*

$$N[\mu(G);x] = N[G;x^2] + (x+2)N[G;x] + 2\sum_{i=0}^{n} n(G,i)x^i + m.$$

Proof. Let v_i be the vertices of $\mu(G)$ corresponding to the vertices u_i of G where $i=1,2,\ldots,n$ and let w be the vertex of $\mu(G)$ which is connected to each v_i by edges. Let (u,v) be any pair of vertices of $\mu(G)$.

Case(i) Let $u,v \in \{u_1,u_2,\ldots,u_n\}$.

If (u,v) has i common neighbors in G, then those neighbors and the vertices corresponding to those neighbors become the neighbors of (u,v) in $\mu(G)$. Thus (u,v) has $2i$ common neighbors in $\mu(G)$. Hence number of vertex pairs (u, v) with i common neighbors in $\mu(G)$ equals $|N(G, \frac{i}{2})|$ where i is always even.

Case(ii) Let $u, v \in \{v_1, v_2, \ldots, v_n\}$.

For $i, j \in \{1, 2, \ldots, n\}$, the common neighbors of (v_i, v_j) in $\mu(G)$ are the common neighbors of (u_i, u_j) in G and the vertex w. Hence number of vertex pairs (u, v) with i common neighbors in $\mu(G)$ equals $|N(G, i-1)|$ where i is always greater than or equal to 1.

Case(iii) Let $u = u_k$ and $v = v_k$ where $k \in \{1, 2, \ldots, n\}$.

Here we are considering the pair (u, v) where v is the vertex in $\mu(G)$ corresponding to the vertex u in G. Then the common neighbors of (u, v) are the neighbors of u in G. Hence number of vertex pairs (u, v) with i common neighbors in $\mu(G)$ equals $n(G, i)$.

Case(iv) Let $u = u_k$ and $v = v_j$ where $j \neq k$ and $k, j \in \{1, 2, \ldots, n\}$.

Here we are considering the pair (u, v) where v is the vertex in $\mu(G)$ corresponding to some vertex of G other than u. Each common neighbor of (u_i, u_j) in G are common neighbors of the pairs (u_i, v_j) and (u_j, v_i). Hence number of pairs (u, v) with i common neighbors in $\mu(G)$ equals $2|N(G, i)|$.

Case(v) Let $u \in \{u_1, u_2, \ldots, u_n\}$ and $v = w$.

The common neighbors of (u, v) are the vertices in $\mu(G)$ corresponding to the neighbors of u in G. Hence number of pairs (u, v) with i common neighbors in $\mu(G)$ equals $n(G, i)$.

Case(vi) Let $u \in \{v_1, v_2, \ldots, v_m\}$ and $v = w$.

There are no common neighbors for the n pairs of vertices under this case.

From the above cases, we can conclude that

$$|N(\mu(G), i)| = |N(G, \frac{i}{2})| + |N(G, i-1)| + 2|N(G, i)| + 2n(G, i) + n\delta_{i0}$$
$$N[\mu(G); x] = N[G; x^2] + (x+2)N[G; x] + 2\sum_{i=0}^{n} n(G, i)x^i + n.$$

This completes the proof. □

3.4 Duplication of a vertex

Duplication of a vertex v of a graph G is the graph G' obtained by adding a vertex v' in G with $N(v') = N(v)$.

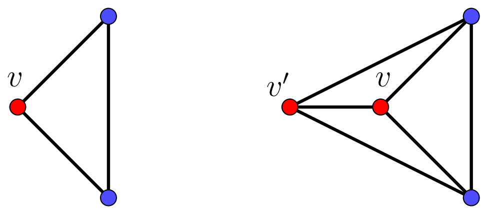

Figure 3.4: Duplication of the vertex v in C_3

Theorem 3.4.1. *The common neighbor polynomial of the graph K_n' obtained by the duplication of one of the vertices of the complete graph K_n is given by*

$$N[K_n'; x] = 2(n-1)x^{n-2} + \left[\binom{n-1}{2} + 1\right]x^{n-1}.$$

Proof. Let x be a vertex of K_n duplication of which produces the graph K_n' and let x' be the corresponding duplicate vertex. Let (u, v) be any pair of vertices of K_n'. We consider two cases:

Case(i) Let $u, v \in V(K_n)$. Then the pairs (u, v) has exactly $n-1$ common neighbors except the case when u or v equals x. The vertex pairs of the form (x, v) has exactly $n-2$ common neighbors. Thus under this case, there are $\binom{n-1}{2}$ pairs with $n-1$ common neighbors and $n-1$ pairs with $n-2$ common neighbors.

Case(ii) Let $u = x'$ and $v \in K_n$. Then the vertex pairs (u, v) has $n-1$ common neighbors if $v = x$ and has $n-2$ common neighbors if $v \neq x$. Thus under this case, there is exactly one pair (x', x) with $n-1$ common neighbors and there are $n-1$ pairs with $n-2$ common neighbors.

It follows that

$$\begin{aligned} N[K_n'; x] &= \binom{n-1}{2} x^{n-1} + (n-1)x^{n-2} + x^{n-1} + (n-1)x^{n-2} \\ &= 2(n-1)x^{n-2} + \left[\binom{n-1}{2} + 1\right] x^{n-1}. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.4.2. *If $K_{m,n}'$ is a graph obtained by duplication of a vertex of $K_{m,n}$ having degree m, then the common neighbor polynomial of $K_{m,n}'$ is given by*

$$N[K_{m,n}'; x] = \binom{m}{2} x^{n+1} + \binom{n+1}{2} x^m + m(n+1).$$

Proof. Note that (see Theorem 2.2.5) $N[K_{m,n}; x] = \binom{m}{2}x^n + \binom{n}{2}x^m + mn$. Then the result follows from the fact that $K_{m,n}' = K_{m,n+1}$. $\square$

Theorem 3.4.3. *If C_n' is a graph obtained by duplication of a vertex of C_n, then*

$$N[C_n'; x] = \begin{cases} 2x^2 + (n+1)x + \frac{(n-3)(n+2)}{2}, & n \neq 4, \\ 4x^2 + 6, & n = 4. \end{cases}$$

Proof. For $n = 4$, it is easy to note that $N[C_4'; x] = 4x^2 + 6$. So we consider the case when $n \neq 4$. Let the vertices of C_n be $u_1, u_2, \ldots, u_n$ and let u_n' be the additional vertex of C_n' which is the duplication of the vertex u_n. Let (u, v) be any pair of vertices of C_n'. We consider two cases:

Case(i) Let $u, v \in V(C_n)$.

Then all the pairs of vertices (u, v) have as many common neighbors in C_n' as in C_n except the pair (u_1, u_{n-1}) which has two common neighbors instead of the one common neighbor in C_n.

Case(ii) Let $u \in C_n$ and $v = u_n'$.

Then the vertex pair (u_n, u_n') has 2 common neighbors u_1 and u_{n-1}, the pair (u_2, u_n') has 1 common neighbor u_1, the pair (u_{n-2}, u_n) has 1 common neighbor u_{n-1} and all other $n - 3$ pairs under this case have no common neighbors.

Using Theorem 2.2.4, it follows that

$$\begin{aligned} N[C_n'; x] &= \Big[N[C_n; x] - x + x^2\Big] + \Big[x^2 + 2x + n - 3\Big] \\ &= 2x^2 + (n+1)x + \frac{(n-3)(n+2)}{2}. \end{aligned}$$

This completes the proof. □

Theorem 3.4.4. *If P_n' is a graph obtained by the duplication of the vertex u_i which is not a pendent vertex of P_n, then*

$$N[P_n'; x] = \begin{cases} 2x^2 + (n-1)x + \frac{(n-2)(n+1)}{2}, & \text{if } n \neq 3, \\ 2x^2 + 4, & \text{if } n = 3. \end{cases}$$

Proof. If $n = 3$, the result follows from the fact that the duplication of the unique non-pendent vertex of P_n produces the graph C_4. So consider the case when $n \neq 3$. Let the vertices of P_n be $u_1, u_2, \ldots, u_n$ and let u_i' be the additional vertex of P_n' which is the duplication of the vertex u_i where $i \in \{2, 3, \ldots, n-1\}$ which is a non pendent vertex of P_n. Let (u, v) be any pair of vertices of P_n'. We consider two cases:

Case(i) Let $u, v \in V(P_n)$.

Then all the pairs of vertices (u, v) have as many common neighbors in P_n' as in P_n except the pair (u_{i-1}, u_{i+1}) which has two common neighbors u_i and u_i' instead of the one common neighbor u_i in P_n.

Case(ii) Let $u \in V(P_n)$ and $v = u_i'$.

Then the vertex pair (u_i, u_i') has 2 common neighbors u_{i-1} and u_{i+1}, the pair (u_{i-2}, u_i') has 1 common neighbor u_{i-1} and the pair (u_{i+2}, u_i') has 1 common neighbor u_{i+1}. All other $n-3$ pairs of vertices under this case have no common neighbors.

Then using Theorem 2.2.2, if follows that

$$\begin{aligned} N[P_n'; x] &= \Big[N[P_n; x] - x + x^2\Big] + \Big[x^2 + 2x + n - 3\Big] \\ &= 2x^2 + (n-1)x + \frac{(n-2)(n+1)}{2}. \end{aligned}$$

This completes the proof. □

Theorem 3.4.5. *If P_n' is a graph obtained by the duplication of the pendent vertex u_n of P_n, then $N[P_n'; x] = nx + n(n-1)$.*

Proof. Let (u, v) be any pair of vertices of P_n'. We consider two cases:

Case(i) Let $u, v \in V(P_n)$. Then all the pairs of vertices (u, v) has as many common neighbors in P_n' as in P_n.

Case(ii) Let $u = u_n'$ and $v \in V(P_n)$. Then the vertex pairs (u_n', u_n) and (u_n', u_{n-2}) have one common neighbor u_{n-1} and all other $n-2$ pairs (u_n', v) have no common neighbors.

Hence using Theorem 2.2.2, it follows that

$$\begin{aligned} N[P_n'; x] =& N[P_n; x] + 2x + (n-2) \\ & - (n-2)x + \binom{n-1}{2} + 1 + 2x + n - 2 \\ =& nx + \binom{n}{2}. \end{aligned}$$

This completes the proof. □

Chapter 4

Common neighbor polynomial of some binary graph operations

Binary graph operations are used to produce new graphs by applying some binary operations on two underlying graphs. Using binary operations, highly complicated graphs may produce using parent graphs with comparatively simpler structures. In this chapter, we discuss common neighbor polynomial of some binary graph operations.

4.1 Main results

Theorem 4.1.1. *If H and K are any two graphs with h and k vertices, then the common neighbor polynomial of the join of H and K is given by*
$N[H \vee K; x] = x^k N[H; x] + x^h N[K; x] + \sum_{i=0}^{h+k-2} \Big[\sum_{m+l=i} n(H, m) n(K, l) \Big] x^i$ *where $n(G, i)$ represents the number of vertices of a graph G with degree i.*

Proof. Let (u, v) be any pair of vertices of $H \vee K$. We consider the following cases:

Case(i) Let $u, v \in V(H)$.

Since all the vertices in $V(K)$ are common neighbors of (u, v) in $H \vee K$, the number of vertex pairs (u, v) with i common neighbors in $H \vee K$ equals the number of pairs (u, v) with $(i - k)$ common neighbors in H.

Case(ii) Let $u, v \in V(K)$.

Since all the vertices in $V(H)$ are common neighbors of (u, v) in $H \vee K$, the number of vertex pairs (u, v) with i common neighbors in $H \vee K$ equals the number of pairs (u, v) with $(i - h)$ common neighbors in K.

Case(iii) Let $u \in H$ and $v \in K$.

In this case all the neighbors of u in H are neighbors of v in $H \vee K$ and all the neighbors of v in K are neighbors of u in $H \vee K$. Hence the number of pairs (u, v) with i common neighbors in $H \vee K$ equals $\sum_{i=m+l} n(H, m).n(K, l)$.

Thus we have

$$|N(H \vee K, i)| = |N(H, i - k)| + |N(K, i - h)| + \sum_{i=m+l} n(H, m).n(K, l).$$

This completes the proof. $\square$

Corollary 4.1.2. *$N[G+w; x] = xN[G; x] + \sum_{i=0}^{n-1} n(G, i)x^i$ where $n(G, i)$ represents the number of vertices of G with degree i.*

Proof. The result follows from the fact that $G + w$ is isomorphic to $G \vee K_1$ where $V(K_1) = \{w\}$. $\square$

Corollary 4.1.3. *If W_n is a wheel graph with n vertices, then*

$$N[W_n; x] = 2(n - 1)x^2 + \frac{(n - 1)(n - 4)}{2}x, \text{ for } n > 5.$$

Proof. Observe that $W_n = C_{n-1} \vee K_1$. Therefore, using Theorems 2.2.4 and 4.1.1,

$$\begin{aligned} N[C_{n-1} \vee K_1; x] &= x[(n-1)x + \frac{(n-1)(n-4)}{2}] + x^{n-1} \times 0 + (n-1)x^2 \\ &= 2(n-1)x^2 + \frac{(n-1)(n-4)}{2}x. \end{aligned}$$

This completes the proof. $\square$

Corollary 4.1.4. *If S_n is the shell graph with $n > 3$ vertices, then*

$$N(S_n; x) = 2(n-3)x^2 + \big[\binom{n-2}{2} + 3\big]x.$$

Proof. Note that $S_n = P_{n-1} \vee K_1$. Then the result follows from theorems 2.2.2 and 4.1.1. $\square$

Theorem 4.1.5. *If K is a graph having k vertices and l edges and H is a graph having h vertices, then the common neighbor polynomial of corona of K and H is given by*

$$\begin{aligned} N[K \circ H; x] &= N[K; x] + kxN[H; x] + 2lhx \\ &\quad + hk(k-1)\left(1 + \frac{h}{2}\right) - 2hl + k\sum_{i=0}^{h} n(H, i)x^i, \end{aligned}$$

where $n(H, i)$ represents the number of vertices of H with degree i.

Proof. Let (u, v) be any pair of vertices of $K \circ H$. We consider the following 5 cases:

Case(i) Let $u, v \in V(K)$.

Then the number of vertex pairs (u, v) with i common neighbors in $K \circ H$ equals the number of pairs (u, v) with i common neighbors in K which equals $|N(K, i)|$.

Case(ii) Let $u, v \in V(H)$.

Let u, v be vertices of the copy of H attached to a vertex w of K. If (u, v) has i common neighbors in H, say, $w_1, w_2, \ldots, w_i$ then, the common neighbors of (u, v) in $K \circ H$ are $w_1, w_2, \ldots, w_i$ and w. Hence there are $|N(H, i-1)|$ pairs of vertices of H with i common neighbors in $K \circ H$. Note that there are k such copies of H.

Case(iii) Let $u \in V(K)$ and $v \in V(H_i)$ where H_i denote the copy of H corresponding to the vertex u.

Then the number of common neighbors of (u, v) in $K \circ H$ equals the degree of v in H. Hence the number of vertex pairs (u, v) with i common neighbors in $K \circ H$ equals $n(H, i)$ and there are k such copies of H in $K \circ H$.

Case(iv) Let $u \in V(K)$ and $v \in V(H_i)$ where H_i denote the copy of H corresponding to the vertex w of K such that $w \neq u$.

Such pair of vertices have a common neighbor if and only if u is a neighbor of w and in such case, the only common neighbor is w. Hence the number of pairs (u, v) under this case with 1 common neighbor equals $h \sum_{w \in V(K)} d(w) = 2lh$. All the remaining pairs under this case have no common neighbors and there are $(hk^2 - hk) - 2lh = h(k^2 - k - 2l)$ such pairs.

Case(v) Let $u \in H_i$ and $v \in H_j$ where H_i and H_j are two distinct copies of H.

Then the pairs (u, v) have no common neighbors and there are $\binom{k}{2} h^2$ such pairs. Thus if

$$\delta_{ij} = \begin{cases} 1 & ; i = j, \\ 0 & ; i \neq j, \end{cases} \quad \text{we have,}$$

$$|N(K \circ H, i)| = |N(K, i)| + k|N(H, i-1)| + kn(H, i) + 2lh\delta_{i1} + [h(k^2 - k - 2l)]\delta_{i0} + \binom{k}{2} h^2 \delta_{i0}.$$

It follows that

$$N[K \circ H; x] = N[K; x] + kxN[H; x] + 2lhx + hk(k-1)(1 + \frac{h}{2}) - 2hl + k\sum_{i=0}^{h} n(H, i)x^i.$$

This completes the proof. □

The graph $Q(m, n)$ is obtained by identifying each vertex of the complete graph K_m with a vertex of a unique K_n where there are m copies of K_n[26].

Corollary 4.1.6. *We have the following:*
$N[Q(m, n); x] = \binom{m}{2} x^{m-2} + m\binom{n}{2} x^{n-2} + m(m-1)(n-1)x + \binom{m}{2}(n-1)^2.$

Proof. The result follows from the fact that $Q(m, n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

Theorem 4.1.7. *If K is a graph with k vertices and e_1 edges and H is a graph with h vertices and e_2 edges, then the common neighbor polynomial of Cartesian product of G and H is given by*

$$N[K \square H; x] = kN[H; x] + hN[K; x] + 2e_1e_2x^2 + 2\left[\binom{k}{2}\binom{h}{2} - e_1e_2\right].$$

Proof. Let $V(K) = \{u_1, u_2, \ldots, u_k\}$ and $V(H) = \{v_1, v_2, \ldots, v_h\}$.
Let the vertices of $K \square H$ be denoted by u_iv_j where $i \in \{1, 2, \ldots, k\}$ and $j \in$

$\{1, 2, \ldots, h\}; k, h \geq 2$. The pairs of vertices of $k\Box H$ can be categorized as in the following cases:

Case(i) Consider the vertex pairs of $K\Box H$ of the form $(u_r v_s, u_r v_t)$ where $s \neq t$, $r \in \{1, 2, \ldots, k\}$, and $s, t \in \{1, 2, \ldots, h\}$.

For each $r = 1, 2, \ldots, k$, the common neighbors of $(u_r v_s, u_r v_t)$ in $K\Box H$ are the common neighbors of v_s and v_t in H. Thus the number of vertex pairs under case(i) with i common neighbors equals $k|N(H, i)|$.

Case(ii) Consider the vertex pairs of $K\Box H$ of the form $(u_s v_r, u_t v_r)$ where $s \neq t$, $s, t \in \{1, 2, \ldots, k\}$ and $r \in \{1, 2, \ldots, h\}$.

For each $r = 1, 2, \ldots, h$, the common neighbors of $(u_s v_r, u_t v_r)$ in $K\Box H$ are the common neighbors of u_s and u_t in K. Thus the number of vertex pairs under case(ii) with i common neighbors equals $h|N(K, i)|$.

Case(iii) Consider the vertex pairs of $K\Box H$ of the form $(u_s v_r, u_t v_l)$; $s \neq t, r \neq l$ where $u_s u_t \in E(K)$ and $v_r v_l \in E(H)$.

Corresponding to any pair of edges $u_s u_t$ of K and $v_r v_l$ of H, the vertices $u_s v_l$ and $u_t v_r$ are common neighbors of $(u_s v_r, u_t v_l)$ in $K\Box H$ and the vertices $u_s v_r$ and $u_t v_l$ are common neighbors of $(u_s v_l, u_t v_r)$ in $K\Box H$. So there are $2e_1 e_2$ pairs in $K\Box H$ with 2 common neighbors.

Case(iv) Consider the vertex pairs of $K\Box H$ of the form $(u_s v_r, u_t v_l)$; $s \neq t, r \neq l$ where either $u_s u_t \notin E(K)$ or $v_r v_l \notin E(H)$ or both.

Since $s \neq t$ and $r \neq l$ there cannot be any common neighbors for $(u_s v_r, u_t v_l)$. Corresponding to each vertex pair (u_s, u_t) of K and (v_r, v_l) of H, there can be two vertex pairs $(u_s v_r, u_t v_l)$ and $(u_s v_l, u_t v_l)$ in $K\Box H$. So there are

$2\binom{k}{2}\binom{h}{2} - 2e_1e_2$ pairs of vertices under case(iv) where there are no common neighbors.

It follows that,

$N[K\square H;x] = kN[H;x] + hN[K;x] + 2e_1e_2x^2 + 2\big[\binom{k}{2}\binom{h}{2} - e_1e_2\big].$ □

A *Ladder graph* L_n is obtained as the cartesian product of two paths one of which has only one edge.

Corollary 4.1.8. $N[L_n;x] = 2(n-1)x^2 + 2(n-2)x + 4\binom{n-1}{2} + n + 2.$

Proof. Since L_n is isomorphic to $P_n\square P_2$, $N[L_n;x] = N[P_n\square P_2;x]$.
Note that(see Theorem 2.2.2) $N[P_n;x] = (n-2)x + \binom{n-1}{2} + 1$. So we obtain

$$\begin{aligned} N[L_n;x] &= n + 2\Big[(n-2)x + \binom{n-1}{2} + 1\Big] + 2(n-1)x^2 + 2\Big[\binom{n}{2}\binom{2}{2} - (n-1)\Big] \\ &= 2(n-1)x^2 + 2(n-2)x + 4\binom{n-1}{2} + n + 2. \end{aligned}$$

This complete the proof. □

A *circular ladder graph* CL_n is obtained as the cartesian product of the cycle graph C_n and the path P_2.

Corollary 4.1.9. *We have the following.*

$$N[CL_n;x] = \begin{cases} 2nx^2 + 2nx + n(2n-5), & \text{if } n \neq 4, \\ 12x^2 + 16, & \text{if } n = 4. \end{cases}$$

Proof. Using the theorem 2.2.4 and using the fact that CL_n is isomorphic to $C_n\square P_2$, it follows that

Case(i) Let $n \neq 4$. Then,

$$N[CL_n; x] = n + 2\Big[nx + \frac{n(n-3)}{2}\Big] + 2nx^2 + 2\Big[\binom{n}{2}\binom{2}{2} - n\Big]$$
$$= 2nx^2 + 2nx + n(2n-5).$$

Case(ii) Let $n = 4$. Then,

$$N[CL_n; x] = 4 + 2[2x^2 + 4] + 8x^2 + 2\Big[\binom{4}{2}\binom{2}{2} - 4\Big]$$
$$= 12x^2 + 16.$$

This completes the proof. $\square$

A $m-$*book graph*[43] is obtained as the cartesian product of the star graph $K_{1,m}$ and the path graph P_2.

Corollary 4.1.10. *If B_m is a $m-$book graph, then we have the following:*

$$N[B_m; x] = 2mx^2 + m(m-1)x + (m+1)^2.$$

Proof. Since B_m is isomorphic to $K_{1,m}\Box P_2$, $N[B_m; x] = N[K_{1,m}\Box P_2; x]$. Note that (see Corollary 2.2.6), $N[K_{1,m}; x] = \binom{m}{2}x + m$ and $N[P_2; x] = 1$. Thus we obtain,

$$N[B_m; x] = (m+1) + 2\Big[\binom{m}{2}x + m\Big] + 2mx^2 + 2\Big[\binom{m+1}{2}\binom{2}{2} - m\Big]$$
$$= 2mx^2 + m(m-1)x + (m+1)^2.$$

This completes the proof. $\square$

Theorem 4.1.11. *Let G be a graph with n vertices and l edges. Let H be a rooted graph with m vertices having a root vertex v_1 with degree d. If $G\prime$ is the rooted product of G and H, then,*

$$N[G\prime; x] = N[G; x] + nN[H; x] + 2ldx + \binom{n}{2}(m^2 - 1) - 2ld.$$

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $V(H) = \{v_1, v_2, \ldots, v_m\}$ where v_1 is the root vertex. Then a vertex of G' can be represented by u_iv_j where $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Let (u, v) be any pair of vertices of G'.We consider the following 4 cases:

Case(i) Let $u, v \in \{u_iv_1; i = 1, 2, \ldots, n\}$.

Then the common neighbors of (u, v) in G' are exactly the common neighbors of (u_i, u_j) in G where $i, j \in \{1, 2, \ldots, n\}$.

Case(ii) For $i = 1, 2, \ldots, n$, let $u, v \in \{u_iv_j; j = 1, 2, \ldots, m\}$ where u_i is a particular vertex of G.

Then for each i, the common neighbors of (u, v) in $G\prime$ are exactly the common neighbors of (u, v) in H.

Case(iii) Let $u = u_iv_1$ and $v = u_rv_j$ where $i, r \in \{1, 2, \ldots, n\}$; $i \neq r$ and $j \in \{2, 3, \ldots, m\}$.

Then if d is the degree of the root vertex v_1, corresponding to each edge of G, there are $2d$ vertex pairs under this case having exactly one common neighbor. And there are $2ld$ such pairs. All other $n(n-1)(m-1) - 2ld$ vertex pairs under this case have no common neighbors.

Case(iv) Let $u = u_iv_j$ and $v = u_rv_s$ where $i, r \in \{1, 2, \ldots, n\}$; $i \neq r$ and $j, s \in \{2, 3, \ldots, m\}$.

Then the vertex pairs (u, v) have no common neighbors and there are $\binom{n}{2}(m-1)^2$ such pairs.

If follows that

$$\begin{aligned}
N[G'; x] &= N[G; x] + nN[H; x] + 2ldx + [n(n-1)(m-1) - 2ld] + \binom{n}{2}(m-1)^2 \\
&= N[G; x] + nN[H; x] + 2ldx + 2\binom{n}{2}(m-1) - 2ld + \binom{n}{2}(m-1)^2 \\
&= N[G; x] + nN[H; x] + 2ldx + \binom{n}{2}(m^2-1) - 2ld.
\end{aligned}$$

This completes the proof. □

Theorem 4.1.12. *If K and H are any two graphs with k and h vertices respectively and if $n(H, a)$ denote the number of vertices of the graph H with degree a, then the common neighbor polynomial of tensor product of K and H is*

$$N[K \times H; x] = \sum_{i=0}^{kh-2} |N(K \times H, i)| x^i,$$

where $|N(K \times H, i)|$ is given by

$\sum_{ab=i} \left\{ \left|N(K, b)\right| n(H, a) + \left|N(H, b)\right| n(K, a) + \left|N(K, a)\right| \left|N(H, b)\right| \right\}$; a, b integers; $k, h \geq 2$ and $i \in \{0, 1, 2, \ldots, (kh-2)\}$.

Proof. Let $V(K) = \{u_1, u_2, \ldots, u_k\}$ and $V(H) = \{v_1, v_2, \ldots, v_h\}$.
Let the vertices of $K \times H$ be denoted by $u_r v_s$ where $r \in \{1, 2, \ldots, k\}$ and $s \in \{1, 2, \ldots, h\}; k, h \geq 2$. The pairs of vertices of $K \Box H$ can be categorized as in the following cases:

Case(i) Consider the vertex pairs of $K \times H$ of the form $(u_r v_s, u_r v_t)$ where $s \neq t$, $r \in \{1, 2, \ldots, k\}$ and $s, t \in \{1, 2, \ldots, h\}$.

For each $r = 1, 2, \ldots, k$, the number of common neighbors of $(u_r v_s, u_r v_t)$

equals degree of u_r in K multiplied by the number of common neighbors of (v_s, v_t) in H. Hence for $i = 1, 2, \ldots, (kh-2)$, the number of vertex pairs under this case with i common neighbors in $K \times H$ equals $\sum_{ab=i} n(K,a)|N(H,b)|$ where the summation is taken over all integers a, b such that $ab = i$.

Case(ii) Consider the vertex pairs of $K \times H$ of the form $(u_s v_r, u_t v_r)$ where $s \neq t$, $s, t \in \{1, 2, \ldots, k\}$ and $r \in \{1, 2, \ldots, h\}$.

For each $r = 1, 2, \ldots, h$, the number of common neighbors of $(u_s v_r, u_t v_r)$ equals degree of v_r in H multiplied by the number of common neighbors of (u_s, u_t) in K. It follows that the number of pairs with i common neighbors equals $\sum_{ab=i} n(H,a)|N(K,b)|$.

Case(iii) Consider the vertex pairs of $K \times H$ of the form $(u_r v_s, u_t v_l)$ where $r \neq t$ and $s \neq$ l, $r, t \in \{1, 2, \ldots, k\}$ and $s, l \in \{1, 2, \ldots, h\}$.

The common neighbors of $(u_r v_s, u_t v_l)$ are of the form (x, y) where x is a common neighbor of (u_r, u_t) in K and y is a common neighbor of (v_s, v_l) in H. It follows that the number of pairs with i common neighbors equals $\sum_{ab=i} [|N(K,a)||N(H,b)|]$.

This completes the proof. $\square$

Chapter 5

CNP equivalent classes of graphs

It is obvious that isomorphic graphs have same common neighbor polynomial. But the existence of isomorphism may not be a criteria for identifying two graphs as equivalent as far as structural equivalence is concerned. From this point of view, CNP-equivalent classes of graphs are defined and studied in the present chapter.

5.1 Main results

We say that two graphs G and H are CNP-equivalent ($G \overset{\mathcal{N}}{\sim} H$) if and only if $N[G;x] = N[H;x]$. For example, the non isomorphic graphs shown in figure 5.1 are CNP-equivalent graphs. In figure 5.1, $N[G;x] = N[H;x] = 3x + 6$.

Obviously, the relation $\overset{\mathcal{N}}{\sim}$ is an equivalence relation on the class $\mathcal{G}$ of all simple finite graphs. The set of all graphs CNP-equivalent to a graph G is denoted as $[G]_{\mathcal{N}}$ and is defined as

$$[G]_{\mathcal{N}} = \{H \in \mathcal{G} : N[H;x] = N[G;x]\}.$$

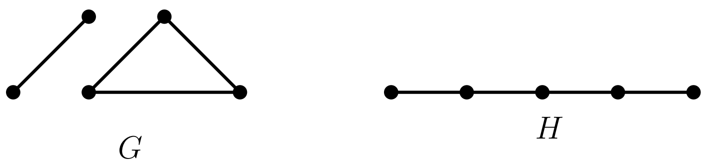

Figure 5.1: Two CNP-equivalent graphs G and H

A graph H is said to be $CNP-$unique if $[H]_{\mathcal{N}} = \{H\}$.

In this chapter we identify some $CNP-$ unique graphs and also some CNP-equivalent graph classes. Through out this chapter, $p(G)$ denotes disjoint union of p copies of the graph G.

Theorem 5.1.1. *Let G be a graph with n vertices and let $\overline{G}$ denotes the complement of G. Then $\overline{G} \in [G]_{\mathcal{N}}$ if and only if there are $|N(G,i)|$ vertex pairs of G which dominate $n-i$ vertices of G.*

Proof. Let $\overline{G} \in [G]_{\mathcal{N}}$. Then $|N(G,i)| = |N(\overline{G},i)|$ for $i = 1,2,\ldots,n-2$. Let $(u,v) \in N(\overline{G},i)$. Then (u,v) has i common neighbors in $\overline{G}$. All the vertices of $G-\{u,v\}$ other than these i vertices are adjacent to either u or v in G. So $\{u,v\}$ dominates $n-i$ vertices of G. Since $|N(\overline{G},i)| = |N(G,i)|$, there are $|N(G,i)|$ vertex pairs of G which dominate $n-i$ vertices of G.

Conversely assume that there are $|N(G,i)|$ vertex pairs of G which dominate $n-i$ vertices of G. For each vertex pair (u,v) of G which dominate $n-i$ vertices of G the remaining i vertices of G are not adjacent to either u or v in G. Thus those i vertices are common neighbors of (u,v) in $\overline{G}$. So $|N(\overline{G},i)|$ equals the number of vertex pairs of G which dominate $n-i$ vertices of G which equals $|N(G,i)|$ by assumption. Thus $N[G;x] = N[\overline{G};x]$. It follows that $\overline{G} \in [G]_{\mathcal{N}}$. $\square$

Corollary 5.1.2. *Let G be a graph with n vertices. If $\overline{G} \in [G]_{\mathcal{N}}$ then $|N(G, 0)|$ gives the number of dominating sets of G of order 2.*

Theorem 5.1.3. *If $H \in [G]_{\mathcal{N}}$ and $N(G, 2) > 0$, then H has a cycle of length 4 if and only if G has a cycle of length 4.*

Proof. If $H \in [G]_{\mathcal{N}}$, then $N[H; x] = N[G; x]$ so that $N(H, i) = N(G, i)$, $\forall i \geq 0$. In particular, $N(H, 2) = N(G, 2)$. Now the proof follows from the fact that, for any graph G, if $N(G, 2) > 0$ then G has a cycle of length 4. □

5.2 Some CNP-unique graph classes

In this section, we identify some CNP-unique graph classes.

Theorem 5.2.1. *For $n > 2$, the complete graph K_n is $CNP-$unique.*

Proof. Let H be any graph such that $H \in [K_n]_{\mathcal{N}}$. Then from Theorem 2.2.1, it follows that $N[H; x] = \binom{n}{2} x^{n-2}$. Let $u, v \in V(H)$ such that $uv \notin E(H)$. Let w be any vertex in H other than u and v. Then u is not a common neighbor of (v, w) since u is not a neighbor of v. So the number of common neighbors of (v, w) is at most $n - 3$. Since $N[H; x] = \binom{n}{2} x^{n-2}$, there are no pairs of vertices in H with number of common neighbors less than $n - 2$. Thus we arrive at a contradiction. Thus $uv \in E(H)$, $\forall u, v \in V(H)$. Then H is isomorphic to K_n. This completes the proof. □

Theorem 5.2.2. *The complete bipartite graph $K_{m,n}$ is $CNP-$unique, for every $n, m > 2$.*

Proof. Let G be any graph such that $G \in [K_{m,n}]_{\mathcal{N}}$ where $m, n > 2$. Then $|V(G)| = m + n$ and from Theorem 2.2.5, $N[G; x] = \binom{m}{2}x^n + \binom{n}{2}x^m + mn$. Therefore G has mn pairs of vertices having no common neighbors. Since G has $m + n$ vertices, the set of vertices of G can be partitioned into two, viz, $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ where (u_i, v_j); $i \in \{1, 2, \ldots, m\}$; $j \in \{1, 2, \ldots, n\}$ have no common neighbors in G and the pairs (u_i, u_j) where $i, j \in \{1, 2, \ldots, m\}$ and (v_i, v_j) where $i, j \in \{1, 2, \ldots, n\}$ have at least one common neighbor in G.

Without loss of generality assume that $m < n$. If possible, assume that all the common neighbors of (u_i, u_j) where $i, j \in \{1, 2, \ldots, m\}$ are in U itself. Since U has only m vertices, (u_i, u_j) can have at most $m - 2$ common neighbors. Then the expression for $N[G; x]$ shows that (u_i, u_j) has no common neighbors in G which is a contradiction to the construction of U. Therefore every pairs of vertices (u_i, u_j) of U has some common neighbors in V.

Now suppose that (u_i, u_j) has a common neighbor u in U and another common neighbor v in V. Then u_i and u_j are common neighbors of (u, v) in G, a contradiction to the construction of U and V. Thus all the common neighbors of (u_i, u_j) are in V.

We will prove that there is no edge in G connecting two vertices in U. Let $e = u_i u_j \in E(G)$ and let $v \in V$ be a common neighbor of (u_i, u_j) in V. Then u_j is a common neighbor of (u_i, v), a contradiction to the construction of U and V. Hence there is no edge in G connecting two vertices in U.

Now we will show that $\forall u \in U$ and $\forall v \in V$, $\exists$ an edge $uv \in E(G)$. Let $e = uv \notin E(G)$. Then v is not a common neighbor of $(u, u_i); i = 1, 2, \ldots, m$

in G. Since all the common neighbors of (u, u_i) lies in V, (u, u_i) can have at most $n-1$ common neighbors in G. Then only possible cases are that either (u, u_i) has no common neighbors or number of common neighbors of (u, u_i) equals m. First case is a contradiction to the construction of U. Therefore number of common neighbors of (u, u_i) equals m for $i = 1, 2, \ldots, m-1$. Then at most $\binom{m}{2} - (m-1) = \binom{m-1}{2}$ vertex pairs in U can have n common neighbors in G. Since there are $\binom{m}{2}$ vertex pairs in G with n common neighbors, at least $\binom{m}{2} - \binom{m-1}{2}$ vertex pairs in V must have n common neighbors. Let (v_r, v_s) be a vertex pair in V with n common neighbors in G. Since V has only n vertices and since $m < n$, (v_r, v_s) must have one common neighbor, say $u_l \in U$ and other common neighbor, say $v_l \in V$. Then v_r and v_s are common neighbors of (u_l, v_l) where $u_l \in U$ and $v_l \in V$ which is a contradiction to the construction of U and V. Therefore $e = uv \in E(G)$.

Finally we will prove that there is no edge in G connecting two vertices of V. Let $e = v_i v_j \in E(G)$ where $v_i, v_j \in V$. Let $u \in U$. Then there exists an edge uv_i in G. Then v_i is a common neighbor of (u, v_j) where $u \in U$ and $v_j \in V$ which is a contradiction to the construction of U and V. Hence there is no edge in G connecting two vertices of V. Then G is isomorphic to $K_{m,n}$ and hence $K_{m,n}$ is $CNP-$unique. □

Lemma 5.2.3 (Friendship theorem [31])**.** *If G_n is a graph in which any two points are connected by a path of length 2 and which does not contain any cycle of length 4, then $n = 2k + 1$ and G_n consists of k triangles which have one common vertex.*

Theorem 5.2.4. *Let F_n be a friendship graph with P vertices. Then the friend-*

ship graph F_n is $CNP-$ unique.

Proof. Let $G \in [F_n]_{\mathcal{N}}$. Then G has P vertices and from Theorem 2.2.17, $N[G; x] = n(2n+1)x$. Since F_n is the friendship graph, number of vertices, $P = 2n+1$. It follows that $N[G; x] = \binom{P}{2}x$. Thus all the $\binom{P}{2}$ pairs of vertices of G has exactly one common neighbor. Then by lemma 5.2.3, G is a friendship graph. □

5.3 CNP-equivalent graph classes

In this section, we prove that well known graph classes like cycle graphs, path graphs, star graphs etc. are not CNP-unique. Here we identify some specific graphs which are CNP-equivalent to these graph classes.

Theorem 5.3.1. *For $n \geq 5$, the cycle graph C_n is not $CNP-$unique. In particular,*

$$N[C_n; x] = \begin{cases} N[T_{(n-2,1)} + K_1;\ x], & \text{if } n \neq 6, \\ N[K_{1,4} + K_1;\ x], & \text{if } n = 6. \end{cases}$$

where $T_{(n,l)}$ is a tadpole graph with $n + l$ vertices and $K_{1,n}$ is a star graph with $n + 1$ vertices.

Proof. Here we consider two cases:

Case(1) Let $n = 6$. We will show that $N[K_{1,4} + K_1; x] = N[C_6; x]$.

From Corollary 2.2.6 and Theorem 2.2.18, it follows that,

$N[K_{1,4} + K_1; x] = N[K_{1,4}; x] + N[K_1; x] + 5 = 6x + 9$.

Hence $K_{1,4} + K_1 \in [C_6]_{\mathcal{N}}$.

Case(2) Let $n \geq 5$ and $n \neq 6$. We will show that $N[T_{(n-2,1)}+K_1;x] = N[C_n;x]$.

From Corollary 2.2.22, we have

$$N[T_{(n,l)};x] = N[C_n;x] + N[P_l;x] + 2x + nl - 2.$$

Hence by Theorem 2.2.18, it follows that

$$\begin{aligned} N[T_{(n-2,1)} + K_1;x] &= N[T_{(n-2,1)};x] + N[K_1;x] + (n-1) \\ &= N[C_{n-2};x] + N[P_1;x] + 2x + 1(n-2) - 2 + (n-1) \\ &= (n-2)x + \frac{(n-2)(n-5)}{2} + 2x + 2n - 5 \\ &= nx + \frac{n(n-3)}{2} \\ &= N[C_n;x]. \end{aligned}$$

It follows that $T_{(n-2,1)} + K_1 \in [C_n]_{\mathcal{N}}$ and thus C_n is not $CNP-$unique. $\square$

Theorem 5.3.2. *For $n > 3$, the path P_n is not $CNP-$unique. In particular,*

$$N[P_n;x] = N[C_3 + P_{n-3};x].$$

Proof. We will show that $N[C_3 + P_{n-3};x] = N[P_n;x]$ for $n > 3$. From Theorems 2.2.2 and 2.2.18, we have

$$\begin{aligned} N[C_3 + P_{n-3};x] &= N[C_3] + N[P_{n-3};x] + 3(n-3) \\ &= 3x + \left[(n-5)x + \binom{n-4}{2} + 1\right] + 3(n-3) \\ &= (n-2)x + \binom{n-1}{2} + 1 = N[P_n;x]. \end{aligned}$$

Since $C_3 + P_{n-3}$ is not isomorphic to P_n and $C_3 + P_{n-3} \in [P_n]_{\mathcal{N}}$, P_n is not $CNP-$unique. $\square$

Theorem 5.3.3. *If $|N(G,1)| = n-2$ with $|V(G)| = n$ and $N(G,i) = 0$ for $i > 1$, then $G \in [P_n]_{\mathcal{N}}$.*

Proof. Since $\sum_{i=1}^{n-2} |N(G,i)| = \binom{n}{2}$, $N(G,0) = \binom{n}{2} - (n-2) = \binom{n-1}{2} + 1$.
So it follows that $N[G;x] = (n-2)x + \binom{n-1}{2} + 1$ and thus $G \in [P_n]_{\mathcal{N}}$. □

Theorem 5.3.4. *If $G \in [P_n]_{\mathcal{N}}$, G has no cycles of length 4.*

Proof. Note that $G \in [P_n]_{\mathcal{N}}$. Hence $N[G;x] = (n-2)x + \binom{n-1}{2} + 1$. So G has no vertex pairs with 2 or more common neighbors. If G has a cycle of length 4, say $uvwxu$, then the number of common neighbors of (u,w) is at least 2, which is a contradiction. Therefore G has no cycles of length 4. □

Theorem 5.3.5. *Let G be a graph of order $n = 3l + r + 2$; $r = 0, 1, 2$ and if $G = l(C_3) + P_{r+2}$, then $G \in [P_n]_{\mathcal{N}}$.*

Proof. Here G is a graph with $l+1$ components. We consider 3 cases each of which uses Corollary 2.2.19 to evaluate the common neighbor polynomial of G.

Case(1) Let $n = 3l + 2$. Then $G = l(C_3) + P_2$.

$$\begin{aligned} N[G;x] &= lN[C_3;x] + N[P_2;x] + 6l + \binom{l}{2}(9) \\ &= 3lx + \frac{3l(3l+1)}{2} + 1 \\ &= (n-2)x + \binom{n-1}{2} + 1 = N[P_n;x]. \end{aligned}$$

Case(2) Let $n = 3l + 3$. Then $G = l(C_3) + P_3$.

$$N[G;x] = lN[C_3;x] + N[P_3;x] + 9l + \binom{l}{2}(9)$$

$$= (3l+1)x + \frac{(3l+2)(3l+1)}{2} + 1$$
$$= (n-2)x + \binom{n-1}{2} + 1 = N[P_n; x].$$

Case(3) Let $n = 3l + 4$. Then $G = l(C_3) + P_4$.

$$\begin{aligned} N[G;x] &= lN[C_3;x] + N[P_4;x] + 12l + \binom{l}{2}(9) \\ &= (3l+1)x + \frac{(3l+3)(3l+2)}{2} + 1 \\ &= (n-2)x + \binom{n-1}{2} + 1 = N[P_n;x]. \end{aligned}$$

This completes the proof. □

Theorem 5.3.6. *If $G = K_{1,r} + K_{1,s} + p(K_1)$ where $rC_2 + sC_2 = n - 2$ and $p = n - r - s - 2$, then $G \in [P_n]_{\mathcal{N}}$.*

Proof. Let $G = K_{1,r} + K_{1,s} + p(K_1)$. From Corollary 2.2.6 and Corollary 2.2.19, it follows that

$$\begin{aligned} N[G;x] &= N[K_{1,r};x] + N[K_{1,s};x] + pN[K_1;x] + (r+1)(s+1) \\ &\qquad + p(r+1) + p(s+1) + \binom{p}{2} \\ &= \binom{r}{2}x + r + \binom{s}{2}x + s + (r+1)(s+1) + p(r+s+2) + \binom{p}{2} \\ &= (n-2)x + \frac{1}{2}\Big[(\binom{r}{2} + \binom{s}{2})^2 + \binom{r}{2} + \binom{s}{2} + 2\Big] \\ &= (n-2)x + \frac{1}{2}[(n-2)^2 + n] \\ &= (n-2)x + \frac{(n-1)(n-2)}{2} + 1 = N[P_n;x]. \end{aligned}$$

This completes the proof. □

Theorem 5.3.7. *If $n = 2k+1$, $k = 1, 2, \ldots$ and if $G = F_k + K_1$ where F_k is the friendship graph with k 3-cycles, then $G \in [K_{1,n}]_{\mathcal{N}}$.*

Proof. Let $G = F_k + K_1$. Then from Theorems 2.2.17 and 2.2.18, we have,

$$\begin{aligned} N[G;x] &= N[F_k + K_1; x] \\ &= N[F_k; x] + N[K_1; x] + |V(F_k)||V(K_1)| \\ &= [3k + 4\binom{k}{2}]x + (2k+1)(1) \\ &= \frac{2k(2k+1)}{2}x + (2k+1) \\ &= \binom{n}{2}x + n = N[K_{1,n}; x]. \end{aligned}$$

This completes the proof. □

Theorem 5.3.8. *If $n = 2k$, $k = 1, 2, \ldots$ then $G \in [K_{1,n}]_{\mathcal{N}}$ where G is the friendship graph F_{k-1} with $(k-1)$ 3-cycles whose center vertex is attached to a path P_2 through a bridge.*

Proof. Let G be the friendship graph F_{k-1} with center vertex u to which a path P_2 is attached through a bridge $e = uv$. Then $e = uv$ is a cutedge of G. From Theorem 2.2.20, it follows that $N[G;x] = N[F_{k-1} + P_2; x] + [2(k-1)+1]x - [2(k-1)+1]$. Then from Theorems 2.2.17 and 2.2.18, we have,

$$\begin{aligned} N[F_{k-1} + P_2; x] &= N[F_{k-1}; x] + N[P_2; x] + |V(F_{k-1})||V(P_2)| \\ &= (k-1)(2k-1)x + 1 + 2(2k-1). \end{aligned}$$

It follows that $N[G;x] = N[K_{1,n}; x]$.

This completes the proof. □

Theorem 5.3.9. *If G_1 and G_2 are two components of a graph G, then G is $CNP-$unique if and only if both G_1 and G_2 are $CNP-$unique.*

Proof. The result follows from Theorem 2.2.18. □

Theorem 5.3.10. *If G is a connected graph and $e = uv$ a cutedge of G, then $G - e$ is $CNP-$unique if and only if G is $CNP-$unique.*

Proof. The result follows from Theorem 2.2.20. □

Corollary 5.3.11. *The tadpole $T_{(n,l)}$ is not $CNP-$unique for $l > 3$ and $n \geq 5$.*

Proof. Removing the bridge e from $T_{n,l}$, the resulting graph is the union of the path P_l and the cycle C_n. Hence from Theorems 5.3.1, 5.3.2 and 5.3.9, $T_{(n,l)} - e$ is not $CNP-$unique for $l > 3$ and $n \geq 5$. Now the result follows from Theorem 5.3.10. □

The n- barbell graph $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge e.

Corollary 5.3.12. *The $n-$barbell graph $B_{n,1}$ is $CNP-$unique for $n > 2$.*

Proof. Note that $B_{n,1} - e = K_n + K_n$. From Theorems 5.2.1 and 5.3.9, $B_{n,1} - e$ is $CNP-$unique. Hence from Theorem 5.3.10, $B_{n,1}$ is $CNP-$unique. □

The lollipop graph $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a path P_n with a bridge.

Corollary 5.3.13. *The lollipop graph $L_{m,n}$ is not $CNP-$unique for $n > 3$.*

Proof. Note that $L_{m,n} - e = K_m + P_n$. Hence from Theorems 5.3.2 and 5.3.9, $L_{m,n} - e$ is not $CNP-$unique for $n > 3$. Hence the result follows from Theorem 5.3.10. □

A bistar graph $B_{m,n}$ is obtained by connecting the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by a bridge e.

Corollary 5.3.14. *The bistar graph $B_{m,n}$ is not $CNP-$unique.*

Proof. Note that $B_{m,n} - e = K_{1,m} + K_{1,n}$. Hence from Theorems 5.3.7, 5.3.8 and 5.3.9, $B_{m,n} - e$ is not $CNP-$unique. Then the result follows from Theorem 5.3.10. $\square$

Chapter 6

Stability of common neighbor polynomial of graphs

A polynomial $f(x_1, \ldots, x_n)$ is said to be stable [28] with respect to a region $\Omega \in \mathbb{C}^n$ if no root of f lies in Ω. Polynomials which are stable with respect to the closed right half plane and with respect to the open unit disk are called Hurwitz polynomial and Schur polynomial respectively. Hurwitz polynomials are important in control systems theory, because they represent the characteristic equations of stable linear systems[15]. A graph polynomial is worthwhile to study only if it models some stable physical systems. In this chapter we study the stability of common neighbor polynomial of graphs with respect to the closed right half plane and thus identify the conditions under which the common neighbor polynomial of certain graph classes become a Hurwitz polynomial.

6.1 Main results

Definition 6.1.1. *A polynomial $f(x_1, x_2, \ldots, x_n)$ is said to be stable with respect to the closed right half plane if and only if all of its non zero roots lie in the open left half plane.*

Theorem 6.1.2. *Let P_n be a path with $n > 2$ vertices. Then $N[P_n; x]$ is stable.*

Proof. From Theorem 2.2.2, it follows that $N[P_n; x]$ has a single root

$$x = \frac{-(n-1)(n-2) - 2}{2(n-2)}$$

which lie in the open left half plane since $n > 2$. Hence the result follows. □

Theorem 6.1.3. *For a cycle C_n, $N[C_n; x]$ is stable unless $n = 4$.*

Proof. From Theorem 2.2.4, we have

$$N[C_n; x] = \begin{cases} nx + \frac{n(n-3)}{2}, & n > 2, n \neq 4. \\ 2x^2 + 4, & n = 4. \end{cases}$$

If $n = 3$, $N[C_3; x] = 3x$. In this case, zero is the only root of $N[C_3; x]$. If $n > 4$, then $N[C_n; x]$ has a single root $x = \frac{-(n-3)}{2}$ which lie in the left half plane. If $n = 4$, the roots of $N[C_4; x]$ are given by $x = \pm\sqrt{2}i$. Hence $N[C_4; x]$ is not stable as it has non zero roots in the closed right half plane. □

Theorem 6.1.4. *Let G be a graph with common neighbor polynomial $N[G; x]$ of degree 2. Then the following hold:*

1. *If $N(G, 0) = \phi$ and $N(G, 1) \neq \phi$, then $N[G; x]$ is a stable polynomial.*

2. *If $N(G,0) \neq \phi$ and $N(G,1) = \phi$, then $N[G;x]$ is not a stable polynomial.*

Proof. Since $N[G;x]$ is of degree 2, $|N(G,2)| \neq 0$. We consider the two cases:

1. Let $N(G,0) = \phi$ and $N(G,1) \neq \phi$. In this case, the roots of $N[G;x]$ are given by $x = 0$ and $x = -\dfrac{|N(G,1)|}{|N(G,2)|}$. It follows that $N[G;x]$ is stable.

2. Let $N(G,0) \neq \phi$ and $N(G,1) = \phi$. Then the roots of $N[G;x]$ are given by $x = \pm\sqrt{\dfrac{|N(G,0)|}{|N(G,2)|}}\, i$. Since $N[G;x]$ has non zero roots in the closed right half plane, $N[G;x]$ is not stable.

This completes the proof. $\square$

Corollary 6.1.5. *If W_n is a wheel graph having n vertices, $N[W_n;x]$ is stable for $n \geq 4$.*

Proof. We have (see 2.2.11),

$$N[W_n;x] = \begin{cases} \frac{(n-1)(n-4)}{2}x + 2(n-1)x^2, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 4x, & \text{if } n = 5. \end{cases}$$

When $n = 4$, $N[W_4;x] = 6x^2$ which has only one root namely zero. When $n = 5$, the common neighbor roots of W_n are $x = 0, -1 \pm i$ which lie in the open left half plane. When $n > 5$, $N[W_n;x]$ is of degree 2 with $|N(W_n,0)| = 0$ and $|N(W_n,1)| \neq 0$. So the result follows from Theorem 6.1.4. $\square$

Corollary 6.1.6. *If S_n is a shell graph with $n \geq 3$ vertices, then $N[S_n;x]$ is stable.*

Proof. We have (see Theorem 2.2.14), $N[S_n;x] = 2(n-3)x^2 + \left(\binom{n-2}{2} + 3\right)x$. Since $n \geq 3$, $N[S_n;x]$ satisfies the conditions of first part of Theorem 6.1.4. Hence it is stable. □

Corollary 6.1.7. *If B_N is a bow graph with $N > 5$ vertices, then $N[B_N;x]$ is stable.*

Proof. From Theorem 2.2.15 we have, $N[B_N;x] = 2(N-5)x^2 + \left[\frac{N(N-5)}{2} + 10\right]x$. Since $n > 5$, $N[B_n;x]$ satisfies the conditions of first part of Theorem 6.1.4. Hence it is stable. □

Here we need the following:

Theorem 6.1.8. *(Routh-Hurwitz Criteria [30]) Given a polynomial, $P(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_0$, where the coefficients a_i are real constants, $i = 1, 2, \ldots, n$ define the n Hurwitz matrices using the coefficients a_i of the above polynomial as*

$$H_1 = [a_1] \qquad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} \quad \cdots \quad H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \cdots & a_n \end{bmatrix}$$

where $a_j = 0$ if $j > n$. All the roots of the polynomial $P(x)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive: det $H_j > 0$, $j = 1, 2, \ldots, n$.

Corollary 6.1.9. *If $L_{n,1}$ is the lollipop graph with $n+1$ vertices, then $N[L_{n,1};x]$ is stable if and only if $n \leq 4$.*

Proof. We have (see Theorem 2.2.10), $N[L_{n,1};x] = \binom{n}{2}x^{n-2} + (n-1)x + 1$. For $n = 1, 2, 3$, the result follows from simple calculations. When $n = 4$, $N[L_{4,1};x] = 6x^2 + 3x + 1$. So considering the polynomial $x^2 + \frac{1}{2}x + \frac{1}{6}$, the determinants of Hurwitz matrices are given by, $|H_1| = \frac{1}{2}$ and $|H_2| = \begin{vmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{6} \end{vmatrix} = \frac{1}{12}$. Since all the determinants are positive, it follows that $N[L_{4,1};x]$ is stable. When $n > 4$, the result follows from the fact that the determinant of first Hurwitz matrix of $N[L_{n,1};x]$ is zero. $\square$

Corollary 6.1.10. *If $B_{n,1}$ is the n-Barbell graph with $2n$ vertices, then $N[B_{n,1};x]$ is stable if and only if $n \leq 4$.*

Proof. From Corollary 2.2.23 we have, $N[B_{n,1};x] = 2\binom{n}{2}x^{n-2} + 2(n-1)x + (n-1)^2 + 1$. For $n = 1, 2, 3$, the result follows from simple calculations. When $n = 4$, $N[B_{4,1};x] = 12x^2 + 6x + 10$. So considering the polynomial $x^2 + \frac{1}{2}x + \frac{5}{6}$, the determinants of Hurwitz matrices are given by, $|H_1| = \frac{1}{2}$ and $|H_2| = \begin{vmatrix} \frac{1}{2} & 1 \\ 0 & \frac{5}{6} \end{vmatrix} = \frac{5}{12}$. Since all the determinants are positive, it follows that $N[B_{4,1};x]$ is stable. When $n > 4$, the result follows from the fact that the determinant of first Hurwitz matrix of $N[B_{n,1};x]$ is zero. $\square$

Corollary 6.1.11. *If B_n, $n \geq 3$ is the bipartite cocktail party graph, then $N[B_n;x]$ is stable if and only if $n = 3$.*

Proof. We have (see Theorem 2.2.26), $N[B_n;x] = 2\binom{n}{2}x^{n-2} + n^2$. When $n = 3$, $N[B_n;x] = 6x + 9$, having root $x = -3/2$ lying in the open left half plane. When

$n > 3$, the determinant of the first Hurwitz matrix becomes zero and hence the result follows. $\square$

Corollary 6.1.12. *If $W_n^{(m)}$, $n \geq 3$ is the windmill graph, then $N[W_n^{(m)}; x]$ is stable if and only if $n = 3, 4$.*

Proof. From Theorem 2.2.27, we have, $N[W_n^{(m)}; x] = m\binom{n}{2}x^{n-2} + \binom{m}{2}(n-1)^2 x$. When $n = 3$, $N[W_n^{(m)}; x] = m(2m+1)x$ having root $x = 0$ only. When $x = 4$, $N[W_n^{(m)}; x] = 6mx^2 + 9\binom{m}{2}x$ having roots $x = 0$ and $x = (-9/12)(m-1)$. Hence for $n = 3, 4$, $N[W_n^{(m)}; x]$ is stable.

For $n > 4$, $N[W_n^{(m)}; x]$ is not stable as the determinant of the first Hurwitz matrix becomes zero. $\square$

Theorem 6.1.13. *Let G be a graph with common neighbor polynomial $N[G; x]$ of degree 2 where $|N(G, i)| > 0$ for $i = 0, 1, 2$. Then $N[G; x]$ is stable. Moreover, $N[G; x]$ has two negative real roots if $|N(G, 1)|^2 \geq 4|N(G, 2)||N(G, 0)|$ and $N[G; x]$ has two complex roots with negative real parts otherwise.*

Proof. Since $N[G; x]$ is a polynomial of degree 2, $N[G; x]$ can be represented in the form $N[G; x] = |N(G, 2)|x^2 + |N(G, 1)|x + |N(G, 0)|$. The Hurwitz matrices of $N[G; x]$ are given by $H_1 = \begin{bmatrix} \frac{|N(G,1)|}{|N(G,2)|} \end{bmatrix}$ and $H_2 = \begin{bmatrix} \frac{|N(G,1)|}{|N(G,2)|} & 1 \\ 0 & \frac{|N(G,0)|}{|N(G,2)|} \end{bmatrix}$. Since $|N(G, i)| > 0$ for $i = 0, 1, 2$; it follows that $\det H_1 > 0$ and $\det H_2 > 0$. Hence by Theorem 6.1.4, $N[G; x]$ is stable so that all the roots of $N[G; x]$ lie in the open left half plane. Moreover, the discriminant of $N[G; x]$ is given by $\Delta = |N(G, 1)|^2 - 4|N(G, 2)||N(G, 0)|$. It follows that $N[G; x]$ has 2 real roots if $\Delta \geq 0$ and has two complex roots if $\Delta < 0$. This completes the proof. $\square$

Corollary 6.1.14. *If H_n, $n > 3$ is a helm with $2n-1$ vertices, then $N[H_n;x]$ is stable. Moreover, $N[H_n;x]$ has two negative real distinct roots if $n > 40$ and has two complex roots with negative real parts if $n \leq 40$.*

Proof. From Theorem 2.2.12 we have,

$$N[H_n;x] = \begin{cases} 2(n-1)x^2 + \frac{(n-1)(n+2)}{2}x + \frac{(n-1)(3n-8)}{2}, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 16x + 14, & \text{if } n = 5. \end{cases}$$

We consider two cases.

1. Let $n = 5$. In this case, $N[H_5;x] = 2x^3 + 4x^2 + 16x + 14$. So considering the equation, $x^3 + 2x^2 + 8x + 7 = 0$, the values of the determinants of corresponding Hurwitz matrices are given by, $|H_1| = 2$, $|H_2| = 9$ and $|H_3| = 63$. Since determinants of all Hurwitz matrices are positive, by Theorem 6.1.8, $N[H_5;x]$ is stable.

2. Let $n \neq 5$. Since $n > 3$, $N[H_n;x]$ satisfies the conditions of Theorem 6.1.13 and hence it is stable. Moreover, the discriminant of $N[H_n;x]$ is given by,

$$\Delta = \frac{(n-1)^2}{4}\Big[n^2 - 44n + 132\Big].$$

   Since $n > 3$, $\Delta \geq 0$ implies that $(n-a)(n-b) \geq 0$ where a, b are the roots of the equation $n^2 - 44n + 132 = 0$. Here $a \simeq 40.76$ and $b \simeq 3.24$. Since $n > 3$ and n is an integer, the case when $n \leq a$ and $n \leq b$ becomes ruled out. Hence the only possible case is $n > a$ and $n > b$ and then $n > 40$. Hence the result follows from Theorem 6.1.13.

This completes the proof. □

Corollary 6.1.15. *If WB_n, $n > 3$ is a web graph with $3(n-1)$ vertices, $N[WB_n;x]$ is stable. Moreover, $N[WB_n;x]$ have two real distinct roots if $n > 241$ and have two complex roots with negative real parts if $n \leq 241$.*

Proof. From Theorem 2.2.13 we have,

$$N[WB_n;x] = \begin{cases} 4(n-1)x^2 + \frac{(n-1)(n+6)}{2}x + (n-1)(4n-10), & \text{if } n \neq 5, \\ 2x^3 + 14x^2 + 20x + 42, & \text{if } n = 5. \end{cases}$$

We consider two cases.

1. Let $n = 5$. In this case, $N[WB_n;x] = 2x^3 + 14x^2 + 20x + 42$. So considering the equation, $x^3 + 7x^2 + 10x + 21 = 0$, the values of the determinants of corresponding Hurwitz matrices are given by, $|H_1| = 7$, $|H_2| = 49$ and $|H_3| = 1029$. Since determinants of all Hurwitz matrices are positive, by Theorem 6.1.4, $N[WB_5;x]$ is stable.

2. Let $n \neq 5$. Since $n > 3$, $N[WB_n;x]$ satisfies the conditions of Theorem 6.1.8 and hence it is stable. Moreover, the discriminant of $N[WB_n;x]$ is given by,

$$\Delta = \tfrac{(n-1)^2}{4}\Big[n^2 - 244n + 676\Big].$$

   Now from a similar proof as in Corollary 6.1.14, it follows that $n > 241$ when $n > 3$ and $\Delta \geq 0$. Hence the result follows from Theorem 6.1.13.

This completes the proof. □

Corollary 6.1.16. *If BF is a butterfly grpah with $N > 7$ vertices, then $N[BF;x]$ is stable.*

Proof. From Theorem 2.2.16 it follows that $N[BF;x]$ $N > 7$, satisfies the conditions of Theorem 6.1.13 and hence it is stable. □

Theorem 6.1.17. *If $K_{m,n}$ is a complete bipartite graph with $m+n$ vertices, where $n > m$ then $N[K_{m,n};x]$ is stable if and only if $n = 2$ and $m = 1$.*

Proof. We have, $N[K_{m,n};x] = \binom{m}{2}x^n + \binom{n}{2}x^m + mn$ where $n, m \geq 2$.
Clearly, the condition is sufficient since, $N[K_{1,2};x] = x + 2$ which has only one root $x = -2$ lying in the open left half plane.
To prove the necessary part, we consider two cases.

Case(i) Let $n - m = 1$

In this case $N[K_{m,n};x]$ can be expressed as,
$N[K_{m,n};x] = \binom{m}{2}[x^n + a_1x^{n-1} + a_n]$, where $a_1 = \frac{\binom{n}{2}}{\binom{m}{2}}$ and $a_n = \frac{mn}{\binom{m}{2}}$. Now, considering the polynomial $x^n + a_1x^{n-1} + a_n$, the determinants of first two Hurwitz matrices are given by $|H_1| = a_1 > 0$ and

$$|H_2| = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = \begin{cases} a_1a_2 > 0 \text{ if } & n = 2 \\ -a_3 < 0 \text{ if } & n = 3 \\ 0 \qquad \text{if } & n \geq 4 \end{cases}$$

Hence determinants of all the Hurwitz matrices are positive only if $n = 2$. Then by the assumption of Case(i), $m = 1$.

Case(ii) Let $n - m > 1$

Then the determinant of first Hurwitz matrix of $N[K_{m,n};x]$ itself is zero for all m, n.

Hence from Theorem 6.1.8, it follows that $N[K_{m,n};x]$ is stable only if $n = 2$ and $m = 1$. □

Chapter 7

Real roots of common neighbor polynomial of graphs

In this chapter we study the real roots of common neighbor polynomial of graphs. In particular, some characterizations on graphs are done based on the roots of common neighbor polynomial of graphs.

7.1 Main results

The roots of common neighbor polynomial of a graph G are called the common neighbor roots of G. The number of real common neighbor roots of a graph G where the multiplicities counted, is denoted by $\mathcal{N}(G)$.

Theorem 7.1.1. *Zero is a common neighbor root of a graph G if and only if any pair of vertices of G has at least one common neighbor.*

Proof. Let G be a graph with n vertices. Let zero be a root of $N[G;x]$. Then $N[G;x] = x\ g(x)$ where $g(x)$ is a polynomial of degree one less than that of

$N[G;x]$. Then the constant term of $N[G;x]$ is zero and hence the result follows. Conversely, assume that any pair of vertices of G has at least one common neighbor. Then $|N(G,0)| = 0$. It follows that x is a factor of $N[G;x] = \sum_{i=0}^{n-2} |N(G,i)|x^i$. $\square$

Theorem 7.1.2. *Zero is the only common neighbor root of a graph G if and only if any two pairs of vertices of G has same number of common neighbors.*

Proof. First assume that zero is the only common neighbor root of a graph G with multiplicity k. Then $N[G;x]$ is of the form Kx^k for some constant K. Since, $\sum_{i=0}^{n-2} |N(G,i)| = \binom{n}{2}$, it follows that $K = \binom{n}{2}$. Thus, $N[G;x] = \binom{n}{2}x^k$ and so all the pairs of vertices of G has k common neighbors. Conversely assume that all the pairs of vertices of G has exactly k common neighbors. Then the result follows from the fact that $N[G;x] = \binom{n}{2}x^k$. $\square$

Example 7.1.3. *Zero is the only common neighbor root of the complete graph K_n with multiplicity $n-2$.*

Theorem 7.1.4. $(0,\infty)$ *is a zero-free interval of the common neighbor polynomial $N[G;x]$ of any graph G.*

Proof. The result follows from the fact that all the coefficients of $N[G;x]$ are non-negative. $\square$

Theorem 7.1.5. *Let G be a graph of order $n \geq 3$. If zero is a common neighbor root of G, then diameter of $G \leq 2$ and G has a spanning subgraph which is a union of triangles.*

Proof. Let G be a graph with $n \geq 3$ vertices which has zero as a common neighbor root. Then $N(G,0) = \phi$ which means that, every pair of vertices (u,v)

has at least one common neighbor. Therefore, any pair of vertices of G are at a distance less than or equal to 2. So it follows that $\operatorname{diam}(G) \leq 2$.

Now we will prove that any vertex $v \in V(G)$ is a common neighbor of some vertex pair (v_i, v_j). Assume the contrary. Then v is not a common neighbor of any pair of vertices of G. Let $v_1 \in V(G)$ such that $v_1 \neq v$. Then by assumption, (v, v_1) has at at least one common neighbor, say v_2 in G. Then the common neighbor of (v, v_2) cannot be v_1, since then v is a common neighbor of (v_1, v_2), a contradiction. Therefore, (v, v_2) has a common neighbor, say, v_3 such that $v_3 \neq v_1, v_2$. Then v is a common neighbor of (v_2, v_3), a contradiction. Thus, every vertex v is a common neighbor of some pair (v_i, v_j).

Now we will prove that G has a spanning subgraph which is a union of triangles. It is enough to show that every vertex of G is a vertex of some triangle in G. Let $v \in V(G)$. Then there exists a vertex pair (v_1, v_2) such that v is a common neighbor of (v_1, v_2). The common neighbor of (v, v_1) may be v_2 or some other vertex v_3. If it is v_2, then v is a vertex of the triangle vv_1v_2v. If it is v_3, then v is a vertex of the triangle vv_1v_3v. Thus every vertex of G is the vertex of some triangle in G. The union of all such triangles in G forms the required spanning subgraph of G. □

Theorem 7.1.6. *Let G be a graph of order $n \geq 3$. Then zero is a common neighbor root of G if and only if diameter of $G \leq 2$ and every edge of G is a part of some triangle in G.*

Proof. Let G be a graph with $n \geq 3$ vertices which has zero as a common neighbor root. Then by Theorem 7.1.5, $\operatorname{diam}(G) \leq 2$. Let $e = uv$ be any edge of G. Since zero is a common neighbor root of G, $N(G, 0) = \phi$. Then the vertex pair (u, v) has at least one common neighbor w in G. Then the edge e is a part of the triangle $uvwu$ which proves the necessary part of the theorem.

To prove the sufficient part, assume that $\text{diam}(G) \leq 2$ and every edge of G is a part of some triangle in G. It is enough to show that any pair of vertices in G has at least one common neighbor in G. Let (u, v) be a pair of vertices of G. Here we consider 2 cases.

Case(i) u is adjacent to v by an edge e.

By assumption, there exist a vertex w in G such that e is the part of some triangle $uvwu$ in G. Then w is a common neighbor of the vertex pair (u, v) in G.

Case(ii) u is not adjacent to v in G.

Since each edge of G is a part of some triangle in G and since G has no isolated vertices, each vertex must also be a part of some triangle in G. Then u and v are vertices of some triangles say, uu_1u_2 and vv_1v_2 in G. Since $\text{diam}(G) \leq 2$, there is a $u, v-$path of length 2 in G. Since u is not adjacent to v, one of the following holds for $i, j \in \{1, 2\}$. (i) u_i and v_j coincides, (ii) u_i and v coincides, (iii) u and v_j coincides, (iv) u_i is adjacent to v, (v) u is adjacent to v_j or (vi)there exists a vertex w in G such that $uwv$is a path in G. All these cases ensures a common neighbor for (u, v).

It follows that $(u, v) \notin N(G, 0)$. Since the vertex pair (u, v) is chosen arbitrary, $N(G, 0) = \phi$. Then zero is a common neighbor root of G. Hence the proof. $\square$

Theorem 7.1.7. *Let G_1 and G_2 be two disjoint graphs with n and m vertices respectively and let $G = G_1 + G_2$. If a is a common neighbor root of both G_1 and*

G_2, then a is a root of $N[G;x]-nm$. The converse is not true.

Proof. The result follows from the fact that $N[G;x]=N[G_1;x]+N[G_2;x]+nm$ (See Theorem 2.2.18). The converse is not true. For example, let G_1 be the path P_2 and G_2 be the cycle C_3. Then $N[G_1;x]=1$, $N[G_2;x]=3x$ and $N[G_1+G_2;x]=3x+7$. Here $x=-\frac{1}{3}$ is a root of $N[G_1+G_2;x]-nm$, but it is not a root of $N[G_1;x]$ or $N[G_2;x]$. □

Theorem 7.1.8. *Let G be a graph and let $e\in E(G)$. Then the common neighbor roots of $N[G-e;x]$ are the roots of $N[G;x]-mx+m$ where m is the number of edges adjacent to e in G.*

Proof. The result follows from the fact that $N[G-e;x]=N[G;x]-mx+m$ (See Theorem 2.2.20). □

Theorem 7.1.9. *For a complete graph K_n, $\mathcal{N}(K_n)=n-2$ for $n>2$.*

Proof. Since (see Theorem 2.2.1) $N[K_n;x]=\binom{n}{2}x^{n-2}$, K_n has zero as a real common neighbor root with multiplicity $n-2$. Hence the result follows. □

Theorem 7.1.10. *For a path graph P_n, $\mathcal{N}(P_n)=1$ for $n>2$.*

Proof. From Theorem 2.2.2, we have $N[P_n;x]=(n-2)x+\binom{n-1}{2}+1$ for $n\geq 2$. It follows that P_n has only one common neighbor root $x=-\frac{\binom{n-1}{2}+1}{n-2}$ which is real. Thus $\mathcal{N}(P_n)=1$. □

Theorem 7.1.11. *For the cycle graph C_n, $n>2$, we have the following:*

$$\mathcal{N}(C_n)=\begin{cases}1 & \text{if} \quad n\neq 4,\\ 0 & \text{if} \quad n=4.\end{cases}$$

Proof. The result follows from the fact that (see Theorem 2.2.4)

$$N[C_n;x] = \begin{cases} nx + \frac{n(n-3)}{2}, & n > 2, n \neq 4, \\ 2x^2 + 4, & n = 4. \end{cases}$$

□

Theorem 7.1.12. *For the complete bipartite graph $K_{2m-1,2n-1}$, where $m, n \geq 1$, we have the following:*

$$\mathcal{N}(K_{(2m-1,2n-1)}) < \begin{cases} 2n-1 & \text{if} \quad m \geq 2n-1, \\ 2m-2n+1 & \text{if} \quad m \leq 2n-1. \end{cases}$$

Proof. For $m = 1$ and $n = 1$, the result trivially follows from the fact that $N[K_{1,1};x] = 1$ which has no zeros. Now let $m, n \geq 2$ where $m > n$.
Then from Theorem 2.2.5,

$$N[K_{m,n};x] = \binom{m}{2}x^n + \binom{n}{2}x^m + mn.$$

It follows that

$$N[K_{(2m-1,2n-1)};x] = \binom{2n-1}{2}x^{2m-1} + \binom{2m-1}{2}x^{2n-1} + (2m-1)(2n-1).$$

This polynomial lacks $2m-2n$ terms between the first two terms and lacks $2n-2$ terms between the last two terms. Hence by Theorem 1.3.1, $N[K_{(2m-1,2n-1)};x] = 0$ has no less than $2m-2n$ imaginary roots if $2m-2n \geq 2n-2$ and has no less than $2n-2$ imaginary roots if $2m-2n \leq 2n-2$. Since the polynomial has $2m-1$ zeros, if follows that the number of real roots is less than $2n-1$ if $m \geq 2n-1$ and is less than $2m-2n+1$ if $m \leq 2n-1$. □

Theorem 7.1.13. *For a complete m-partite graph $K_{2n,2n,\ldots,2n}$ where $2n$ repeats m times, we have $\mathcal{N}(K_{2n,2n,\ldots,2n}) < 2n(m-2)$.*

Proof. For a complete m partite graph $K_{n,n,\ldots,n}$ where n repeats m times,we have (see Corollary 2.2.9)

$$N[K_{n,n,\ldots,n}; x] = m\binom{n}{2}x^{n(m-1)} + n^2\binom{m}{2}x^{n(m-2)}.$$

It follows that

$$N[K_{2n,2n,\ldots,2n}; x] = m\binom{2n}{2}x^{2n(m-1)} + 4n^2\binom{m}{2}x^{2n(m-2)}.$$

This polynomial lacks $2n$ consecutive terms in between the two terms. Hence by Theorem 1.3.1, it has no less than $2n$ imaginary zeros. Since the polynomial has $2n(m-1)$ zeros, the number of real roots of the polynomial equation become less than $2n(m-2)$. This completes the proof. □

Theorem 7.1.14. *The lollipop graph $L_{2n-1,1}$, $n \geq 2$ has only one real common neighbor root and it lies in the disc $|z| \leq 1$.*

Proof. From Theorem 2.2.10, we have, $N[L_{n,1}; x] = \binom{n}{2}x^{n-2} + (n-1)x + 1$.
It follows that

$$N[L_{2n-1,1}; x] = \binom{2n-1}{2}x^{2n-3} + (2n-2)x + 1.$$

As this polynomial lacks $2n-4$ terms between first two terms, by Theorem 1.3.1, it has no less than $2n-4$ imaginary roots. It follows that $\mathcal{N}(L_{2n-1,1}) \leq 1$. Since the complex roots of a real polynomial always occurs in conjugate pairs, $N[L_{2n-1,1}; x] = 0$ has exactly one real root. Since $N[L_{2n-1,1}; x]$ satisfies the hypothesis of Theorem 1.3.2, the root lies in $|z| \leq 1$. □

Theorem 7.1.15. *For the wheel graph W_n, we have the following:*

$$\mathcal{N}(W_n) = \begin{cases} 2 & \textit{if} \quad n \neq 5, \\ 1 & \textit{if} \quad n = 5. \end{cases}$$

Proof. For a wheel graph W_n, we have, (see Theorem 2.2.11),

$$N[W_n; x] = \begin{cases} \frac{(n-1)(n-4)}{2}x + 2(n-1)x^2, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 4x, & \text{if } n = 5. \end{cases}$$

Here we consider two cases:

Case (i) $n \neq 5$.

Then the polynomial equation $N[W_n; x] = 0$ has two real roots given by $x = 0$ and $x = -\frac{n-4}{4}$.

Case (ii) $n = 5$.

Then the polynomial equation $N[W_n; x] = 2x^3 + 4x^2 + 4x = 0$ has only one real root $x = 0$ and has two imaginary roots $x = -1 \pm i$.

Hence the result follows. □

Theorem 7.1.16. *For the helm $H_n, n > 3$, we have the following:*

$$\mathcal{N}(H_n) = \begin{cases} 2 & \textit{if} \quad n \geq 41, \\ 1 & \textit{if} \quad n = 5, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For a helm graph H_n we have (see Theorem 2.2.12),

$$N[H_n; x] = \begin{cases} 2(n-1)x^2 + \frac{(n-1)(n+2)}{2}x + \frac{(n-1)(3n-8)}{2}, & \text{if } n \neq 5, \\ 2x^3 + 4x^2 + 16x + 14, & \text{if } n = 5. \end{cases}$$

Here we consider two cases:

Case(i) $n = 5$.

Then the polynomial equation $N[H_5; x] = 2x^3 + 4x^2 + 16x + 14 = 0$ has only one real root $x = -1$ and has two imaginary roots $x = \frac{-1 \pm \sqrt{27}i}{2}$.

Case(ii) $n \neq 5$.

It is enough to consider the equation

$$4x^2 + (n+2)x + (3n-8) = 0$$

which has either two real roots or has no real roots according as the discriminant $\Delta = n^2 - 44n + 132$ is non negative or non positive respectively. Let the roots of $\Delta = 0$ be x and y. Then $x \simeq 40.76$ and $y \simeq 3.24$. Then $\Delta = (n-x)(n-y)$ is positive if either $n \geq x$ and $n \geq y$ or $n \leq x$ and $n \leq y$. Since n is an integer, it follows that, $\Delta \geq 0$ if $n \geq 41$ or $n \leq 3$. It follows that if $n \neq 5$, H_n has two real common neighbor roots if $n \geq 41$ and has no real common neighbor root otherwise.

This completes the proof. $\square$

Theorem 7.1.17. *For a web graph $WB_n, n > 3$, we have the following:*

$$\mathcal{N}(WB_n) = \begin{cases} 2 & \text{if} \quad n \geq 242, \\ 1 & \text{if} \quad n = 5, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Theorem 2.2.13, for a web graph $WB_n, n > 3$,

$$N[WB_n; x] = \begin{cases} 4(n-1)x^2 + \frac{(n-1)(n+6)}{2}x + (n-1)(4n-10), & \text{if } n \neq 5, \\ 2x^3 + 14x^2 + 20x + 42, & \text{if } n = 5. \end{cases}$$

Here we consider two cases:

Case(i) $n = 5$.

Considering the equation $f(x) = x^3+7x^2+10x+21 = 0$, we have $f(-5) = 0$ and $f(-6) = 0$. Hence by intermediate value theorem, $f(x) = 0$ has a real root between -5 and -6. Now, using Theorem 1.3.3 the discriminant of the cubic equation $f(x) = 0$ is given by $\Delta = -13359 < 0$ which implies that the cubic equation has only one real root.

Case(ii) $n \neq 5$.

Since $n > 1$, consider the equation $8x^2 + (n + 6)x + 2(4n - 10) = 0$. This equation has either two real roots or has no real roots according as the discriminant $\Delta = n^2 - 244n + 676$ is non negative or non positive respectively. Let the roots of $\Delta = 0$ be x and y. Then $x \simeq 241.2$ and $y \simeq 2.8$. Then $\Delta = (n-x)(n-y)$ is non negative if either $n \geq x$ and $n \geq y$ or $n \leq x$ and $n \leq y$. Since n is an integer, it follows that, $\Delta \geq 0$ if $n \geq 242$ or $n \leq 2$. It follows that if $n \neq 5$, WB_n has two real common neighbor roots if $n \geq 242$ and has no real common neighbor root otherwise.

This completes the proof. $\square$

Theorem 7.1.18. *For the* $(2n + 1)$*-barbell graph* $B_{2n+1,1}$, $\mathcal{N}(B_{2n+1}) = 1$.

Proof. For the n-barbell graph $B_{n,1}$ we have (see Corollary 2.2.23),

$$N[B_{n,1}; x] = 2\binom{n}{2}x^{n-2} + 2(n-1)x + (n-1)^2 + 1.$$

It follows that, for the $(2n + 1)$-barbell graph $B_{2n+1,1}$,

$$N[B_{2n+1,1}; x] = 2\binom{2n+1}{2}x^{2n-1} + 2(2n)x + 4n^2 + 1.$$

This polynomial lacks $2n-2$ terms between the first two terms. Thus it has no less than $2n-2$ imaginary roots. It follows that $N[B_{2n+1,1};x]=0$ has at most one real root.

Now, considering the equation $f(x)=N[B_{2n+1,1};x]=0$, we have, $f(0)=4n^2+1>0$ and $f(-1)=-6n+1<0$. Then by intermediate value theorem, there is a real root for $f(x)=0$ between 0 and -1. This completes the proof. $\square$

Chapter 8

Generalized common neighbor polynomial of graphs

In this chapter we generalize the concepts of common neighbor sets and common neighbor polynomial of graphs. In the previous chapters, focus was laid on the pairs of vertices of a graph with common neighbors. In this chapter r-tuple of vertices of a graph having a common neighbor is considered and thereby generalizes the concept. The definition of generalized i-common-neighbor set *is introduced and the* generalized common neighbor polynomial *of a graph is defined. Moreover, we discuss some properties of generalized i-common neighbor sets and also derive the generalized common neighbor polynomial of some well known graph classes. Also we express generalized common neighbor sets using the theory of simplicial complexes in order to deduce some interesting properties of the generalized common neighbor sets.*

8.1 Generalized common neighbor sets and common neighbor polynomial of graphs

Definition 8.1.1. *Let $G(V, E)$ be a graph of order n. Let $\mathscr{L}_r$ denotes the set of all unordered r-tuples of distinct elements of V. For $0 \leq i \leq n - r$, the generalized i-common neighbor set of G is defined as follows:*

$$N_r(G, i) = \{(u_1, u_2, \ldots, u_r) \in \mathscr{L}_r : |\cap_{k=1}^{r} N(u_k)| = i\}.$$

Definition 8.1.2. *Let G be a graph of order n. For $0 < r \leq n$ the generalized common neighbor polynomial , $N_r[G; x]$, of G is defined as*

$$N_r[G; x] = \sum_{i=0}^{(n-r)} |N_r(G, i)| x^i.$$

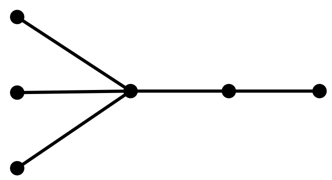

Figure 8.1: The graph G

Example 8.1.3. *For the graph G shown in figure 8.1, we have the following:*
$N_1[G; x] = x^4 + x^2 + 4x.$

$N_2[G; x] = 7x + 8.$

$N_3[G; x] = 4x + 16.$

$N_r[G; x] = \binom{6}{r}$ *for* $r \geq 4.$

Throughout this chapter, r denotes an integer such that $1 \leq r \leq n$. We observe the following simple properties of $N_r[G; x]$:

(i) $N_2[G;x] = N[G;x]$, the common neighbor polynomial of the graph G.

(ii) $N_r[G;x]$ is a polynomial of degree at most $(n-r)$.

(iii) Isomorphic graphs have same generalized common neighbor polynomials.

(iv) $N_r(G,i) = \phi$ for $n-r+1 \leq i \leq n$.

(v) $N_r[G;1] = \sum_{i=1}^{n-r} |N_r(G,i)| = \binom{n}{r}$.

(vi) $N_r[G;0]$ gives the number of elements in $\mathscr{L}_r$ having no common neighbors.

Theorem 8.1.4. *For any graph G, we have $|N_r(G,0)| \leq |N_s(G,0)|$ if $r \leq s \leq n$.*

Proof. It is enough to show that corresponding to each r-tuple of vertices in $N_r(G,0)$, there are one or more s-tuples of vertices in $N_s(G,0)$. Let $(u_1,u_2,\ldots,u_r) \in N_r(G,0)$. Let $u_{r+1},u_{r+2},\ldots,u_s$ be any $s-r$ vertices in $V(G)-\{u_1,u_2,\ldots,u_r\}$. Then the s-tuple of vertices $(u_1,u_2,\ldots,u_r,u_{r+1},\ldots,u_s)$ have no common neighbors in G since the first r vertices have no common neighbors in G. Then $(u_1,u_2,\ldots,u_r,u_{r+1},\ldots,u_s) \in N_s(G,0)$. This completes the proof. □

Theorem 8.1.5. *For any graph G, we have the following:*
If $(u_1,u_2,\ldots,u_r) \in N_r(G,i)$, then $(u_1,u_2,\ldots,u_r,u_{r+1},\ldots,u_s) \notin N_s(G,j)$ where $r < s$ and $0 < i < j$.

Proof. Let $(u_1,u_2,\ldots,u_r) \in N_r(G,i)$. Let $u_{r+1},u_{r+2},\ldots,u_s$ be any $s-r$ vertices in $V(G)-\{u_1,u_2,\ldots,u_r\}$ such that $(u_1,u_2,\ldots,u_r,u_{r+1},\ldots,u_s) \in N_s(G,j)$ where $r < s$ and $0 < i < j$. Then the vertices $u_1,u_2,\ldots,u_r,\ldots,u_s$ have j common neighbors in G where $j > i$. In particular, the vertices $u_1,u_2,\ldots,u_r$ have at least j common neighbors in G, a contradiction since $j > i$ and $(u_1,u_2,\ldots,u_r) \in N_r(G,i)$. □

Theorem 8.1.6. *For a complete graph K_n $(n \geq r)$, we have*

$$N_r[K_n; x] = \binom{n}{r} x^{n-r}.$$

Proof. The proof follows from the fact that any r-tuple of vertices of K_n have $(n - r)$ common neighbors and there are $\binom{n}{r}$ such r-tuples of vertices. □

Theorem 8.1.7. *For a path graph P_n, we have $N_r[P_n; x] = \binom{n}{r}$, $r \geq 3$.*

Proof. The result follows from the fact that no r-tuple of vertices in P_n where $r \geq 3$, having common neighbors in P_n. □

Theorem 8.1.8. *For a cycle graph C_n, we have $N_r[C_n; x] = \binom{n}{r}$, $r \geq 3$.*

Proof. The result follows from the fact that no r-tuple of vertices in C_n where $r \geq 3$ have common neighbors in C_n. □

Theorem 8.1.9. *For a complete bipartite graph $K_{m,n}$, we have the following:*

$$N_r[K_{m,n}; x] = \binom{m}{r} x^n + \binom{n}{r} x^m + \sum_{j=1}^{r-1} \binom{m}{j} \binom{n}{r-j}.$$

Proof. Let M, N be the bipartite sets of vertices of $K_{m,n}$ where $|M| = m$ and $|N| = n$. Let $(u_1, u_2, \ldots, u_r)$ be any r-tuple of vertices of $K_{m,n}$. We consider the following 3 cases according to the selection of vertices in the $r-$tuple $(u_1, u_2, \ldots, u_r)$.

Case(i) Let $u_k \in M$ for $1 \leq k \leq r$.

In this case, each of the $r-$tuple of vertices $(u_1, u_2, \ldots, u_r)$ have n common neighbors contributing the term $\binom{m}{r} x^n$ in the generalized common neighbor polynomial of $K_{m,n}$.

Case(ii) Let $u_k \in N$ for $1 \leq k \leq r$.

In this case, each of the $r-$tuple of vertices $(u_1, u_2, \ldots, u_r)$ have m common neighbors contributing the term $\binom{n}{r}x^m$ in the generalized common neighbor polynomial of $K_{m,n}$.

Case(iii) After a sufficient rearrangement of terms, let $u_k \in M$ for $1 \leq k \leq j$ and $u_k \in N$ for $j+1 \leq k \leq r$.

For each j where $1 \leq j \leq r-1$, the $r-$tuple of vertices $(u_1, u_2, \ldots, u_r)$ has no common neighbor in $K_{m,n}$ and there are $\binom{m}{j}\binom{n}{r-j}$ such r- tuples.

This completes the proof. $\square$

Corollary 8.1.10. *For a star graph $K_{1,n}$, we have $N_r[K_{1,n}; x] = \binom{n}{r}x + \binom{n}{r-1}$ for $r \geq 2$.*

Theorem 8.1.11. *For a bistar graph $B_{n,n}$ we have the following:*

$$N_r[B_{n,n}; x] = 2\binom{n+1}{r}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1}\binom{n}{m}\binom{n}{r-m} + \delta_{r2},$$

where $\delta_{rj} = \begin{cases} 1 & ; r = j, \\ 0 & ; r \neq j. \end{cases}$

Proof. Let $S = \{s_1, s_2, \ldots, s_n\}$ and $T = \{t_1, t_2, \ldots, t_n\}$ be the pendent vertices of the star graphs with center vertices u and v respectively, which together with the edge uv constitute the bistar graph $B_{n,n}$. Let $(u_1, u_2, \ldots, u_r)$ be any r-tuple of vertices of $B_{n,n}$. We consider the following cases according to the selection of vertices in the r-tuple where $r > 2$.

Case(i) Let $u_i \in S$ or $u_i \in T$ for all $i \in \{1, 2, \ldots, r\}$.

All the r-tuple of vertices under this case have exactly one common neighbor u or v according as $u_i \in S$ or $u_i \in T$. Hence this case contribute the term $2\binom{n}{r}x$ to the generalized common neighbor polynomial of $B_{n,n}$.

Case(ii) For $i \in \{1, 2, \ldots, r\}$, $u_i = v$ for exactly one i and all other $u_i \in S$.

The r-tuple of vertices under this case have exactly one common neighbor u and there are $\binom{n}{r-1}$ such r-tuples thereby contributing the term $\binom{n}{r-1}x$ to $N_r[B_{n,n}; x]$.

Case(iii) For $i \in \{1, 2, \ldots, r\}$, $u_i = u$ for exactly one i and all other $u_i \in T$.

By a similar argument as in Case(ii), the r-tuples in this case also contributes the term $\binom{n}{r-1}x$ to $N_r[B_{n,n}; x]$.

Case(iv) For $i \in \{1, 2, \ldots, r\}$, $u_i = u$ or $u_i = v$ for exactly one i where all other u_i belongs to S or T respectively.

All the r-tuple of vertices under this case have no common neighbors and there are $2\binom{n}{r-1}$ such r-tuples.

Case(v) After an appropriate rearrangement of terms of the r-tuple of vertices, let $u_1, u_2, \ldots, u_m \in S$ and $u_{m+1}, u_{m+2}, \ldots, u_r \in T$ where $1 \le m \le r - 1$.

All the r-tuple of vertices under this case have no common neighbors and this case contribute the term $\sum_{m=1}^{r-1} \binom{n}{m}\binom{n}{r-m}$ to $N_r[B_{n,n}; x]$.

It follows that

$$\begin{aligned} N_r[B_{n,n}; x] &= 2\binom{n}{r}x + 2\binom{n}{r-1}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1} \binom{n}{m}\binom{n}{r-m} \\ &= 2\binom{n+1}{r}x + 2\binom{n}{r-1} + \sum_{m=1}^{r-1} \binom{n}{m}\binom{n}{r-m}. \end{aligned}$$

This completes the proof with a sufficient remark that when $r = 2$, the pair of vertices (u, v) have no common neighbors. $\square$

Theorem 8.1.12. *Every graph G contains $|N_r(G, i)|$ number of complete bipartite subgraphs $K_{i,r}$ where $1 \leq i \leq n - r$.*

Proof. Note that corresponding to each $r-$tuples of vertices $(u_1, u_2, \ldots, u_r) \in N_r(G, i)$, the vertices $u_1, u_2, \ldots, u_r$ together with their i common neighbors constitute a complete bipartite subgraph $K_{i,r}$. Hence the result follows. $\square$

Theorem 8.1.13. *The generalized common neighbor polynomial of a graph G is non constant if and only if there exists a star $K_{1,r}$ in G where $1 \leq r \leq n$.*

Proof. Let $N_r[G; x]$ be a non constant polynomial of degree $m \geq 1$. Then there exists an r-tuple of vertices $(u_1, u_2, \ldots, u_r)$ in G which has at least one common neighbor, say w in G. Then w together with the vertices $u_1, u_2, \ldots, u_r$ produces a star $K_{1,r}$ in G.

Conversely let there exists a star $K_{1,r}$ in G where $1 \leq r \leq n$. Let $u_1, u_2, \ldots, u_r$ be the pendent vertices of $K_{1,r}$. Then the center of the star graph $K_{1,r}$ is a common neighbor of the r-tuple $(u_1, u_2, \ldots, u_r)$. Now the result follows from the fact that $N_r(G, i) \neq \phi$ for some $i \geq 1$. $\square$

Corollary 8.1.14. *If a graph G doesn't contain any star graph $K_{1,r}$ as a subgraph where $1 \leq r \leq n$, then the generalized common neighbor polynomial $N_r[G; x] = \binom{n}{r}$.*

Theorem 8.1.15. *The generalized common neighbor polynomial $N_r[G; x]$ of a graph G is of degree $k \geq 1$ if and only if k is the largest integer such that G has a complete bipartite subgraph $K_{r,k}$.*

Proof. Assume that $N_r[G;x]$ of a graph G is of degree $k \geq 1$. Then, $|N_r(G,k)| \neq \phi$. Take $(u_1, u_2, \ldots, u_r) \in N_r(G,k)$. Then the vertices $u_1, u_2, \ldots, u_r$ together with their k common neighbors constitute a complete bipartite subgraph $K_{r,k}$ of G. Now let j be the largest integer such that G contains $K_{r,j}$ as a bipartite subgraph. If possible, let $j \geq k+1$. Then G contains an r-tuple of vertices having j common neighbors where $j \geq k+1$ which is a contradiction since $N_r[G;x]$ is of degree k. This proves the necessary part of the theorem.

Conversely, we assume that k is the largest integer such that G has a complete bipartite subgraph $K_{r,k}$. If possible, let $N_r[G;x]$ is of degree $j \geq k+1$. Then G contains an r-tuple of vertices having at least $k+1$ common neighbors. These r vertices together with their $k+1$ common neighbors constitute a complete bipartite subgraph $K_{r,k+1}$ of G which is a contradiction to the assumption. □

Definition 8.1.16. *Two graphs G and H are said to be CNP_r equivalent if $N_r[G;x] = N_r[H;x]$. The set of all graphs which are CNP_r equivalent to G is denoted by $[G]_{\mathcal{N}_r}$.*

Theorem 8.1.17. *For any graph G, $\overline{G} \in [G]_{\mathcal{N}_r}$ if and only if there are $|N_r(G,i)|$ number of $r-$tuple of vertices in G which dominate $n-i$ vertices of G.*

Proof. First suppose that $\overline{G} \in [G]_{\mathcal{N}_r}$. Then $|N_r(G,i)| = |N_r(\overline{G},i)|$ for $0 \leq i \leq n-r$. Let $(u_1, u_2, \ldots, u_r) \in N_r(\overline{G},i)$. Since the vertices $u_1, u_2, \ldots, u_r$ have only i common neighbors in $\overline{G}$, all the remaining $n-i$ vertices in G are adjacent to at least one of the vertices in $\{u_1, u_2, \ldots, u_r\}$. Then $\{u_1, u_2, \ldots, u_r\}$ dominate exactly $n-i$ vertices of G. Since $|N_r(G,i)| = |N_r(\overline{G},i)|$, it follows that G has $|N_r(G,i)|$ number of $r-$tuples of vertices which dominate $n-i$ vertices of G.

Conversely assume that there are $|N_r(G,i)|$ number of $r-$tuple of vertices in G which dominate $n-i$ vertices of G. Let $\{u_1, u_2, \ldots, u_r\}$ be a set of r vertices of G which dominate exactly $n-i$ vertices of G. Then the remaining i vertices in G are not equal or adjacent to any of the vertices in $\{u_1, u_2, \ldots, u_r\}$. Hence these i vertices becomes the common neighbor of the r-tuple $(u_1, u_2, \ldots, u_r)$ in $\overline{G}$. So the set of r vertices $\{u_1, u_2, \ldots, u_r\}$ in G which dominate exactly $n-i$ vertices of G forms an r-tuple of vertices $(u_1, u_2, \ldots, u_r)$ which belongs to $N_r(\overline{G}, i)$. It follows that $|N_r(G,i)| = |N_r(\overline{G}, i)|$ and hence $N_r[G;x] = N_r[\overline{G};x]$. This completes the proof. $\square$

Corollary 8.1.18. *Let G be a graph of order n. If $\overline{G} \in [G]_{\mathcal{N}_r}$, then $|N_r(G,0)|$ gives the number of dominating sets in G of order r.*

Lemma 8.1.19. *Let G be a connected graph with $n > 3$ vertices. If all the pairs of edges of G have a common end vertex, then G is a star graph.*

Proof. Since $n > 3$ and G is connected, the number of edges m should be greater than or equal to 3. We will prove the result by using method of induction on the number of edges m of G. Clearly the result is true for $m = 3$. Let the result be true for all graphs G with less than m edges. And let G be a graph with m edges such that all the pairs of edges have a common end vertex. By deleting any edge e from G, we have a graph with $m-1$ edges. Clearly all the pairs of edges of $G-e$ are incident to a common vertex. Hence by induction assumption, $G-e$ is a star. Let v be the center vertex of the star so that the edges of $G-e$ be represented by $e_i = vv_i$ where $i = 1, 2, \ldots, m-1$. Since the edges e and e_1 of G are incident to a common vertex, either $e = vw$ or $e = v_1w$ for some vertex $w \in V(G)$. In the first case G is a star and the proof is complete. And in the second case, there are two possibilities according as w belongs to $\{v_1, v_2, \ldots, v_{m-1}\}$ or not. If w belongs to the set, let $w = v_i$ where $i \in \{2, 3, \ldots, m-1\}$. Then the edges v_1w and and vv_{i+1} have no common end vertex which ruled out the possibility of $w \in \{u_1, u_2, \ldots, u_{m-1}\}$. If w doesn't belong to the set, then the edges v_1w and vv_3 have no common end vertex. Hence by the induction assumption, the second possibility is also ruled out. Hence the result follows. □

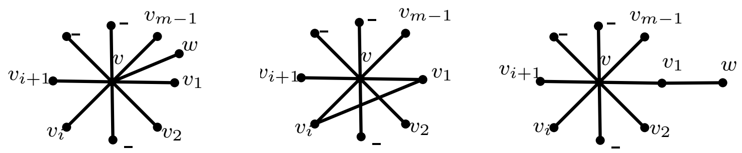

Figure 8.2: Figure showing different cases of lemma 8.1.19

The line graph $L(G)$ of a graph G is the graph with vertex set the set of all edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are incident to a common vertex.

Theorem 8.1.20. *Let G be a connected graph of order $n > 3$. The number k of cliques of size $r > 1$ in the line graph of G is given by* $k = \sum_{i=1}^{n-r} i|N_r(G, i)|$.

Proof. Let S be the collection of all r-tuples of vertices $(u_1, u_2, \ldots, u_r)$ of G which have at least one common neighbor in G. Also let the r-tuple $(u_1, u_2, \ldots, u_r)$ repeat as many times in S as its number of common neighbors. Then, $|S| = \sum_{i=1}^{n-2} i|N_r(G, i)|$. Let P be the collection of all cliques of size r in the line graph $L(G)$ of G. Let the vertices of $L(G)$ be denoted by uv where u, v are adjacent vertices of G. Define $\phi : S \to P$ as follows.

Let $u = (u_1, u_2, \ldots, u_r) \in S$ which repeats i-times in S. Let these i members be represented by $u_k = (u_1, u_2, \ldots, u_r)^{(k)}$ where $k = 1, 2, \ldots, i$. Then each $(u_1, u_2, \ldots, u_r)^{(k)}$ can be assigned to exactly one common neighbor w_k of $(u_1, u_2, \ldots, u_r)$ in G. It follows that all the pairs of vertices $u_l w_k$ and $u_m w_k$ where $l, m \in \{1, 2, \ldots, r\}$ and $l \neq m$ are adjacent vertices of $L(G)$ which forms a clique C_{uk} of size r in $L(G)$.

Now define $\phi : S \to P$ as $\phi((u_1, u_2, \ldots, u_r)^{(k)} = C_{uk}$. Clearly ϕ is one-one. we claim that $\phi : S \to P$ is onto. Let C be a clique of size r in the line graph $L(G)$ of G. Since any pair of vertices of C are adjacent in $L(G)$, all the pairs of edges in G which constitute the vertex set of C, have a common end vertex in G. Hence by lemma 8.1.19, those edges form a star in G whose pendent vertices forms an r-tuple $(u_1, u_2, \ldots, u_r) \in S$ such that $\phi(u_1, u_2, \ldots, u_r) = C$. Thus ϕ is onto.

It follows that ϕ is a bijection from S to P and $|S| = |P|$. This completes the proof. □

Corollary 8.1.21. *Let G be a graph of order n. Then the number of edges of the line graph $L(G)$ of G equals* $\displaystyle\sum_{i=1}^{n-2} i|N(G, i)|$.

Proof. The result follows from the fact that the 2-cliques of any graph are the edges of the graph. □

Theorem 8.1.22. *(Schwartz 1969 and Ghirlanda 1963) A graph is isomorphic to its line graph if and only if it is regular of degree two.*

Corollary 8.1.23. *If a graph G is regular of degree two, then the number of edges of G equals* $\displaystyle\sum_{i=1}^{n-2} i|N(G, i)|$.

8.2 Simplicial complexes of graphs and common neighbor sets

A family Δ of finite subsets of a set V is an (abstract) **simplicial complex**[41] if it satisfy the condition that whenever $\sigma \in \Delta$ and $\tau \subset \sigma$ then, $\tau \in \Delta$. If $\sigma \in \Delta$ is of cardinality $k+1$, then σ is called a **k-simplex** and every $\tau \subset \sigma$ is a **face** of the simplex. **The dimension of a simplex** is one less than its cardinality. Thus a k-simplex has dimension k. If a simplex is not a proper subset of any other simplexes in the complex, then it is a **facet** of the complex.

As we represent an abstract simplicial complex geometrically, a k-simplex is the convex hull of $k+1$ points which constitute the simplex. In the graphical representation, 0-simplexes are vertices, 1-simplexes are edges, 2-simplexes are triangles and so on. A face of the simplex is identified as the convex hull of a subset of the vertices in the simplex.

In this section, we first define the simplicial complex of a graph G and introduce the cluster of a vertex $v \in G$ as a simplicial complex of G. Then we incorporate the concept of generalized i-common neighbor set of a graph with the cluster of vertices in it, to deduce some interesting properties of generalized i-common neighbor sets.

Definition 8.2.1. *Let $G(V, E)$ be a graph and let Δ be a collection of subsets of V. The elements of Δ are called simplexes. A simplex of cardinality $k+1$ is called a k-simplex. Let τ be an element in Δ. Then the subsets of τ are called its faces. We say that Δ is a simplicial complex of G if for every τ in Δ, all its faces are in Δ.*

Let G be a simple finite graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For each vertex v_i, the cluster of v_i is defined as

$$\mathrm{clr}(v_i) =: \{W \subset V : v_i \in \cap_{v\in W} N(v)\}.$$

Then each $\mathrm{clr}(v_i)$ where $i \in \{1, 2, \ldots, n\}$ is a simplicial complex of G. We may consider $clr(v_i)$ as a simplicial complex of G generated by the vertex v_i. Note that each simplex W of $clr(v_i)$ spans a subgraph of G which is a star graph with center vertex v_i. So these simplexes are called the stars of v_i denoted by $str(v_i)$. The facets of $clr(v_i)$ are the maximal stars in $clr(v_i)$.

Lemma 8.2.2. *Let v be a vertex of the graph G having degree d. Then the cluster of v contains $\binom{d}{r}$ number of $(r-1)$-simplexes.*

Proof. Let S be the set of all neighbors of the vertex v such that $|S| = d$. Any subset S_1 of S with cardinality $r \leq d$ will act as a r-tuple of vertices with v as a common neighbor. There are exactly $\binom{d}{r}$ distinct subsets of S with cardinality r and these subsets are exactly the $(r-1)$-simplexes of the cluster of v. Hence the result follows. □

Theorem 8.2.3. *Let $G(V, E)$ be a simple graph and let $v \in V$. Let f_i, $i = 1, 2, \ldots, m$ be the facets of the simplicial complex $clr(v)$. If the facet f_i is of cardinality d_i, then $clr(v)$ contains $\displaystyle\sum_{i=1}^{m}\sum_{r=1}^{d_i}\binom{d_i}{r}$ distinct simplexes.*

Proof. According to the definition of a simplicial complex, all the subsets of its facets must also be simplexes of the complex. If the facet f_i of $clr(v)$ is of cardinality d_i, there are $\binom{d_i}{r}$ simplexes of dimension r in $clr(v)$. Thus corresponding to each facet f_i, there are $\displaystyle\sum_{r=1}^{d_i}\binom{d_i}{r}$ distinct simplexes in $clr(v)$. As there are m facets, the result follows. □

Theorem 8.2.4. *If G is a graph having degree sequence $(d_1, d_2, \ldots, d_n)$, then we have the following:*

$$\sum_{i=1}^{n-r} i|N_r(G,i)| = \sum_{i=1}^{n} \binom{d_i}{r}.$$

Proof. Let $clr(v_i)$, $i = 1, 2, \ldots, n$ be the simplicial complexes generated by the vertices $v_1, v_2, \ldots, v_n$ of the graph G. We will show that the expression on both sides of the equation equates the total number of $(r-1)$ simplexes of the clusters $clr(v_i)$, $i = 1, 2, \ldots, n$.

By lemma 8.2.2, the number of $(r-1)$-simplexes in $clr(v_i)$ is given by $\binom{d_i}{r}$ where d_i is the degree of the vertex v_i which generates $clr(v_i)$. Hence if all the simplicial complexes $clr(v_i)$, $i \in \{1, 2, \ldots, n\}$ are taken into account, there are altogether $\sum_{i=1}^{n} \binom{d_i}{r}$ number of $(r-1)$-simplexes.

Now, for a fixed $i \in \{1, 2, \ldots, n\}$, the $(r-1)$-simplexes of $clr(v_i)$ are exactly r-tuples of vertices with v_i as a common neighbor. Hence the total number of $(r-1)$-simplexes of $clr(v_i)$, $i = 1, 2, \ldots, n$ equals the number of r-tuples of vertices with at least one common neighbor where the r-tuple with i common neighbors has to be counted i times. From the definition of generalized i-common neighbor set of G, the number of such r-tuple of vertices is given by $\sum_{i=1}^{n-r} i|N_r(G,i)|$. This completes the proof. $\square$

Theorem 8.2.5. *The generalized i-common neighbor set $N_r(G,i)$ is the set of all $(r-1)$-simplexes which belongs to the intersection of exactly i of the clusters of vertices of G.*

Proof. Let W be a $(r-1)$-simplex which belongs to a simplicial complex $clr(v_j)$, for some $j \in \{1, 2, \ldots, n\}$. From the definition of $clr(v_j)$, it is clear that the members of W constitute a r-tuple of vertices of G having v_j as a common neighbor. Now fix an integer i such that $1 \leq i \leq n-2$. W belongs to exactly i of the $clr(v_j)$, if and only if the corresponding r-tuple of vertices has exactly i common neighbors. It follows that $W \in N_r(G, i)$. $\square$

Remark 8.2.6. *We observe the following properties of the simplicial complexes $clr(v_i)$ generated by the vertices v_i of a simple graph G.*
For $i, j, k \in \{1, 2, \ldots, n\}$,

1. *If a simplicial complex $clr(v_i)$ is generated by a vertex v_i, then, $\{v_i\} \notin clr(v_i)$.*

2. *$clr(v_i)$ contains all possible unions of the 0-simplexes containing in it.*

3. *If $\{v_i\} \in clr(v_j)$, then $\{v_j\} \in clr(v_i)$.*

The first statement follows from the fact that a vertex cannot be adjacent to itself as we are considering only simple graphs. The second and third statements directly follows from the definition of $clr(v_i)$.

The following theorem shows that these are the sufficient conditions for a collection of simplicial complexes $\{clr(v_i)\}, i \in \{1, 2, \ldots, n\}$ on a set of cardinality n to be generated by a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ of a simple graph G.

Theorem 8.2.7. *Let $V = \{v_1, v_2, \ldots, v_n\}$ be any set of n elements. If $clr(v_i), i \in \{1, 2, \ldots, n\}$ are simplicial complexes on the set V satisfying the conditions (1),(2) and (3) stated in above remark, then there exists a simple graph G with vertex set V where $clr(v_i)$ is the simplicial complex generated by the vertex v_i of G.*

Proof. Given a set of elements $V = \{v_1, v_2, \ldots, v_n\}$ and a collection of simplicial complexes $\{clr(v_i)\}, i \in \{1, 2, \ldots, n\}$ on the set V which satisfies the conditions (1),(2) and (3) stated in remark 8.2.6. Construct a graph with vertex set V and edge set E where an edge $v_iv_j \in E$ if and only if $\{v_j\} \in clr(v_i)$.

By condition (1), $\{v_i\} \notin clr(v_i)$ which implies that G has no loops. Also by condition (3), if $\{v_i\} \in clr(v_j)$, then $\{v_j\} \in clr(v_i)$ which implies that the adjacency of vertices of the graph is well defined in the sense that whenever v_i is adjacent to v_j, v_j is also adjacent to v_i.

Now we will prove that $\{clr(v_i)\}$ are the simplicial complexes generated by the vertices $\{v_i\}$ of the graph G. Let V_1 be a subset of V which belongs to $clr(v_i)$. Then $V_1 = \{v_{j_1}, v_{j_2}, \ldots, v_{j_m}\}$ where each of the vertices in the set are adjacent to a vertex $v_i \in V$ in G. Then $v_iv_{j_k} \in E$ and $\{v_{j_k}\} \in clr(v_i)$ for all $k \in \{1, 2, \ldots, m\}$. Hence by condition(3), all the subsets of V_1 are in $clr(v_i)$. It follows that $clr(v_i)$ is a simplicial complex on V. And by definition of edge set of G, it is generated by v_i. This completes the proof. □

Chapter 9

Significance of common neighbor polynomial of graphs

Graph polynomials serves as a tool to unveil the structural properties of graphs. Starting from the edge difference polynomial introduced by J.J. Sylvester in 1878, many graph polynomials were introduced each of which has its own significance in the applied areas of graph theory such as network theory and chemical graph theory. In the present work, the common neighbor polynomial of graphs is introduced by emphasizing on the structural similarity of pair of nodes in a graph. In this chapter we discuss the relevance of the common neighbor polynomial of graphs in the fields of chemical graph theory and network theory.

9.1 Chemical graph theory

A dendrimer is a branched nano structure comprising of three major components the core, the branches and the end groups. In each stage of its growth, new branches are attached to the core. Dendrimers are considered as one of the major building blocks of nanotechnology. Because of its wide range of uses in industrial and pharmaceutical fields, dendrimers has attracted the interest of both Chemists and Mathematicians and many works has done to explore the topological properties of these structures [For example see [27, 45]].

In the first two subsections, we study the common neighbor polynomial of some dendrimer structures. In the third subsection, we establish a relation between the common neighbor polynomial and hosoya polynomial of triangle free graphs of diameter not more than three.

9.1.1 Common neighbor polynomial of Nanostar Dendrimers

The nanostar dendrimer of third generation $D_3[n]$ has a core(leaf) whose molecular graph is as shown in figure 9.1. Graphically, it is a cycle C_6 with two pendent edges attached to diagonally opposite vertices of C_6. The primary structure $D_3[0]$ of nanostar dendrimer is depicted in figure 9.1. During the synthesis of $D_3[n]$, for $n = 1, 2, \ldots$ two leaf graphs are attached to each branch of $D_3[n-1]$.

The number of leafs in a nanostar dendrimer grown upto n levels is given by Zeinab Foruzanfar [45] as $k = 3(2^{n+1} - 1)$. Also the number of copies of primary structures in $D_3[n]$ is given by $l = 1 + 3(2^n - 1)$. Figure 9.2 shows the nanostar

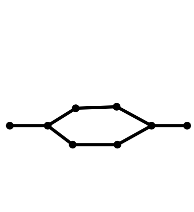

Figure 9.1: The core(leaf) and primary structure of nanostar dendrimer $D_3[n]$

dendrimer $D_3[n]$ for $n = 3$.

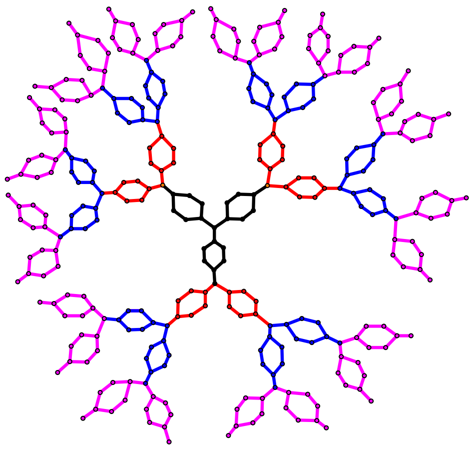

Figure 9.2: Nanostar dendrimer $D_3[n]$ for $n = 3$

In this subsection we study the common neighbor polynomial of nanostar dendrimer $D_3[n]$ with n^{th} level growth.

Theorem 9.1.1. *The common neighbor polynomial of nanostar dendrimer grown upto n levels is given by*

$$N[D_3[n]; x] = k(10x + 18) + 3l\ x + 49\binom{k}{2} - 3l,$$

where $k = 3(2^{n+1} - 1)$ and $l = 1 + 3(2^n - 1)$.

Proof. At the n^{th} level of growth of $D_3[n]$, let the core structure appears k times and let the primary structure appears l times in the molecular graph. In each core structure, there are 10 pairs of vertices having one common neighbor and 18 pairs of vertices having no common neighbor. Hence the pairs of vertices in a particular core structure contribute the term $(10x+18)$ to the common neighbor polynomial of $D_3[n]$ and there are k number of such core structures.

Let (u, v) be an unordered pair of vertices of $D_3[n]$ where u and v belong to distinct copies of core structures. There are $49\binom{k}{2}$ distinct choices for the pair (u, v). None of these pairs have a common neighbor unless both u and v are adjacent to the central vertex of the same primary structure in which case they have exactly one common neighbor. Since there are l copies of primary structure, each of which has a central vertex adjacent to 3 vertices from distinct copies of core structures, there are $\binom{3}{2}l$ pairs (u, v) having exactly one common neighbor. Hence the common neighbor polynomial of $D_3[n]$ is given by

$$N[D_3[n]; x] = k(10x + 18) + 3l\ x + 49\binom{k}{2} - 3l,$$

where $k = 3(2^{n+1} - 1)$ and $l = 1 + 3(2^n - 1)$. □

9.1.2 Common neighbor polynomial of PAMAM dendrimers

Polyamidoamine(PAMAM) dendrimers, sometimes referred as Starburst are hyper branched molecules with repeated branches of amide and amine functionality. Here we discuss the common neighbor polynomial of PAMAM dendrimer D_k of k^{th} generation whose core(leaf) and primary structures are depicted in figure 9.3. The primary structure may be considered as zero-generation of the dendrimer. During the synthesis of PAMAM dendrimer D_k, for $k = 1, 2, \ldots$ leaf structures are attached at each of the pendent vertices of D_{k-1}. Ethylenediamine (EDA)

Figure 9.3: The core and primary structures of PAMAM dendrimer.

molecule contributes towards the primary structure which have 4 possible pendent vertices to bind with the amidoamine repeating units of core structure [29]. Figure 9.4 shows the PAMAM dendrimer grown upto 3^{rd} generation.

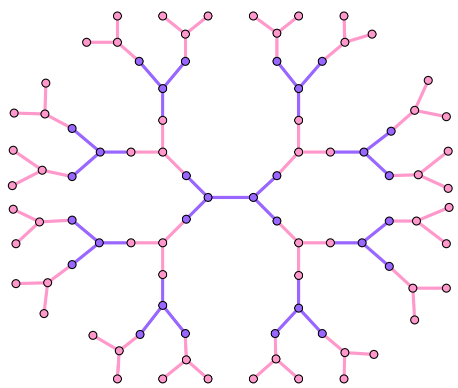

Figure 9.4: PAMAM dendrimer of k^{th} generation D_k for $k = 3$.

Theorem 9.1.2. *The common neighbor polynomial of PAMAM dendrimer D_k of k^{th} generation is given as follows:*

$$N[D_k; x] = N[D_{k-1}; x] + 2^{k+1}[4x + 3f_{k-1} - 1] + 9\binom{2^{k+1}}{2},$$

where f_k denote the number of vertices of D_k given by $f_k = 3(2^{k+2}) - 6$.

Proof. The primary structure of PAMAM dendrimer D_k consists of two star graphs $K_{1,2}$ with the center vertices attached through a bridge. Whenever the dendrimer grows to the next generation, all the pendent vertices of the preceding generation are identified with one of the pendent vertices of the star graph $K_{1,3}$. Since the primary structure contains 6 vertices, the number of vertices in the k^{th} generation is given by

$$f_k = 6 + 3[2^2 + 2^3 + \ldots + 2^{k+1}] = 3[2^{k+2}] - 6.$$

The common neighbor polynomial of the primary structure of PAMAM dendrimer is $6x + 9$. Let $N[D_{k-1}; x]$ be the common neighbor polynomial of the dendrimer grown upto $(k-1)^{th}$ generation. Then the common neighbor polynomial of the k^{th} generation dendrimer can be constructed by considering the number of common neighbors of newly formed pairs of vertices (u, v) under the following cases:

Case(i) Let u, v be the vertices of one of the newly attached stars. Such pairs of vertices (u, v) contribute the term $3x + 3$ to the common neighbor polynomial of D_k and there are 2^{k+1} number of such new stars.

Case(ii) Let u be the center vertex of one of the newly attached stars and let v be a vertex of the $(k-1)^{th}$ generation other than the vertex to which u is adjacent. (The excluded pair has already considered in Case(i)). For a particular star, only one pair (u, v) of such vertices has a common neighbor and all other $3(2^{k+1}) - 8$ pairs of vertices have no common neighbors.

Case(iii) Let u be one of the pendent vertices of k^{th} generation dendrimer and let v be a vertex defined as in Case(ii). Here the pair of vertices (u, v) have no common neighbors where there are 2^{k+2} choices for the vertex u and $3(2^{k+1}) - 7$ choices for v.

Case(iv) Let u and v be vertices of two distinct stars which are newly attached during the growth of k^{th} generation, except those which are identified with the pendent vertices of $(k-1)^{th}$ generation. There are $9\binom{2^{k+1}}{2}$ such pairs of vertices (u, v) each having no common neighbors.

From the above cases, it follows that,

$$\begin{aligned} N[D_k; x] &= N[D_{k-1}; x] + 2^{k+1}[3x+3] + 2^{k+1}[x + 3(2^{k+1}) - 8] \\ &\qquad + 2^{k+2}[3(2^{k+1}) - 7] + 9\binom{2^{k+1}}{2} \\ &= N[D_{k-1}; x] + 2^{k+1}[4x + f_{k-1} + 1] + 2^{k+2}[f_{k-1} - 1] + 9\binom{2^{k+1}}{2} \\ &= N[D_{k-1}; x] + 2^{k+1}[4x + 3f_{k-1} - 1] + 9\binom{2^{k+1}}{2}. \end{aligned}$$

This completes the proof. □

9.1.3 Hosoya Polynomial of graphs with diameter not more than three

The hosoya polynomial of graphs introduced by H.Hosoya[26] as early in 1988 received wide attention due to its association with the famous Wiener Index and Hyper Wiener Index of molecules in chemical graph theory. The Wiener Index and Hyper Wiener Index of of a graph G are defined as follows:

$$W(G) = \sum_{u,v \in V(G)} d(u,v) \quad \text{and} \quad WW(G) = \frac{1}{2} \sum_{u,v \in V(G)} [d(u,v) + d(u,v)^2]. \tag{9.1}$$

Only some elementary calculations are enough to show that the Wiener Index $W(G)$ of a graph G is equal to $H'(G, 1)$ where $H'(G, x)$ denotes the first derivative of Hosoya polynomial of G. The relation between Hosoya polynomial $H(G)$ and

Hyper Wiener Index $WW(G)$ of a graph G was established by G. G. Cash [17] as follows:

$$WW(G) = H'(G,1) + \frac{1}{2}H''(G,1). \tag{9.2}$$

For connected graphs of diameter 1, the hosoya polynomial is given by $H(G,x) = \binom{n}{2}x$ since all the pairs of vertices are at a distance 1 from each other. Similarly for connected graphs of diameter 2, it is given by $H(G,x) = mx + [\binom{n}{2} - m]x^2$ where m is the number of edges of G.

Some of the consequences of this result are given as follows:

1. The well known Petersen graph $\mathcal{P}$ has 10 vertices and 15 edges. Since it is of diameter 2, its Hosoya polynomial is given by $H(\mathcal{P},x) = 30x^2 + 15x$.

2. A windmill graph $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common. It is of diameter 2 with $mn-m+1$ vertices and mn edges. The hosoya polynomial of the windmill graph $W_n^{(m)}$ is given by $H(W_n^{(m)},x) = \left[\binom{mn-m+1}{2} - mn\right]x^2 + mn\ x$.

3. Hoffman-Singleton graph $\mathcal{H}$ was constructed by A.J. Hoffman and R.R. Singleton [1] which is a 7-regular graph of diameter 2 with 50 vertices and 175 edges. The hosoya polynomial of Hoffman-Singleton graph is given by $H(\mathcal{H},x) = 1050x^2 + 175x$.

For graphs with diameters 3 or more, no general formulae are available to find their hosoya polynomial. Theorem 9.1.3 shows that, for connected triangle free graphs of diameter 3, the hosoya polynomial can be derived from their common neighbor polynomial.

Theorem 9.1.3. *Let G be a connected triangle free graph with n vertices and m edges. If G has diameter not more than three, the hosoya polynomial of G is given by*

$$H(G,x) = \Big[N[G;0] - m\Big]x^3 + \Big[\binom{n}{2} - N[G;0]\Big]x^2 + mx. \tag{9.3}$$

where $N[G;0]$ is the common neighbor polynomial of G equated at $x = 0$.

Proof. Let G be a connected triangle free graph with n vertices and m edges having diameter less than or equal to 3. The Hosoya polynomial of G is defined as[26]

$$H(G,x) = \sum_{j=1}^{l} d(G,j)x^j \tag{9.4}$$

where $d(G,j)$ denote the number of pairs of vertices in G having distance j apart and l denote the diameter of the graph. Clearly $d(G,1)$ is the number of edges of G. From the definition of i-common neighbor set of G, all the pairs of vertices in G which are at a distance 2 apart have at least one common neighbor and hence lie in $N(G,i)$ where $1 \leq i \leq (n-2)$. Since G is triangle free, pairs of vertices having a common neighbor cannot have an edge connecting them. It follows that $d(G,2) = \sum_{i=1}^{n-2} |N(G,i)|$ which equals $\binom{n}{2} - N[G;0]$. Now among the remaining pairs of vertices which lie in $N(G,0)$, the pairs of vertices which are end points of edges are at a distance 1 apart and hence $d(G,1) = m$ and $d(G,3) = |N(G,0)| - m$=$N[G;0] - m$. Now the result follows from equation 9.4. □

Corollary 9.1.4. *If G is a connected triangle free graph having diameter not more than 3, the Wiener index and and the Hyper Wiener index of G are given* *by the following:*

$$W(G) = N[G;0] + 2\binom{n}{2} - 2m,$$

$$WW(G) = 3N[G;0] + 3\binom{n}{2} - 5m,$$

where n and m are the order and size of the graph G respectively.

Proof. The result follows from theorem 9.1.3 and the fact that $W(G) = H'(G,1)$ and $WW(G) = H'(G,1) + \frac{1}{2}H''(G,1)$. □

The bipartite Cocktail party graph B_n is the graph obtained by removing a perfect matching from the complete bipartite graph $K_{n,n}$.

Corollary 9.1.5. *The Hosoya polynomial of bipartite cocktail party graph B_n is given by*

$$H(B_n, x) = n(n-1)x + n(n-1)x^2 + nx^3.$$

Proof. Bipartite cocktail party graph is a triangle free graph with diameter 3 having $2n$ vertices and $n^2 - n$ edges. Its common neighbor polynomial is given by (see 2.2.26),

$$N[B_n; x] = 2\binom{n}{2}x^{n-2} + n^2$$

so that $N[G;0] = n^2$. In the light of theorem 9.1.3, it follows that

$$\begin{aligned} H(B_n, x) &= [n^2 - (n^2 - n)]x^3 + \left[\binom{2n}{2} - n^2\right]x^2 + (n^2 - n)x \\ &= n(n-1)x + n(n-1)x^2 + nx^3. \end{aligned}$$

This completes the proof. □

A bistar graph $B_{m,n}$ is obtained by connecting the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ by a bridge.

Corollary 9.1.6. *The hosoya polynomial of bistar graph $B_{m,n}$ is given by*

$$H(B_{m,n}, x) = (m+n+1)x + \frac{1}{2}[m(m+1) + n(n+1)]x^2 + mnx^3.$$

Proof. Bistar graph $B_{m,n}$ is a triangle free graph with diameter 3 having $m+n+2$ vertices and $m+n+1$ edges. Its common neighbor polynomial is given by (see 2.2.25)

$$N[B_{m,n}; x] = \left[\binom{m}{2} + \binom{n}{2} + m + n\right]x + m + n + mn + 1$$

so that $N[B_{m,n}; 0] = m + n + mn + 1$. Hence the result follows from theorem 9.1.3. □

9.2 Network theory

In network data analysis, clustering of data is widely used as a tool to group the data in such a way that items in the same group share some similarity. For example, in document clustering, documents which are most similar to each other are clustered even though they belong to different classes or subjects.

Shared Nearest Neighbor(SNN) clustering is one of the most common clustering technique which produces clusters of data within a huge network according to their structural similarity. The concept of Shared Nearest Neighbor approach was first introduced by R. A. Jarvis and E. A. Patrick [32] in which two nodes are placed in the same cluster by examining the number of nearest neighbors shared by both of them. During the clustering process, a Shared Nearest Neighbor(SNN) graph is constructed whose formal definition is given in [44] as follows:

"The SNN graph of a graph G is derived from G in such a way that the neighbors of the node v_i are the nodes v_j, $j \neq i$ if v_i and v_j have at least k neighbors in common, or equivalently, if there exist at least k distinct paths of length two between v_i and v_j in G."

Observe that the value of k is called the threshold value of similarity. As the threshold value varies, the resulting SNN graph also varies and as a result, final clusters differ in their size, shape and density. So it is important to fix the threshold value in such a way that the resulting clusters are suitable for further analysis. If the threshold value is set too high, then the resulting clustering patterns with very few links may be very poor in its significance. In such a situation, a significant cluster may split into small groups of clusters. In turn, if the threshold value is set too low, unnecessary links(noises) are appeared in the final clusters. These discussion pointed towards the importance of fixing the appropriate threshold value in order to produce a SNN graph worthwhile to generate meaningful clusters.

The common neighbor polynomial of graphs provides a clear idea about the density of SNN graphs by providing the number of links in the SNN graphs for various threshold values. In first subsection, we discuss the significance of common neighbor polynomial of a graph in the area of network data clustering. In the second subsection, we establish a relation between adjacency matrix of a graph and its common neighbor polynomial. Using this we produce graphs which are CNP- equivalent.

9.2.1 Significance of Common neighbor polynomial in network data clustering

Theorem 9.2.1. *The number of edges in a SNN graph with threshold value k is given by $\sum_{i \geq k} |N(G, i)|$ where $N(G, i)$ is the i-common neighbor set of G. In particular, the SNN graph of G is empty if the threshold value exceeds the degree of common neighbor polynomial of G.*

Proof. In the SNN graph of G with threshold value k, two nodes v_i and v_j are adjacent if and only if they share at least k neighbors in common. Since $|N(G, i)|$ gives the number of pairs of vertices in G with exactly i common neighbors, the result follows. If the threshold value k exceeds the degree of common neighbor polynomial of G, $N(G, i) = \phi$ for $i \geq k$ and the corresponding SNN graph contain no edges. □

The following observations are straight forward from the above theorem.

1. The Windmill graph $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common. The common neighbor polynomial of W_n^m is given by (see 2.2.27) $N[W_n^{(m)}; x] = m\binom{n}{2}x^{n-2} + \binom{m}{2}(n-1)^2 x$. By theorem 9.2.1, the SNN graph of W_n^m contains $m\binom{n}{2}$ edges if the threshold value k is chosen in such a way that $3 \leq k \leq (n-2)$.

2. A web graph $WB_n, n > 3$ is obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. For $n \geq 6$, the common neighbor polynomial of web graph is given by (see 2.2.13), $N[WB_n; x] = 4(n-1)x^2 + \frac{(n-1)(n+6)}{2}x +$

$(n-1)(4n-10)$. From theorem 9.2.1, it follows that the SNN graph of web graph WB_n where $n \geq 6$ is empty for the threshold value $k \geq 3$.

Remark 9.2.2. *The congraph[2] $C(G)$ of a given graph G has the same vertex set as that of G in which two vertices are adjacent if they share at least one common neighbor in the graph G. The SNN graph of a graph G with threshold value $k = 1$ becomes the congraph of the graph G.*

Theorem 9.2.3. *Let i be the degree of the common neighbor polynomial of a graph G which has only zero as the common neighbor root. Then the SNN graph of G with threshold value k is either a complete graph or an empty graph according as $k \leq i$ or $k > i$ respectively.*

Proof. Let G be a graph on n vertices. If zero is the only common neighbor root of G, then the common neighbor polynomial of G is of the form $N[G; x] = \binom{n}{2}x^i$. It follows that any pair of vertices of G has exactly i common neighbors. Hence if the threshold value k is less than or equal to i, all the pairs of vertices are adjacent in SNN(G) and it becomes a complete graph. If $k > i$, the result follows from theorem 9.2.1. $\square$

Theorem 9.2.4. *If the SNN graph of a graph G with threshold value k contains a clique of size n, then the graph G contains at least $k(k-1)^{(n-1)}\ \frac{(n-1)!}{2}$ distinct $2n$- circuits which share the n alternate vertices of the circuits in common.*

Proof. Assume that the SNN graph of a graph G contains a clique C of size n. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of the clique C. There are $\frac{(n-1)!}{2}$ distinct ways for the cyclic arrangement of the vertices $v_1, v_2, \ldots, v_n$. We will consider the cycles in G produced by concatenating the 2-paths from v_i to v_{i+1} where

$i = 1, 2, \ldots, n-1$ and the 2-paths from v_n to v_1 ensuring the non repetition of edges. Now, due to the construction of SNN graph, each pair of vertices (v_i, v_j) of C has at least k common neighbors in G. Hence there are at least k distinct 2-paths in G from v_1 to v_2. Ensuring the non repetition of edges, there are at least $(k-1)$ paths of length 2 from v_i to v_{i+1} where $i = 2, 3, \ldots, n-1$ and also from v_n to v_1. Note that distinct pairs of vertices in G may have same common neighbors and hence concatenating the 2-paths between the vertex pairs (v_{i-1}, v_i) and (v_i, v_{i+1}) results in repetition of vertices. It follows that G contains at least $k(k-1)^{(n-1)}\ \frac{(n-1)!}{2}$ circuits of length $2n$ which share the alternate vertices $v_1, v_2, \ldots, v_n$ in common. This completes the proof. □

Theorem 9.2.5. *If the SNN graph of a graph G with threshold value k contains a uv- path of length m then the graph G contain at least $k(k-1)^{m-2}$ distinct uv-trials which share exactly the alternate vertices of the trials.*

Proof. Assume that SNN graph of a graph G with threshold value k contains a uv-path $u\ u_1\ u_2\ \ldots u_{m-2}\ v$ of length m. By the definition of SNN(G), there are at least k common neighbors in G for each pair of adjacent vertices of this path. In particular, there are at least k distinct 2-paths from u to u_1 and avoiding the possibility of repetition of edges, there are at least $(k-1)$ distinct 2-paths between each pair of vertices (u_i, u_{i+1}) where $i = 1, 2, \ldots, (m-1)$ and between (u_{m-2}, v). Concatenating these 2-paths results in required number of uv-trials which may contain repeated vertices. This completes the proof. □

9.3 The common neighbor polynomial and adjacency matrix of a graph

The adjacency matrix of a graph is a well celebrated concept in graph theory due to its connection with the characteristic polynomial of the graph and its spectra which have wide range of applications. This section derive a relation between adjacency matrix of a graph and its common neighbor polynomial.

Theorem 9.3.1. *Let A be the adjacency matrix of a simple graph G of order n and let $A^2 = B$. Then the common neighbor polynomial of G is given by*

$$N[G;x] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} x^{B_{ij}},$$

where B_{ij} denote the ij^{th} entry of the matrix B.

Proof. If A is the adjacency matrix of a graph G, the ij^{th} entry of A^2 is the number of walks of length 2 from the vertex v_i to v_j in G which is equal to the number of common neighbors of v_i and v_j. Hence for $i, j \in \{1, 2, \ldots, n\}$, the matrix $B = [B_{ij}]$ is given by

$$B_{ij} = \begin{cases} n_{ij} & \text{if } i \neq j, \\ d(v_i) & \text{if } i = j. \end{cases}$$

where n_{ij} denote the number of common neighbors of the vertices v_i and v_j and $d(v_i)$ denote the degree of the vertex v_i. Any pair of distinct vertices in G can be uniquely represented in the form (v_i, v_j) where $i \in \{1, 2, \ldots, n\}$ and $j \in \{i+1, i+2, \ldots, n\}$ which has exactly B_{ij} common neighbors. It follows that each vertex pair (v_i, v_j) in G contribute the term $x^{B_{ij}}$ to the common neighbor polynomial of G. Hence the result follows. $\square$

The bipartite double cover or kronecker double cover of a graph G is constructed as the tensor product of G and K_2. Corresponding to each vertex u_i of G, there are two vertices v_i and w_i in the kronecker double cover $\mathcal{K}(G)$ of G. Two vertices v_i and w_j are adjacent in $\mathcal{K}(G)$ if and only if the vertices u_i and u_j are adjacent in G.

Theorem 9.3.3 explore the existence of plenty of non isomorphic CNP- equivalent graphs. To prove the theorem, we use the following result from [6].

Theorem 9.3.2. *(see[6]) A graph G with adjacency matrix A is bipartite if and only if the matrices* $\begin{bmatrix} A & O \\ O & A \end{bmatrix}$ *and* $\begin{bmatrix} O & A \\ A & O \end{bmatrix}$ *are similar.*

Theorem 9.3.3. *The disjoint union of two copies of a non bipartite graph G and the Kronecker cover of G are non isomorphic CNP- equivalent graphs. Moreover, the common neighbor polynomial of the kronecker double cover $\mathcal{K}(G)$ of G is*

$$N[\mathcal{K}(G); x] = 2N[G; x] + n^2,$$

where n is the order of G.

Proof. Let G be a graph with adjacency matrix A. The adjacency matrices of the disjoint union $G \cup G$ and the kronecker double cover $\mathcal{K}(G)$ of G are given by $\begin{bmatrix} A & O \\ O & A \end{bmatrix}$ and $\begin{bmatrix} O & A \\ A & O \end{bmatrix}$ respectively. From Theorem 9.3.1, it follows that $G \cup G$ and $\mathcal{K}(G)$ have same common neighbor polynomial as the square of their adjacency matrices are equal. Since G is non bipartite, Theorem 9.3.2 guarantees that $\begin{bmatrix} A & O \\ O & A \end{bmatrix} \nsim \begin{bmatrix} O & A \\ A & O \end{bmatrix}$. It follows that $G \cup G$ and $\mathcal{K}(G)$ are non isomorphic

CNP-equivalent graphs. Now, using Lemma 2.2.18,

$$N[\mathcal{K}(G);x] = N[G \cup G;x] = 2N[G;x] + n^2.$$

This completes the proof. □

A crown graph is obtained by deleting a perfect matching from the complete bipartite graph $K_{n,n}$. It is the kronecker double cover of the complete graph K_n.

Corollary 9.3.4. *The disjoint union of two copies of K_n and the crown graph of order $2n$ are CNP-equivalent graphs.*

Chapter 10

Conclusion and further scope of research

This chapter includes a summary of the thesis and some guidelines which helps to explore the topic further.

10.1 Summary of the thesis

In the thesis, a new graph polynomial called "Common Neighbor Polynomial of graphs" is introduced and studied in a detailed manner. The common neighbor polynomial of many well known graph classes are identified. Moreover, the common neighbor polynomial of some tree structures and some special graph constructions are also discussed. The thesis provides explicit formulae to find common neighbor polynomial of some well known graph products such as join, corona, cartesian product, rooted product and tensor product of graphs in terms of the common neighbor polynomial of the parent graphs. The common neighbor polynomial of graphs obtained by graph operations such as splitting graph, shadow graph or mycielsky graph of a given graph are also studied.

The concept of CNP-equivalent classes of graphs and $CNP-$unique graphs are introduced and some CNP-equivalent classes of graphs and CNP-unique graphs are identified. A study on the roots of common neighbor polynomial are conducted. The stability of common neighbor polynomial of graphs are studied and the number of real common neighbor roots of some well known graph classes are found.

The generalized i-common neighbor sets and generalized common neighbor polynomial of graphs are defined and the generalized common neighbor polynomial of some well known graph classes are identified. Moreover, some characterizations on graphs in terms of generalized common neighbor polynomial of graphs are also discussed. The concept of simplicial complexes of graphs and cluster of a vertex in a graphs are introduced and in the light of these concepts, generalized i-common neighbor sets of graphs is studied.

Common neighbor polynomial of graphs are studied incorporated with chemical graph theory. The common neighbor polynomial of nanostar dendrimers and PAMAM dendrimers are studied. The hosoya polynomial of graphs with diameter not more than three is derived using the common neighbor polynomial of corresponding graphs. In order to establish the significance of common neighbor polynomial of graphs in network theory, the Shared Nearest Neighbor(SNN) clustering is discussed and explains the way in which the common neighbor polynomial of graphs is useful in the formation of meaningful clusters. A relation which connects common neighbor polynomial of a graph with its adjacency matrix is identified. Making use of this relation, a C^{++} program is developed for generating coefficients of common neighbor polynomial of a graph.

10.2 Further scope of research

1. Identify CNP-unique graph classes.
2. Identify CNP-equivalent graphs.
3. Characterize the properties of CNP-equivalent and CNP-unique graphs.
4. Identify CNP_r-equivalent graphs.
5. Identify CNP_r-unique graph classes.
6. Characterize the properties of CNP_r-equivalent graphs.
7. Characterize the properties of CNP_r-unique graphs.
8. Explore the formulae for generalized common neighbor polynomial of graphs obtained from various graph operations.
9. Characterize the polynomials over the set of integers which may be the common neighbor polynomial of some simple finite graphs.
10. Characterize the polynomials over the set of integers which may be the generalized common neighbor polynomial of some simple finite graphs.

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APPENDIX I

A C^{++} program for generating coefficients of Common Neighbor Polynomial of a graph

The following $C++$ program generates the coefficients $|N(G,i)|$ of the common neighbor polynomial of a graph of order n, for $i = n-2, n-3, \ldots, 2, 1, 0$. In this program, the cardinality of i-common neighbor set $|N(G,i)|$ is denoted simply as $N[i]$.

```
#include <iostream>
using namespace std;
int main()
{
int a[10][10], N[10];
int i, j, n, k, m, s;
cout <<"Enter the order of the adjacency matrix of the graph";
cin >>n;
```

cout<<“Enter the elements of the adjacency matrix row wise”;

$for(i = 1; i <= n; i + +)$

$\{for(j = 1; j <= n; j + +)$

$\{cin >> a[i][j];\}$

$\}$

$for(k = 1; k <= n - 2; k + +)$

$\{N[k] = 0;\}$

$for(i = 1; i <= n - 1; i + +)$

$\{for(m = i + 1; m <= n; m + +)$

$\{k = 0;$

$for(j = 1; j <= n; j + +)$

$\{if(a[i][j] * a[m][j] == 1)$

$k = k + 1;\}$

$N[k] = N[k] + 1;\}$

$\}$

$s = 0;$

$for(k = 1; k <= n - 2; k + +)$

$\{s = s + N[k];\}$

$N[0] = n * (n - 1)/2 - s;$

$for(k = 0; k <= n - 2; k + +)$

$\{cout << N[k] << "\backslash t";\}$

$\}$

APPENDIX II

List of publications

1. Shikhi M. and Anil Kumar V., *Common neighbor polynomial of graphs*, Far East Journal of mathematical sciences,Volume 102, Number 6, 2017, Pages 1201-1221.

2. Shikhi M. and Anil Kumar V., *Common neighbor polynomial of graph operations*, Far East Journal of mathematical sciences, Volume 102, Issue 11, 2017,Pages 2629 - 2641.

3. Shikhi M. and Anil Kumar V., *CNP-equivalent Classes of Graphs*, South East Asian Journal of Mathematics and Mathematical Sciences, Vol.13, No.2, 2017, pages 75-84.

4. Shikhi M. and Anil Kumar V., *On the Stability of Common Neighbor Polynomial of some Graphs*, South East Asian Journal of Mathematics and Mathematical Sciences, Vol.14, No.1, 2018, pages 95-102.

5. Shikhi M. and Anil Kumar V., *Common neighbor polynomial of some graph constructions*, International Journal of Research in Advent Technology, Vol.6, No.11, 2018, pages 3330-3334.

6. Shikhi M. and Anil Kumar V., *On the real roots of common neighbor polynomial of graphs*, Journal of Applied Science and Computations, Vol.6,Issue 4, 2019, pages 1424-1431.

7. Shikhi M. and Anil Kumar V., *Generalized common neighbor polynomial of graphs*, International Journal of Mathematical Combinatorics, Vol.3, 2019, pages 80-89.

8. Shikhi M. and Anil Kumar V., *Common neighbor polynomial of some special trees*, Aegaeum Journal, Vol. 8, Issue 5, 2020.

9. Shikhi M. and Anil Kumar V.,*Common neighbor polynomial of graphs and its significance in network data clustering*, Journal of xidian university, Vol. 14, Issue 4, 2020,857-862.

10. Shikhi M. and Anil Kumar V., *Common neighbor polynomial of some dendrimer structures*, South East Asian Journal of Mathematics and Mathematical Sciences (Communicated).

APPENDIX III

Paper Presentation

Presented a paper on ‘Common neighbor polynomial of some graph constructions’ in the international Conference on Discrete Mathematics and its Applications to Network Science held at the Department of Mathematics, Birla Institute of Technology and Science(BITS) Pilani, Goa.

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