Ph.D. THESIS **MATHEMATICS**

CYCLE TRACKING SET OF A GRAPH

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CERTIFICATE

I hereby certify that the thesis entitled "**Cycle Tracking Set of a Graph**" is a bona fide work carried out by **Smt. Jalsiya M. P.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Dr. Raji Pilakkat

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DECLARATION

I hereby declare that the thesis, entitled "**Cycle Tracking Set of a Graph**" is based on the original work done by me under the supervision of **Dr. Raji Pilakkat**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

University of Calicut,

September 04, 2019. Jalsiya M. P.

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Contents

List of Symbols

G	Graph
$V(G)$	Vertex set of G
$E(G)$	Edge set of G
$n(G)$	Order of G
$m(G)$	Size of G
$\Delta(G)$	Maximum degree of G
$\delta(G)$	Minimum degree of G
$\cong$	Isomorphic
$\overline{G}$	Complement of G
$d_G(v)$	Degree of the vertex v in G
P_n	Path on n vertices
C_n	Cycle on n vertices
K_n	Complete graph
$K_{m,n}$	Complete bipartite graph
$G_1 \vee G_2$	Join of G_1 and G_2
$G_1 \circ G_2$	Corona of G_1 and G_2
$\mathcal{P}(X)$	Power set of a set X
$\langle S \rangle$	Subgraph of G induced by S

$T_G(v)$	Cycle trace of a vertex v
$T_G(A)$	Cycle trace of a vertex set A
$\tau_c(G)$	Cycle tracking number of a graph G
$T_c(G)$	Upper cycle tracking number of a graph G
$pt(u, S)$	Open S-private trace of v
$pt[u, S]$	S-private trace of v
$ept[v, S]$	S-external private trace of v
$ept(v, S)$	Open S-external private trace of v
$ipt[v, S]$	S-internal private trace of v
$ipt[v, S]$	Open S-internal private trace of v
T	Maximum tracing number of a graph G
t	Minimum tracing number of a graph G
$V^0_{\tau_c}$	$\{v \in V : \tau_c(G - v) = \tau_c(G)\}$
$V^-_{\tau_c}$	$\{v \in V : \tau_c(G - v) < \tau_c(G)\}$
$V^+_{\tau_c}$	$\{v \in V : \tau_c(G - v) > \tau_c(G)\}$.
$TC(G)$	Cycle track completion number of a graph G
$OTB(S)$	Open trace boundary of S
$\tau_i(G)$	Independent cycle tracking number of a graph G
$T_i(G)$	Upper independent cycle tracking number of a graph G
$pt(S)$	Private trace set of S
$ptc(S)$	Private trace count of S
$\tau_{ir}(G)$	Trace irredundance number of a graph G
$T_{ir}(G)$	Upper trace irredundance number of a graph G
$T(G, i)$	The family of all cycle tracking sets of a graph G with cardinality i

List of Symbols

$t(G, i)$	Number of cycle tracking sets of a graph G of size i
$T(G, x)$	Cycle tracking polynomial of a graph G
$T_i(G, j)$	The family of independent cycle tracking sets of a graph G with cardinality j
$t_i(G, j)$	Number of cycle tracking sets of a graph G of size j
$T_i(G, x)$	Independent tracking polynomial of a graph G
$TM(G)$	Cycle tracking matrix of a graph G
$\chi_{\tau_c}(G, \lambda)$	τ_c- characteristic polynomial of a graph G
$Aut(G)$	The class of all automorphism of G
$\tau_t(G)$	Total cycle tracking number of a graph G
$T_t(G)$	Upper total cycle tracking number of a graph G
$\tau_f(G)$	Fractional cycle tracking number of a graph G
$T_f(G)$	Upper fractional cycle tracking number of a graph G
$\mathcal{T}_G$	Trace sigma algebra of a graph G
M_v^G	Smallest measurable set containing v.
μ	Measure
$\langle v \rangle$	Intersection of all $T_G(x)$ containing v
$(V(G), tc)$	Track closure space associated with a graph G
$\mathcal{T}_t^G$	Track topology of a graph G

Introduction

During the last few decades, graph theory experienced a tremendous development being the most important and interesting area of mathematics. Many real life situations can be explored by means of graph theory. Its results have wide application in many areas of Physics, Chemistry, Genetics, Computer science, Psychology etc.

We all know the unbroken loops which allow the flow of electrons called electric circuits are the heart of any electronic device. Electric network topology similar to mathematical topology is often used in analysis of electric circuits as an application of graph theory. Here network nodes are represented by vertices and branches are represented by edges of a graph. Graphs as abstract representation of electric circuit was successfully formulated by Gustav Kirchhoff in 1847 in loop analysis of resistive circuits at the moment of Kirchhoff's law formulation [24]. Later different researchers extended the use of graph theory in various electrical networks [34, 30, 28].

The goal of circuit analysis is to determine branch current and voltage in the network. Voltage and current in a network are related by its transfer function. Therefore solution of a network is obtained either in current or voltage. A general approach is solving loop current rather than branch current and deriving branch current from loop current. In a closed circuit of an electrical network, there exists a node which defines current at each branch of the closed circuit. This motivated us to introduce the concept of cycle tracking set. A complex circuit can be simplified by defining a cycle tracking set for each loop in a complex circuit. Each cycle tracking set will exhibit a unique property defining the entire loop.

0.1 Outline of the Thesis

Our work entitled "Cycle Tracking Set of a Graph" introduces the concept of cycle tracking set and various graph parameters related to it and studies various properties of it. In this thesis we consider only those graphs which are simple, finite and undirected.

Apart from this introductory chapter we have described our work in ten chapters.

In the **first chapter**, we gather the preliminary ideas that we need in our study of cycle tracking set to make the thesis self contained. This chapter includes necessary definitions and concepts in basic graph theory, measure theory and topology.

In the **second chapter**, we introduce the concept of cycle tracking set of a given graph G and studies its properties. In the first section a necessary and sufficient condition for a cycle tracking set to be minimal is also obtained. Graphs G with $\tau_c(G) = 1, n, n-1, n-2, n-3$ and $n-4$ are characterized, where n denotes the order of G.

In the second section another concept called transitively tracked graph is introduced and characterized. Furthermore some behavioral aspects of cycle tracking sets in a transitively tracked graph are studied.

In section 3, various bounds for the cycle tracking number $\tau_c(G)$ of a graph G of order n in terms of a variety of other graph parameters are presented. For any graph G of order n, $1 \leq \tau_c(G) \leq n$. Both of these bounds are sharp. The upper bound is attained if and only if G is a forest and the lower bound is attained if and only if G is either a track connected graph or a track connected floral graph.

In section 4, the cycle tracking number of some graphs are determined.

In the **third chapter**, we examined the effect on $\tau_c(G)$ when we add or remove vertex or edge from the given graph G. In the first section of this chapter, the effect on $\tau_c(G)$ on vertex removal is studied. It is proved that removal of a vertex can increase the cycle tracking number by more than one, but can decrease by at most one.

The second section of the chapter discusses the effect on $\tau_c(G)$ on edge removal.

In the **fourth chapter**, we consider the minimum number of edges required to be added to a graph to make it track connected. This problem is defined as the problem of finding the cycle track completion number $TC(G)$ of the given graph G.

In the second section of this chapter, bounds for the cycle track completion number $TC(G)$ of a graph G of order n in terms of various parameters are determined.

In the **Fifth chapter**, we introduce the concept of trace independence and trace irredundance, and discuss the close relationships among cycle tracking sets, trace independent sets and trace irredundance sets in a graph in two sections.

In the first section, necessary and sufficient condition for a trace independent set to be maximal is obtained. Independent cycle tracking number $\tau_i(G)$ and upper independent cycle tracking number $T_i(G)$ are defined. And proved that every maximal trace independent set in a graph G is a minimal cycle tracking set of G.

In section 2, we introduce the trace irredundant set and derived the condition for a vertex set to be a trace irredundant set. A necessary and sufficient condition for a trace irredundant set to be maximal is determined. It is also proved that a cycle tracking set S is a minimal cycle tracking set if and only if it is cycle tracking and trace irredundant. Trace irredundant number $\tau_{ir}(G)$ and upper trace irredundant number $T_{ir}(G)$ are also defined.

In the **Sixth chapter**, we introduce a new type of graph polynomial based on cycle tracking set, called cycle tracking polynomial $T(G,x)$ and studied its properties. The cycle tracking polynomial of Firefly graph $F_{s,t,n-2s-2t-1}$, Lollipop graph $L_{n,m}$, Tadpole $T_{(n,l)}$, Helm graph H_n, Web graph WB_n, Friendship graph F_n and Armed crown $C_n \bigodot P_m$ are derived. Also independent cycle tracking polynomial $T_i(G,x)$ is introduced and some of its properties are studied in this chapter. Further independent cycle tracking polynomial of some graphs are derived.

In the **Seventh chapter**, we introduce cycle tracking matrix $TM(G)$ associated with a graph. For a graph G of order n, $TM(G)$ is a real symmetric matrix with trace n. We characterize matrices which are cycle tracking matrix for some graph and graphs with nonsingular cycle tracking matrices. Furthermore a study on the spectral properties of $TM(G)$ which are invariant under permutations of its rows and columns is carried out. Also a discussion on graph automorphisms and corresponding cycle tracking matrices is done here.

Eighth chapter introduces total cycle tracking set in graphs and establishes its fundamental properties. General bounds relating the total cycle tracking number to other parameters are presented in this chapter, and properties of minimum total cycle tracking set are listed.

Nineth chapter introduces the concept of cycle tracking function and studies its properties. Moreover it is established that the computation of the cycle tracking number $\tau_c(G)$ of a given graph G is a constrained optimization problem, which is in fact an integer programming problem given below.

$$\tau_c(G) = min \sum_{i=1}^{n} x_i$$

$$subject\ to\ TM(G).X \geq 1$$

$$with\ X \in \{0,1\}^{n\times 1}.$$

The linear programming version of cycle tracking problem motivated us to introduce a new concept called a cycle tracking function which is in fact a generalization of the existing concept of dominating function [20]. A necessary and sufficient condition for a cycle tracking function to be minimal is also obtained. Fractional cycle tracking number $\tau_f(G)$ and upper fractional cycle tracking number $T_f(G)$ are defined and bounds for fractional cycle tracking number $\tau_f(G)$ are derived.

By introducing the concept of trace sigma algebra $\mathcal{T}_G$ of a graph G and by studying and analyzing its properties we extend this notion to introduce measurable cycle tracking function of finite graphs. A necessary and sufficient condition for a measurable cycle tracking function to be minimal is also obtained.

For every pair of vertices $u, v \in V$, we say that u is related to v $(u \sim v)$ if $u \in T_G(v)$. Then $' \sim \ '$ is a reflexive and symmetric relation. In the **Tenth chapter** we define and investigate a new closure operator with respect to the relation $' \sim \ '$ on the vertex set V of a graph G. The topology associated with this closure operator is studied.

Some of the problems that were thought about and where further research is possible are briefly mentioned in the epilogue.

Chapter 1

Preliminaries

1.1 Introduction

This chapter reviews the basic ideas, definitions and terminologies that we need in the discussion of our study. It includes the basics of graph theory, the concept of domination in graphs, measure theory and topology. For definitions, notations and terminologies, we follow mainly [7], [4] and [20].

1.2 Basics of Graph Theory

This section focuses on the definitions and terminologies of Graph theory, which are needed for the discussion of the topics in the forthcoming chapters.

A (undirected) *graph* [7] G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to *join* u and v, and the vertices u and v are called the *ends* of e. In this case we also denote the edge by uv. In the diagramatic representation of a graph each vertex is indicated by a point, and each edge by a line segment joining the points representing its ends [7].

Throughout this thesis the letter G denotes the graph with vertex set V and edge set E, unless otherwise specified.

The number of vertices of the graph G is called the *order* [4] of G, denoted by $n(G)$ and the number of edges is called the *size* [4] of G, denoted by $m(G)$. A *finite graph*[7] is one in which both vertex set and edge set are finite. A graph having exactly one vertex and no edges is called a *trivial graph*[7] and all other graphs are called *nontrivial* graphs or simply graphs.

If u and v are distinct vertices and if $e = uv$ is an edge of the graph G, then u and v are said to *adjacent vertices*, the edge e is said to *incident with* u and v [15] and the vertices u and v are called the *end vertices* of the edge e [4]. Two adjacent vertices are referred to as neighbors of each other. Two edges are said to be adjacent if they have a common vertex[7]. In a graph G with vertex v, the set of neighbors of v is called the *open neighborhood* [15] of v and it is denoted by $N_G(v)$ or by $N(v)$ if there is no confusion. The set $N_G(v) \bigcup \{v\}$ is called the *closed neighborhood* [15] of v and it is denoted by $N_G[v]$ (or simply $N[v]$ if there is no confusion).

An edge with identical ends is called a *loop*[7]. Two or more edges with the same pair of ends are said to be *parallel* edges or multiple edges and graph having multiple edges is usually called a *multigraph*[7].

A graph having no loops or multiple edges is called a *simple graph*[7].

The *degree* [7] of a vertex v in a graph G, denoted by $d_G(v)$ (or $d(v)$), is the number of edges of G incident with v, each loop counting as two edges. In particular, if G is a simple graph, $d(v)$ is the number of neighbors of v in G. A vertex of degree zero is called an *isolated vertex* [29] . A vertex of degree one is called a *pendant vertex* or an *end vertex*[4] . A vertex adjacent to a pendant vertex is called a *support vertex*[29]. A *pendant edge* [4] is the edge incident with a pendant vertex. The minimum (respectively, maximum) of the degrees of the vertices of a graph G is denoted by $\delta(G)$ (respectively, $\Delta(G)$) [4].

Every graph mentioned in this thesis is simple, finite and undirected.

The *complement*[7] of a simple graph G is the simple graph $\bar{G}$ whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G.

Two graphs G and H are said to be *disjoint*[11] if they have no vertex in common.

A *subgraph*[18] of a graph G is a graph having all of its vertices and edges are in G. If G_1 is a subgraph of G, then G is a *supergraph* [18] of G_1. A *spanning subgraph*[18] is a subgraph containing all vertices of G. For any set S of points of G, the *induced subgraph*[18] $\langle S \rangle$ is the maximal subgraph of G with vertex set S. A *Hamiltonian graph* [35] is a graph with a spanning cycle, also called a Hamiltonian cycle.

A graph G is *connected*[7] if, for every partition of its vertex set into two nonempty set X and Y, there is an edge with one end in X and the other end in

Y; otherwise the graph is *disconnected*[7]. *Components*[4] of a graph G are the maximal connected subgraphs of G.

A *cut edge* (or *a cut vertex*) [35] of a graph is an edge or vertex whose deletion increases the number of components. We write $G-e$ (or $G-M$) for the subgraph of G obtained by deleting an edge e (or a set M of edges). We write $G-v$ (or $G-S$) for the subgraph of G obtained by deleting a vertex v (or a set of vertices S).

Two graphs G and H are *isomorphic* [7], written $G \cong H$, if there are bijections $\theta : V(G) \longrightarrow V(H)$ and $\phi : E(G) \longrightarrow E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$; such a pair of mappings is called an *isomorphism* between G and H. Here the bijection θ satisfies the condition that u and v are end vertices of an edge e of G if and only if $\theta(u)$ and $\theta(v)$ are end vertices of the edge $\phi(e)$ in H [4]. An isomorphism from a graph G to itself is called an *automorphism* of G [10] and the class of all automorphism of G is usually denoted by Aut(G). If $g \in Aut(G)$ and Y is a subgraph of G, then we define Y^g to be the graph with $V(Y^g) = \{g(u) : u \in V(Y)\}$ and $E(Y^g) = \{g(u)g(v) : uv \in E(Y)\}$ [10]. A graph G is *vertex transitive*[10] if its automorphism group acts transitively on $V(G)$, ie; for any two distinct vertices of G there is an automorphism mapping one to the other.

A *walk*[4] in a graph G is an alternating sequence $W : v_0e_1v_1e_2v_2 \dots e_nv_n$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 is the *origin* and v_n is the *terminus* of W. The walk W is said to join v_0 and v_n. A walk is called a *trail*[4] if all the edges appearing in the walk are distinct. It is called a *path*[4] if all the vertices are distinct. Thus a path in G is automatically a trail in G. Two distinct paths are *internally disjoint*[7] if they have no internal vertices in common. When writing a path, we usually omit the edges. A *cycle*[4] is a closed trail in which the vertices are all distinct. The number of edges in a walk is called its *length*[4]. In a graph G which has at least one cycle, the length of a longest cycle is called its *circumference* [7] and the length of a shortest cycle is its *girth*[7]. A cycle of length n is denoted by C_n and P_n denotes a path on n vertices [4].

Theorem 1.2.1. *[35] An edge is a cut edge if and only if it belongs to no cycle.*

A *vertex cut* [35] of a graph G is a set $S \subset V(G)$ such that $G - S$ has more than one component. The *connectivity*[35] of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is *k-connected*[35] if its connectivity is at least k.

An *acyclic*[7] graph is one that contains no cycle. A connected acyclic graph is called a *tree*[7]. Acyclic graphs are called *forests*[7].

Given a vertex x and a set U of vertices, an *x, U-fan* is a set of paths from x to U such that any two of them share only the vertex x.

Theorem 1.2.2. *(Fan Lemma, Dirac)[35]. A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an x, U-fan of size k.*

1.3 Some Special Graphs

There are several classes of graphs which are used in this thesis.

A *complete graph* [11] is a simple graph in which each pair of distinct vertices is joined by an edge. A complete graph on n vertices is denoted by K_n.

A graph is said to be *bipartite*[4] if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a *bipartition*[4] of the bipartite graph. The bipartite graph with bipartition (X, Y) is denoted by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is *complete*[4] if each vertex of X is adjacent to all the vertices of Y. A complete bipartite graph $G(X, Y)$ with $|X| = r$ and $|Y| = s$, is denoted by $K_{r,s}$.

A *Firefly graph* $F_{s,t,n-2s-2t-1}$ $(s \geq 0, t \geq 0$ and $n - 2s - 2t - 1 \geq 0)$[25] is a graph of order n that consists of s triangles, t pendant paths of length 2 and $n - 2s - 2t - 1$ pendant edges sharing a common vertex.

A *Lollipop graph* $L_{n,m}$[13] is obtained by joining K_n to a path P_m of length m with a bridge.

A *Tadpole* $T_{(n,l)}$[32] is the graph obtained by attaching a path P_l to a cycle C_n.

A graph with the vertex set $V = \{u_0, u_1, u_2, ..., u_n\}$ for $n \geq 3$ and the edge set $E = \{u_0u_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ is called *Wheel graph*[26] of length n and is denoted by W_n. The vertex u_0 is called the axial vertex of the wheel graph. The *Helm graph* H_n[26] is obtained from the wheel graph W_n by attaching a pendant edge at each vertex of the n-cycle of the wheel.

For a positive integer $n > 3$, a *Web graph* WB_n[26] is obtained by joining the pendant vertices of a helm H_n to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. It has $3n + 1$ vertices and $5n$ edges.

The *Friendship graph* F_n[21] is a collection of n triangles with a common vertex.

An *Armed crown* $C_n \bigodot P_m$[14] is a graph obtained by attaching paths P_m to every vertex of the cycle C_n.

1.4 Domination in Graph Theory

The study of domination is the fastest growing area in graph theory. This section discusses the concept of dominating set, dominating function and domination polynomial in a graph.

For a graph G a set $S \subseteq V(G)$ is called a *dominating set*[20] of G if every vertex $u \in V(G)$ is either an element of S or is adjacent to an element of S. If S is a dominating set of a graph, then every superset of S is also a dominating set. On the other hand, not every subset of S is necessarily a dominating set. A dominating set S of G is a *minimal dominating set*[20] if no proper subset of S is a dominating set. The *domination number*[20] of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G and a γ-set is a dominating set with cardinality $\gamma(G)$.

A function $f : V(G) \to \{0, 1\}$ is called a *dominating function*[20] of G if $\sum\limits_{u \in N[v]} f(u) \geq 1$ for all $v \in V(G)$.

A function $f : V(G) \longrightarrow [0, 1]$ is called a *fractional dominating function*[20] of G if $\sum\limits_{u \in N[v]} f(u) \geq 1$ for all $v \in V(G)$.

The concept of *domination polynomial* of a graph was introduced by S. Alikhani in 2009. For a graph G, let $\mathcal{D}(G,i)$ be the family of dominating sets with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$[1]. The *domination polynomial*[1] $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$.

1.5 Operations on Graphs

This section includes some graph operations used in this thesis.

The *union*[35] of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

The *corona*[8] of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph formed from one copy of G_1 and $\mid V(G_1) \mid$ copies of G_2, such that the i^{th} vertex of the copy of G_1 is adjacent to every vertex in the i^{th} copy of G_2 for $i = 1, 2, ..., |V(G_1)|$.

The *join*[8] of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \colon u \in V(G_1)$ and $v \in V(G_2)\}$.

Edge addition [18] is a local operation on a graph. Let G be a graph. If u and v are nonadjacent vertices of G, then $G+e$ where $e = uv$ denotes the graph obtained from G by adding edge e. Let $X \subset \overline{E(G)}$, the set of edges which are not in $E(G)$, then $G + X$ denote the graph obtained from G by adding all edges in the set X.

Duplication[33] of a vertex v_k by a new edge in a graph G produces a new graph G' by adding an edge $e' = u'v'$ to G such that $N(v') = \{v_k, u'\}$ and $N(u') = \{v_k, v'\}$.

Duplication[33] of an edge $e = uv$ by a new vertex in a graph G produces a new graph G' by adding a vertex v' to G such that $N(v') = \{u, v\}$.

In a graph G, *subdivision*[35] of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w. A *subdivision*[35] of H is a graph obtained from the graph H by successive edge subdivisions.

The process of deletion of an edge e of a graph G and the amalgamation of the ends of this edge to form a single vertex is called *shorting an edge e*[33].

Let G be a graph with n vertices. If there are two non-adjacent vertices u_1 and v_1 in G such that $deg(u_1) + deg(v_1) \geq n$, join u_1 and v_1 by an edge to form the super graph G_1. Now, if there are two non-adjacent vertices u_2 and v_2 in G_1 such that $deg(u_2) + deg(v_2) \geq n$, join u_2 and v_2 by an edge to form the super graph G_2. Continue in this way, recursively joining pairs of non-adjacent vertices whose degree sum is at least n until no such pair remains. The final super graph thus obtained is called the *closure* [11] of G and is denoted by $c(G)$.

1.6 Matrices

This section focuses on some basic concepts related to matrices. For further details refer [5].

A *matrix*[5] of order $m \times n$, called an $m \times n$ matrix consists of mn real numbers arranged in m rows and n columns. The entry in row i and column j

of the matrix A is denoted by a_{ij}. An $m \times 1$ matrix is called a *column vector* of order m; similarly, a $1 \times n$ matrix is a *row vector* of order n. An $m \times n$ matrix is called a *square matrix*[5] if $m = n$.

A matrix A of order $n \times n$ is said to be *nonsingular*[5] if $rank\ A = n$; otherwise the matrix is *singular*.

Let A be an $n \times n$ matrix. The determinant $det(A - \lambda I)$ is a polynomial in the (complex) variable λ of degree n and is called the *characteristic polynomial* [5] of A. The equation, $det(A - \lambda I) = 0$ is called the *characteristic equation*[5] of A. By the fundamental theorem of algebra the equation $det(A - \lambda I) = 0$ has n complex roots counting multiplicities and these roots are called the *eigenvalues*[5] of A. A square matrix A is called *symmetric*[5] if $A = A'$, where A' denotes the transpose of A. The eigenvalues of a symmetric matrix are real.

Let G be a loop less graph with vertex set $V(G) = \{v_1, ..., v_n\}$ and edge set $E(G) = \{e_1, ..., e_n\}$. The *adjacency matrix* [35] of G, written $A(G)$, is the $n \times n$ matrix in which entry a_{ij} is the number of edges in G with end points $\{v_i, v_j\}$. The *incident matrix* [35] $M(G)$ is the $n \times m$ matrix in which the entry m_{ij} is 1 if v_i is an end point of e_j and otherwise is 0.

1.7 Measure Theory and Topology

This section focuses on some basic concepts of measure theory and topology. For further details refer [31] and [17].

A distinguished collection $\mathcal{R}$ of subsets of a set X is called an *algebra* [17] if the following axioms are satisfied.

(i) If $E \in \mathcal{R}$ and $F \in \mathcal{R}$, then $E \bigcup F \in \mathcal{R}$

(ii) If $E \in \mathcal{R}$, then $E^c \in \mathcal{R}$, where $E^c := X \setminus E$ is the complement of E in X.

An algebra $\mathcal{R}$, of subsets of a set X is called a *sigma algebra* [31] if $\bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$, whenever $E_1, E_2, \ldots \in \mathcal{R}$.

Proposition 1.7.1. *[31] If $\mathcal{F}$ is any family of subsets of a set X, there exists a smallest sigma algebra containing $\mathcal{F}$, called the sigma-algebra generated by $\mathcal{F}$.*

A set X together with a sigma algebra $\mathcal{R}$ of subsets of X is called a *measurable space*[31], and the members of $\mathcal{R}$ are called the *measurable sets*[31] in X.

Let $(X, \mathcal{R})$ be a measurable space. A *measure*[31] is a function μ, defined on the sigma algebra $\mathcal{R}$, whose range is in $[0, \infty]$ and which is countably additive.

Another distinguished family of subsets f a set X is called a topology of X. More specifically, a *topology*[27] on a set X is a collection τ of subsets of X having the following properties:

(1) $\emptyset$ and X are in τ.

(2) The union of the elements of any subcollection of τ is in τ.

(3) The intersection of the elements of any finite subcollection of τ is in τ.

A set X for which a topology τ has been specified is called a *topological space*[27].

Properly speaking, a topological space is an ordered pair $(X,\ \tau)$ consisting of a set X and a topology τ on X, but we often omit specific mention of τ if no confusion will arise [27].

If X is a topological space with topology τ, we say that a subset U of X is an *open set* [27] of X if U belongs to the collection τ.

Let X be a measurable space and Y be a topological space [27]. A mapping f from X into Y is said to be *measurable*[31] if $f^{-1}(S)$ is a measurable set in X for every open set S in Y.

Chapter 2

Cycle Tracking Sets of a Graph

In this chapter we introduce the concept of cycle tracking set and discuss some basic results on cycle tracking sets. A necessary and sufficient condition for a vertex set to be a minimal cycle tracking set of a graph is derived and some bounds on cycle tracking number, $\tau_c(G)$ are determined. Also transitively tracked graphs are introduced and characterized.

2.1 Cycle Tracking Sets of a Graph

An electrical circuit is a network consisting of a closed loop, giving a return path for the current. When faced with a new circuit, the software first tries to find a steady state solution, that is, one where all nodes conform to Kirchhoff's current law and the voltages across and through each element of the circuit conform to the voltage/current equations governing that element[9].

Once the steady state solution is found, the operating points of each element in the circuit are known, the circuit can be analyzed by employing graph theory. In a closed circuit of an electrical network, there exists a vertex which defines current at each edge of the closed circuit. The collection of all such vertices is defined as the cycle tracking set. Therefore a complex circuit can be analyzed through graph theory by considering nodes as the vertices and branches as the edges in a graph and by defining a cycle tracking set for it.

This section introduces the concept of cycle tracking set and studies various properties of it.

Definition 2.1.1. *Let G be a graph. For $v \in V(G)$, the cycle trace (simply trace) of v is the set of all vertices $u \in V$ such that u and v belong to same cycle of G and is denoted by $T_G(v)$.*
ie; $T_G(v) = \{u \in V : \text{ u and v belong to same cycle}\}$.

Clearly $v \in T_G(v)$ for every vertex in a graph G.

Definition 2.1.2. *Let G be a graph and let $u, v \in V$. Then v is said to be cycle traced (traced) by u if $v \in T_G(u)$.*

From the definition of $T_G(v)$ it is immediate that for every pair of vertices u and v, $u \in T_G(v)$ if and only if $v \in T_G(u)$.

Definition 2.1.3. *Let G be any graph, $A \subseteq V(G)$. The cycle trace (simply trace), $T_G(A)$ of A is the union of all subsets $T_G(v)$ of $V(G)$, where v varies over A. That is, $T_G(A) = \cup_{v \in A} T_G(v)$.*

Definition 2.1.4. *A vertex of the graph G is said to be trace $free$ vertex if it is not a vertex of any non trivial cycle in G.*

Definition 2.1.5. *A set S of vertices in a graph G is called a cycle tracking set if for every vertex $v \in V \setminus S$, there exists a vertex $u \in S$ such that $v \in T_G(u)$. That is $T_G(v) \cap S \neq \emptyset$ for every $v \in V \setminus S$.*

A straight forward consequence of the definition is that every superset of a cycle tracking set of graph G is again a cycle tracking set.

Definition 2.1.6. *A cycle tracking set is a minimal cycle tracking set if no proper subset S' of S is a cycle tracking set.*

Definition 2.1.7. *The cycle tracking number $\tau_c(G)$ of a graph G is the minimum cardinality of a minimal cycle tracking set of G.*

Definition 2.1.8. *The upper cycle tracking number $T_c(G)$ of a graph G is the maximum cardinality of a minimal cycle tracking set of G.*

Definition 2.1.9. *A cycle tracking set with minimum cardinality is called a $\tau_c - set$ of G.*

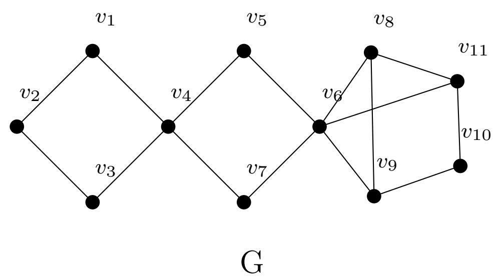


Figure 2.1: $\tau_c(G) = 2$.

In figure 2.1 the sets $\{v_1, v_5, v_8\}$ and $\{v_4, v_8\}$ are two cycle tracking sets of the graph G and for this graph $\tau_c(G) = 2$ and $T_c(G) = 3$. Moreover $\{v_4, v_8\}$ is a $\tau_c - set$ of G.

- The cycle tracking problem of a graph G is the problem of finding a minimal cycle tracking set of G.
- If G is a graph of order n, then we have $1 \leq \tau_c(G) \leq n$.

Definition 2.1.10. *A graph G is said to be track connected if for every pair of vertices u, v in G there exists two internally disjoint paths connecting u and v.*

Remark 2.1.11. *1. A trivial graph is trivially track connected.*

2. There is no track connected graph with 2 vertices.

3. Every track connected non trivial graph is 2-connected . Hence such graphs have no cut vertices or cut edges.

4. An induced subgraph of a graph G is said to be maximal track connected if it is not a proper subgraph of any track connected subgraph of G. Clearly each subgraph induced by any trace free vertex of a graph G is maximal track connected. Also any maximal 2-connected subgraph of G is maximal track connected.

Since a track connected graph having more than two vertices is 2-connected, by Fan Lemma, Dirac[35], we have:

Theorem 2.1.12. *Let G be a graph of order $n \geq 3$. Then G is track connected if and only if for every triple of vertices u,v and w there exists two internally disjoint paths one from w to u and the other from w to v.*

Theorem 2.1.13. *Two maximal track connected subgraphs of a graph share at most one vertex.*

Proof. Let G be a graph and H_1 and H_2 be two maximal track connected subgraphs of G. If possible let v_1 and v_2 be two common vertices of H_1 and H_2. Let $x, y \in V(H_1 \cup H_2)$. Since H_1 is track connected by Theorem 2.1.12 there exist two paths P_1 from x to v_1 and P_2 from x to v_2 such that P_1 and P_2 are internally disjoint. Similarly since H_2 is track connected again by Theorem 2.1.12 there exist two paths P_3 from v_1 to y and P_4 from v_2 to y such that P_3 and P_4 are internally disjoint. The paths P_1, P_2, P_3, P_4 altogether forms a cycle containing x and y in $H_1 \cup H_2$. Hence $H_1 \cup H_2$ is track connected, a contradiction to the maximality of H_1 and H_2. □

Corollary 2.1.14. *If two maximal track connected subgraphs of G share a vertex, then it must be a cut vertex of G.*

Corollary 2.1.15. *Let G be a graph. Let u and v be two vertices in V such that* $u \notin T_G(v)$. *Then* $|T_G(u) \cap T_G(v)| \leq 1$.

Proposition 2.1.16. *Let G be a graph and v be a vertex of G. Then,*

1. $|T_G(v)| = 1$ *if and only if v is trace free.*
2. *for any vertex v,* $|T_G(v)| \neq 2$.
3. $|T_G(v)| = 3$ *if and only if the subgraph induced by* $T_G(v)$ *is a triangle.*
4. $|T_G(v)| = 4$ *if and only if the subgraph induced by* $T_G(v)$ *is* C_4*, kite[18] or* K_4.
5. $|T_G(v)| = 5$ *if and only if the subgraph induced by* $T_G(v)$ *is a track connected subgraph with 5 vertices or a bowtie[18].*

Definition 2.1.17. *Let S be a set of vertices of a graph G, and let $u \in S$. A vertex v of G is said to be a private trace of u with respect to S if $T_G(v) \cap S = \{u\}$. The $S -$ private trace, $pt[u, S]$ of a vertex $u \in S$ is the subset $\{v : T_G(v) \cap S = \{u\}\}$.*

Theorem 2.1.18. *A cycle tracking set $S \subset V$ in a graph G is a minimal cycle tracking set if and only if every vertex in S has at least one private trace, that is $pt[u, S] \neq \emptyset$, for every $u \in S$.*

Proof. Assume that S is a minimal cycle tracking set of G. Then every vertex in V is traced by some vertex in S and for every vertex $u \in S$, $S \setminus \{u\}$ is not a cycle tracking set. Fix $u \in S$. Then there exists a vertex $v \in (V \setminus S) \cup \{u\}$, which is not traced by any vertex in $S \setminus \{u\}$. Therefore $T_G(v) \cap S = \{u\}$.

Conversely suppose that S is a cycle tracking set and for each vertex $u \in S$, $pt[u, S] \neq \emptyset$. We show that S is minimal cycle tracking set. Suppose that S is not a minimal cycle tracking set. Then there exists a vertex $u \in S$ such that $S \setminus \{u\}$ is a cycle tracking set. Hence every vertex in $(V \setminus S) \cup \{u\}$ is traced by at least one vertex in $S \setminus \{u\}$, that is $pt[u, S] = \emptyset$, which contradicts the assumption. □

Theorem 2.1.19. *Let G be any graph. Then the following statements are equivalent.*

(i) A vertex v is in every cycle tracking set.

(ii) $v \in S$ for all cycle tracking set S and $pt[v, S] = \{v\}$.

(iii) v is a trace free vertex.

Proof. Let G be any graph. We first assume that $v \in V(G)$ is in every cycle tracking set. If possible let $|pt[v, S]| \geq 2$ for some cycle tracking set $S = \{v, u_1, u_2, ..., u_m\}$. Let $pt[v, S] = \{v_1, v_2, ..., v_k\}$, $k \geq 2$. Then $\{v_1, v_2, ..., v_k, u_1, u_2, ..., u_m\}$ forms a cycle tracking set of G, a contradiction. So $pt[v, S] = \{v\}$ for all cycle tracking set. So (i) implies (ii).

Now we prove that (ii) implies (iii).

Suppose the condition (ii) holds. If possible let v is not a trace free vertex. Then $|T_G(v)| \geq 3$. Then, $S \cup T_G(v) \setminus \{v\}$ forms a cycle tracking set, a contradiction. So v is a trace free vertex. Hence (ii) implies (iii).

The implication of (i) from (iii) follows directly from the definition of trace free vertices. □

Theorem 2.1.20. *If G is a graph without trace free vertices, then the complement $V \setminus S$ of every minimal cycle tracking set is a cycle tracking set.*

Proof. Let S be a minimal cycle tracking set of G. Assume that the vertex $u \in S$ is not traced by any vertex in $V \setminus S$. This is possible only if $T_G(u) \subset S$. Which is possible only if $T_G(u) = \{u\}$. That is u is a trace free vertex of G, a contradiction. Thus every vertex in S is traced by at least one vertex in $V \setminus S$ and hence $V \setminus S$ is a cycle tracking set. □

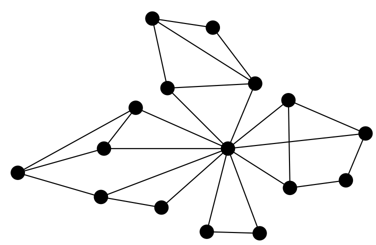
Theorem 2.1.21. *Let G be a graph $\tau_c(G) = |V|$ if and only if G is a forest.*

Proof. Let G be a forest. Since every cycle tracking set of G contains all trace free vertices of G, $\tau_c(G) = |V|$.

Conversely suppose that $\tau_c(G) = |V|$. If G contains a non trivial cycle C. Let u be a vertex in C. Then $(V \setminus V(C)) \cup \{u\}$ forms a cycle tracking set for G. Hence

$\tau_c(G) \leq |V| - |V(C)| + 1 < |V|$, a contradiction. So G is a forest. $\square$

Definition 2.1.22. *Let G be a graph with exactly one cut vertex. Let v be the cut vertex of G and $G_1, G_2, ..., G_k$ be the components of $G - v$. If the order of G_i is greater than or equal to two and the graphs induced by $V(G_i) \cup \{v\}, i = 1, 2, ..., k$ are track connected then G is called a track connected $floral$ graph. For $i = 1, 2, ..., k$ the graph induced by $V(G_i) \cup \{v\}$ is called a petal of G.*



G

Figure 2.2: A track connected floral graph with four petals.

Theorem 2.1.23. *For a graph G, $\tau_c(G) = 1$ if and only if G is track connected or G is a track connected floral graph.*

Proof. For a track connected or track connected floral graph G, $\tau_c(G)$ is clearly 1.

To prove the converse let G be any graph with $\tau_c(G) = 1$. Then there exists a vertex $v \in V(G)$ such that every vertex belongs to $T_G(v)$. Then G may be track connected because for track connected graph $\tau_c(G) = 1$.

Now suppose that G is not track connected. Then there exist two vertices $x, y \in V(G)$ such that they are not connected by two internally disjoint paths. Since $\tau_c(G) = 1$ there is a path from x to y which passes through v. Clearly v is a cut vertex. Hence G is a track connected floral graph. $\square$

Definition 2.1.24. *The tracing number of a vertex v in a graph G is the cardinality of $T_G(v)$.*

Definition 2.1.25. *A vertex v in a graph G is said to be a maximum tracing vertex of G if $|T_G(v)| \geq |T_G(u)|$ for all $u \in V$. If v is a maximum tracing vertex then $|T_G(v)|$ is called the maximum tracing number and it is denoted by T.*

Definition 2.1.26. *A vertex v in a graph G is said to be minimum tracing vertex of G if $|T_G(v)| \leq |T_G(u)|$ for all $u \in V$. If v is a minimum tracing vertex then $|T_G(v)|$ is called the minimum tracing number and it is denoted by t.*

Theorem 2.1.27. *Let G be a graph of order n. Then $1 \leq t \leq T \leq n$.*

Proof. For every vertex v, v traces itself so $|T_G(v)| \geq 1$ and v can trace at most n vertices so $|T_G(v)| \leq n$. Hence $1 \leq t \leq T \leq n$. □

Proposition 2.1.28. *For a graph G,*

1. *$t = 1$ if and only if G contains a trace free vertex.*

2. *$t = n$ if and only if G is a track connected graph.*

3. *$T = n$ if and only if G is a track connected graph or a track connected floral graph.*

4. *$T = 1$ if and only if G a forest.*

Theorem 2.1.29. *Let G be a graph without trace free vertices. Then there exists at least two vertices $u, v \in V$ such that $T_G(u) = T_G(v)$.*

Proof. Let G be a graph without trace free vertices. Then there exist two vertices v_i and v_j such that $v_i \in T_G(v_j)$. Without loss of generality assume that $v_1 \in$

$T_G(v_2)$, v_1 has minimum tracing number among all vertices of G and v_2 has minimum tracing number among all vertices in $T_G(v_1) \setminus \{v_1\}$. If $T_G(v_1) \neq T_G(v_2)$, then v_2 is a cut vertex. Then there exist at least two vertices in $T_G(v_2)$ which are not in $T_G(v_1)$. Let v_3 have minimum tracing number among all such vertices and let v_4 be a vertex having minimum tracing number among all vertices in $T_G(v_2) \cap T_G(v_3) \setminus \{v_2, v_3\}$. If $T_G(v_3) \neq T_G(v_4)$, then v_4 is a cut vertex and we can repeat the process again. As G is a finite graph this process cannot be repeated indefinitely. The process will be terminated at, say k^{th} stage only if there exists one vertex v_{2k-1} with minimum tracing number among the vertices $T_G(v_{2k-2}) \setminus T_G(v_{2k-3})$ and another vertex v_{2k} with minimum tracing number among all vertices in $T_G(v_{2k-2}) \cap T_G(v_{2k-1}) \setminus \{v_{2k-2}, v_{2k-1}\}$, such that v_{2k-1} and v_{2k} have the same cycle tracking set. That is $T_G(v_{2k-1}) = T_G(v_{2k})$. □

Theorem 2.1.30. *For any graph G of order n, $\tau_c(G) \neq n-1$.*

Proof. If G is a forest $\tau_c(G) = n$. If G is not a forest, G contains at least one cycle. That is there exist a vertex in G which traces at least three vertices of G and hence $\tau_c(G) \leq n-2$. Hence the result. □

Theorem 2.1.31. *Let G be a graph of order $n \geq 3$. Then $\tau_c(G) = n-2$ if and only if G is a unicyclic graph[18] and the cycle in G is a triangle.*

Proof. Suppose $\tau_c(G) = n-2$. Then by Theorem 2.1.21 G is not a forest. Therefore G contains at least one cycle. So there exists at least one vertex which is not trace free. Since every non trace free vertex traces at least three vertices and $\tau_c(G) = n-2$, every $\tau_c - set$ contains one vertex v with $|T_G(v)| = 3$ and all other $n-3$ vertices are trace free vertices. That is G is a graph having exactly one cycle of length 3.

The converse of the theorem is obvious. $\square$

Theorem 2.1.32. *Let G be a graph of order $n \geq 4$. Then $\tau_c(G) = n - 3$ if and only if G is a graph having $n - 4$ trace free vertices and a track connected subgraph of cardinality* 4.

Proof. Suppose $\tau_c(G) = n - 3$. Then by Theorem 2.1.21 there exist one vertex in G, which is not trace free. We claim that there exists one vertex v in V with $|T_G(v)| = 4$ all other vertices in $V \setminus T_G(v)$ are trace free.

We prove this result in two steps.

Step I: If $v \in V$, then $|T_G(v)| \leq 4$.

If for some $v \in V$, $|T_G(v)| \geq 5$ then v traces 5 vertices including v. Then $(V \setminus T_G(v)) \cup \{v\}$ forms a cycle tracking set of cardinality $n-4$, a contradiction. Thus $|T_G(v)| \leq 4$.

Step II: If S is any τ_c-set of G, then for any $v \in S$, $|T_G(v)|$ is 1 or 4.

Let S be a $\tau_c - set$ of G. Then S contains a vertex which is not trace free. Let $v \in S$ be such that $|T_G(v)|$ is maximum among vertices in S. Then $|T_G(v)|$ is either 3 or 4. First of all suppose that $|T_G(v)| = 3$. Then the vertex v traces only 3 vertices. Since $\tau_c(G) = n - 3$ the $n - 3$ vertices of $V \setminus T_G(v)$ must be traced by $n - 4$ vertices. Thus at least one vertex in $S \setminus T_G(v)$, say w must have $|T_G(w)| = 3$. Then we have $T_G(\{w, v\}) = 4,\ 5\ or\ 6$. But $|T_G(\{v, w\})|$ is 4 only if $|T_G(v) \cap T_G(w)| = 2$, which is not possible by Corollary 2.1.15. Therefore $|T_G(\{v, w\})| = 5\ or\ 6$. $|T_G(\{v, w\})| = 5$ only if $T_G(v) \cap T_G(w)$ is a singleton set. Let $x \in T_G(v) \cap T_G(w)$. Then $|T_G(x)| = 5$, a contradiction to step I.

If $|T_G(\{v, w\})| = 6$ then $T_G(v) \cap T_G(w) = \emptyset$ and v and w together traces 6 vertices of G. Therefore $(V \setminus T_G(\{v, w\})) \cup \{v, w\}$ forms a cycle tracking set of G of cardinality $n-4$, a contradiction. Therefore $|T_G(v)| = 4$. In this case v traces 4 vertices. As $\tau_c(G) = n-3$ the vertices in $V \setminus T_G(v)$ must be trace free vertices. The converse part of the theorem is obvious. □

Theorem 2.1.33. *Let G be a graph of order $n \geq 5$. Then $\tau_c(G) = n-4$ if and only if G is one of the following graphs,*

1. *a graph having $n-6$ trace free vertices and two triangles having no common vertices.*

2. *a graph having $n-5$ trace free vertices and a track connected subgraph of order 5.*

3. *a graph having $n-5$ trace free vertices and a bow tie[18] graph.*

Proof. Suppose $\tau_c(G) = n-4$. Let S be any τ_c-set. Then by Theorem 2.1.21 there exist a vertex in S which is not trace free. Let $v \in S$ be such that $|T_G(v)|$ is maximum among vertices in S. Since $\tau_c(G) = n-4$, $3 \leq |T_G(v)| \leq 5$.

Case(i) $|T_G(v)| = 3$.

In this case v traces all vertices in $T_G(v)$. The $n-5$ vertices of $S \setminus \{v\}$ traces all the vertices of $V \setminus T_G(v)$. Then there exists another vertex $u \in S \setminus \{v\}$ of V such that $|T_G(u)| = 3$ and $T_G(u) \cap T_G(v) = \emptyset$ (If $x \in T_G(u) \cap T_G(v)$, then the graph induced by $T_G(u) \cup T_G(v)$ is a bow tie graph with central vertex, say x and x traces all vertices in $T_G(\{u, v\})$, so that $|T_G(v)| \geq 4$ which contradicts the maximality of v.). Therefore $T_G(\{u, v\}) = 6$ and that implies G is a graph having $n-6$ trace free vertices and two triangles having no common vertices.

Case(ii) $|T_G(v)| = 4$.

In this case v traces all the vertices of $T_G(v)$. To trace the remaining $n-4$ vertices there are only $n-5$ vertices in S. Thus there should exist another vertex $u \in S \setminus \{v\}$ such that $1 < |T_G(u)| \leq 4$ and $|T_G(\{u,v\})| = 6$. Then $T_G(u) \cap T_G(v) \neq \emptyset$. Otherwise $|T_G(\{u,v\}| \geq 7$ which would lead to that $\tau_c(G) \leq n-6$, a contradiction. Let $x \in T_G(u) \cap T_G(v)$. Then the graph induced by $T_G(u) \cup T_G(v)$ is a track connected floral graph with central vertex x and x traces all the vertices in $T_G(\{u,v\})$. Then $(V \setminus T_G(\{u,v\})) \cup \{x\}$ forms a cycle tracking set with $n-5$ vertices, which is a contradiction. So $|T_G(v)| \neq 4$ for any maximum tracing vertex in S.

Case(iii) $|T_G(v)| = 5$.

Here v traces 5 vertices of $T_G(v)$. The remaining $n-5$ vertices in $V \setminus T_G(v)$ are traced by the other $n-5$ vertices in S. Since $\tau_c(G) = n-4$, all the vertices of $V \setminus T_G(v)$ must be in S. By Theorem 2.1.19 this is true only if each vertex in $V \setminus T_G(v)$ is trace free. Thus G is a graph having n-5 trace free vertices and a track connected subgraph of order 5, or G is a graph having $n-5$ trace free vertices and a bow tie graph. □

Removal of all cut edges will not make any difference of cycle tracking number.

Proposition 2.1.34. *Let G' be the graph formed by removing all cut edges of a graph G. Then a subset S of $V(G)$ is a cycle tracking set of G if and only if S is a cycle tracking set of G'.*

Theorem 2.1.35. *Let G be a connected graph with at least one cut vertex. Then for every cut vertex c of G one of the following statements holds.*

1 $T_G(c) = \{c\}$, ie; c is a trace free cut vertex.

2 There exist $u, v \in T_G(c)$ *such that* $u \notin T_G(v)$.

3 There exists a vertex u such that the edge cv is a cut edge of G.

Proof. Let G be a connected graph and c be a cut vertex, which is not trace free. Then there exists a cycle C containing c in G. Since c is a cut vertex, $G \setminus \{c\}$ is disconnected. That is there exist two vertices u and v adjacent to c such that every path from u to v contains c. If $u, v \in T_G(c)$ the second condition holds. If one of them, say $v \notin T_G(c)$, then there exists only one path connecting c and u. Hence (3) holds. □

Theorem 2.1.36. *If a non trace free vertex v is in every* τ_c-*set then it is a cut vertex.*

Proof. If possible, let v be not a cut vertex. Then the graph induced by $T_G(v)$ is track connected. Let $u \in T_G(v)$. Then $T_G(v) \subset T_G(u)$. Let S be a τ_c-set of G. Then $(S \setminus \{v\}) \cup \{u\}$ is a τ_c- set of G. Hence the theorem. □

Theorem 2.1.37. *Let G be a connected graph with at least one cut vertex and C(G) be the set of all cut vertices of G. Then C(G) together with pendant vertices form a cycle tracking set. Moreover* $\tau_c(G) \leq |C(G)| + P_G$ *where* P_G *is the total number of pendant vertices of G.*

Proof. Let S be the set of all cut vertices and pendant vertices of G. Then S contains all trace free vertices. Let $v \in V$ such that v is neither a cut vertex nor a pendant vertex. Then v is not a trace free vertex. So $T_G(v)$ contains at least one cut vertex. Otherwise $T_G(v) = V(G)$. Hence G is track connected and has no cut vertex, a contradiction. □

Corollary 2.1.38. *Let G be a connected graph without trace free vertices then $\tau_c(G) \leq |C(G)|$, where $C(G)$ is the set of all cut vertices.*

Theorem 2.1.39 follows from the fact that every hamiltonian graph is track connected.

Theorem 2.1.39. *Let G be a hamiltonian graph. Then $\tau_c(G)$=1.*

2.2 Transitively Tracked Graph

A graph G is track connected, then $\tau_c(G) = 1$. A generalization of this result is possible for a particular type of graphs called transitively tracked graphs. Such graphs are introduced by introducing a relation '$\sim$' on the vertex set V of a graph G as follows: For every pair of vertices $u, v \in V$, we say that u is related to v ($u \sim v$) if $u \in T_G(v)$.

Definition 2.2.1. *For every pair of vertices $u, v \in V$, we say that u is related to v ($u \sim v$) if $u \in T_G(v)$.*

Then '$\sim$' is a reflexive and symmetric relation.

Definition 2.2.2. *A graph G is said to be transitively tracked graph if the relation '$\sim$' is transitive on V(G). ie; for every triple of vertices $u, v, w \in V$, $w \in T_G(u)$ and $u \in T_G(v)$ implies $w \in T_G(v)$.*

Every track connected graph is transitively tracked.

Theorem 2.2.3. *A graph G is a transitively tracked graph if only if $V(G)$ can be partitioned into $V_1, V_2, ..., V_k$ such that the graph $\langle V_i \rangle$ induced by each V_i is a maximal track connected subgraph of G.*

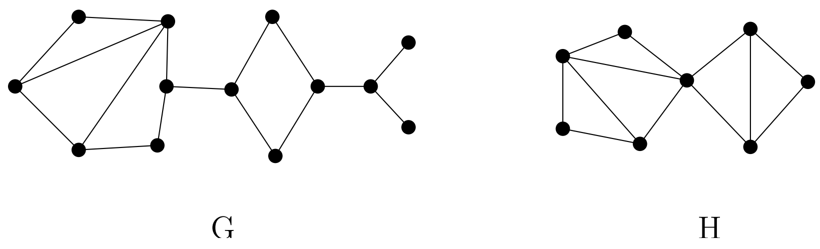


Figure 2.3: G is a transitively tracked graph but H is not.

Proof. Let G be a transitively tracked graph. Then the relation '$\sim$' defined on the vertex set V of G by $u \sim v$ if $v \in T_G(u)$ is an equivalence relation. Hence there exists a partition of V(G) into equivalence classes. We denote the equivalence class containing v by $[v]$.

Let $v \in V$. Suppose $u \in [v]$. Then $u \sim v$ and hence $v \in T_G(u)$. ie; u and v belong to same cycle. That is there exist two internally disjoint paths from u to v. Therefore the graph induced by $[v]$ is track connected.

If the graph induced by $[v]$ is not a maximal track connected subgraph. Then there exists a vertex $w \notin [v]$ such that w and v belong to same cycle. Therefore $w \in T_G(v)$. Hence $w \sim v$, a contradiction. Therefore the graph induced by $[v]$ is a maximal track connected subgraph.

Conversely suppose that $V(G)$ can be partitioned into $V_1, V_2, ..., V_k$ such that the graph induced by each V_i is a maximal track connected subgraph of G. Let $u, v, w \in V(G)$ such that $u \in T_G(v)$ and $w \in T_G(u)$. Then $u, v \in V_i$ for some i and $u, w \in V_j$ for some j. Since V_i and V_j are disjoint if $i \neq j$, we conclude that $i = j$. That is u, v, w belong to the same V_i and hence $w \in T_G(v)$. Therefore G is a transitively tracked graph. $\square$

Proposition 2.2.4. *Let G be a transitively tracked graph. Then the components* *of G obtained by deleting all cut edges of G are precisely the maximal track connected subgraph of G.*

Transitively tracked graphs can be characterized as follows.

Theorem 2.2.5. *Let G be a graph. A necessary and sufficient condition for G to be a transitively tracked graph is that for any two vertices u and v of G, $v \in T_G(u)$ if and only if $T_G(v) = T_G(u)$.*

Proof. Suppose G is a transitively tracked graph and $u, v \in V(G)$ such that $v \in T_G(u)$. Let $w \in T_G(v)$. Then $w \in T_G(u)$, since $v \in T_G(u)$ and G is a transitively tracked graph. Therefore $T_G(v) \subseteq T_G(u)$.
On other hand, let $x \in T_G(u)$. Since $v \in T_G(u)$, we have $u \in T_G(v)$. Therefore $x \in T_G(v)$. Therefore $T_G(u) \subseteq T_G(v)$.
Thus $T_G(v) = T_G(u)$.
Conversely suppose that $v \in T_G(u)$ if and only if $T_G(v) = T_G(u)$ for all $u, v \in G$. Let $u, v, w \in V(G)$ such that $u \in T_G(v)$ and $w \in T_G(u)$. Then $T_G(v) = T_G(u) = T_G(w)$. Hence $w \in T_G(v)$. Therefore G is a transitively tracked graph. □

Corollary 2.2.6. *Let S be a τ_c-set of a transitively tracked graph. Then $\sum_{u \in S} |T_G(u)| = |V|$.*

Lemma 2.2.7. *If the graph induced by $T_G(v)$ is track connected then it is maximally track connected.*

Proof. Suppose the graph induced by $T_G(v)$ is track connected. If possible assume that $T_G(v)$ is not maximally track connected. Then there exists a track connected subgraph H of G such that $T_G(v) \subset V(H)$. Let $u \in V(H) \setminus T_G(v)$.

Since H is track connected there exist two internally disjoint paths form u to v. That is $u \in T_G(v)$, a contradiction. $\square$

Theorem 2.2.8. *Let G be a transitively tracked graph. Then for every vertex v in G, the graph induced by $T_G(v)$ is a maximal track connected subgraph of G.*

Proof. Let G be a transitively tracked graph. Let $v \in V(G)$. If possible assume that $T_G(v)$ not track connected. Then there exist two vertices $u, w \in T_G(v)$ such that $u \notin T_G(w)$. Since G is transitively tracked, $u \in T_G(v)$ and $v \in T_G(w)$ implies that $u \in T_G(w)$, a contradiction. Hence $T_G(v)$ is track connected. By Lemma 2.2.7 the graph induced by $T_G(v)$ is a maximal track connected subgraph of G. $\square$

Corollary 2.2.9. *Let G be a transitively tracked graph and let H be a maximal track connected subgraph of G. Then for every $v \in V(H)$, $T_G(v) = V(H)$.*

Proof. Let G be a transitively tracked graph. Let H be a maximal track connected subgraph of G. Let $v \in V(H)$. Since H is track connected, H is a subgraph of the graph induced by $T_G(v)$. Since G is transitively tracked the graph induced by $T_G(v)$ is a track connected subgraph of G. By maximality of H, $V(H) = T_G(v)$. Since v is arbitrary $T_G(v) = V(H)$ for every vertex in $V(H)$. $\square$

Corollary 2.2.10. *Let G be a transitively tracked graph. Then for every vertex v in G there exists a cycle tracking set containing v.*

Proof. Let G be a transitively tracked graph and let S be any cycle tracking set of G. Let v be a vertex of G. Then there exists a vertex u in S such that

$v \in T_G(u)$. Then by theorem 2.2.5 $T_G(v) = T_G(u)$. Let $w \in V$.

case(i)$w \in T_G(u)$.

Then $w \in T_G(v)$.

case(ii) $w \notin T_G(u)$.

Then there exists a vertex $x \in S \setminus \{u\}$ such that $w \in T_G(x)$.

So $S \setminus \{u\} \cup \{v\}$ forms a cycle tracking set for G containing v. $\square$

Theorem 2.2.11. *For a transitively tracked graph G, $\tau_c(G)$ is the number of maximal track connected subgraph of G.*

Proof. Let G be a transitively tracked graph. Then $V(G)$ can be partitioned into $V_1, V_2, ..., V_k$ such that the graph $\langle V_i \rangle$ induced by each V_i is a maximal track connected subgraph of G. Let S be any τ_c-set of G. Let $v_i \in V_i$. Then there exist a $u_i \in S$ such that $u_i \in T_G(v_i)$. Since G is transitively tracked $T_G(u_i) = T_G(v_i) = V_i$. So for every $i, 1 \leq i \leq k$ there exist $u_i \in S$ such that $T_G(u_i) = V_i$. So $|S| \geq k$. But since $S^* = \{v_i : v_i \in V_i \ 1 \leq i \leq k\}$ is a cycle tracking set of G and S being a τ_c- set we have $|S| \leq k$. Hence $|S| = k$. That is $\tau_c(G) = k$. $\square$

Theorem 2.2.12. *Let G be transitively traced and let $V(G)$ be partitioned into $V_1, V_2, ..., V_k$ such that the graph $\langle V_i \rangle$ induced by each V_i is maximal track connected subgraph of G. If $V_1, V_2, ..., V_k$ have $m_1, m_2, ..., m_k$ vertices respectively, then G has $m_1 m_2 ... m_k$ different τ_c-sets.*

Proof. Consider a set S of vertices which contains exactly one element from each V_i. Then S forms a τ_c- set and it can be chosen in $m_1 m_2 ... m_k$ ways. $\square$

Theorem 2.2.13. *Let G be a transitively tracked graph. Then $pt[v, S^*] = \{v\}$*

for a cycle tracking set S^* containing the vertex v of G if and only if v belongs to every cycle tracking set S and $pt[v, S] = \{v\}$.

Proof. Let G be a transitively tracked graph and $v \in V$. Let S^* be a cycle tracking set containing v with $pt[v, S^*] = \{v\}$. Let S be any cycle tracking set. Then there exists a vertex $u \in S$ such that $v \in T_G(u)$. Since S^* is a cycle tracking set there exists a vertex $w \in S^*$ such that $u \in T_G(w)$. Since G is transitively tracked $v \in T_G(w)$. This is possible if and only if $w = u = v$. Hence $v \in S$ and $pt[v, S] = \{v\}$. □

If G is not transitively tracked then the conclusion of the theorem may not hold. In figure 2.4, $S = \{v_4, v_5, v_9, v_{14}\}$ is a cycle tracking set of G and $pt[v_5, S] = v_5$. But $S^* = \{v_2, v_4, v_9, v_{14}\}$ is another cycle tracking sets of G not containing v_5.

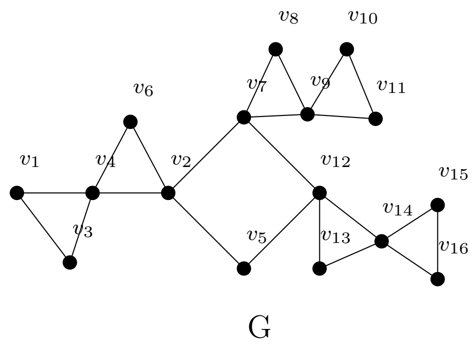


Figure 2.4: Graph G.

By Theorem 2.1.19 and Theorem 2.2.13 we have the Corollary

Corollary 2.2.14. *In a transitively tracked graph G, $pt[v, S] = \{v\}$ for any cycle tracking set S if and only if v is a trace free vertex.*

Theorem 2.2.15. *Let G be a graph. If $t = T$ then G is transitively tracked.*

Proof. Let G be a graph. Suppose $t = T$. Let $v_1 \in T_G(v_2)$. We have $|T_G(v_1)| = |T_G(v_2)| = t$. We claim that $T_G(v_1) = T_G(v_2)$. If not, v_2 is a cut vertex. Therefore there exists a vertex $v_3 \in T_G(v_2)$ such that $v_3 \notin T_G(v_1)$. But since $|T_G(v_2)| = |T_G(v_3)|$ and $T_G(v_2) \neq T_G(v_3)$ there exist a vertex $v_4 \in T_G(v_3)$ such that $v_4 \notin T_G(v_2)$. Then $v_4 \notin T_G(v_1)$ (Otherwise there exist two internally disjoint paths from v_1 to v_4. Since there is a path from v_1 to v_4 containing both v_2 and v_3, and since v_2 and v_3 are cut vertices it is not possible.). Continuing in this manner we get a sequence of vertices $v_1, v_2, ..., v_n, ...$ such that $v_n \in T_G(v_{n-1})$, $v_n \notin T_G(v_i)$ for $1 \leq i \leq n-2$. But it is not possible as G is a finite graph. So $T_G(v_1) = T_G(v_2)$. □

Theorem 2.2.16. *Let G be a graph. Then $t = T = n$ if and only if G is track connected.*

Proof. Let G be a graph. Suppose $t = T = n$. Then $T_G(v) = V$ for all $v \in V$. That is for every $u, v \in G$, u and v belong to same cycle and hence there exist two distinct paths from u to v. So G is track connected.

Conversely suppose that G is track connected. Then there exist two distinct paths from any two vertices u to v. That is u and v belong to a common cycle and hence $T_G(v) = V$ for all $v \in V$. Thus $t = T = n$. □

Definition 2.2.17. *A graph G is well tracked if $\tau_c(G) = T_c(G)$.*

Theorem 2.2.18. *Let G be a graph. If V can be partitioned into maximal track connected subset of G, then G is well tracked.*

Proof. Let $V_1, V_2, ..., V_k$ be a partition of V into maximal track connected sets. Let $v_i \in V_i$, $i = 1, 2, ..., k$. Since V_i is the only maximal track connected set containing v_i, v_i can track all elements of V_i and no element of V_j for $i \neq j$. So $\{v_1, v_2, ..., v_k\}$ forms a minimal cycle tracking set. In fact any such minimal cycle tracking set is obtained like this. Hence $\tau_c(G) = T_c(G)$ and G is well tracked. □

Corollary 2.2.19. *If a graph G is transitively tracked then G is well tracked.*

The converse of Corollary 2.2.19 need not be true as $C_5 \circ K_2$ is well tracked but not transitively tracked.

2.3 Bounds for Cycle Tracking Sets

In this section we describe bounds for the cycle tracking number $\tau_c(G)$ of a graph G of order n. For any graph G of order n, $1 \leq \tau_c(G) \leq$n . Both of these bounds are sharp. The upper bound is attained if and only if G is a forest and the lower bound is attained if and only if G is either track connected graph or track connected floral graph.

Theorem 2.3.1. *If G is a graph without trace free vertices, then* $\tau_c(G) \leq \frac{n}{2}$.

Proof. Let G be a graph without trace free vertices. Then by Theorem 2.1.20 the complement $V \setminus S$ of every minimal cycle tracking set S is a cycle tracking set. So either $|S|$ or $|V \setminus S|$ is less than or equal to $\frac{n}{2}$. □

Conjecture 2.3.2. *For a graph G without trace free vertices,* $\tau_c(G) \leq \frac{n}{3}$.

Theorem 2.3.3. *Let G be a graph. Then* $\frac{n}{T} \leq \tau_c(G) \leq n - T + 1$.

Proof. Let G be a graph. Let S be a τ_c- set of G. First we consider the lower bound. Each vertex can trace at most T vertices. Hence $\tau_c(G) \geq \frac{n}{T}$.
For the upper bound let v be a vertex of maximum trace T. Then v traces all the vertices in $T_G(v)$. Hence $(V \setminus T_G(v)) \cup \{v\}$ is a cycle tracing set of cardinality $n - T + 1$. So $\tau_c(G) \leq n - T + 1$. □

Theorem 2.3.4. *For any connected graph G of order n, $\tau_c(G) = n - T + 1$ if and only if G is obtained by attaching a tree to vertices of a track connected graph or a track connected floral graph of order T.*

Proof. Let G be a graph with $\tau_c(G) = n - T + 1$. Let v be the vertex with $T_G(v) = T$. Then v traces T vertices and the graph induced by the vertices in trace of v is track connected or track connected floral graph. Since $\tau_c(G) = n - T + 1$ the vertices which are not in $T_G(v)$ must be trace free. Hence G is the graph obtained by attaching a tree to vertices of a track connected graph or track connected floral graph. □

Theorem 2.3.5. *If G is a simple graph in which every vertex has degree at least k, where $k \geq 2$ $(ie; \delta(G) \geq k)$, then $\tau_c(G) \leq n - k$.*

Proof. Let G be a simple graph in which every vertex has degree at least k. Then G contains a cycle C of length at least k+1[35]. A vertex in that cycle can trace all vertices in C. So $\tau_c(G) \leq n - k$. □

Theorem 2.3.6. *For a graph G, $\tau_c(G) \leq |V(G)| - c(G) + 1$ where $c(G)$ is the circumference of G.*

Proof. Let C be a cycle with $c(G)$ vertices. Since every vertex v in C traces all vertices in C. Hence the vertex v together with all vertices in $V \setminus V(C)$ form a cycle tracking set. Thus $\tau_c(G) \leq |V(G)| - c(G) + 1$. $\square$

Theorem 2.3.7. *For a graph G, $\tau_c(G) = |V(G)| - c(G) + 1$ if and only if G contains only one non trivial maximal track connected component which is hamiltonian of order $c(G)$.*

Proof. Suppose $\tau_c(G) = |V(G)| - c(G) + 1$. Since circumference of G is $c(G)$ there exist a cycle C of length $c(G)$ in G. Then a vertex v in C traces all vertices in C. To trace the remaining vertices in $V \setminus T_G(v)$ we have $|V(G)| - c(G)$ vertices. It is possible only if $|T_G(v)| = |c(G)|$ and the remaining vertices are trace free vertices. Therefore G contains only one non trivial maximal track connected component. The converse is trivial. $\square$

It is obvious that the cycle tracking number $\tau_c(G)$ of a graph G is always greater than or equal to the number of components of G.

If G is a graph with exactly p cut edges then the graph G^* obtained by removing these p cut edges from G has exactly p components. Thus $\tau_c(G) = \tau_c(G^*) \geq p + 1$. We can summarize these results as follows.

Theorem 2.3.8. *For a graph G with p cut edges $\tau_c(G) \geq p + 1$.*

2.4 Cycle Tracking Number of Some Graphs

In this section we derive cycle tracking number of some graphs. Cycle tracking number of some more graphs are derived in section 6.2.

Theorem 2.4.1. *For a bipartite graph $K_{m,n}$, $\tau_c(K_{m,n}) = 1$ if and only if $m > 1$ and $n > 1$.*

Proof. Let $K_{m,n}$ be any complete bipartite graph. Then $\tau_c(K_{m,n}) = 1$ if and only if $K_{m,n}$ is track connected. That is if and only if $m > 1$ and $n > 1$. □

Theorem 2.4.2. *If H is a connected graph of order $n \geq 2$ then $\tau_c(G \circ H) = |V(G)|$.*

Proof. Let H be a connected graph and G be any graph. Then $G \circ H$ is a graph formed from one copy of G and $|V(G)|$ copies of H, where i^{th} vertex v_i of G is joined to every vertex in the i^{th} copy of H. We claim that the subgraph (say H_i^*)induced by the i^{th} vertex (say v_i) of G together with the i^{th} copy of H is track connected. Let $x, y \in H_i^*$. Since H is connected there exist a path P from x to y. Then P together with the edges yv_i and v_ix in $G \circ H$ form a cycle containing both x and y. Therefore we have $H_i^*, i = 1, 2, ..., n$ are track connected. Hence a vertex $v_i \in H_i$ traces all vertices of H_i. So $\tau_c(G \circ H) \leq |V(G)|$. Since for $i \neq j$, the i^{th} vertex of G cannot trace the vertices of j^{th} copy of H, $\tau_c(G \circ H) \geq |V(G)|$. Thus $\tau_c(G \circ H) = |V(G)|$. □

Corollary 2.4.3. *If G_m denotes a connected graph with $m(m > 1)$ vertices. Then $\tau_c(K_1 \circ G_m) = 1$.*

Theorem 2.4.4. *If G and H are connected graphs, then $\tau_c(G \times H) = 1$.*

Proof. Let G and H are connected graphs. Then $G \times H$ is track connected and hence $\tau_c(G \times H) = 1$. □

Theorem 2.4.5. *Let G_1, G_2 be two graphs with $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$ then $\tau_c(G_1 \vee G_2) = 1$.*

Proof. To prove the theorem we show that $V(G_1 \vee G_2)$ is track connected. Let $x, y \in V(G_1 \vee G_2)$.

Case 1: $x, y \in V(G_1)$ or $x, y \in V(G_2)$.

First of all suppose that $x, y \in V(G_1)$. Let $u, v \in V(G_2)$. Then there exist two internally disjoint paths xuy and xvy from x to y.

Case 2: $x \in V(G_1)$ and $y \in V(G_2)$.

Let $u(\neq x)$ be a vertex of G_1 and $v(\neq y)$ be a vertex in G_2. Then there exist two internally disjoint paths namely xy and $xvuy$ from x to y.

So $G_1 \vee G_2$ is track connected and hence $\tau_c(G_1 \vee G_2) = 1$. □

Chapter 3

Changing and Unchanging Cycle Tracking

It is often of interest to know whether the value of cycle tracking number of a graph is effected when a change is made in a graph, for example vertex or edge removal, edge addition, edge subdivision, edge contraction, etc. This chapter includes some results in this direction.

3.1 Changing and Unchanging Cycle Tracking on Vertex Removal

Removing a vertex from a graph can cause its cycle tracking number to increase, to decrease, or to remain the same. This section examines the effect on

$\tau_c(G)$ when G is modified by deleting a vertex .

The vertices of G can be partitioned into three sets,

$V^0_{\tau_c}, V^-_{\tau_c}$ and $V^+_{\tau_c}$, where

$V^0_{\tau_c} = \{v \in V : \tau_c(G - v) = \tau_c(G)\}$,

$V^-_{\tau_c} = \{v \in V : \tau_c(G - v) < \tau_c(G)\}$ and

$V^+_{\tau_c} = \{v \in V : \tau_c(G - v) > \tau_c(G)\}$.

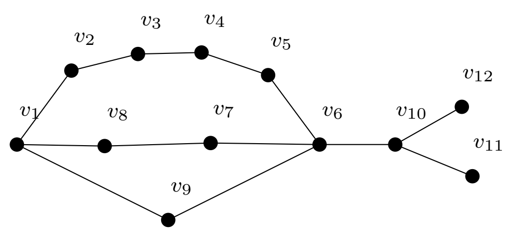


Figure 3.1: Graph G.

Definition 3.1.1. *A graph G is τ_c- changing if $\tau_c(G-v) \neq \tau_c(G)$ for any vertex $v \in V(G)$, while a graph G is τ_c- stable if $\tau_c(G - v) = \tau_c(G)$ for every vertex $v \in V(G)$.*

Thus a graph G is τ_c-changing if the removal of any vertex from G either increases or decreases the cycle tracking number, that is, $V(G) = V^-_{\tau_c} \cup V^+_{\tau_c}$. A graph G is τ_c-stable if $V(G) = V^0_{\tau_c}$.

For the graph G in figure 3.1

- $V^0_{\tau_c} = \{v_9\}$,
- $V^-_{\tau_c} = \{v_{10}, v_{11}, v_{12}\}$ and

- $V_{\tau_c}^{+} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.

If a vertex v of a graph G belongs to $V_{\tau_c}^{-}$ then no τ_c-set S^* of $G - v$ contain a vertex of $T_G(v)$. Otherwise S^* forms a cycle tracking set of G. Thus, $pt[v, S] = \{v\}$. On the other hand if $pt[v, S] = \{v\}$ then v necessarily belongs to $V_{\tau_c}^{-}$. We summarize these results as follows.

Theorem 3.1.2. *Let G be a graph. A vertex $v \in V_{\tau_c}^{-}$ if and only if $pt[v, S] = \{v\}$ for some τ_c- set S of G containing v.*

Proposition 3.1.3. *Removal of a vertex from any graph G can*

1. *decrease the cycle tracking number by at most one and*

2. *increase the cycle tracking number by more than one.*

Proof. Let G be a graph. Let $v \in V$. If possible assume that $\tau_c(G - v) \leq \tau_c(G) - 2$. Let $S = \{v_1, v_2, ..., v_{\tau_c(G)-2}\}$ be a τ_c- set of $G - v$. Then $S^* = \{v_1, v_2, ..., v_{\tau_c(G)-2}, v\}$ form a cycle tracking set for G of cardinality $\tau_c(G) - 1$, a contradiction. So $\tau_c(G - v) \geq \tau_c(G) - 1$. □

For the graph G in figure 3.1 $\tau_c(G) = 4$ and $\tau_c(G - v_3) = 7$.

Corollary 3.1.4. *For $v \in V_{\tau_c}^{-}$, $\tau_c(G - v) = \tau_c(G) - 1$.*

Since for every trace free vertex v and cycle tracking set S $pt[v, S] = \{v\}$, we have:

Corollary 3.1.5. *Every trace free vertex of a graph belongs to $V_{\tau_c}^{-}$.*

Corollary 3.1.6. *Every vertex of a forest belongs to $V_{\tau_c}^{-}$.*

Corollary 3.1.7. *Removal of a trace free vertex in a graph G will decrease cycle tracking number by 1.*

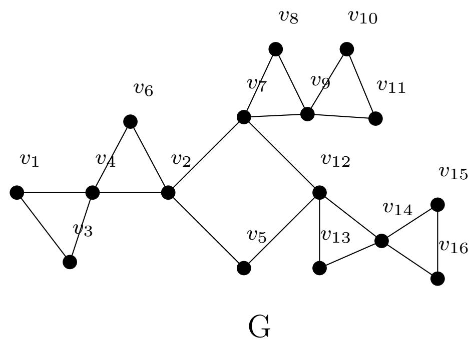


Figure 3.2: Graph G.

Remark 3.1.8. *Removal of a non trace free vertex in a graph G may or may not decrease cycle tracking number.*
For the graph G in figure 3.1, $\tau_c(G - v_9) = \tau_c(G) = 4$, and for the graph G in figure 3.2, $\tau_c(G) = 4$ and $\tau_c(G - v_5) = 3$.

Theorem 3.1.9. *For a transitively tracked graph G, a vertex v belongs to $V^-_{\tau_c}$ if and only if v is trace free vertex.*

Proof. Let G be a transitively tracked graph. suppose $v \in V^-_{\tau_c}$. Then by Theorem 3.1.2, $pt[v, S] = \{v\}$ for some τ_c-set S containing v. By Theorem 2.2.13 and Corollary 2.2.14 v is a trace free vertex.
Converse follows from Corollary 3.1.4 and 3.1.5. □

Corollary 3.1.10. *Every vertex of a transitively tracked graph G is in $V^-_{\tau_c}$ if and only if it is a forest.*

Corollary 3.1.11. *Let G be a transitively tracked graph without trace free vertices. Then for every vertex v in G, $\tau_c(G - v) \geq \tau_c(G)$.*

Theorems 3.1.12 and 3.1.13 follow from the definitions of $V^+_{\tau_c}$ and $V^0_{\tau_c}$ respectively.

Theorem 3.1.12. *A vertex $v \in V^+_{\tau_c}$ if and only if no vertex u in $T_G(v) \setminus \{v\}$ can trace the graph induced by $T_G(v) \setminus \{v\}$.*

Theorem 3.1.13. *Let G be a graph and $v \in V$. If the graph induced by $T_G(v) \setminus \{v\}$ is track connected or track connected floral graph then $v \in V^0_{\tau_c}$.*

The converse of Theorem 3.1.13 is not true. For example, $\tau_c(G) = \tau_c(G-v) = 2$ for the graph G in figure 3.3. Here $v \in V^0_{\tau_c}$ but the graph induced by $T_G(v) \setminus \{v\}$ is neither a track connected graph nor a track connected floral graph.

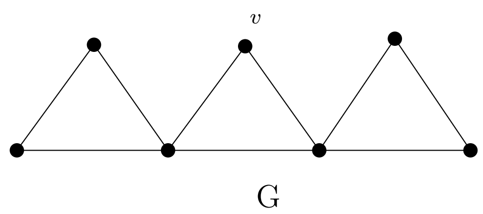


Figure 3.3: graph G.

Theorem 3.1.14. *Let G be a transitively tracked graph and let H be the maximal track connected subgraph of G containing v then $\tau_c(G-v) = \tau_c(G) - 1 + \tau_c(H \setminus v)$.*

Proof. Let G be a transitively tracked graph. Then V(G) can be partitioned into $V_1, V_2, ..., V_k$ such that each $\langle V_i \rangle$ is maximal track connected subgraph of G. Then $H = \langle V_i \rangle$ for some i. Let $v \in V_i$. Then every τ_c−set of $G - v$ contains exactly one vertex from each $V_j, j \neq i$ and $\tau_c(H \setminus \{v\})$ vertices from $V_i \setminus \{v\}$. So $\tau_c(G - v) = \tau_c(G) - 1 + \tau_c(V_i \setminus v)$. □

Theorem 3.1.15. *Let G be a transitively tracked graph and let v be a non trace free vertex of G. Let D be maximal track connected component of G containing v. Then $v \in V^0$ if and only if $D - v$ is track connected or track connected floral graph.*

Proof. Suppose $v \in V^0_{\tau_c}$. Since G is transitively tracked each τ_c-set of G has exactly one vertex from each maximal track connected component of G. So each τ_c-set of $G - v$ has exactly one vertex from each maximal track connected component of G except from D. Since $\tau_c(G) = \tau_c(G - v)$ there must exist a vertex w in any τ_c-set of $G - v$ such that $T_G(w) = D - v$. Which is possible if and only if $D - v$ is track connected or track connected floral graph. The converse follows from Theorem 3.1.13. □

By Theorems 3.1.9 and 3.1.15 it is clear that a non trace free vertex v of a transitively tracked graph G is in $V^+_{\tau_c}$ if and only if $D - v$ is neither a track connected graph nor a track connected floral graph, where D is the maximal track connected component of G.

Remark 3.1.16. *For any graph G, $0 \leq |V^*_{\tau_c}| \leq n$, where $* = 0$ or $-$ or $+$. There are graphs for which $|V^*_{\tau_c}| = 0$ and $|V^*_{\tau_c}| = n$. For example; for any cycle C_n with $n \geq 3$, $|V^0_{\tau_c}| = 0$, and $|V^+_{\tau_c}| = n$. For complete graph K_n with $n \geq 4$, $|V^0_{\tau_c}| = n$, $|V^-_{\tau_c}| = 0$ and $|V^+_{\tau_c}| = 0$. For path P_n with $n \geq 2$, $|V^-_{\tau_c}| = n$.*

Theorem 3.1.17. *For a graph G, $V = V^-_{\tau_c}$ if and only if G is a forest.*

Proof. Let G be a graph. Suppose $V_{\tau_c}^{-} = V$. Let $v \in V$. Then $v \in V_{\tau_c}^{-}$. Then by Theorem 3.1.2 there exists a τ_c- set S_v containing v such that $pt[v, S_v] = \{v\}$. Let S_v^* be a τ_c- set of $G-v$. Then $S = S_v^* \cup \{v\}$ is a τ_c-set of G. If S_v^* contains a vertex of $T_G(v)$, then S_v^* is a cycle tracking set of G, contradicting the assumption that $v \in V_{\tau_c}^{-}$. So S_v^* does not contain any vertex of $T_G(v)$. So for every vertex $u \in T_G(v)$ there exists a vertex $w_u \in S_v^*$ such that $u \in T_G(w_u)$ and $v \notin T_G(w_u)$. Hence every vertex u in $T_G(v) \setminus \{v\}$ is cut vertex of G and $T_G(u)$ is not track connected. Since v is arbitrary, for every vertex v in G, $u \in T_G(v) \setminus \{v\}$ implies u is a cut vertex of G and $T_G(u)$ is not track connected.

If $T_G(v) = \{v\}$. Then v is a trace free vertex. So suppose that $T_G(v)$ contains another vertex u_1. Then u_1 is cut vertex of G and $T_G(u_1)$ is not track connected. Therefore there exists a vertex $u_2 \in T_G(u_1)$ such that $u_2 \notin T_G(v)$. Then u_2 is cut vertex of G and $T_G(u_2)$ is not track connected. Therefore there exists a vertex $u_3 \in T_G(u_2)$ such that $u_3 \notin T_G(u_1)$. Then $u_3 \notin T_G(v)$. Continuing in this manner we get a sequence of vertices $v = u_0, u_1, u_2, ..., u_n, ...$ such that $u_n \in T_G(u_{n-1})$ and $u_n \notin T_G(u_i)$ for $0 \leq i \leq n-2$. Which is absurd as G is a finite graph.
So every vertex in G is a trace free. Hence G is a forest.
The converse follows from Corollary 3.1.6. □

3.2 Changing and Unchanging Cycle Tracking on Edge Removal

As in the case of vertex removal from a graph G edge removal also effect the cycle tracking number of G. This section deals with the effect on $\tau_c(G)$ when G is modified by deleting an edge.

Theorem 3.2.1. *For a graph G and an edge e in G, $\tau_c(G) \leq \tau_c(G-e)$.*

Proof. The number of cycles in G is always less than or equal to the number of cycles in $G-e$. So more vertices are needed to trace $V(G-e)$. □

Definition 3.2.2. *An edge e is said to be a weak edge if $\tau_c(G-e) = \tau_c(G)$. Otherwise it is called a strong edge.*

Every edge in any cycle graph C_n, $n \geq 3$ is a strong edge. Every cut edge is a weak edge, but a weak edge need not be a cut edge, for complete graph K_n with $n \geq 4$ every edge is a weak edge.

Though the process of shorting of an edge in a triangle free graph decreases the number of vertices it will not affect the number of its cycles. Therefore we have:

Theorem 3.2.3. *If G^* is a graph obtained by shorting of an edge of G, then $\tau_c(G^*) = \tau_c(G)$, provided G is triangle free.*

A similar result, we get in the case of subdivision of edges.

Theorem 3.2.4. *Let G be a graph with no cut edge and G' be a subdivision of G. Then $\tau_c(G) = \tau_c(G')$.*

Proof. Let G' be the graph obtained by replacing the edge $e_i = a_ib_i$ of G by the path $(a_i, x_{i1}, ..., x_{ik}, b_i)$ where $x_{i1}, ..., x_{ik}$ are new vertices. Suppose S be a τ_c- set of G'. Then for every vertex $v \in V(G')$ there exists a vertex $u \in S$ such that $v \in T_{G'}(u)$. Clearly S contains at most one x_{ij}. Let S^* be S if no $x_{ij} \in S$,

$(S \setminus \{x_{ij}\}) \cup \{a_i\}$ otherwise. Since G has no cut edge, $T_G(x_{ij}) = T_G(a_i)$ and hence S^* forms a cycle tracking set of G. Therefore $\tau_c(G) \leq \tau_c(G')$.

The reverse inequality, $\tau_c(G') \leq \tau_c(G)$ is obvious because no edge of G is a cut edge. Hence the theorem. $\square$

Chapter 4

Cycle Track Completion Number

This chapter considers the problem of determining two distinct paths between every pair of vertices u, v of a graph G. Recall that a graph G is *track connected* if every pair of its vertices is connected by two paths. When no such paths can be found for a pair u, v of vertices in G, it may be appropriate to consider the introduction of new edges to enable the construction of such paths. Each new edge added will have a cost overhead so that the number of new edges should kept to be minimum. In this chapter, we consider the minimum number of edges required to be added to a graph to make it track connected. This problem is the problem of finding the cycle track completion number $TC(G)$ of the given graph.

4.1 Cycle Track Completion Number

Definition 4.1.1. *The cycle track completion number of a graph G, denoted by $TC(G)$, is the minimum number of edges that need to be added to G to make it* *track connected.*

If G is a graph with two vertices then $TC(G)$ does not exist. So graphs which are distinct from K_2 and $\overline{K}_2$ are only considered in this chapter.

The cycle track completion problem consists of determining $TC(G)$ of a given graph G and explicitly constructing a track connected graph through the addition of $TC(G)$ new edges. A graph G itself is track connected if and only if $TC(G) = 0$.

Theorem 4.1.2. *For a graph G of order $n(n > 2)$, $TC(G) = n$ if and only if G is totally disconnected.*

Proof. Let G be a graph of order $n(n > 2)$. If G is not totally disconnected then there exists an edge $uv \in E(G)$. Then uv together with a suitable choice of $n-1$ edges form a cycle in G. It follows that $TC(G) \leq n-1$.

Conversely if G is totally disconnected then at least n edges to be added to G to make it into a track connected graph. That is $TC(G) = n$ if and only if $|E(G)| = 0$. □

Theorem 4.1.3. *Let G_1 and G_2 be two disjoint subgraphs of a graph G such that $G = G_1 \cup G_2$. Then $TC(G) \leq TC(G_1) + TC(G_2) + 2$, provided $|V(G_1)|$ or $|V(G_2)|$ is greater than or equal to two.*

Proof. Let G be a graph and G_1, G_2 be two disjoint subgraphs of G such that $G = G_1 \cup G_2$ and $|V(G_1)| \geq 2$ or $|V(G_2)| \geq 2$. Then addition of $TC(G_1)$ edges makes G_1 track connected and $TC(G_2)$ edges makes G_2 track connected. So addition of $TC(G_1)$ edges to G_1, $TC(G_2)$ edges to G_2 and two more distinct edges which join vertices of G_1 and G_2 in a suitable way to graph G will make the graph into a track connected graph. So $TC(G) \leq TC(G_1) + TC(G_2) + 2$. $\square$

The inequality is sharp if G_1 and G_2 are track connected. Strict inequality holds for union of paths $P_n \cup P_m$, where $n, m \geq 2$.

Lemma 4.1.4. *Let G be a graph. Let u, v be non adjacent vertices in G such that $deg(u) + deg(v) \geq n$. Then $u \in T_G(v)$.*

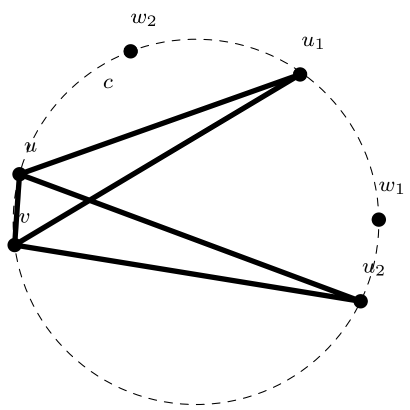
Proof. Let G be a graph. Let u, v be non adjacent vertices in G such that $deg(u) + deg(v) \geq n$. That is $|N_G(u)| + |N_G(v)| \geq n$, where $N_G(u)$ and $N_G(v)$ denote the set of all neighbors of u and v in G respectively. As $N_G(u) \cup N_G(v) \subset V \setminus \{u, v\}$, the vertices u and v have at least two common neighbors say p and q. Thus upv and uqv form two distinct $u - v$ paths in G. Hence $u \in T_G(v)$. $\square$

Theorem 4.1.5. *A graph G is track connected if and only if its closure c(G) is track connected.*

Proof. Let G be a graph. It is enough to prove the sufficient part. For, assume that $c(G)$ is track connected. If possible, let G be not track connected. Choose a pair of non adjacent vertices $u, v \in V$ with $deg(u) + deg(v) \geq n$. Let us suppose that the addition of the new edge uv to G will result in the track connected graph $G_1 \subset c(G)$. So there exist vertices w_1, w_2 in G such that every cycle containing w_1 and w_2 contains the new edge uv in G_1. Let $C : ux_1x_2...x_kvu$ be a cycle containing w_1 and w_2 in G_1. As in Lemma 4.1.4 there exist two vertices u_1 and u_2 in V such that both of them are adjacent to both u and v in G.

Case(i) Either u_1 or u_2 not belong to C.

First of all suppose that $u_1 \notin V(C)$. Then $C' = ux_1x_2...x_kvv_1u$ is a cycle in G



containing both w_1 and w_2, a contradiction. A similar contradiction arises when $u_2 \notin V(C)$.

Case(ii) $u_1, u_2 \in V(C)$.

Let $u_1 = x_i, u_2 = x_j, w_1 = x_m, w_2 = x_n$. Without loss of generality assume that $i < j$ and $m < n$. If $i < m$ then $C' = vx_ix_{i+1}...x_kv$ is a cycle in G containing both w_1 and w_2, a contradiction. If $n < j$ then $C'' = ux_1x_2...x_ju$ is a cycle in G containing both w_1 and w_2, a contradiction. If $m < i < j < n$ then $C' = ux_1x_2...x_ivx_kx_{k-1}...x_ju$ is a cycle in G containing both w_1 and w_2, a contradiction. Hence the theorem. □

Corollary 4.1.6. *Let G be a graph with $n \geq 3$ vertices. If $c(G)$ is complete, then G is track connected.*

4.2 Bounds for Cycle Track Completion Number of a Graph

For a graph G on n vertices, the addition of n edges which together form a cycle on n vertices will make G into a track connected graph. So for any graph G of order n, $0 \leq TC(G) \leq n$. Both of these bounds are sharp. The upper bound is attained if and only if G is a track connected graph and the lower bound is attained if and only if G is totally disconnected. In this section we derive some bounds for cycle track completion number of a graph in terms of various graph parameters.

Proposition 4.2.1. *If G is a graph with n vertices then $0 \leq TC(G) \leq \binom{n}{2} - |E(G)|$.*

Proposition 4.2.2. *A track connected graph of order n has at least n edges.*

Theorem 4.2.3. *Let G be a graph of order $n \geq 3$. Then $TC(G) = n - 1$ if and only if $|E(G)| = 1$.*

Proof. Let G be any graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. Suppose $|E(G)| = 1$. Without loss of generality assume that v_1v_2 is the only edge in G. Then the addition of $n-1$ edges, v_2v_3, $v_3v_4, ..., v_{n-1}v_n, v_nv_1$ makes G track connected. As the minimum number of edges in any track connected graph is n, $TC(G) = n - 1$.

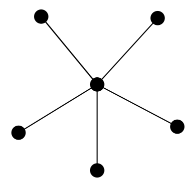
For the converse, suppose that $|E(G)| \neq 1$.
If $|E(G)| = 0$, then $TC(G) = n$ by Theorem 4.1.2.
If $|E(G)| > 1$ then there exist two distinct edges say e_1 and e_2 in G.

First of all assume that e_1 and e_2 are adjacent. By renaming the vertices, if necessary, we can suppose that $e_1 = v_1v_2$ and $e_2 = v_2v_3$. Then the addition of $n-2$ edges v_3v_4, $v_4v_5, ..., v_{n-1}v_n, v_nv_1$ makes G track connected.
Now suppose that e_1 and e_2 are not adjacent. By renaming the vertices, if necessary, we can suppose that $e_1 = v_1v_2$ and $e_2 = v_3v_4$. Then the addition of $n-2$ edges $v_2v_3, v_4v_5, v_5v_6, ..., v_{n-1}v_n, v_nv_1$ makes G track connected.
Hence $TC(G) \leq n-2$. □

In the proof of Theorem 4.2.3 we have shown that $TC(G) \leq n-2$ if $|E(G)| \geq 2$. As the minimum number of edges in a track connected graph is n, we have in fact shown that $TC(G)$ is $n-2$ if $|E(G)| = 2$.

Theorem 4.2.4. *Let G be a graph of order $n \geq 3$. If $|E(G)| = 2$ then, $TC(G) = n-2$.*

The converse of Theorem 4.2.4 is not true.



G

Figure 4.1: In Graph G, $|E(G)| = 6$ and $TC(G) = 4$.

Corollary 4.2.5. *For a graph G of order $n \geq 3$, $TC(G) \leq n-2$ if and only if $|E(G)| \geq 2$.*

Lemma 4.2.6. *Let T be a tree. Then for every pair of distinct vertices w_1, w_2 in T there exists a pendant vertex $v \in V(T)$ such that the path from w_2 to v contains w_1.*

Proof. We prove the result by induction on the order of T. If the order of T is 1 or 2, then there is nothing to prove. Let us suppose that the result is true for any tree of order $k-1$, where $k \geq 3$. Let T be a tree of order k. Let w_1 and w_2 be two distinct vertices of T. If w_1 or w_2 is a pendant vertex then there is nothing to prove. So suppose that w_1 is not a pendant vertex of T. Let v be a pendant vertex of T. Such a vertex exists as every tree contains at least two pendant vertices. Consider $T-v$, which is a tree of order $k-1$. Now by induction hypothesis there exists a pendant vertex v' in $T-v$ such that the path P from w_2 to v' contains w_1. If v' is also a pendant vertex of T then there is nothing to prove. Otherwise v' is a support vertex of v and $P+v'v$ is a path in T from w_2 to v which contains the vertex w_1. Hence the lemma. □

Lemma 4.2.7. *Let T be a tree. Then for every pair of vertices w_1, w_2 in T there exists two pendant vertices $v_1, v_2 \in V(T)$ such that the path from v_1 to v_2 contain both w_1 and w_2.*

Proof. Let T be a tree and $w_1, w_2 \in V(T)$. By Lemma 4.2.6 there exist a pendant vertex $v_1 \in V(T)$ such that the path P_1 from w_2 to v_1 contains w_1 and a pendant vertex $v_2 \in V(T)$ such that the path P_2 from w_1 to v_2 contains w_2. Thus both P_1 and P_2 contain subpaths joining w_1 and w_2. As any two vertices of a tree are connected by exactly one path, the subpath of P_1 joining w_1 to w_2 is the same as the subpath of P_2 joining w_1 and w_2. Hence $P_1 \cup P_2$ is a path from v_1 to v_2 containing both w_1 and w_2. □

Theorem 4.2.8. *Let T be a tree with p pendant vertices. Then* $\lceil \frac{p}{2} \rceil \leq TC(T) \leq p-1$.

Proof. Let T be a tree with p pendant vertices. Let $V' = \{v_1, v_2, ..., v_p\}$ be the set of all pendant vertices of T. Let $G = T + \{v_1v_2, v_2v_3, ..., v_{p-1}v_p\}$. We claim that G is a track connected graph.

Let $w_1, w_2 \in V(G)$.

Case(i) $w_1, w_2 \in V'$.

Let $w_1 = v_i$ and $w_2 = v_j$ with $i < j$. Then there exist two distinct paths, one from v_i to v_j in T (since T is connected) and another one is the path $v_iv_{i+1}v_{i+2}...v_{j-1}v_j$ in G.

Case(ii) $w_1 \notin V'$ and $w_2 \in V'$.

Let $w_2 = v_i$. Since T is a tree, by Lemma 4.2.6 there exists a pendant vertex $v_j \in V'$ such that the path P_1 from w_2 to v_j contains the vertex w_1. Let P' be the path joining w_1 to v_j and let $P_2 = v_iv_{i+1}v_{i+2}...v_{j-1}v_j$ in G. By renaming the vertices, if necessary, we can suppose that $i < j$. So there exist two distinct paths, one from w_1 to w_2 in T and another path from w_1 to w_2 formed by joining P' and P_2.

Case(iii) $w_1, w_2 \notin V'$.

Then, by Lemma 4.1.4 there exist two vertices $v_i, v_j \in V'$, $i < j$ such that the path in T from v_i to v_j contain both w_1 and w_2. Without loss of generality assume that $d(v_i, w_1) < d(v_i, w_2)$ in T. Let P_1 be the path joining w_1 to v_i in T and P_2 be the path joining v_j to w_2 in T. Also let P_3 be the path $v_iv_{i+1}v_{i+2}...v_{j-1}v_j$ in G. So there exist two distinct paths, one from w_1 to w_2 in T and another path from w_1 to w_2 formed by joining the three paths P_1, P_3 and P_2. So G is track connected. Hence $TC(T) \leq p - 1$.

If a graph G is track connected then, the degree of each vertex is at least two. So to make degree of each pendant vertex to be two we have to add at least$\lceil \frac{p}{2} \rceil$ edges between pendant vertices, since there are p pendant vertices. So $\lceil \frac{p}{2} \rceil \leq TC(T)$. $\square$

Remark 4.2.9. *The inequalities in Theorem* 4.2.8 *may or may not be sharp. For example,*

1. *the right hand side inequality is sharp for* $P_n, \ n > 2$ *and* $K_{1,t}, \ t > 1$.

2. *the left hand side inequality is sharp for the graph G in figure* 4.2. *Here, addition of the four edges* $v_1v_6, v_4v_8, v_8v_{12}, v_{10}v_{14}$ *makes G track connected. So* $TC(G) = 4$.

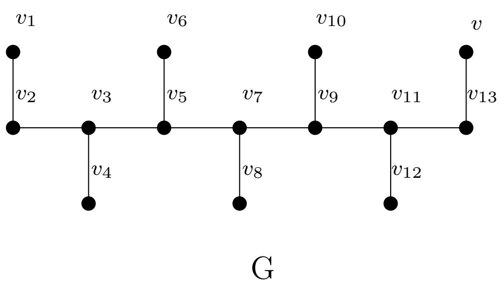


Figure 4.2: $TC(G) = 4$.

3. *the inequalities are strict for the graph H in figure* 4.3. *Here addition of the seven edges* $v_1v_8, v_7v_{11}, v_8v_9, v_9v_6, v_6v_5, v_5v_4, v_4v_{10}$ *makes H track connected. So* $TC(H) = 7$.

Theorem 4.2.10. *For a tree T,* $TC(T) \leq n - L(P) + 1$ *where* $L(P)$ *is the length of a longest path in T.*

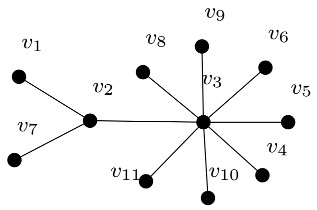


Figure 4.3: $TC(H) = 7$.

Proof. Let P be one of the longest paths in T. Then we can construct a cycle containing P and the other $n - L(P)$ vertices which are not in P choosing $n - L(P) + 1$ edges in a suitable way. Therefore $TC(T) \leq n - L(P) + 1$. $\square$

Theorem 4.2.11. *Let G be a graph with a pendant vertex u. If its support vertex v has degree 2 then* $TC(G) = TC(G - u)$.

Proof. Let G be a graph with a pendant vertex u. Let its support vertex v has degree 2. Suppose TC(G) = p. Let $A = \{e_1, e_2, ..., e_p\}$ be a set of edges in $\overline{G}$ such that $G' = G + A$ is track connected. Let $B = \{e'_1, e'_2, ..., e'_p\}$, where

$$e'_i = \begin{cases} e_i & if\ e_i\ not\ incident\ with\ u, \\ w_i v & if\ e_i = w_i u\ for\ some\ w_i \in V. \end{cases}$$

Since u is a pendant vertex in G, B is nonempty. Let $G'' = G - u + B$ and let $v_1, v_2 \in V(G'')$. Then $v_1, v_2 \in V(G')$. Hence there exist a cycle C containing both v_1 and $v_2 \in G'$.

Case(i) $u \notin V(C)$.

Then C lies in G''.

Case(ii) $u \in V(C)$.

Let $C = uu_1u_2...u_ku$. If neither u_1 nor u_k is v, then $C' = vu_1u_2...u_kv$ is a cycle containing both v_1 and v_2 in G''. If $u_1 = v$ then $C' = vu_2u_3...u_kv$ is a cycle containing both v_1 and v_2 in G''. If $u_k = v$ then $C' = vu_1u_2...u_{k-1}v$ is a cycle containing both v_1 and v_2 in G''. Therefore G'' is track connected. Thus an addition of p edges makes $G - u$ track connected. So $TC(G-u) \leq TC(G)$.

For the reverse inequality suppose $TC(G-u) = q$. Let $D = \{e_1, e_2, ..., e_q\}$ be a set of q edges in $\overline{G-u}$ such that $H' = G - u + D$ is track connected. Let $E' = \{e'_1, e'_2, ..., e'_q\}$, where

$$e'_i = \begin{cases} e_i & if\ e_i\ not\ incident\ with\ v, \\ w_iu & if\ e_i = w_iv\ for\ some\ w_i \in V. \end{cases}$$

Since v is a pendant vertex in $G-u$, E' is nonempty. Let $H'' = G + E'$ and let $v_1, v_2 \in V(G)$.

Case(i) Either $v_1 = u$ or $v_2 = u$.

Let $v_1 = u$. Since H' is track connected there exists a cycle $C = vu_1u_2...u_kv$ containing both v and v_2 in H'. Since v is a pendant vertex in $G-u$ either vu_1 or u_kv belongs to D. If both vu_1 and u_kv are in D then the cycle $C' = uu_1u_2...u_ku$ is in H'' and contains both v_1 and v_2. If $vu_1 \notin D$ and $vu_k \in D$ then $C' = vu_1u_2...u_kuv$ is a cycle in H'' which contain both v_1 and v_2. If $vu_k \notin D$ and $vu_1 \in D$ then $C' = vuu_1u_2...u_kv$ is a cycle in H'' which contain both v_1 and v_2.

Case(ii) Neither $v_1 = u$ nor $v_2 = u$.

Then $v_1, v_2 \in V(H')$. Hence there exist a cycle C in H' which contain both v_1 and v_2.

If $v \notin V(C)$, then C lies in H''. Suppose $v \in V(C)$. Let $C = vu_1u_2...u_kv$. If, neither vu_1 nor u_kv is in $E(G)$, then $C' = uu_1u_2...u_ku$ is a cycle containing both v_1 and v_2 in H''. If $vu_1 \in E(G)$ then $u_kv \notin E(G)$ as v is a pendant vertex in $G-u$. Then $C' = uvu_1u_2u_3...u_ku$ is a cycle containing both v_1 and v_2 in H''. If $u_kv \in E(G)$ then $vu_1 \notin E(G)$ as v is a pendant vertex in $G-u$. Then $C' = uu_1u_2u_3...u_kvu$ is a cycle containing both v_1 and v_2 in H''. Therefore H'' is track connected. Thus an addition of q edges makes G track connected. So $TC(G) \leq TC(G-u)$. Hence the theorem. $\square$

Theorem 4.2.12. *Let G be a connected graph of order $n > 3$ and u be an isolated vertex of G. Then*

$$TC(G) = \begin{cases} TC(G-u)+1 & \textit{if } G-u \textit{ is not track connected,} \\ TC(G-u)+2 & \textit{if } G-u \textit{ is track connected.} \end{cases}$$

Proof. Let G be a graph of order $n > 3$ with an isolated vertex u.

Case(i) $G-u$ is track connected.

Since u is isolated at least two edges are required to make G track connected. Let v_1, v_2 be two distinct vertices of $G-u$. Since $G-u$ is track connected $(G-u)+uv_1+uv_2$ is track connected. So $TC(G) = 2 = TC(G-u)+2$.

Case(ii) $G-u$ is not track connected.

Let $TC(G-u) = p$, where $p \geq 1$. Let $A = \{e_1, e_2, ..., e_p\}$ be a collection of edges in $\overline{G-u}$ such that $G-u+A$ is track connected. Let $e_1 = w_1w_2$. Let $B = \{w_1u, uw_2, e_2, ..., e_p\}$. Then $G+B$ is track connected. So $TC(G) \leq TC(G-u)+1$. For the reverse inequality, let $TC(G) = q$, where $q \geq 2$. Let $C = \{e_1, e_2, ..., e_q\}$ be a collection of edges in $\overline{G}$ such that $G' = G+C$ is track connected. Since u is an isolated vertex there exist two edges in C which are incident with u. Let $e_i = w_iu$ and $e_j = w_ju$. Let $D = C \setminus \{e_i, e_j\} \cup \{w_iw_j\}$. Let $G'' = G-u+D$ and let $v_1, v_2 \in V(G'')$. Then $v_1, v_2 \in V(G')$. Hence there exist a cycle containing both v_1 and v_2 in G'.

If $u \notin V(C)$, then C lies in G''.

If $u \in V(C)$, let $C = uw_iu_1u_2...u_kw_ju$. Then $C' = w_iu_1u_2...u_kw_jw_i$ is a cycle containing both v_1 and v_2 in G''. Therefore G'' is track connected. Thus addition of $q-1$ edges makes $G-u$ track connected. So $TC(G-u) \leq TC(G)-1$. Hence the theorem. $\square$

A similar result holds for isolated edges also.

Theorem 4.2.13. *If G has an isolated edge uv then*

$$TC(G) = \begin{cases} TC(G-uv)+1 & if\ G-uv\ is\ not\ track\ connected, \\ TC(G-uv)+2 & if\ G-uv\ is\ track\ connected. \end{cases}$$

Theorem 4.2.14. *Let G be a connected graph with a cut edge e and G_1, G_2 be two components of $G-e$. Then $TC(G) \leq TC(G_1)+TC(G_2)+1$.*

Proof. Let G be a connected graph with cut edge e and G_1, G_2 be two components of $G-e$. Then, addition of $TC(G_1)$ edges makes G_1 track connected and $TC(G_2)$ edges makes G_2 track connected. So addition of an edge joining G_1 and G_2 distinct form e together with the above $TC(G_1)+TC(G_2)$ edges make G track connected. So $TC(G) \leq TC(G_1)+TC(G_2)+1$. $\square$

The inequality in Theorem 4.2.14 is sharp if and only if G_1 and G_2 are track connected.

Theorem 4.2.15. *Let G be a connected graph with cut vertex u and let $G_1, G_2, ..., G_r$ be the connected components of $G-u$. Then $TC(G) \leq TC(G_1 \cup \{u\})+TC(G_2 \cup \{u\})+...+TC(G_r \cup \{u\})+r-1$.*

Proof. Let G be a connected graph with cut vertex u and $G_1, G_2, ..., G_r$ be the connected components of $G-u$. Then the addition of $TC(G_i \cup \{u\})$ edges makes $G_i \cup \{u\}$ track connected for all $i = 1, 2, ..., r$. So the addition of $r-1$ edges say $e_1, e_2, ..., e_{r-1}$ such that e_i joins $G_i \cup \{u\}$ and $G_{i+1} \cup \{u\}$, makes G track connected. So $TC(G) \leq TC(G_1 \cup \{u\}) + TC(G_2 \cup \{u\}) + ... + TC(G_r \cup \{u\}) + r - 1$. □

Theorem 4.2.16. *For transitively tracked graph G with r maximal track connected components, $TC(G) \leq r$.*

Proof. Let G be transitively tracked and let V (G) be partitioned into $V_1, V_2, ..., V_r$ such that the graph $\langle V_i \rangle$ induced by each V_i is a maximal track connected subgraph of G.

Let $v_{1i}, v_{2i} \in V_i$, $i = 1, 2, ..., r$ and $A = \{v_{21}v_{12}, v_{22}v_{13}, v_{23}v_{14}, ..., v_{2(r-1)}v_{1r}, v_{2r}v_{11}\}$.

Let $G' = G + A$ and let $u_1, u_2 \in V(G')$.

Case(i) $u_1, u_2 \in V_i$.

Then there is a cycle containing both u_1 and u_2 in V_i.

Case(ii) $u_1 \in V_i$ and $u_2 \in V_j$, $i \neq j$.

Since each V_p, $p = 1, 2, ..., r$ is track connected there exists a path P_i form v_{1i} to v_{2i} containing u_1, a path P_j from v_{1j} to v_{2j} containing u_2 and a path P_p from v_{1p} to v_{2p} for each $p \neq i, j$. Let C be the cycle formed by joining the paths P_is $i = 1, 2, ..., r$ with the edges $v_{2i}v_{1(i+1)}$ $i = 1, 2, ..., r-1$ and $v_{2r}v_{11}$. Then C is a cycle containing both u_1 and u_2. Since u_1 and u_2 are arbitrary, G' is track connected. Thus addition of r edges makes G track connected. So $TC(G) \leq r$. □

For a transitively tracked graph G, the inequality in Theorem 4.2.16 is sharp if and only if G has no cut edge.

Theorem 4.2.17. *Let G be a graph with a cut vertex v. If b is the number of components of $G - v$ then $TC(G) \geq b - 1$.*

Proof. Let G be a graph with a cut vertex v. Let b =number of components of $G - v$. If possible let $E = \{e_1, e_2, ..., e_{b-2}\} \subset E(\overline{G})$ be such that $G' = G + E$ is track connected. Then $G' - v$ is connected and $(G' - E) - v$ has b components. But the removal of the edges in E split $G' - v$ into at most $b - 1$ components, a contradiction. □

Theorem 4.2.18 follows from Theorems 4.2.15 and 4.2.17.

Theorem 4.2.18. *For a track connected floral graph G with k petals $TC(G) = k - 1$.*

Chapter 5

Independent and Irredundant Cycle Tracking Sets in a Graph

This chapter introduces independent and irredundant cycle tracking sets in a graph. Some basic results on trace independent sets and trace irredundant sets of a graph, bounds on $\tau_i(G)$, its exact values for some standard graphs are also included and it discusses the relation between cycle tracking set, trace independent set and trace irredundance set.

5.1 Independent Cycle Tracking Sets

In [20] Teresa W. Haynes, Stephen Hedetniemi and Peter Slater had defined Independent dominating set. This concept can be extended in the case of cycle tracking sets.

Definition 5.1.1. *Two vertices u, v of a graph G are said to be trace independent if $u \notin T_G(v)$.*

Alternatively two vertices u, v of a graph G are said to be trace independent if u and v are not vertices of same cycles of G.

Definition 5.1.2. *A set S of vertices in a graph G is called a trace independent set if any two vertices of S are trace independent in G.*

Definition 5.1.3. *A trace independent set is a maximal trace independent set if no proper superset S' of S is a trace independent set.*

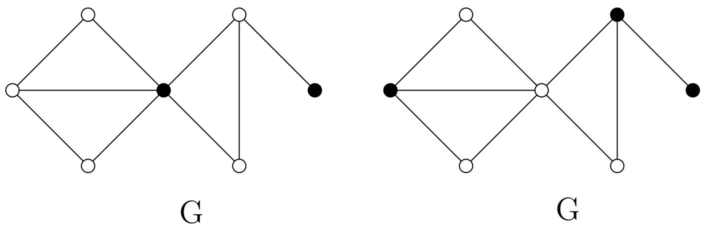


Figure 5.1: Two different maximal trace independent sets of the graph G.

Theorem 5.1.4. *A trace independent set S is maximal trace independent if and only if it is trace independent and cycle tracking.*

Proof. Suppose S is a maximal trace independent set. Then S is trace independent.

Let $u \in V \setminus S$. Then the set $S \cup \{u\}$ is not trace independent. Therefore there exists a $v \in S$ such that $u \in T_G(v)$. Hence S is a cycle tracking set.

Conversely suppose that S is trace independent and cycle tracking. Suppose that S is not maximal trace independent. Then there exists a vertex $u \in V \setminus S$ for which $S \cup \{u\}$ is trace independent. Thus $u \notin T_G(v)$ for every $v \in S$. This implies that S is not a cycle tracking set, a contradiction. $\square$

Corollary 5.1.5. *A trace independent set S is maximal trace independent if and only if for every vertex $v \in V \setminus S$, there is a vertex $u \in S$ such that $v \in T_G(u)$ and for every pair of vertices $u, v \in S$, $u \notin T_G(v)$.*

Remark 5.1.6. *Every maximal trace independent set is a cycle tracking set.*

Definition 5.1.7. *If a cycle tracking set S is trace independent then S is called independent cycle tracking set.*

Definition 5.1.8. *The minimum cardinality of an independent cycle tracking set of G is the independent cycle tracking number and it is denoted by $\tau_i(G)$.*

Definition 5.1.9. *The maximum cardinality of independent cycle tracking set of G is called the upper independent cycle tracking number and is denoted by $T_i(G)$.*

Theorem 5.1.10. *Every maximal trace independent set in a graph G is a minimal cycle tracking set of G.*

Proof. Let S be a maximal trace independent set in a graph G. Then S is a cycle tracking set. Suppose S is not a minimal cycle tracking set of G. Then there exists a vertex $v \in S$ for which $S \setminus \{v\}$ is a cycle tracking set. But if $S \setminus \{v\}$ is a cycle tracking set then, there exists a vertex $u \in S \setminus \{v\}$ such that $v \in T_G(u)$, a contradiction. Thus S must be a minimal cycle tracking set. $\square$

The converse of Theorem 5.1.10 is not true in general. In figure 5.1 Two minimal cycle tracking sets of a graph are indicated in figure by darkened vertices. The first one is a maximal trace independent set of G but the second is not.

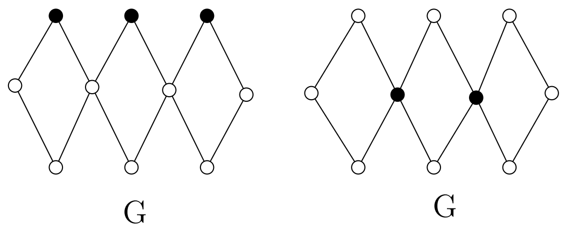


Figure 5.2: A graph G and its two minimal cycle tracking set.

Corollary 5.1.11. *For any graph G,* $\tau_c(G) \leq \tau_i(G) \leq T_i(G) \leq T_c(G)$.

Theorem 5.1.12. *A set S of vertices in a graph is an independent cycle tracking set if and only if S is a maximal trace independent set.*

Proof. We have already seen that every maximal trace independent set of vertices is a cycle tracking set. Conversely, suppose that S is an independent cycle tracking set. Then S is trace independent and every vertex not in S is traced by a vertex of S. Therefore by Theorem 5.1.4 S is maximal trace independent set. □

Theorem 5.1.13. *Let G be a graph. Then* $\tau_i(G) = 1$ *if and only if G is a track connected graph or a track connected floral graph.*

Proof. If G is a track connected graph or a track connected floral graph then, $\tau_i(G) = 1$.

Let G be any graph with $\tau_i(G) = 1$. Then there exists a vertex $v \in V(G)$ such that every vertex of G belongs to $T_G(v)$. ie; $T_G(v) = V(G)$.

If G is not track connected, there exist a pair of vertices $x, y \in V(G)$ such that they are not connected by two internally disjoint $x - y$ path. But as G is connected there is a $x - y$ path in G. Since $T_G(v) = V(G)$, there exist a cycle C_1 containing x and v and another cycle C_2 containing y and v in G such that these two cycles have only v as the common vertex and no other vertices of C_1 is connected to a vertex in C_2 by a path in G. Which implies that v is a cut vertex of G. If $G_1, G_2, ..., G_k$ are the components of $G \setminus \{v\}$ then the graph induced by $V(G_i) \cup \{v\}$, $i = 1, 2, ..., k$ are track connected. Hence G is a track connected floral graph. □

Theorem 5.1.14. *A graph G is track connected if and only if* $T_i(G) = 1$.

Proof. Let G be any graph with $T_i(G) = 1$. Then $\tau_i(G) = 1$ and by Theorem 5.1.13 G is a track connected graph or a track connected floral graph.
If G is a track connected floral graph then G has at least two petals. In this case the set S consisting of exactly one vertex from every petal will form a trace independent cycle tracking set. Hence $T_i(G) \geq 2$, a contradiction. So G must be track connected. □

Remark 5.1.15. *Let G be a track connected floral graph with k petals. Then the cardinality of every maximal trace independent set is either 1 or k.*

Theorem 5.1.16. *Let G be a graph of order n. Then* $\tau_i(G) = n$ *if and only if G is a forest.*

Proof. Let G be a forest. Since every trace independent cycle tracking set of G contains all trace free vertices of G, $\tau_i(G) = n$.
Conversely suppose that $\tau_i(G) = n$. If G contains a non trivial cycle C, then the vertices in C are not trace independent and hence $\tau_i(G) < n$. So G must be a forest. □

Definition 5.1.17. *A graph G is said to be well track covered if every maximal trace independent set of G is a maximum trace independent set.*

All trees and forests are well track covered.

Theorem 5.1.18. *Every well tracked graph is well track covered. Moreover* $\tau_c(G) = \tau_i(G)$.

Proof. If a graph G is well tracked then every minimal cycle tracking set has the same cardinality. By Theorem 5.1.10 every maximal track independent set is a minimal cycle tracking set. Therefore every maximal track independent set has the same cardinality. Thus G is well track covered and $\tau_c(G) = \tau_i(G)$. □

Definition 5.1.19. *A graph G is said to be track perfect if* $\tau_i(G) = \tau_c(G)$.

Remark 5.1.20. *Theorem 5.1.18 shows that every well tracked graph is track perfect.*

Theorem 5.1.21. *Every transitively tracked graph is track perfect.*

Proof. For a transitively tracked graph all minimal cycle tracking sets S are formed by choosing exactly one vertex from each equivalence class. But then S is trace independent. For any transitively tracked graph $\tau_c(G) = T_c(G)$. By Corollary 5.1.11 $\tau_c(G) \leq \tau_i(G) \leq T_c(G)$. Therefore $\tau_c(G) = \tau_i(G)$ for transitively tracked graphs. □

Theorem 5.1.22 follows from Corollaries 5.1.11 and 2.2.19.

Theorem 5.1.22. *For a transitively tracked graph G,* $\tau_c(G) = \tau_i(G) = T_i(G) = T_c(G)$.

The following theorem is a consequence of the fact that addition of a cut edge will not increase the independent tracking number of a graph .

Theorem 5.1.23. *If G is a graph obtained by attaching a graph H to one of the vertices of another graph k using a bridge. Then* $\tau_i(G) = \tau_i(H) + \tau_i(K)$.

Proposition 5.1.24. *If* T' *is the tree obtained by duplicating each edge of a tree T by a new vertex, then* $\tau_c(T') = \tau_i(T') = |V(G)| + |E(G)|$.

Theorem 5.1.25. *Let G be a graph. If* G' *is the graph obtained by the duplication of each vertex of G by a new edge, then* $\tau_c(G') = \tau_i(G') = |V(G)|$.

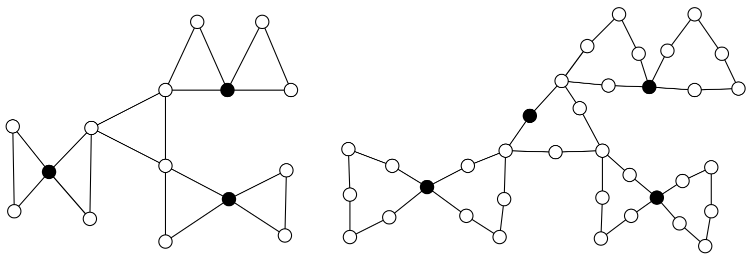
Proof. Let G be any graph with $|V(G)| = n$ and $|E(G)| = m$ and let each vertex of the graph G be duplicated by a new edge. Then the resultant graph G' will have $3n$ vertices, $3n + m$ edges and n vertex disjoint cycles of length three. To trace these n disjoint cycles, at least n distinct vertices of G', one from each cycle, are required. These n vertices in fact traces all vertices of G'. Hence, $\tau_c(G') = n$.

Consider the set $S^* \subset V(G')$ consists of exactly one end vertex u'_k or v'_k of each new edge $u'_kv'_k$ corresponding to the vertex v_k of G. Then S^* is a minimal cycle tracking set of G and is track independent. Therefore $\tau_i(G') = n$. Thus, $\tau_c(G') = \tau_i(G') = n$. □

Theorem 5.1.26. *Let* G' *be the graph obtained by the duplication of each edge of a graph G by a new vertex and let p be the number of cut edges of G, then* $\tau_i(G') \geq \tau_i(G) + p$.

Proof. Let uv be an edge in G. Consider the duplication of uv by a vertex, say u'. If uv is a cut edge then u' is a trace free vertex and belong to all cycle tracking sets. So $\tau_i(G') \geq \tau_i(G) + p$. □

This inequality is strict for the graph G in figure 5.3



G $\qquad$ G'

Figure 5.3: $\tau_i(G') = 4$ and $\tau_i(G) = 3$.

5.2 Trace Irreduntance

Let G be a graph, $S \subset V$ and $u \in S$. Recall that a vertex v is a private trace of u with respect to S if $T_G(v) \cap S = \{u\}$ and $pt[u, S]$ is the set of all such vertices.

In this section we introduce the concept of trace irredudance and examine the relation between cycle tracking set, trace independent set and trace irredundant set.

Definition 5.2.1. *Let $S \subset V$. The subset consisting of all vertices of S having at least one private trace is called a private trace set of S and it will be denoted by $pt(S)$. ie; $pt(S) = \{u \in S : pt[u, S] \neq \emptyset\}$.*
The cardinality $ptc(S)$ of $pt(S)$ is called the private trace count of S.

Remark 5.2.2. *The vertex u may or may not be in $pt[u, S]$. In figure 5.4, let*

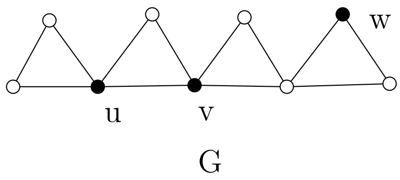


Figure 5.4: Graph G.

$S = \{u, v, w\}$. Then $u \notin pt[u, S]$ but $w \in pt[w, S]$.

Definition 5.2.3. *A set S of vertices of a graph G is trace irredundant if for every vertex $v \in S$, $pt[v, S] \neq \emptyset$.*

Remark 5.2.4.

1. *If a set S of vertices of a graph G is trace irredundant then, every vertex $v \in S$ has at least one private trace. That is $pt(S) = S$ and $ptc(S) = |S|$.*
2. *The property of being trace irredundant is hereditary.*

Theorem 5.2.5. *A set S of vertices in a graph G is trace irredundant if and only if every vertex v in S satisfies at least one of the following two properties:*

1. *There exists a vertex w in $V(G) \setminus S$ such that $T_G(w) \cap S = \{v\}$.*
2. *$T_G(v) \cap (S \setminus \{v\}) = \emptyset$.*

Proof. First, let S be a set of vertices of G such that for every vertex $v \in S$, at least one of the above properties is satisfied. If first holds, then $w \in pt[v, S]$. If the second holds then $v \in pt[v, S]$. So S is trace irredundant.

Conversely, let S be a trace irredundant set of vertices in G, and let $v \in S$. Since S is trace irredundant, there exists a vertex $w \in pt[v, S]$. If $w \neq v$ then the first property is satisfied, and if $w = v$ then the second. □

Theorem 5.2.6. *A cycle tracking set S of a graph G is a minimal cycle tracking set if and only if it is cycle tracking and trace irredundant.*

Proof. Suppose S is a minimal cycle tracking set then every vertex in S has at least one private trace. That is for every $u \in S$, $pt[u, S] \neq \emptyset$. That is S is a trace irredundant set.

Conversely suppose that S is a cycle tracking and trace irredundant. We have to show that it is a minimal cycle tracking set. Suppose that S is not a minimal cycle tracking set. Then there exists a vertex, say $v \in S$ such that $S \setminus \{v\}$ is a cycle tracking set. But since S is trace irredundant, $pt[v, S] \neq \emptyset$. Let $w \in pt[v, S]$. Then w is not traced by any vertex in $S \setminus \{v\}$, that is, $S \setminus \{v\}$ is not a cycle tracking set, which is a contradiction. $\square$

Definition 5.2.7. *A trace irredundant set S of a graph G is maximal trace irredundant set if no proper superset S' of S is a trace irredundant set.*

Theorem 5.2.8. *A trace irredundant set S of a graph G is maximal trace irredundant if and only if for every vertex $u \in V \setminus S$, there exists a vertex $v \in S \cup \{u\}$ for which $pt[v, S \cup \{u\}] = \emptyset$.*

Proof. Assume that S is a maximal trace irredundant set of G. Then for every vertex $u \in V \setminus S$, $S \cup \{u\}$ is not a trace irredundant set. This means that there exist at least one vertex $v \in S \cup \{u\}$ which does not have a private trace. That is $pt[v, S \cup \{u\}] = \emptyset$.

Conversely suppose that S is a trace irredundant set and for each vertex $u \in V \setminus S$, there exists a vertex $v \in S \cup \{u\}$ for which $pt[v, S \cup \{u\}] = \emptyset$. We will show that S is maximal trace irredundant set. Suppose that S is not a maximal trace irredundant set then there exist a vertex $u \in V$ such that $S \cup \{u\}$ is a trace irredundant set. Hence for every vertex $v \in S \cup \{u\}$ we have $pt[v, S \cup \{u\}] \neq \emptyset$, which contradicts our assumption. $\square$

Theorem 5.2.9. *A trace irredundant set S of a graph G is a maximal trace irredundant if and only if for every vertex $u \in V \setminus S$, $ptc(S \cup \{u\}) \leq ptc(S)$.*

Proof. Assume that S is a maximal trace irredundant set of G. Then for every vertex $u \in V \setminus S$, $S \cup \{u\}$ is not a trace irredundant set. This means that there exists at least one vertex $v \in S \cup \{u\}$ which does not have a private trace. So $ptc(S \cup \{u\}) \leq |S| = ptc(S)$.
Conversely suppose that for every vertex $u \in V \setminus S$, $ptc(S \cup \{u\}) \leq ptc(S)$. We show that S is maximal trace irredundant set. Suppose that S is not a maximal trace irredundant set, that is, there exist a vertex $v \in V$ such that $S \cup \{v\}$ is a trace irredundant set. Then $ptc(S \cup \{v\}) = |S| + 1 > |S| = ptc(S)$, which contradicts our assumption. $\square$

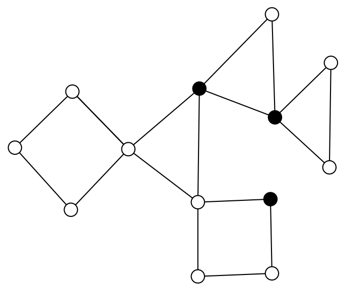
Definition 5.2.10. *The minimum of cardinalities of maximal trace irreduntant sets of a graph G is called the trace irredundance number of G, and it is denoted by $\tau_{ir}(G)$.*

Definition 5.2.11. *The maximum of cardinalities of trace irredundant sets of a graph G is called the upper trace irredundance number of G, and it is denoted by $T_{ir}(G)$.*

Theorem 5.2.12. *Every minimal cycle tracking set in a graph G is a maximal trace irredundant set of G.*

Proof. Theorem 5.2.6 shows that every minimal cycle tracking set S is trace irredundant. Suppose S is not maximal trace irredundant. Then there exists a vertex $u \in V$ such that $S \cup \{u\}$ is a trace irredundant set. Hence for every vertex $v \in S \cup \{u\}$, $pt[v, S \cup \{u\}] \neq \emptyset$. In particular $pt[u, S \cup \{u\}] \neq \emptyset$, that is, there exist at least one vertex w which is a private trace of u with respect to $S \cup \{u\}$. This means that no vertex in S traces w. That is, S is not a cycle tracking set. This contradicts our assumption that S is a cycle tracking set. □

The converse of Theorem 5.2.12 is not true. In figure 5.5 the darkened vertices form a maximal trace irredundant set of the graph G but it is not a cycle tracking set.



G

Figure 5.5: A graph G and a maximal trace irredundant set.

Theorem 5.1.10 implies that every maximal trace independent set in a graph G is a minimal cycle tracking set of G. Therefore by Theorem 5.2.12, every maximal trace independent set is a maximal trace irredundant set of G. Thus the preceding arguments may be summarized as follows.

Theorem 5.2.13. *Every maximal trace independent set in a graph G is a maximal trace irredundant set of G.*

Theorem 5.2.14. *For any graph G, $\tau_{ir}(G) \leq \tau_c(G) \leq \tau_i(G) \leq T_c(G) \leq T_{ir}(G)$.*

Proof. Since every minimal cycle tracking set in a graph G is a maximal trace irredundant set of G $\tau_{ir}(G) \leq \tau_c(G) \leq T_c(G) \leq T_{ir}(G)$. By Corollary 5.1.11 $\tau_c(G) \leq \tau_i(G) \leq T_c(G)$. Hence $\tau_{ir}(G) \leq \tau_c(G) \leq \tau_i(G) \leq T_c(G) \leq T_{ir}(G)$. □

Definition 5.2.15. *A maximal trace irredundant set with minimum cardinality is called a $\tau_{ir} - set$ of G.*

Theorem 5.2.16. *For any graph G, $\tau_c(G)/2 < \tau_{ir}(G) \leq \tau_c(G) \leq 2\tau_{ir}(G) - 1$.*

Proof. Let $\tau_{ir}(G) = m$ and let $S = \{v_1, v_2, ..., v_m\}$ be a $\tau_{ir} - set$ of G. Since S is trace irredundant, $pt[v_i, S] \neq \emptyset$, $1 \leq i \leq m$. Let $S^* = \{u_1, u_2, ..., u_m\}$, where $u_i \in pt[v_i, S]$, $1 \leq i \leq m$. Then $|S \cup S^*| \leq 2m = 2\tau_{ir}(G)$.
We claim that the set $S^{**} = S \cup S^*$ is a cycle tracking set. If not there must exist at least one vertex $w \in V \setminus S^{**}$ which is not traced by any vertex in S^{**}. That is $w \notin T_G(v)$ for any vertex $v \in S^{**}$. Therefore $w \in pt[w, S \cup \{w\}]$ and hence $pt[w, S \cup \{w\}] \neq \emptyset$. Since $u_i \notin T_G(w)$ for any vertex $u_i \in S^*$, $pt[v_i, S \cup \{w\}] \neq \emptyset$. Thus, $S \cup \{w\}$ is an irredundant set, which contradicts the maximality of S. Therefore S^{**} is a cycle tracking set. But by Theorem 5.2.12, S^{**} is not minimal. Hence $\tau_c(G) < 2m$. Therefore $\frac{\tau_c(G)}{2} < m$ and $\tau_c(G) \leq 2m - 1$. And by Theorem 5.2.14, $\tau_{ir}(G) \leq \tau_c(G)$. Thus $\tau_c(G)/2 < \tau_{ir}(G) \leq \tau_c(G) \leq 2\tau_{ir}(G) - 1$. □

Theorem 5.2.17. *Suppose that S is a maximal trace irredundant set of a graph G. If a vertex u of G is not traced by S then $T_G(u) \supseteq pt[x, S]$ for some $x \in S$.*

Proof. By maximality of S, $S \cup \{u\}$ is not trace irredundant in G. So $pt[x, S \cup \{u\}] = \emptyset$ for some $x \in S \cup \{u\}$. Since u is not traced by S, $u \in pt[u, S \cup \{u\}]$.

Therefore $x \neq u$. Further, since $pt[x, S \cup \{u\}] = T_G(x) \setminus T_G(S \cup \{u\} \setminus \{x\}) = [T_G(x) \setminus T_G(S \setminus \{x\})] \setminus T_G(u) = pt[x, S] \setminus T_G(u)$. Since $pt[x, S \cup \{u\}] = \emptyset$, $T_G(u) \supseteq pt[x, S]$. $\square$

Chapter 6

Cycle Tracking polynomial

The concept of cycle tracking polynomial and independent cycle tracking polynomial in graphs is introduced and discussed in this chapter. Such polynomials of certain graphs are also determined.

6.1 Cycle tracking Polynomial

Definition 6.1.1. *Let G be a graph of order n. Let $T(G,i)$ be the family of all cycle tracking sets of a graph G with cardinality i and let $t(G,i) = |T(G,i)|$. Then the cycle tracking polynomial $T(G,x)$ of G is defined as*

$$T(G,x) = \sum_{i=\tau_c(G)}^{n} t(G,i)x^i$$

where $\tau_c(G)$ is the cycle tracking number of G.

The path P_3 on three vertices has only one cycle tracking set with cardinality 3 ($\tau_c(G) = 3$), so its tracking polynomial is $T(P_3,x) = x^3$. In the case of the

cycle C_n on n vertices,
$T(C_n, x) = \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n}x^n = (1+x)^n - 1.$

Theorem 6.1.2. *If a graph G has m components $G_1, G_2, ..., G_m$ then $T(G, x) = T(G_1, x)T(G_2, x)...T(G_m, x)$.*

Proof. It is enough to prove the theorem for n=2.

For $k \geq \tau_c(G)$, a cycle tracking set of k vertices in G arises by choosing a cycle tracking set of j vertices in G_1 for some j such that $\tau_c(G) \leq j \leq |V(G)|$ and a cycle tracking set of $k - j$ vertices in G_2. The number of ways of doing this over all $j = \tau_c(G_1), ..., |V(G_1)|$ is exactly the coefficient of x^k in $T(G_1, x)T(G_2, x)$. So $T(G, x) = T(G_1, x)T(G_2, x)$. □

Theorem 6.1.3. *Let G be a graph of order n. Then*

1. *$t(G, n) = 1$.*
2. *$t(G, i) = 0$ if and only if $i < \tau_c(G)$ or $i > n$.*
3. *$T(G, x)$ has no constant term.*
4. *$T(G, x)$ is a strictly increasing function on $(0, \infty)$.*
5. *for any subgraph H of G, $deg(T(G, x)) \geq deg(T(H, x))$.*
6. *zero is a root of $T(G, x)$ with multiplicity $\tau_c(G)$.*
7. *$\tau_c(G) = n$ if and only if $T(G, x) = x^n$.*

Theorem 6.1.4. *Let G be a graph of order n. Then $T(G, x) = x^n$ if and only if G is a forest.*

Proof. $T(G,x) = x^n$ if and only if $V(G)$ is the only cycle tracking set for G. That is if and only if $\tau_c(G) = n$. That is if and only if G is a forest (by Theorem 2.1.21). $\square$

Theorem 6.1.5. *Let G be a graph of order n. Then $T(G,x) = (1+x)^n - 1$ if and only if G is track connected.*

Proof. If G is track connected then $\tau_c(G) = 1$ and every vertex traces all vertices of G. So coefficient of x is n and $t(G,p) = \binom{n}{p}$. So $T(C_n,x) = \binom{n}{1}x^1 + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n}x^n = (1+x)^n - 1$.
Conversely if $T(G,x) = (1+x)^n - 1$, the coefficient of x is n. That is, every vertex traces all vertices of G. Hence G is track connected. $\square$

Theorem 6.1.6. *Let G be a track connected floral graph with k petals. If the petals respectively having $m_1, m_2, ..., m_k$ vertices then*

$$t(G,p) = \binom{m_1+m_2+...+m_k}{p-1} + \sum_{i_1=1}^{p-k+1} \binom{m_1}{i_1} \left[\sum_{i_2=1}^{p-k-i_1+2} \binom{m_2}{i_2} \left[... \sum_{i_j=1}^{p-k-i_1-i_2-...+j} \binom{m_j}{i_j} \left[... \left[\sum_{i_{k-1}=1}^{p-k-i_1-i_2-...-i_{k-1}+k-1} \binom{m_{k-1}}{i_{k-1}} \binom{m_k}{p-i_1-i_2-...-i_{k-1}} \right] ... \right] \right] \right],$$

when $k \le p \le n$. *And*

$$T(G,x) = \sum_{p=1}^{k-1} \binom{m_1+m_2+...+m_k}{p-1} x^p + \sum_{p=k}^{n} \left[\binom{m_1+m_2+...+m_k}{p-1} + \sum_{i_1=1}^{p-k+1} \binom{m_1}{i_1} \left[\sum_{i_2=1}^{p-k-i_1+2} \binom{m_2}{i_2} \left[... \sum_{i_j=1}^{p-k-i_1-i_2-...+j} \binom{m_j}{i_j} \left[... \left[\sum_{i_{k-1}=1}^{p-k-i_1-i_2-...-i_{k-1}+k-1} \binom{m_{k-1}}{i_{k-1}} \binom{m_k}{p-i_1-i_2-...-i_{k-1}} \right] ... \right] ... \right] \right] \right] x^p.$$

Proof. Case(1) $1 \le p \le k-1$.

Then any cycle tracking set S contains the central vertex. So the central vertex together with $p-1$ vertices constitute a cycle tracking set S and it can be chosen in $\binom{m_1+m_2+...+m_k}{p-1}$ ways.

Case(2) $k \leq p \leq n$.

Here the central vertex together with $p-1$ vertices constitute a cycle tracking set S and a set of vertices S of cardinality p having at least one element from each petal is also form a cycle tracking set and it can be chosen in $\binom{m_1+m_2+...+m_k}{p-1}+$

$$\sum_{i_1=1}^{p-k+1}\binom{m_1}{i_1}\left[\sum_{i_2=1}^{p-k-i_1+2}\binom{m_2}{i_2}\left[\ldots\sum_{i_j=1}^{p-k-i_1-i_2-...+j}\binom{m_j}{i_j}\right.\right.$$
$$\left.\left.\left[\ldots\left[\sum_{i_{k-1}=1}^{p-k-i_1-i_2-...-i_{k-1}+k-1}\binom{m_{k-1}}{i_{k-1}}\binom{m_k}{p-i_1-i_2-...-i_{k-1}}\right]\right]\right]\ldots\right] \text{ ways.} \qquad \square$$

Let G be a transitively tracked graph then its vertex set can be partitioned into $V_1, V_2, ..., V_k$ of cardinality $m_1, m_2, ..., m_k$ respectively such that the graph $\langle V_i \rangle$ induced by each V_i is maximal track connected subgraph of G. Then a set S of vertices which contains at least one element from each V_i form a cycle tracking set. So a cycle tracking set of G with cardinality p can be chosen in

$$\sum_{i_1=1}^{p-k+1}\binom{m_1}{i_1}\left[\sum_{i_2=1}^{p-k-i_1+2}\binom{m_2}{i_2}\left[\ldots\sum_{i_j=1}^{p-k-i_1-i_2-...+j}\binom{m_j}{i_j}\right.\right.$$
$$\left.\left.\left[\ldots\left[\sum_{i_{k-1}=1}^{p-k-i_1-i_2-...-i_{k-1}+k-1}\binom{m_{k-1}}{i_{k-1}}\binom{m_k}{p-i_1-i_2-...-i_{k-1}}\right]\right]\right]\ldots\right] \text{ ways.}$$

So $$t(G,p) = \sum_{i_1=1}^{p-k+1}\binom{m_1}{i_1}\left[\sum_{i_2=1}^{p-k-i_1+2}\binom{m_2}{i_2}\left[\ldots\sum_{i_j=1}^{p-k-i_1-i_2-...+j}\binom{m_j}{i_j}\right.\right.$$
$$\left.\left.\left[\ldots\left[\sum_{i_{k-1}=1}^{p-k-i_1-i_2-...-i_{k-1}+k-1}\binom{m_{k-1}}{i_{k-1}}\binom{m_k}{p-i_1-i_2-...-i_{k-1}}\right]\right]\right]\ldots\right], k \leq p \leq n.$$

The above discussion may be summarized as follows.

Theorem 6.1.7. *Let G be a transitively tracked graph. Let $V_1, V_2, ..., V_k$ be the partition of V(G) such that each $\langle V_i \rangle$ induced by each V_i is a maximal track* *connected component of G with* $|V_i| = m_i$. *Then,*

$$T(G,x) = \sum_{p=k}^{n} \left[\sum_{i_1=1}^{p-k+1} \binom{m_1}{i_1} \left[\sum_{i_2=1}^{p-k-i_1+2} \binom{m_2}{i_2} \left[\dots \sum_{i_j=1}^{p-k-i_1-i_2-\dots+j} \binom{m_j}{i_j} \right.\right.\right.$$
$$\left.\left.\left.\left[\dots \left[\sum_{i_{k-1}=1}^{p-k-i_1-i_2-\dots-i_{k-1}+k-1} \binom{m_{k-1}}{i_{k-1}} \binom{m_k}{p-i_1-i_2-\dots-i_{k-1}} \right]\right]\right] \dots \right]\right] x^p.$$

Theorem 6.1.8. *Let G be transitively tracked graph and let its vertex set* $V(G)$ *be partitioned into* $V_1, V_2, ..., V_k$ *such that the graph* $\langle V_i \rangle$ *induced by each* V_i *is a maximal track connected subgraph of G. Then,*

$$T(G,x) = T(\langle V_1 \rangle, x)T(\langle V_2 \rangle, x)...T(\langle V_m \rangle, x).$$

Proof. Let G be transitively tracked and let $V(G)$ be partitioned into $V_1, V_2, ..., V_k$ such that the graph $\langle V_i \rangle$ induced by each V_i is maximal track connected subgraph of G. Let G' be the graph formed by removing all cut edges of G. Then by Proposition 2.1.34 $T(G,x) = T(G',x)$ and by Proposition 2.2.4 and Theorem 6.1.2 $T(G,x) = T(\langle V_1 \rangle, x)T(\langle V_2 \rangle, x)...T(\langle V_m \rangle, x)$. □

Corollary 6.1.9. *Let G be transitively tracked graph. If its vertex set* $V(G)$ *can be partitioned into* $V_1, V_2, ..., V_k$ *of cardinality* $m_1, m_2, ..., m_k$ *respectively such that the graph* $\langle V_i \rangle$ *induced by each* V_i *is maximal track connected subgraph of G. Then*

$$T(G,x) = ((x+1)^{m_1} - 1)((x+1)^{m_2} - 1)...((x+1)^{m_k} - 1).$$

Theorem 6.1.10. *For a graph G,* $t(G,1) = 1$ *if and only if G is a track connected floral graph.*

Proof. Let G be any graph with $t(G,1) = 1$. Then there exists one and only one cycle tracking set S with $|S| = 1$. That is there exist a vertex $v \in V$ such that

$T_G(v) = V$ and no other vertex can trace G. And since $\tau_c(G) = 1$, G must be a track connected floral graph. □

Remark 6.1.11. *For any graph G,*

$$t(G,1) = \begin{cases} 1 & if\ G\ is\ track\ connected\ floral\ graph, \\ |V| & if\ G\ is\ track\ connected, \\ 0 & otherwise. \end{cases}$$

Theorem 6.1.12. *Let G be a graph of order n with r trace free vertices. If $T(G,x) = \sum_{i=\tau_c(G)}^{n} t(G,i)x^i$ is its cycle tracking polynomial, then $r = n - t(G, n-1)$.*

Proof. Suppose that $A \subset V(G)$ is the set of all trace free vertices. Then by hypothesis, $|A| = r$. For a vertex $v \in V(G)$, the set $V(G) \setminus \{v\}$ is a cycle tracking set of G if and only if $v \in V(G) \setminus A$. Therefore $t(G, n-1) = |V(G \setminus A)| = n - r$. Hence the theorem. □

Theorem 6.1.13. *Let G be a graph of order n. Then,*

$$t(G,1) = |\{v \in V(G) : T_G(v) = V(G)\}|.$$

Proof. For every $v \in V(G)$, $\{v\}$ is a cycle tracking set if and only if v traces all vertices. ie; $T_G(v) = V(G)$. □

6.2 Cycle Tracking Polynomial for Some Graphs

In this section we determine the cycle tracking number and cycle tracking polynomial for some graphs.

Theorem 6.2.1. *For Firefly graph $F_{s,t,n-2s-2t-1}$, $\tau_c(F_{s,t,n-2s-2t-1}) = n - 2s$.*

Proof. The graph $F_{s,t,n-2s-2t-1}$ has $n-2s-1$ trace free vertices and the common vertex traces all s triangles. So the $n-2s-1$ trace free vertices together with the common vertex form a τ-set. Hence $\tau_c(F_{s,t,n-2s-2t-1}) = n-2s$. $\square$

Theorem 6.2.2. $t(F_{s,t,n-2s-2t-1}, p) = \binom{2s}{p-n+2s}$ *if* $n-2s \leq p \leq n-s-2$ *and* $t(F_{s,t,n-2s-2t-1}, p) = \binom{2s}{p-n+2s} + \sum_{i_1=1}^{p-n+s+2} \binom{2}{i_1} \left[\sum_{i_2=1}^{p-n+s-i_1+3} \binom{2}{i_2} \left[\cdots \sum_{i_j=1}^{p-n+s-i_1-i_2-\ldots+j+1} \binom{2}{i_j} \left[\cdots \left[\sum_{i_{s-1}=1}^{p-n+s-i_1-i_2-\ldots-i_{s-1}+s} \binom{2}{i_{s-1}} \binom{2}{p-n+2s+1-i_1-i_2-\ldots-i_{s-1}} \right] \cdots \right] \cdots \right] \right]$ *if* $n-s-1 \leq p \leq n$.

Proof. By Theorem 6.2.1, $\tau_c(F_{s,t,n-2s-2t-1}, x) = n-2s$.

case(1) $n-2s \leq p \leq n-s-2$.

Then any cycle tracking set S contains the common vertex. So the central vertex together with $n-2s-1$ trace free vertices and $p-n+2s$ other vertices constitute a cycle tracking set S and it can be chosen in $\binom{2s}{p-n+2s}$ ways.

case(2) $n-s-1 \leq p \leq n$.

Here the central vertex together with $n-2s-1$ trace free vertices and $p-n+2s$ other constitute a cycle tracking set S and a set of vertices S of cardinality p having at least one element from each triangle is also form a cycle tracking set and it can be chosen in

$$\binom{2s}{p-n+2s} + \sum_{i_1=1}^{p-n+s+2} \binom{2}{i_1} \left[\sum_{i_2=1}^{p-n+s-i_1+3} \binom{2}{i_2} \left[\cdots \sum_{i_j=1}^{p-n+s-i_1-i_2-\ldots+j+1} \binom{2}{i_j} \left[\cdots \left[\sum_{i_{s-1}=1}^{p-n+s-i_1-i_2-\ldots-i_{s-1}+s} \binom{2}{i_{s-1}} \binom{2}{p-n+2s+1-i_1-i_2-\ldots-i_{s-1}} \right] \cdots \right] \right] \right]$$

ways. $\square$

Theorem 6.2.3. $T(F_{s,t,n-2s-2t-1}, x) = \sum_{p=n-2s}^{n-s-2} \binom{2s}{p-n+2s} + \sum_{p=n-s-2}^{n} \left[\binom{2s}{p-n+2s} \right.$

$$+\sum_{i_1=1}^{p-n+s+2}\binom{2}{i_1}\left[\sum_{i_2=1}^{p-n+s-i_1+3}\binom{2}{i_2}\left[\ldots\sum_{i_j=1}^{p-n+s-i_1-i_2-\ldots+j+1}\binom{2}{i_j}\right.\right.$$
$$\left.\left.\left[\ldots\left[\sum_{i_{s-1}=1}^{p-n+s-i_1-i_2-\ldots-i_{s-1}+s}\binom{2}{i_{s-1}}\binom{2}{p-n+2s+1-i_1-i_2-\ldots-i_{s-1}}\right]\ldots\right]\right]\right].$$

Theorem 6.2.4. *For a Lollipop graph $L_{n,m}$, $\tau_c(L_{n,m}) = m+1$ and $T(L_{n,m}, x) = ((1+x)^n - 1)x^m$.*

Proof. Since a vertex in K_n can trace all vertices in it and all vertices of P_m are trace free vertices we need at least $m+1$ vertices to trace $L_{n,m}$. Hence $\tau_c(L_{n,m}) = m+1$ and $T(L_{n,m}, x) = ((1+x)^n - 1)x^m$. □

Theorem 6.2.5. *For a Tadpole $T_{(n,l)}$, $\tau_c(T_{(n,l)}) = l+1$ and $T(T_{(n,l)}, x) = ((1+x)^n - 1)x^l$.*

Proof. Since a vertex in C_n can trace all vertices in it and all vertices of P_l are trace free vertices we need at least $m+1$ vertices to trace $T_{(n,l)}$. Hence $\tau_c(T_{(n,l)}) = l+1$ and $T(T_{(n,l)}, x) = ((1+x)^n - 1)x^l$. □

Theorem 6.2.6. *For a helm graph H_n, $\tau_c(H_n) = n$ and $T(H_n, x) = ((1+x)^n - 1)x^{n-1}$.*

Proof. Since H_n contains $n-1$ pendant vertices, all these vertices belong to every cycle tracking set. Since W_n is track connected, a vertex of W_n together with the pendant vertices form a minimal cycle tracking set. So $\tau_c(H_n) = n$ and $T(H_n, x) = ((1+x)^n - 1)x^{n-1}$. □

Theorem 6.2.7. *For a web graph $WB_n, n > 3$, $\tau_c(WB_n) = n$ and $T(WB_n, x) = ((1+x)^{2n-1} - 1)x^{n-1}$.*

Since F_n is a track connected floral graph with n petals each having 3 vertices we have:

Theorem 6.2.8. *For a friendship graph F_n, $\tau_c(F_n) = 1$.*

Using Theorem 6.1.6 we have;

Theorem 6.2.9. $t(F_n, i) = \begin{cases} \binom{2n}{i-1} & 1 \leq i \leq n-1, \\ \binom{2n}{i-1} + \binom{n}{i-n}2^{2n-i} & n \leq i \leq 2n. \end{cases}$

and $T(F_n, x) = x + 2nx^2 + ... + \binom{2n}{i-1}x^i + ... + \binom{2n}{n-2}x^{n-1}$
$+ \left[\binom{2n}{n-1} + 2^n\right]x^n + ... + \left[\binom{2n}{j-1} + \binom{n}{j-n}2^{2n-j}\right]x^j + ... + x^{2n+1}.$

Theorem 6.2.10. *For an Armed crown $C_n \bigodot P_m$, $\tau_c(C_n \bigodot P_m) = mn + 1$ and $T(C_n \odot P_m, x) = ((1 + x)^n)x^{mn}$.*

Proof. Since a vertex in C_n can trace all vertices in it and the remaining mn vertices are trace free vertices. So we need at least $mn + 1$ vertices to trace $C_n \bigodot P_m$. Hence $\tau_c(C_n \bigodot P_m) = mn+1$ and $T(C_n \odot P_m, x) = ((1+x)^n)x^{mn}$. □

Theorem 6.2.11. *For any graph G of order n, $\tau_c(G \circ K_1) = n + \tau_c(G)$ and $T(G \circ K_1, x) = T(G, x)x^n$.*

Theorem 6.2.12. *Let G be any graph of order n and H be a connected graph of order m. Then $\tau_c(G \circ H) = n$.*

In particular if $G = K_1$, then $\tau_c(K_1 \circ H) = 1$.

Corollary 6.2.13. *For a connected graph H of order m, $T(K_1 \circ H, x) = (1 + x)^{m+1} - 1$.*

Proof. Since $K_1 \circ H$ is a track connected graph with $m+1$ vertices, $T(K_1 \circ H, x) = (1+x)^{m+1} - 1$. $\square$

Theorem 6.2.14. *Let G be any graph of order n and H be a connected graph of order m. Then $T(G \circ H, x) = [(1+x)^{m+1} - 1]^n$.*

Proof. We prove the theorem for $n = 2$. General case will follow from it.
For $k \geq \tau_c(G \circ H) = 2$, a cycle tracking set of k vertices in $G \circ H$ is chosen by selecting j $(1 \leq j \leq k-1)$ vertices from first copy of $K_1 \circ H_m$ and $k - j$ vertices from second copy of $K_1 \circ H_m$. The number of way of doing this over all $k = 2, 3, ..., mn$ is exactly the coefficient of x^k in $[(1+x)^{m+1} - 1]^2$. So $T((G \circ H, x) = [(1+x)^{m+1} - 1]^2$. $\square$

6.3 Independent Cycle Tracking Polynomial

In this section we introduce a new type of graph polynomial called independent cycle tracking polynomial $T_i(G, x)$ and studies some of its properties.

Definition 6.3.1. *Let $T_i(G, j)$ be the family of independent cycle tracking sets of a graph G with cardinality j and let $t_i(G, j) = |T_i(G, j)|$. Then the independent tracking polynomial $T_i(G, x)$ of G is defined as*

$$T_i(G, x) = \sum_{j=\tau_i(G)}^{|V(G)|} t_i(G, j) x^j$$

where $\tau_i(G)$ is the independent cycle tracking number of G. The roots of the polynomial $T_i(G, x)$ are called the independent tracking roots of G.

Remark 6.3.2. *As $t_i(G,j)=0$ for $j>T_i(G)$, the independent tracking polynomial $T_i(G,x)$ of G, in fact is,*

$$T_i(G;x)=\sum_{j=\tau_i(G)}^{T_i(G)} t_i(G,j)x^j$$

where $T_i(G)$ is the independent cycle tracking number of G as $t_i(G,j)=0$ for $j>T_i(G)$.

The path P_3 on three vertices has only one independent cycle tracking set with cardinality 3 ($\tau_i(G)=3$) its independent tracking polynomial is then $T_i(P_3,x)=x^3$. Similarly every independent cycle tracking set of C_n contains one and only one vertex and for every vertex $v\in V$, $\{v\}$ forms an independent cycle tracking set. So $T_i(C_n,x)=nx$.

Theorem 6.3.3. *Let the graph $G=G_1\cup G_2$ be the union of two graphs with disjoint vertex set. Then $T_i(G,x)=T_i(G_1,x)T_i(G_2,x)$.*

Proof. For $k\geq\tau_i(G)$, an independent cycle tracking set of k vertices in G arises by choosing an independent cycle tracking set of p vertices in G_1 (for some p such that $\tau_i(G_1)\leq p\leq|V(G_1)|$ and an independent cycle tracking set of $k-j$ vertices in G_2. The number of ways of doing this over all $j=\tau_i(G_1),...,|V(G_1)|$ is exactly the coefficient of x^k in $T_i(G_1,x)T_i(G_2,x)$. So $T_i(G,x)=T_i(G_1,x)T_i(G_2,x)$. □

Corollary 6.3.4. *If a graph G has m components $G_1,G_2,...,G_m$ then $T_i(G,x)=T_i(G_1,x)T_i(G_2,x)...T_i(G_m,x)$.*

Proof. We prove this result by mathematical induction on m. For $m=1$ the result is trivial. The case $m=2$ holds by theorem 6.3.3. Suppose that the result is true for $m = k$. Now we have to prove that the result is true for $m = k + 1$. So let G consist of $k + 1$ components $G_1, G_2, ..., G_{k+1}$. Then $T_i(G, x) = T_i(G_1 \cup G_2 \cup ... \cup G_k, x)T_i(G_{k+1}, x)$ by theorem 6.3.3. Is equal to $T_i(G_1, x)T_i(G_2, x)...T_i(G_{k+1}, x)$, by induction hypothesis. □

Theorem 6.3.5. *Let G be a graph of order n. Then*

1. *$t_i(G, i) = 0$ for $i < \tau_i(G)$ or $i > T_i(G)$.*
2. *$T_i(G, x)$ has no constant term.*
3. *$T_i(G, x)$ is a strictly increasing function on $(0, \infty)$.*
4. *zero is a root of $T_i(G, x)$ with multiplicity $\tau_i(G)$.*

Theorem 6.3.6. *Let G be a graph of order n. Then $T_i(G, x) = x^n$ if and only if G is a forest.*

Proof. The independent cycle tracking polynomial of G is x^n if and only if $V(G)$ is the only cycle tracking set for G. That is if and only if $\tau_i(G) = n$. That is if and only if G is a forest (by Theorem 2.1.21). □

Theorem 6.3.7. *For a graph G of order n, $T_i(G, x) = nx$ if and only if G is track connected.*

Proof. Suppose $T_i(G, x) = nx$. Then every independent cycle tracking set contains only one vertex. That is every vertex in G traces all other vertices and hence the graph is track connected.
Conversely suppose that G is track connected. Then $T_i(G) = 1$ and every vertex trace all other vertices. So $T_i(G, x) = nx$. □

Theorem 6.3.8. *Let G be a track connected floral graph with k petals having $m_1, m_2, ..., m_k$ vertices respectively then $T_i(G, x) = x + m_1 m_2 ... m_n\ x^n$.*

Proof. Let G be a track connected floral graph with k petals with each petal having $m_1, m_2, ..., m_k$ vertices respectively. Then every maximal trace independent set contains exactly 1 or n vertices. Here the central vertex is the only vertex that can trace all other vertices, and a set with one vertex from each petal form an independent cycle tracking set. So there are $m_1 m_2 ... m_k$ trace independent sets which contain n vertices. Therefore $T_i(G, x) = x + m_1 m_2 ... m_n\ x^n$. □

Theorem 6.3.9. *Let G be a transitively tracked and let $V(G)$ be partitioned into $V_1, V_2, ..., V_k$ of cardinality $m_1, m_2, ..., m_k$ respectively such that the graph $\langle V_i \rangle$ induced by V_i is maximal track connected subgraph of G for $i = 1, 2, ..., k$. Then $T_i(G, x) = m_1 m_2 ... m_k\ x^k$.*

Proof. G be a transitively tracked graph then its vertex set can be partitioned into $V_1, V_2, ..., V_k$ of cardinality, say $m_1, m_2, ..., m_k$ respectively such that the graph $\langle V_i \rangle$ induced by each V_i is maximal track connected subgraph of G. Then the independent cycle tracking set of G are precisely the sets of vertices which contains exactly one element from each V_i. So an independent cycle tracking set of G with cardinality k can be chosen in $m_1 m_2 ... m_k$ ways. Therefore $T_i(G, x) = m_1 m_2 ... m_k\ x^k$. □

Theorem 6.3.10. *For a graph G, $t_i(G, 1) = 1$ if and only if G is a track connected floral graph.*

Proof. Let G be any graph with $t_i(G, 1) = 1$. Then there exists one and only one independent cycle tracking set S with $|S| = 1$. That is there exist a vertex $v \in V$

such that $T_G(v) = V$ and no other vertex can trace G. And since $\tau_i(G) = 1$, G must be a track connected floral graph. □

6.4 Independent Cycle Tracking Polynomial of Some Graphs

In this section independent cycle tracking polynomial $T_i(G, x)$ of some graphs are derived.

Theorem 6.4.1. *For Firefly graph $F_{s,t,n-2s-2t-1}$, $\tau_i(F_{s,t,n-2s-2t-1}) = n - 2s$.*

Proof. The graph $F_{s,t,n-2s-2t-1}$ has $n-2s-1$ trace free vertices and the common vertex traces all s triangles. So the $n - 2s - 1$ trace free vertices together with the common vertex form a τ_i-set. Hence $\tau_i(F_{s,t,n-2s-2t-1}) = n - 2s$. □

Theorem 6.4.2. *For Firefly graph $F_{s,t,n-2s-2t-1}$,*

$$t_i(F_{s,t,n-2s-2t-1}, p) = \begin{cases} 1 & if\ p = n - 2s, \\ 2^s & if\ p = n - s - 1, \\ 0 & otherwise. \end{cases}$$

and $T_i(F_{s,t,n-2s-2t-1}, x) = x^{n-2s} + 2^s x^{n-s-1}$.

Proof. Let S be an independent cycle tracking set of $F_{s,t,n-2s-2t-1}$. Then S contain all the $n - 2s - 1$ trace free vertices. Suppose v be the common vertex shared by the triangles, the pendant paths of length 2 and the pendant edges.

Case(i) $v \in S$.

Then the other vertices in the s triangles does not belong to S. So $|S| = n - 2s$ and S has only one choice.

Case(ii) $v \notin S$.

Then S contains exactly one vertex from each triangle other than v. So $|S| = n - s - 1$ and there are 2^s choices for S

Hence $t_i(F_{s,t,n-2s-2t-1}, p) = \begin{cases} 1 & if\ p = n - 2s, \\ 2^s & if\ p = n - s - 1, \\ 0 & otherwise. \end{cases}$

Hence $T_i(F_{s,t,n-2s-2t-1}, x) = x^{n-2s} + 2^s x^{n-s-1}$. □

Theorem 6.4.3. *For a Lollipop graph $L_{n,m}$, $\tau_i(L_{n,m}) = m + 1$.*

Proof. Since a vertex in K_n can trace all vertices in it and all vertices of P_m are trace free vertices we need at least $m + 1$ vertices to trace $L_{n,m}$ and hence $\tau_i(L_{n,m}) = m + 1$. □

Theorem 6.4.4. *For a Lollipop graph $L_{n,m}$, $T_i(L_{n,m}, x) = nx^{m+1}$.*

Proof. Let S be an independent cycle tracking set of $L_{n,m}$. Then S contain all the m trace free vertices and exactly one vertex from K_n. So $|S| = m + 1$ and S can be chosen in n ways. So $t(L_{n,m}, p) = \begin{cases} n & if\ p = m + 1, \\ 0 & otherwise. \end{cases}$

Hence $T_i(L_{n,m}, x) = nx^{m+1}$. □

Theorem 6.4.5. *For a Tadpole $T_{(n,l)}$, $\tau_i(T_{(n,l)}) = l + 1$.*

Proof. Since a vertex in C_n can trace all vertices in it and all vertices of P_l are trace free vertices we need at least $m + 1$ vertices to trace $T_{(n,l)}$ and these $m + 1$ vertices are independent. Hence $\tau_i(T_{(n,l)}) = l + 1$. □

Theorem 6.4.6. *For a Tadpole $T_{(n,l)}$, $T_i(T(n,l), x) = nx^{l+1}$.*

Proof. Let S be an independent cycle tracking set of $T_{(n,l)}$. Then S contain all the l trace free vertices and exactly one vertex from C_n. So $|S| = l + 1$ and S can be chosen in n distinct ways. So,

$$t_i(T_{(n,l)}, p) = \begin{cases} n & if\ p = l + 1, \\ 0 & otherwise. \end{cases}$$

Hence $T_i(T_{(n,l)}, x) = nx^{l+1}$. □

Theorem 6.4.7. *For a Helm graph H_n, $\tau_i(H_n) = n$.*

Proof. Since H_n contains $n - 1$ pendant vertices, all these vertices belong to every independent cycle tracking set. Since W_n is track connected, a vertex of W_n together with the pendant vertices form an independent cycle tracking set. So $\tau_c(H_n) = n$. □

Theorem 6.4.8. *For a Helm graph H_n, $T_i(H_n, x) = nx^n$.*

Proof. Let S be an independent cycle tracking set of H_n. Then S contain all the $n - 1$ trace free vertices and exactly one vertex from W_n. So $|S| = n$ and S can be chosen in n distinct ways. So, $t_i(H_n, p) = \begin{cases} n & if\ p = n, \\ 0 & otherwise. \end{cases}$

Hence $T_i(H_n, x) = nx^n$. □

Theorem 6.4.9. *For a Web graph WB_n, $\tau_i(WB_n, x) = n$ and $T_i(WB_n, x) = (2n - 1)x^n$.*

Since a Friendship graph F_n is a track connected floral graph with n petals each having 3 vertices, we have:

Theorem 6.4.10. $\tau_i(F_n) = 1$ *and* $T_i(F_n, x) = x + 2^n x^n$.

Theorem 6.4.11. *For an Armed crown $C_n \bigodot P_m$, $\tau_i(C_n \bigodot P_m) = mn + 1$ and $T_i(C_n \bigodot P_m, x) = nx^{mn+1}$.*

Proof. Since a vertex in C_n can trace all vertices in it and the remaining mn vertices are trace free vertices. So we need at least $mn + 1$ independent vertices to trace $C_n \bigodot P_m$ and hence $\tau_c(C_n \bigodot P_m) = mn + 1$. □

By Theorem 6.3.3, we have :

Theorem 6.4.12. *For any graph G of order n, $\tau_i(G \circ K_1) = n + \tau_c(G)$ and $T_i(G \circ K_1, x) = T_i(G, x)x^n$.*

More generally we have;

Theorem 6.4.13. *$\tau_i(G_n \circ H_m) = n$, where G_n denote a graph with n vertices and H_m denote a connected graph with $m(m > 1)$ vertices.*

Corollary 6.4.14. *If G denote a connected graph of order m. Then $\tau_i(K_1 \circ G) = 1$ and $T_i(K_1 \circ G, x) = (m + 1)x$.*

Proof. Since $K_1 \circ G$ is a track connected graph with $m+1$ vertices, $T_i(K_1 \circ G, x) = (m + 1)x$. □

Chapter 7

Cycle Tracking Matrix of a Graph

A graph G can be represented by various binary matrices such as adjacency matrix, incidence matrix, cut matrix, circuit matrix etc. The concept of cycle tracking sets introduces a new type of matrix called cycle tracking matrix. This is the matter of study of this chapter.

7.1 Cycle Tracking Matrices

A graph G is said to be an ordered graph if its vertex set is a finite sequence. In the case of an ordered graph G we denote the vertex set $V(G)$ with elements $v_1, v_2, ..., v_n$ as $(v_1, v_2, ..., v_n)$ in order to indicate v_1 as the first vertex, v_2 as the second vertex,..., v_n as the n^{th} vertex.

Only ordered graphs are considered throughout this chapter.

Definition 7.1.1. *Let G be an ordered graph with $V(G) = (v_1, v_2, ..., v_n)$ as its vertex set. The cycle tracking matrix of G, denoted by $TM(G)$, is the $n \times n$ binary matrix defined as follows. The rows and columns of $TM(G)$ are indexed by the ordered set $(v_1, v_2, ..., v_n)$. The (i, j)th entry of $TM(G)$ is 1 if and only if $v_i \in T_G(v_j)$.*

Example 7.1.2. *The cycle tracking matrix $TM(G)$ of the graph G in figure 7.1 is*

$$TM(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Proposition 7.1.3. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Then,*

1. *TM(G) is a symmetric matrix with 1 on the diagonal.*

2. *the number of ones in the i^{th} row or i^{th} column of $TM(G)$ is equal to $|T_G(v_i)|$.*

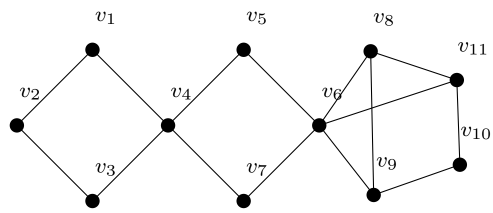


Figure 7.1: Graph G.

3. *the row or column corresponding to each trace free vertex v_i has 1 at i^{th} position and 0 else where.*

4. *there is no row or column having exactly 2 ones.*

Theorem 7.1.4 follows directly from the definition of cycle tracking matrix.

Theorem 7.1.4. *The cycle tracking matrix of a track connected graph is an $n \times n$ matrix with all entries 1.*

Theorem 7.1.5. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. If the $(i, j)^{th}$ entry of $TM(G)$ is zero then $TM(G)_{ik} = TM(G)_{jk} = 1$ for at most one k.*

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Suppose $TM(G)_{ij} = 0$. Then v_i and v_j lie in different maximal track connected components of G. By Theorem 2.1.13 two different maximal track connected components share at most one vertex in G. Therefore there is at most one vertex, say $v_k \in T_G(v_i) \cap T_G(v_j)$. That is $TM(G)_{ik} = TM(G)_{jk} = 1$ for at most one k. □

Theorem 7.1.6. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. If $TM(G)_{ij} = 1$ for some $i \neq j$ then there exists a $k \neq i, j$ such that $TM(G)_{ik} = TM(G)_{jk} = 1$.*

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Suppose $TM(G)_{ij} = 1$ for some $i \neq j$. Then $v_i \in T_G(v_j)$. Therefore there exists a cycle C which contains both v_i and v_j. Since each cycle contains at least 3 vertices, $|V(C)|$ is greater than or equal to 3. Let v_k a vertex of C distinct from v_i and v_j. So $TM(G)_{ik} = TM(G)_{jk} = 1$. □

Theorem 7.1.7. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let B be any $k \times k$ submatrix of $TM(G)$ of the form*

$$B = \begin{bmatrix} 1 & 1 & b_{13} & b_{14} & \dots & b_{1(k-1)} & 1 \\ 1 & 1 & 1 & b_{24} & \dots & b_{2(k-1)} & b_{2k} \\ b_{31} & 1 & 1 & 1 & \dots & b_{3(k-1)} & b_{3k} \\ b_{41} & b_{42} & 1 & 1 & \dots & b_{3(k-1)} & b_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ b_{(n-1)1} & b_{(n-1)2} & b_{(n-1)3} & b_{(n-1)4} & \dots & 1 & 1 \\ 1 & b_{n2} & b_{n3} & b_{n4} & \dots & 1 & 1 \end{bmatrix} \tag{7.1}$$

where $B_{((i-1)(mod k))i} = B_{ii} = B_{((i+1)(mod k))i} = 1$. Then $B_{ij} = 1$ for every $i, j = 1, 2, ..., k$.

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let B be the given $k \times k$ submatrix of TM(G) of the form in (7.1). Without loss of generality assume that B is formed by the subset $(v_1, v_2, ..., v_k)$ of $V(G)$.

Consider the graph H induced by $(T_G(v_1) \cap T_G(v_2)) \cup (T_G(v_2) \cap T_G(v_3)) \cup ... \cup (T_G(v_{k-1}) \cap T_G(v_k)) \cup (T_G(v_k) \cap T_G(v_1))$. We claim that H is track connected so that every entry of B is 1.

If H is not track connected then there exists a cut vertex v in H. Let $D_1, D_2, ..., D_q$

be the components of $H - v$. Then $V(D_i) \cap \{v_1, v_2, ..., v_k\} \neq \emptyset$ as H is formed by union of intersection of traces of vertices. Since the induced graph formed by non empty intersection of traces of vertices is track connected (provided the intersection is non empty), there exist vertices v_i and v_{i+1}(where $1 \leq i \leq k$, with $v_{k+1} = v_1$) which lie in two different components of $H - v$. Since there is only one path in H from each vertex in D_j to any vertex in $D_k(k \neq j)$, there exists only one path from v_i to $v_i + 1$, a contradiction. So H is track connected. Hence $B_{ij} = 1$ for every $i, j = 1, 2, ..., k$. □

Theorem 7.1.8. *Let A be a square, symmetric, binary matrix of order n satisfying the following conditions*

1. *All diagonal entries are 1.*
2. *If $A_{ij} = 1$ for some $i \neq j$ then $A_{ik} = A_{jk} = 1$ for some $k \neq i, j$.*
3. *If ij^{th} element of A is zero then $A_{ik} = A_{jk} = 1$ for at most one k.*
4. *If B is any $k \times k$ submatrix of A of the form*

$$B = \begin{bmatrix} 1 & 1 & b_{13} & b_{14} & \dots & b_{1(k-1)} & 1 \\ 1 & 1 & 1 & b_{24} & \dots & b_{2(k-1)} & b_{2k} \\ b_{31} & 1 & 1 & 1 & \dots & b_{3(k-1)} & b_{3k} \\ b_{41} & b_{42} & 1 & 1 & \dots & b_{3(k-1)} & b_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ b_{(n-1)1} & b_{(n-1)2} & b_{(n-1)3} & b_{(n-1)4} & \dots & 1 & 1 \\ 1 & b_{n2} & b_{n3} & b_{n4} & \dots & 1 & 1 \end{bmatrix}$$

where $B_{((i-1)(modk))i} = B_{ii} = B_{((i+1)(modk))i} = 1$*, then* $B_{ij} = 1$ *for every* $i, j = 1, 2, ..., n$.

Then there exists a graph G *of order* n *whose cycle tracking matrix is* A.

Proof. We need only to prove the necessary part. Let A be a square, symmetric, binary matrix of order n satisfying all the hypothesis of the theorem.
Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$ whose adjacency matrix is $A - I_n$, where I_n is the identity matrix of order n. Then every vertex in V is either isolated or a vertex of a triangle. And for every non adjacent vertices $v_i, v_j \in V$ there exists at most one vertex v_k adjacent to both v_i and v_j.
We claim that $TM(G) = A$. That is for $i \neq j$ $A_{ij} = 1$ if and only if $v_i \in T_G(v_j)$.
Case(i) $A_{ij} = 1$.
Then v_i adjacent to v_j. Then by condition 2 there exists a vertex v_k such that $A_{ik} = A_{jk} = 1$. So $v_i v_j v_k$ is a triangle in G. Thus there exist two distinct paths from v_i to v_j. Hence $v_i \in T_G(v_j)$.
Case(ii)$A_{ij} = 0$.
Then v_i is not adjacent to v_j. If possible let there be two distinct paths from v_i to v_j. These two paths together form the cycle $v_i, u_1, u_2, ..., u_r, v_j, u_{r+1}, ..., u_s, v_i$. Then the submatrix of A corresponding to these vertices is of the form described in condition 4. Then by condition 4, $A_{ij} = 1$ for every i, j, a contradiction. Hence $v_i \notin T_G(v_j)$. □

The Lemma 7.1.9 follows from the fact, whether $v_i \in T_G(v_j)$ or not.

Lemma 7.1.9. *Let* G *be a graph with vertex set* $(v_1, v_2, ..., v_n)$*. Then the* 2×2 *principal submatrix of TM(G) formed by the rows and columns corresponding to*

v_i, v_j $1 \leq i \neq j \leq n$, is either an identity matrix or a matrix with all entries one.

Theorem 7.1.10. *Let G be a graph. Then the 2×2 principal submatrix of TM(G) formed by the rows and columns corresponding to v_i, v_j $1 \leq i \neq j \leq n$, is non singular if and only if $v_i \notin T_G(v_j)$.*

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let B be the 2×2 principal submatrix of TM(G) formed by the rows and columns corresponding to v_i, v_j $1 \leq i \neq j \leq n$. Then B is either the 2×2 identity matrix I_2 or 2×2 matrix with all entries 1. Thus B is non singular if and only if $B = I_2$. That is if and only if $v_i \notin T_G(v_j)$. □

Theorem 7.1.11. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let B^{ij} be the 2×2 principal submatrix of TM(G) formed by the rows and columns corresponding to v_i, v_j. For a fixed j, $\sum_{i \neq j} det B^{ij} = |V(G)| - |T_G(v_j)|$.*

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let B be the 2×2 principal submatrix of TM(G) formed by the rows and columns corresponding to v_i, v_j. For a fixed j, consider the vertex $v_i, i = 1, 2, ..., n$. If $v_i \notin T_G(v_j)$, then B^{ij} is the 2×2 identity matrix I_2. Hence $det(B^{ij}) = 1$.

If $v_i \in T_G(v_j)$, then B^{ij} is the 2×2 matrix with all entries 1. Hence $det(B^{ij}) = 0$. Therefore $\sum_{i \neq j} det(B^{ij}) = \sum_{v_i \notin T_G(v_j)} det(B^{ij}) + \sum_{v_i \in T_G(v_j)} det(B^{ij}) = n - |T_G(v_j)|$. □

Theorem 7.1.12. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. The 3×3 principal submatrix of TM(G) formed by the rows and columns corresponding to the distinct vertices v_i, v_j, v_k is singular if and only if one of the following conditions holds*

1. *$v_i, v_j \in T_G(v_k)$ and $v_i \in T_G(v_j)$ (ie; v_i, v_j and v_k belong to a subgraph of G which is track connected),*

2. *$v_i \in T_G(v_j)$ and $v_k \notin T_G(\{v_i, v_j\})$.*

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let B_{ijk} be the 3×3 principal submatrix of TM(G) formed by the rows and columns corresponding to v_i, v_j, v_k.

Case (i) $v_i, v_j \in T_G(v_k)$ and $v_i \in T_G(v_j)$.

Then B_{ijk} is a 3×3 matrix with all entries 1. Therefore $det(B_{ijk}) = 0$.

Case (ii) $v_i \in T_G(v_j)$ and $v_k \notin T_G(v_i, v_j)$.

Then,

$$B_{ijk} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore $det(B_{ijk}) = 0$.

Case (iii) $v_i, v_j \in T_G(v_k)$ and $v_i \notin T_G(v_j)$.

Then,

$$B_{ijk} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore $det(B_{ijk}) = -1 \neq 0$.

case (iv)$v_i \notin T_G(v_j)$, $v_j \notin T_G(v_k)$ and $v_k \notin T_G(v_i)$.

Then,

$$B_{ijk} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore $det(B_{ijk}) = 1 \neq 0$.

Hence the theorem. □

In any graph G, a vertex in a component of G cannot be traced by a vertex in another component of G. Hence we have the following theorem;

Theorem 7.1.13. *A graph G is the disjoint union of 2 maximal track connected components G_1 and G_2 if and only if the matrix $TM(G)$ is partitioned as*

$$TM(G) = \begin{bmatrix} TM(G_1) & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & TM(G_2) \end{bmatrix}$$

where $TM(G_1)$ and $TM(G_2)$are the cycle tracking matrix of G_1 and G_2 respectively.

Corollary 7.1.14. *Let G be a transitively tracked graph with maximal track connected component $\langle V_1 \rangle, \langle V_2 \rangle, ..., \langle V_k \rangle$ induced by vertex subsets $V_1, V_2, ..., V_k$ with cardinality $m_1, m_2, ..., m_k$ respectively. Then $TM(G)$ takes the block form*

$$\begin{bmatrix} J_{m_1} & 0 & 0 & \dots & 0 \\ 0 & J_{m_2} & 0 & \dots & 0 \\ 0 & 0 & J_{m_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_{m_k} \end{bmatrix},$$

where $J_p, p = m_1, m_2, ..., m_k$ are $p \times p$ matrices with all entries 1.

Corollary 7.1.15. *For transitively tracked graph G, $\tau_c(G) =$ rank of $TM(G)$.*

Theorem 7.1.16. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Then $TM(G)^2_{ij}$, the (i, j)th entry of $TM(G)^2$ is equal to $|T_G(v_i) \cap T_G(v_j)|$.*

Proof. Let G be the graph with $V(G) = (v_1, v_2, ..., v_n)$. Suppose

$$TM(G) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}.$$

Then $a_{ij} = a_{ji}$ and $(TM(G))^2_{ij} = a_{i1}a_{1j} + a_{i2}a_{2j} + ... + a_{in}a_{nj}$. And for $k = 1, 2, ..., n$, $a_{ik}a_{kj} = 1$ if and only if $a_{ik} = 1$ and $a_{kj} = 1$. That is if and only if $v_k \in T_G(v_i)$ and $v_k \in T_G(v_j)$. That is if and only if $v_k \in T_G(v_i) \cap T_G(v_j)$. Therefore $|T_G(v_i) \cap T_G(v_j)| = a_{i1}a_{1j} + a_{i2}a_{2j} + ... + a_{in}a_{nj} = (TM(G)^2)_{ij}$. □

Corollary 7.1.17. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Then $|T_G(v_i)| = (TM(G)^2)_{ii}$.*

For almost all graph G, $TM(G)$ is singular, the only exception is forest.

Theorem 7.1.18. *For a graph G, determinant of $TM(G)$ is non zero if and only if G is a forest. Moreover determinant of $TM(G) = 1$ for any forest G.*

Proof. Let G be a graph. Suppose G is not a forest. In this case there exist two vertices v_i and v_j such that $T_G(v_i) = T_G(v_j)$. Without loss of generality assume that $v_1 \in T_G(v_2)$, v_1 has minimum tracing number among all vertices of G and v_2 has minimum tracing number among all vertices in $T_G(v_1) \setminus \{v_1\}$. If $T_G(v_1) \neq T_G(v_2)$, then v_2 is a cut vertex. Then there exist at least two vertices in $T_G(v_2)$ which are not in $T_G(v_1)$. Let v_3 have minimum tracing number among all such vertices and let v_4 be a vertex having minimum tracing number among all vertices in $T_G(v_2) \cap T_G(v_3) \setminus \{v_2, v_3\}$. If $T_G(v_3) \neq T_G(v_4)$, then v_4 is a cut vertex and we can repeat the process again. As G is a finite graph this process cannot be repeated indefinitely. The process will be terminated at, say k^{th} stage only if there exists one vertex v_{2k-1} with minimum tracing number among the vertices $T_G(v_{2k-2}) \setminus T_G(v_{2k-3})$ and another vertex v_{2k} with minimum tracing number among all vertices in $T_G(v_{2k-2}) \cap T_G(v_{2k-1}) \setminus \{v_{2k-2}, v_{2k-1}\}$, such that v_{2k-1} and v_{2k} have the same cycle tracking set.

Thus if G is not a forest there exist at least two vertices v_{2k-1} and v_{2k} such that $T_G(v_{2k-1}) = T_G(v_{2k})$. Hence the rows in $TM(G)$ corresponding to v_{2k-1} and v_{2k} are identical. Therefore $det(TM(G)) = 0$.

Conversely for a forest G, $TM(G)$ is the $n \times n$ identity matrix. Hence $det(TM(G)) = 1$. □

For a graph G of order n with vertex set $(v_1, v_2, ..., v_n)$, $TM(G)$ is an $n \times n$ matrix over $\mathbb{R}$. Any such matrix define a linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$ which carries e_i to $(a_1, a_2, ..., a_n)$ for $i = 1, 2, ..., n$ where $\{e_1, e_2, ..., e_n\}$ is the standard ordered basis for $\mathbb{R}^n$ and $a_j = \begin{cases} 1 & if\ v_i \in T_G(v_j) \\ 0 & otherwise \end{cases}$.

Let us denote this transformation again by $TM(G)$. Then $TM(G)e_i = (a_1, a_2, ..., a_n)$.

Definition 7.1.19. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let S be any subset of $V(G)$. The characteristic function of S $\chi_S : V \to \{0, 1\}$ defined by $\chi_S(v_i) = \begin{cases} 1 & if\ v_i \in S \\ 0 & otherwise \end{cases}$. The vector $(\chi_S(v_1), \chi_S(v_2), ..., \chi_S(v_n))$ is called the trace vector corresponding to S.*

Theorem 7.1.20. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let S be any subset of G and t be the trace vector corresponding to S. Then i^{th} entry of $TM(G)t = |T_G(v_i) \cap S|$.*

Proof. Let S be any subset of G. Let t be the trace vector corresponding to S. Then $TM(G)t = \chi_S(v_1)TM(G)e_1 + \chi_S(v_2)TM(G)e_2 + ... + \chi_S(v_n)TM(G)e_n$. The i^{th} entry of $TM(G)e_j = 1$ if and only if $v_i \in T_G(v_j)$, and $\chi_S(v_j) = 1$ if and only if $v_j \in S$. So the i^{th} entry of $\chi_S(v_j)TM(G)e_j = 1$ if and only if $v_i \in T_G(v_j)$ and $v_j \in S$. That is if and only if $v_j \in T_G(v_i)$ and $v_j \in S$. That is if and only if $v_j \in T_G(v_i) \cap S$. That is $\chi_S(v_j)TM(G)e_j = 1$ if and only if $v_j \in T_G(v_i) \cap S$. So i^{th} entry of $TM(G)t = |T_G(v_i) \cap S|$. □

Theorem 7.1.21. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let S be a cycle tracking set of G and t be the trace vector corresponding to S. Then $TM(G)t \geq 1$. That is all the entries of $TM(G)t$ are greater than or equal to 1.*

Proof. Let S be a cycle tracking set of G. Let t be a trace vector corresponding to S. For any vertex v_i in V there exist a vertex $v_j \in S$ such that $v_j \in T_G(v_i)$. So i^{th} entry of $TM(G)t = |T_G(v_i) \cap S| \geq 1$. □

Corollary 7.1.22. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let S be any cycle tracking set of G and t be the trace vector corresponding to S. Then the i^{th} entry of $TM(G)t$ is 1 if and only if v_i is in private trace of some vertex in S.*

The following theorem gives the necessary and sufficient condition for a subset S of $\tau_c(G)$ vertices in a graph G to be a $\tau_c - set$ of G.

Theorem 7.1.23. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let $\tau_c(G) = k$ and let S be a subset of V with k vertices. Let $t = (t_1, t_2, ..., t_n)$ be the trace vector corresponding to S. Then $TM(G)t \geq 1$ if and only if S is a τ_c- set.*

Proof. Suppose $TM(G)t \geq 1$. Then each entry of $TM(G)t \geq 1$. That is $TM(G)_{i1}t_1 + TM(G)_{i2}t_2 + ... + TM(G)_{in}t_n \geq 1$ for every i. Therefore $TM(G)_{ik}t_k =$

1 for every i and for some k . That is $TM(G)_{ik} = 1$ and $t_k = 1$ for every i and for some k. This is possible only if $v_i \in T_G(v_k)$ and $v_k \in S$ for every i and for some k. Which implies that S is a cycle tracking set. Since $\tau_c(G) = k$, S is a τ_c- set.

The converse follows from Theorem 7.1.21. □

7.2 τ_c -Spectrum of a Graph

For a graph G of order n, $TM(G)$ is a real symmetric matrix of order n, with trace n. Since the rows and columns of $TM(G)$ correspond to the labeling of the vertices of G, we are interested in those properties of $TM(G)$ which are invariant under permutations of rows and columns of $TM(G)$. One such a property is the spectral property of $TM(G)$.

Since $TM(G)$ is a real symmetric matrix, $TM(G)$ has real eigenvalues and eigenvectors corresponding to distinct eigenvalues are orthonormal[5]. Every eigenvalue λ of $TM(G)$ is a root of the polynomial $det(\lambda I - TM(G))$. In fact the multiplicity of λ as a root of the polynomial $det(\lambda I - TM(G))$ is the dimension of the space of eigenvectors corresponding to λ.

Definition 7.2.1. *The spectrum of the cycle tracking matrix $TM(G)$ of the graph G is called the τ_c − spectrum of G. If the distinct eigenvalues of $TM(G)$ are $\lambda_1 > \lambda_2 > ... > \lambda_p$ with multiplicities $m(\lambda_1), m(\lambda_2), ..., m(\lambda_p)$ respectively, then we write*

$$\tau_c - Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \vdots & \lambda_p \\ m(\lambda_1) & m(\lambda_2) & \vdots & m(\lambda_p) \end{pmatrix}.$$

For example, for the complete graph K_n, $TM(K_n)$ is the $n \times n$ matrix with all entries 1 and $\tau_c - Spec(K_n) = \begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}$.

The eigenvalues of $TM(G)$ are called τ_c-*eigen values* of G and the characteristic polynomial $det(\lambda I - TM(G)) = 0$ is called the τ_c-*charecteristic polynomial* of G and is denoted by $\chi_{\tau_c}(G, \lambda)$.

Theorem 7.2.2. *Let G be a graph of order n. If the τ_c- characteristic polynomial $\chi_{\tau_c}(G, \lambda)$ of G is,*

$$\chi_{\tau_c}(G, \lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n$$

then the coefficient of τ_c-charecteristic polynomial satisfies the following properties,

1. $a_1 = trace\ of\ TM(G) = n.$
2. $a_2 = |A|$, *where* $A = \{\{u, v\} \subset V : u \notin T_G(v)\}$.
3. $a_3 = |D| - |C|$, *where* $C = \{\{u, v, w\} \subset V$ *:no two of them lie on a common cycle*$\}$ *and* $D = \{\{u, v, w\} \subset V : u, w \in T_G(v)$ *but* $w \notin T_G(u)\}$.

Proof. Let G be a graph of order n. Let $\chi_{\tau_c}(G, \lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n$.

For each $i, i = 1, 2, ..., n$, the number $(-1)^i a_i$ is the sum of those principal minors of $TM(G)$ which have i rows and columns. Thus we have the following results.

1) Since diagonal elements of $TM(G)$ are all one, $a_1 = n$.

2) A principal minor with two rows and columns, must be of the form $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ or $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. The determinant is non zero only if it is of the form $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. There is one such minor for each pair of vertices u, v of G such that $u \notin T_G(v)$, and each has determinant 1. Hence $a_2 = |A|$, where $A = \{\{u, v\} : u \notin T_G(v)\}$.

3) There are four possibilities for 3×3 principal minors. They are $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$. Of these the last two have non zero determinant -1 and 1 respectively. The principal minor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ corresponds to three mutually non tracing vertices in G and the principal minor $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ corresponds to the triple of vertices u, v, w such that $u, w \in T_G(v)$ and $u \notin T_G(w)$. Thus $(-1)^3 a_3 = |C| - |D|$, where $C = \{\{u, v, w\}$:no two of them lie on a common cycle$\}$ and $D = \{\{u, v, w\} : u, w \in T_G(v)\ but\ w \notin T_G(u)\}$. □

Corollary 7.2.3. *Let G be a graph of order n and let the n eigenvalues of $TM(G)$, counting multiplicities be $\lambda_1, \lambda_2, ..., \lambda_n$. Then*

1. $\lambda_1 + \lambda_2 + ... + \lambda_n = n$ *and*

2. $\lambda_1 \lambda_2 ... \lambda_n = det(TM(G))$.

Proposition 7.2.4. *Let G be a graph. Zero is an eigenvalue of $TM(G)$ if and only if G is not a forest.*

Theorem 7.2.5. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. If G has a trace free vertex then 1 is an eigenvalue of $TM(G)$.*

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$ and let v_1 be a trace free vertex of G. Then

$$TM(G) = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix},$$

where $A = TM(G - v_1)$. Let $\mathbf{u} = [1, 0, ..., 0]^T$, the transpose of $[1, 0, ..., 0]^T$. Then $TM(G)\mathbf{u} = \mathbf{u}$. Hence 1 is an eigenvalue of $TM(G)$. □

Theorem 7.2.6. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. If G is a track connected graph of order n, then the eigenvalues of $TM(G)$ are n and 0 with respective multiplicities 1 and $n - 1$.*

Proof. If G is a track connected graph of order n, then the cycle tracking matrix of G is the $n \times n$ matrix with all entries 1. It is a symmetric matrix of rank 1. Hence it has only one non zero eigenvalue, which is the trace of $TM(G)$. Thus the eigenvalues of $TM(G)$ are$n$ and 0 with respective multiplicities 1 and $n - 1$. □

Theorem 7.2.7. *Let G be a transitively tracked graph with maximal track connected components $G_1, G_2, ..., G_k$ with their respective vertex set $V_1, V_2, ..., V_k$ of* *cardinality* $m_1, m_2, ..., m_k$. *Then* $m_1, m_2, ..., m_k$ *are the non zero eigenvalues of* G.

Proof. Let G be a graph which satisfies the hypothesis of the theorem. Let $\mathbf{u_i} = [0, ..., 0, 1, 1, ..., 1, 0, ..., 0]^T$, where
the j^{th} entry $\mathbf{u_i}(j)$ of $\mathbf{u_i}$ is,

$$\mathbf{u_i}(j) = \begin{cases} 1 & for\ m_1 + m_2 + ... + m_{i-1} < j \leq m_1 + m_2 + ... + m_i \\ 0 & otherwise. \end{cases}$$

By Corollory 7.1.14

$$TM(G) = \begin{bmatrix} J_{m_1} & 0 & 0 & \dots & 0 \\ 0 & J_{m_2} & 0 & \dots & 0 \\ 0 & 0 & J_{m_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_{m_k} \end{bmatrix},$$

where $J_p (p = m_1, m_2, ..., m_k)$ is a $p \times p$ matrix with all entries 1.

Then $TM(G)\mathbf{u_i} = m_i\mathbf{u_i}$, so that m_i is an eigenvalue of $TM(G)$. Thus $m_1, m_2, ..., m_k$ are eigenvalues of $TM(G)$.

□

Corollary 7.2.8. *Let* G *be a transitively tracked graph with maximal track connected components* $G_1, G_2, ..., G_k$ *with their respective vertex set* $V_1, V_2, ..., V_k$ *of cardinality* $m_1, m_2, ..., m_k$. *Then the eigenvalues are precisely* $m_1, m_2, ..., m_k$ *each with multiplicity 1 and 0 with multiplicity* $n - k$.

Proof. Let G be a graph as stated in the corollary. By Theorem 7.2.7, $m_1, m_2, ..., m_k$ are eigenvalues of $TM(G)$. Since rank of $TM(G) = k$, there are only k non zero eigenvalues for $TM(G)$. Therefore the eigenvalue are precisely $m_1, m_2, ..., m_k$ with multiplicity 1 and 0 with multiplicity $n-k$. $\square$

Theorem 7.2.9. *For any eigenvalue λ of $TM(G)$ we have $|\lambda| \leq T$, the maximal tracing number.*

Proof. Suppose that $TM(G)\mathbf{u} = \lambda\mathbf{u}$, $\mathbf{u} \neq 0$ and let $\mathbf{u}_j$ denote an entry of $\mathbf{u}$ which is largest in absolute value. Note that $\lambda\mathbf{u}_j = (TM(G)\mathbf{u})_j = \sum' \mathbf{u}_i$, where the summation is over those i for which $v_i \in T_G(v_j)$. Therefore $|\lambda||\mathbf{u}_j| = |\sum' \mathbf{u}_i| \leq |T_G(v_j)||\mathbf{u}_j|$. Hence $|\lambda| \leq |T_G(\mathbf{u}_j)| \leq T$. $\square$

Theorem 7.2.10. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$. Let λ be an eigenvalue of $TM(G)$ and $\mathbf{x} = (x_1, x_2, ..., x_n)^T$, the transpose of $(x_1, x_2, ..., x_n)$ be an eigenvector corresponding to it. Then $\lambda x_i = \sum\limits_{v_j \in T_G(v_i)} x_j$, for every $i = 1, 2, ..., n$.*

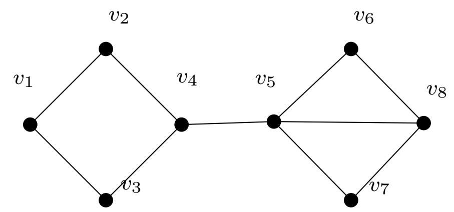
7.3 Automorphism and Cycle Tracking Matrix of a Graph

Theorem 7.3.1. *If v is a vertex of the graph G and g an automorphism of G then $|T_G(g(v))| = |T_G(v)|$.*

Proof. Let $T(v)$ denote the subgraph of G induced by $T_G(v)$ in G. Then $g(T(v))$ is isomorphic to $T(v)$, the maximal track connected subgraph of G containing v. Hence $g(T(v))$ is the maximal track connected subgraph of G containing $g(v)$. So $|T_G(v)| = |T_G(g(v))|$. $\square$

This shows that the automorphism group of a graph permutes the vertices of equal tracking number among themselves.

The converse of Theorem 7.3.1 is not true.



G

Figure 7.2: Graph G.

In the graph G, in figure 7.3 consider the mapping $g : V \longrightarrow V$ defined by

$g(v_1) = v_8$, $g(v_2) = v_6$, $g(v_3) = v_7$, $g(v_4) = v_5$, $g(v_5) = v_4$, $g(v_6) = v_2$, $g(v_7) = v_3$, $g(v_8) = v_1$.

Then $|T_G(g(v))| = |T_G(v)|$ for every $v \in V$, but g is not an automorphism.

Corollary 7.3.2. *If G is a connected transitive graph, then $|T_G(v)| = |T_G(u)|$ for every pair of vertices $u, v \in V(G)$.*

Lemma 7.3.3 follows from the fact that, an automorphism of a graph is a permutation of the vertex set that preserves the graph structure.

Lemma 7.3.3. *Let G be a graph and let $u, v \in V(G)$. Then $u \in T_G(v)$ if and only if $g(u) \in T_G(g(v))$ for any $g \in Aut(G)$, where $Aut(G)$ denotes the class of all automorphism of G.*

Theorem 7.3.4. *Cycle tracking sets of a graph G are preserved under automorphisms. ie; If $\{v_1, v_2, ..., v_k\}$ is a cycle tracking set of G and g an automorphism of G then $\{g(v_1), g(v_2), ..., g(v_k)\}$ is a cycle tracking set of G.*

Proof. Let G be any graph. Let $S = \{v_1, v_2, ..., v_k\}$ be a cycle tracking set of G and g an automorphism of G. Let $S^* = \{g(v_1), g(v_2), ..., g(v_k)\}$.
Let $u \in G$. Then there exists a $v \in V(G)$ such that $u = g(v)$. Since S is a cycle tracking set of G, there exist $v_i \in S$ such that $v \in T_G(v_i)$. By Lemma 7.3.3, $u \in T_G(g(v_i))$. That is for every $u \in V(G)$ there exists a vertex $g(v_i) \in S^*$ such that $u \in T_G(g(v_i))$. Therefore $\{g(v_1), g(v_2), ..., g(v_k)\}$ is a cycle tracking set of G. □

Let G be an ordered graph with $V(G) = (v_1, v_2, ..., v_n)$ and let g be an automorphism of G. Then the permutation matrix P representing g is defined by,
$P = (p_{ij})$, where $p_{ij} = \begin{cases} 1 & if\ v_i = g(v_j) \\ 0 & otherwise. \end{cases}$

Theorem 7.3.5. *Let $TM(G)$ be the cycle tracking matrix of a graph G and let g be an automorphism of G. Then $P(TM(G)) = (TM(G))P$, where P is the permutation matrix representing g.*

Proof. Let $TM(G) = (a_{ij})$ and let $v_h = g(v_i)$ and $v_k = g(v_j)$. Then
$(P(TM(G)))_{hj} = \sum_l p_{hl}a_{lj} = a_{ij}$
and $(TM(G)P)_{hj} = \sum_l a_{hl}p_{lj} = a_{hk}$.

So $(P(TM(G)))_{hj} = ((TM(G))P)_{hj}$ if and only if $a_{ij} = a_{hk}$. That is if and only if $v_i \in T_G(v_j) \iff v_h \in T_G(v_k)$. Which is true by Theorem 7.3.3.
Since h and j are arbitrary $P(TM(G)) = (TM(G))P$.

□

Corollary 7.3.6. *If G_1 and G_2 are isomorphic graphs then there exists a permutation matrix P such that $TM(G_1) = P^T(TM(G_2))P$, where P^T is the transpose of P.*

Theorem 7.3.7. *Let λ be a simple eigenvalue of $TM(G)$, and let $\mathbf{x}$ be an eigenvector corresponding to it. Let g be an automorphism of G and P be the permutation matrix representing g. Then $P\mathbf{x} = \pm\mathbf{x}$.*

Proof. Let λ be a simple eigenvalue of $TM(G)$, and let $\mathbf{x}$ be a corresponding eigenvector. Since $TM(G)$ is real and symmetric λ is real and $\mathbf{x}$ has real coordinates. Let g is an automorphism of G and P be the permutation matrix representing g. Then $(TM(G))P\mathbf{x} = P(TM(G))\mathbf{x} = P\lambda\mathbf{x} = \lambda P\mathbf{x}$. Hence $P\mathbf{x}$ is an eigenvector of $TM(G)$ corresponding to λ. Since λ is a simple eigenvalue of $TM(G)$, $\mathbf{x}$ and $P\mathbf{x}$ are linearly dependent. Therefore $P\mathbf{x} = \alpha\mathbf{x}$ for some real number α. The theorem now follows from the fact that $P^s = 1$ for some positive integer s. □

Chapter 8

Total Cycle Tracking Sets of a Graph

This chapter introduces total cycle tracking sets of a graph, which has many applications. For example, distribution of service centers in a locality can be analyzed through graph theory by considering the service centers as the vertices, reachability between them as the edges, and by defining a cycle tracking set for it. A τ_c- set of vertices can reach all other vertices in the locality in two distinct ways. Therefore selection of a cycle tracking set increases the efficiency of a service providing network. And in the case of total cycle tracking set every service center is reachable from at least one service center different from it in two distinct ways. So to increase the efficiency the concept of total cycle tracking set is more relevant.

8.1 Total Cycle Tracking Set

Definition 8.1.1. *A set $S \subset V$ is a total cycle tracking set if every vertex v in V is traced by some vertex of $S \setminus \{v\}$. In this case we say that S totally tracks V.*

Definition 8.1.2. *A total cycle tracking set is a minimal total cycle tracking set if no proper subset S' of S is a total cycle tracking set.*

Definition 8.1.3. *The total cycle tracking number $\tau_t(G)$ of a graph G is the minimum cardinality of a minimal total cycle tracking set of G.*

Definition 8.1.4. *The upper total cycle tracking number $T_t(G)$ of a graph G is the maximum cardinality of a minimal total cycle tracking set of G.*

Definition 8.1.5. *A total cycle tracking set with minimum cardinality is called a $\tau_t - set$ of G.*

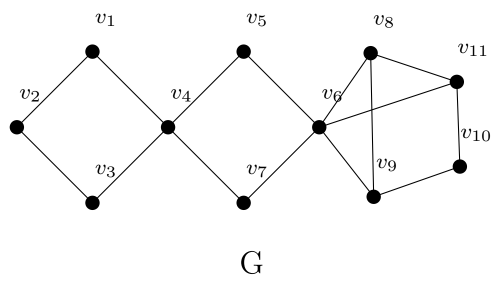


Figure 8.1: $\tau_t(G) = 2$.

The sets $\{v_1, v_3, v_5, v_7, v_8, v_9\}$ and $\{v_4, v_6\}$ are two minimal total cycle tracking sets of the graph G in figure 8.1 and for this graph $\tau_t(G) = 2$ and $T_t(G) = 6$. Moreover $\{v_4, v_6\}$ is a $\tau_t - set$ of G.

Note that these parameters are defined only for graphs without trace free vertices. So throughout this chapter, by a graph G, we mean the graph without trace free vertices.

Let $S \subset V$ and let v be a vertex in S. The *S-private trace* of v denoted by $pt[v, S]$ is defined by $pt[v, S] = \{w \in V : T_G(w) \cap S = \{v\}\}$(Definition 2.1.17), while its *open S-private trace* is defined as $pt(v, S) = \{w \in V : (T_G(w) \cap S) \setminus \{w\}) = \{v\}\}$. The sets $pt[v, S] \setminus S$ and $pt(v, S) \setminus S$ are one and the same. The *S-external private trace* $ept[v, S]$ of v is defined by $ept[v, S] = pt[v, S] \setminus S$. It is also denoted by $ept(v, S)$. The *S-internal private trace* $ipt[v, S]$ of v is defined by $ipt[v, S] = pt[v, S] \cap S$ and its *open S-internal private trace* $ipt(v, S)$ is defined by $ipt(v, S) = pt(v, S) \cap S$. We define an *S-external private trace* of v to be a vertex in $ept(v, S)$ and an *S-internal private trace* of v to be a vertex in $ipt(v, S)$.

In figure 8.1, consider the subset $S = \{v_4, v_6\}$ of V. Then $pt[v_4, S] = \{v_1, v_2, v_3\}$, $pt(v_4, S) = \{v_1, v_2, v_3, v_6\}$, $ept[v_4, S] = \{v_1, v_2, v_3\}$, $ipt[v_4, S] = \emptyset$ and $ipt(v_4, S) = \{v_6\}$.

Theorem 8.1.6. *A total cycle tracking set $S \subset V$ of vertices in a graph G is a minimal total cycle tracking set if and only if every vertex v in S has at least one element in S-private trace (that is for every $v \in S$, $pt[v, S] \neq \emptyset$) or there exists a vertex $u \in S$ such that $u \in pt[v, S \setminus \{u\}]$.*

Proof. Assume that S is a minimal total cycle tracking set of G. Then for every vertex $u \in S$, $S \setminus \{u\}$ is not a total cycle tracking set. This means that some vertex $v \in V \setminus \{u\}$ is not traced by any vertex in $S \setminus \{u, v\}$. If v is not traced by $S \setminus \{u, v\}$ but traced by $S \setminus \{v\}$, then the vertex v is traced only by u in $S \setminus \{v\}$. That is $T_G(v) \cap S \setminus \{v\} = \{u\}$.

Conversely suppose that S is a total cycle tracking set and for each vertex $u \in S$,

$pt[u, S] \neq \emptyset$. We show that S is minimal total cycle tracking set. Suppose that S is not a minimal total cycle tracking set. Then there exists a vertex $u \in S$ such that $S \setminus \{u\}$ is a total cycle tracking set. Then every vertex in $V \setminus S \cup \{u\}$ is traced by at least one vertex in $S \setminus \{u\}$, that is $pt[u, S] = \emptyset$, which contradicts the assumption. $\square$

Theorem 8.1.7. *Let S be a total cycle tracking set in a graph G. Then, S is a minimal total cycle tracking set in G if and only if $|ept(v, S)| \geq 1$ or $|ipn(v, S)| \geq 1$ for each $v \in S$.*

Proof. Let S be a minimal total cycle tracking set in G and let $v \in S$. If $|ept(v, S)| = |ipt(v, S)| = 0$, then every vertex $x \in V(G)$ must be traced by a vertex in $S \setminus \{v\}$, that is $T_G(x) \cap (S \setminus \{v\}) \neq \emptyset$. Hence, $S \setminus \{v\}$ is a total cycle tracking set of G, contradicting the minimality of S. Therefore, $|epn(v, S)| \geq 1$ or $|ipn(v, S)| \geq 1$ for each $v \in S$. Conversely, if $|epn(v, S)| \geq 1$ or $|ipn(v, S)| \geq 1$ for each $v \in S$, then $S \setminus \{v\}$ could not be a total cycle tracking set. $\square$

Definition 8.1.8. *For a subset S of vertices in a graph G, the open trace boundary $OTB(S)$ of S is defined as $OTB(S) = \{v \in V : |(T_G(v) \cap S) \setminus \{v\}| = 1\}$; that is, OTB(S) is the set of vertices traced by exactly one vertex in S other than itself.*

Theorem 8.1.9. *A total cycle tracking set S in a graph G is a minimal total cycle tracking set if and only if for every $v \in S$ there exists a vertex $u \in OTB(S)$ such that $v \in T_G(u) \setminus \{u\}$.*

Proof. Suppose first that for every $v \in S$ there exists a vertex $u \in OTB(S)$ such that $v \in T_G(u) \setminus \{u\}$. Then, $T_G(u) \cap S \setminus \{u\} = \{v\}$. If $u \notin S$, then $u \in ept(v, S)$.

If $u \in S$, then $u \in ipt(v,S)$. Hence, $ept(v,S) \neq \emptyset$ or $ipt(v,S) \neq \emptyset$ for every vertex $v \in S$. Thus, by Theorem 8.1.7, S is a minimal total cycle tracking set.
To prove the necessary part, suppose that S is a minimal total cycle tracking set. Let $v \in S$. By Theorem 8.1.7, $ept(v,S) \neq \emptyset$ or $ipt(v,S) \neq \emptyset$. If $ept(v,S) \neq \emptyset$, then there exists a vertex $u \in V \setminus S$ such that $T_G(u) \cap S = \{v\}$. Therefore $u \in OTB(S)$. On the other hand, if $ipt(v,S) \neq \emptyset$, then there exists a vertex $u \in S \setminus \{v\}$ such that $(T_G(u) \cap S) \setminus \{u\} = \{v\}$, and so $u \in OTB(S)$. So for every $v \in S$ there exists a vertex $u \in OTB(S)$ such that $v \in T_G(u) \setminus \{u\}$. □

Theorem 8.1.10. *Let S be a τ_t-set of a graph G. Then there exist at most two vertices u, v in S with $T_G(v) = T_G(u)$.*

Proof. Let S be a $\tau_t - set$ of the graph G. If possible let $u, v, w \in S$ be such that $T_G(u) = T_G(v) = T_G(w)$. If that is the case then $S \setminus \{u\}$ is a total cycle tracking set. It will lead to a contradiction.
Let $y \in V$. Since S is a τ_t- set, there exists a vertex $x \in S \setminus \{y\}$ such that $y \in T_G(x)$.
Case(1): $x = u$ and $y \neq v$.
In this case $y \in T_G(u) = T_G(v)$. That is there exists a vertex named by v such that $v \in (S \setminus \{u\}) \setminus \{y\}$ such that $y \in T_G(v)$.
Case(2): $x = u$ and $y = v$.
Then $y \in T_G(x) = T_G(u) = T_G(w)$. That is there exists a vertex, namely $w \in (S \setminus \{u\}) \setminus \{y\}$ such that $y \in T_G(w)$.
Case(3): $x \neq u$.
Then x serves the purpose, because $y \in S$
Hence in all the cases the vertex $y \in V$ is traced by some vertex of $(S \setminus \{u\}) \setminus \{y\}$.

As y is arbitrary, every vertex y in V is traced by $(S \setminus \{u\}) \setminus \{y\}$. □

Theorem 8.1.11. *Let G be a transitively tracked graph and S be any minimal total cycle tracking set of G. Then for every vertex $v \in S$ there exists a unique vertex $u \in S \setminus \{v\}$ such that $u \in T_G(v)$.*

Proof. Let G be a transitively tracked graph and S be any minimal total cycle tracking set of G. Let $v \in S$. Since S is a total cycle tracking set there exists a vertex $u \in S \setminus \{v\}$ such that $v \in T_G(u)$.

To prove the uniqueness, assume that there exist two vertices $u, w \in S \setminus \{v\}$ such that $u, w \in T_G(v)$. Since G is transitively tracked $T_G(v) = T_G(u) = T_G(w)$, a contradiction to Theorem 8.1.10. □

Theorem 8.1.12. *Let G be a transitively tracked graph and S be any minimal total cycle tracking set of G. Then $OTB(S) = S$.*

Proof. Let G be a transitively tracked graph and S be any minimal total cycle tracking set of G. Let $v \in S$. Then by Theorem 8.1.11 there exists a unique vertex $u \in S \setminus \{v\}$ such that $u \in T_G(v)$. That is $|(T_G(v) \cap S) \setminus \{v\}| = 1$. So $v \in OTB(S)$. Therefore $S \subset OTB(S)$.

Let $w \in V \setminus S$. Since S is a total cycle tracking set there exists a vertex $v \in S$ such that $w \in T_G(v)$ and a vertex $u \in S \setminus \{v\}$ such that $v \in T_G(u)$. Since G is transitively tracked $w \in T_G(u)$. So $|(T_G(w) \cap S) \setminus \{w\}| \geq 2$. Therefore $w \notin OTB(S)$. That is $V \setminus S \cap OTB(S) = \emptyset$. Thus $OTB(S) = S$. □

8.2 Bounds for Total Cycle Tracking Number of a Graph

In this section we examine the bounds for total cycle tracking number $\tau_t(G)$ of a graph G of order n. For any graph G, every total cycle tracking set is a cycle tracking set. Therefore $\tau_c(G) \leq \tau_t(G)$.

Theorem 8.2.1. *Let G be a graph of order n. Then $\tau_t(G) \leq \frac{2n}{3}$.*

Proof. Let G be a graph . For every vertex v in G there exists a non trivial cycle containing v . So $|T_G(v)| \geq 3$ for every $v \in V$ and we need only at most 2 vertices to form a total cycle tracking set of G from each cycle. Hence $\tau_t(G) \leq \frac{2n}{3}$. □

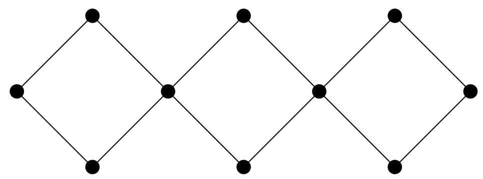
If G is C_3, $\tau_t(G) = \frac{2n}{3}$. So the bound in Theorem 8.2.1 is sharp.

If G is a graph with a cycle C of length 4 and if u, v are two vertices in C, by theorem 8.2.1 $\tau_t(\langle V \setminus T_G(u,v) \rangle) \leq \frac{2(n-4)}{3}$. Then a minimal total cycle tracking set of $T_G(V \setminus T_G(u,v))$ together with the vertices u and v form a total cycle tracking set of G, $\tau_t(G) \leq \frac{2(n-4)}{3} + 2 < \frac{2n}{3}$. Therefore if $\tau_t(G)$ takes the value $\frac{2n}{3}$ then $n \equiv 0(mod3)$ and every cycle in G is a triangle. These triangles may or may not be connected by cut edges.

We can summarize this result as follows;

Theorem 8.2.2. *Let G be a graph of order n. Then $\tau_t(G) = \frac{2n}{3}$ if and only if G is a graph having $n/3$ disjoint triangles and the edges other than the edges of the triangles are all cut edges.*

Theorem 8.2.3. *For any graph G, $\tau_t(G) \leq 2\tau_c(G)$.*



G

Figure 8.2: graph G.

Proof. Let G be any graph and $S = \{u_1, u_2, ..., u_m\}$ be a τ-set of G. Choose v_i such that $v_i \in T_G(u_i)$ for $1 \leq i \leq m$. Then $S^* = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$ forms a total cycle tracking set of G. Hence $\tau_t(G) \leq 2\tau_c(G)$. □

Thus we come across the conclusion: for any graph G, $\tau_c(G) \leq \tau_t(G) \leq 2\tau_c(G)$. Both of these bounds are sharp. The upper bound is attained if G is either track connected graph or track connected floral graph and the lower bound is attained for the graph G in figure 8.2.

Proposition 8.2.4. *If G is any graph with $\tau_c(G) = 1$ then $\tau_t(G) = 2$ so that $\tau_t(G) = 2\tau_c(G)$.*

This result can be generalized as follows.

Theorem 8.2.5. *If G is a transitively tracked graph then $\tau_t(G) = 2\tau_c(G)$.*

Proof. Let G be a transitively tracked graph. Then $V(G)$ can be partitioned into $V_1, V_2, ..., V_k$ such that for each i the graph induced by V_i is a maximal track connected subgraph of G and $\tau_c(G) = k$. Let if possible $\tau_t(G) < 2k$. Let S be a $\tau_t - set$. Then there exist vertices $u, v, w \in S$ such that $u, v \in T_G(w)$. Therefore

$T_G(v) = T_G(u) = T_G(w)$ (since G is transitively tracked), a contradiction to the theorem 8.1.10. $\square$

The converse of Theorem 8.2.5 need not be true. Though floral graphs are not transitively tracked, $\tau_t(G) = 2 = 2\tau_c(G)$.

Theorem 8.2.6. *For a graph G of order n,* $\lceil \frac{n}{T} \rceil \leq \tau_t(G) \leq n - T + 2$.

Proof. Let S be a τ_t- set of G. As each vertex can trace at most T vertices, $\tau_c(G) \geq \lceil \frac{n}{T} \rceil$.
For the upper bound let v be a vertex of maximum trace T. Then v traces $T_G(v)$ and each vertex in $V \setminus T_G(v)$ traces itself and at least two other vertices. By Theorem 2.1.13 two maximal track connected subgraphs of a graph share at most one vertex. So each vertex in $V \setminus T_G(v)$ traces at least one vertex in $V \setminus T_G(v)$ other than itself. Let $u \in T_G(v) \setminus \{v\}$. Then $V \setminus T_G(v) \cup \{u, v\}$ is a total cycle tracking set of cardinality $n - T + 2$. So $\tau_t(G) \leq n - T + 2$. $\square$

The left inequality is sharp for $C_5 \circ K_2$ and the right inequality is sharp for all track connected graph.

Theorem 8.2.7. *Let G be a graph of order n. If* $T < n$, *then* $\tau_t(G) \leq n - T + 1$.

Proof. Let G be a graph of order n. Let v be a vertex of G of maximum trace T. Suppose $T < n$. Then there exists a vertex $u \in V \setminus T_G(v)$. Since G is trace free, there exist vertices $x_1, x_2 \in V$ such that u, x_1 and x_2 belong to same cycle.
Case(i) $x_1 \in T_G(v)$.
Since two maximal track connected subgraphs of a graph share at most one vertex, $x_2 \notin T_G(v)$. So $V \setminus (T_G(v) \cup \{u\}) \cup \{v, x_1\}$ forms a a total cycle tracking

set of cardinality $n - T + 1$.

Case(ii) $x_2 \in T_G(v)$.

As in case(i) $V \setminus (T_G(v) \cup \{u\}) \cup \{v, x_2\}$ forms a a total cycle tracking set of cardinality $n - T + 1$.

Case(iii) $x_1, x_2 \notin T_G(v)$.

Let $w \in T_G(v)$. Then $V \setminus (T_G(v) \cup \{x_1, x_2\}) \cup \{v, w\}$ forms a total cycle tracking set of cardinality $n - T$. □

Corollary 8.2.8. *Let G be a graph of order n. Then $\tau_t(G) = n - T + 2$ if and only if G is track connected or track connected floral graph.*

Proof. From Theorem 8.2.7, $\tau_t(G) = n - T + 2$ implies $T = n$. That is there is a vertex $v \in V$ such that $T_G(v) = V$. Therefore G is track connected or track connected floral graph.

Conversely suppose that G is track connected or track connected floral graph. If G is track connected then $T = n$ and any subset of V consisting of two vertices form a total cycle tracking set and $\tau_t(G) = 2 = n - T + 2$. If G is a track connected floral graph, then the central vertex v together with any other vertex form a minimal total cycle tracking set and the result follows. □

Chapter 9

Cycle Tracking Function of a Graph

For a subset S of $V(G) = (v_1, v_2, ..., v_n)$ the characteristic function $\chi_S : V(G) \rightarrow \{0, 1\}$ defines a unique $n \times 1$ column vector $X_S = [x_i]$, where $x_i = \chi_S(v_i)$ for $1 \leq i \leq n$. A subset S of $V(G)$ is a cycle tracking set if and only if $|T_G(v_i) \cap S| \geq 1$ for $1 \leq i \leq n$ or $TM(G).X_S \geq 1$.

Note that the computation of the cycle tracking number $\tau_c(G)$ of a graph is a constrained optimization problem, which is in fact an integer programming problem given below.

$$\tau_c(G) = min \sum_{i=1}^{n} x_i$$

$$subject\ to\ TM(G).X \geq 1$$

$$with\ X \in \{0, 1\}^{n \times 1}.$$

The linear programming version of cycle tracking problem, motivated us to introduce a new concept called a cycle tracking function which is in fact a generalization of the existing concept of dominating function [20].

9.1 Cycle Tracking Function of a Graph

Definition 9.1.1. *For a graph $G = (V, E)$ and for a real-valued function $f : V \to \mathbb{R}$, the weight $w(f)$ of f is defined as $w(f) = \sum_{v \in V} f(v)$, and for $S \subset V$ we define $f(S) = \sum_{v \in S} f(v)$.*

Remark 9.1.2. *Note that $w(f) = f(V)$.*

To define a cycle tracking function we must require, for any vertex $v \in V(G)$, the sum over the trace of v of the values of f in $T_G(v)$ must be at least one.

Definition 9.1.3. *Let G be a graph and $f : V \to [0, 1]$ be a function which assigns to each vertex of G a value in the interval $[0, 1]$. Then f is said to be a cycle tracking function of G if for every $v \in V$, $f(T_G(v)) \geq 1$.*

Definition 9.1.4. *A cycle tracking function f of a graph G is said to be a minimal cycle tracking function if there does not exist a cycle tracking function $g : V \to [0, 1]$ such that $f \neq g$ and for which $g(v) \leq f(v)$ for every $v \in V$.*

Example 9.1.5. *For any $v \in V(G)$ of the graph G, define a function $g : V(G) \to [0, 1]$ by $g(v) = \frac{1}{t}$, where t is the minimum tracing number of G. Then $g(T_G(v)) = |T_G(v)|\frac{1}{t} \geq \frac{t}{t} \geq 1$. So $g(T_G(v)) \geq 1$ for all $v \in V(G)$ and g is a cycle tracking function.*

Proposition 9.1.6. *Let S be a cycle tracking set of a graph G. Then the characteristic function χ_S is a cycle tracking function.*

Proof. Let S be a cycle tracking set of a graph G. Then the characteristic function χ_S is a function from $V(G)$ to $[0,1]$. Let $v \in V(G)$. Then there exist at least one vertex $u \in S$ such that $u \in T_G(v)$. Therefore

$$\chi_S(T_G(v)) = \sum_{u \in T_G(v)} \chi_S(u) = \sum_{u \in T_G(v) \cap S} \chi_S(u) \geq 1.$$

Hence χ_S is a cycle tracking function. □

Theorem 9.1.7. *Let f and g be two cycle tracking functions of a graph G. Then all convex linear combinations of f and g are cycle tracking functions of G.*

Proof. Let $\alpha \in \mathbb{R}$ be such that $0 \leq \alpha \leq 1$ and let $h = \alpha f + (1-\alpha)g$. Then for $v \in V(G)$,

$$\begin{aligned} \sum_{u \in T_G(v)} h(u) &= \sum_{u \in T_G(v)} [\alpha f + (1-\alpha)g](u) \\ &= \alpha \sum_{u \in T_G(v)} f(u) + (1-\alpha) \sum_{u \in T_G(v)} g(u) \\ &\geq \alpha + (1-\alpha) \\ &= 1 \end{aligned}$$

Therefore, h is a cycle tracking function of G. Hence the theorem. □

Theorem 9.1.8. *Let S be a minimal cycle tracking set of a graph G. Then the characteristic function χ_S of S is a minimal cycle tracking function.*

Proof. Let S be a minimal cycle tracking set of G. Then by Proposition 9.1.6 χ_S is a cycle tracking function. Suppose χ_S is not minimal. Then there exists a cycle tracking function $f : V(G) \to [0,1]$ such that $f(v) < \chi_S(v)$ for every $v \in V$. Therefore $f(v) = 0$ for every $v \notin S$ and there exist a $u \in S$ such that $f(u) < 1$. Since S is a minimal cycle tracking set of G by theorem 2.1.18, u has at least one private trace. Let $w \in V(G)$ be a private trace of u with respect to S. Therefore $T_G(w) \cap S = \{u\}$. Hence

$$f(T_G(w)) = \sum_{x \in T_G(w)} f(x) = \sum_{x \in T_G(w) \cap S} f(x) = f(u) < 1,$$

a contradiction. □

Theorem 9.1.9. *A cycle tracking function f is minimal if and only if for every vertex v such that $f(v) > 0$, there exists a vertex $u \in T_G(v)$ for which $f(T_G(u)) = 1$.*

Proof. Let G be a graph and let f be a cycle tracking function. Suppose that f is a minimal cycle tracking function in G. If possible let there be a vertex $v \in V$ such that $f(v) > 0$ and for every vertex u in the trace of v, $f(T_G(u)) > 1$. Let $s = min\{f(T_G(u)) : u \in T_G(v)\}$ and let

$$g(u) = \begin{cases} f(u) & if\ u \neq v \\ 1 + f(v) - s & if\ u = v\ and\ 1 + f(v) - s \geq 0 \\ 0 & if\ u = v\ and\ 1 + f(v) - s < 0 \end{cases} .$$

Then $g : V(G) \to [0,1]$ and $g(v) < f(v)$. Let $w \in V(G)$.

$$g(T_G(w)) = \begin{cases} f(T_G(w)) & if\ v \notin T_G(w) \\ f(T_G(w)) + 1 - s & if\ v \in T_G(w)\ and\ 1 + f(v) - s \geq 0 \\ f(T_G(w)) - f(v) & if\ v \in T_G(w)\ and\ 1 + f(v) - s < 0 \end{cases} .$$

Since $f(T_G(w)) \geq s$ for every $w \in T_G(v)$, $g(T_G(w)) \geq 1$. Since w is arbitrary $g(T_G(w)) \geq 1$ for every $w \in V(G)$. Hence g is a cycle tracking function. So f is not a minimal cycle tracking function, a contradiction.

Conversely suppose that for every vertex v such that $f(v) > 0$, there exists a vertex $u \in T_G(v)$ for which $f(T_G(u)) = 1$. If possible, suppose that there is a cycle tracking function $g : V \to [0,1]$, $f \neq g$, such that $g(v) \leq f(v)$ for every $v \in V$. Then there exists a vertex $w \in V(G)$ such that $g(w) < f(w)$. By our hypothesis there exists a vertex $x \in T_G(w)$ for which $f(T_G(x)) = 1$. Since g is a cycle tracking function

$$1 \leq g(T_G(x)) = \sum_{y \in T_G(x)} g(y) \leq \sum_{y \in T_G(x)} f(y) = f(T_G(x)) = 1.$$

So $f(y) = g(y)$ for every vertex $y \in T_G(x)$. In particular $g(w) = f(w)$, a contradiction. □

Theorem 9.1.10. *Let G be a graph. Then*

(i) the cycle tracking number $\tau_c(G) \geq min\{w(f) : f : V(G) \to [0,1]$ is a minimal cycle tracking function on $G\}$.

(ii) the upper cycle tracking number $T(G) \leq max\{w(f) : f : V(G) \to [0,1]$ is a minimal cycle tracking function on $G\}$.

Proof. (i) As the characteristic function χ_S of a τ_c-set of a graph G is a minimal cycle tracking function, we have $\tau_c(G) = w(\chi_S) \geq min\{w(f)$: f is a minimal cycle tracking function on G$\}$.

(ii) Let $M = max\{w(f) : f$ is a minimal tracking function on G$\}$. Let G be a graph and let S be a minimal cycle tracking set of G with cardinality $T_c(G)$. Then the characteristic function χ_S is a minimal cycle tracking function by Theorem 9.1.8 and $w(\chi_S) = T_c(G)$. Therefore $T_c(G) \leq M$. □

If we restrict the co-domain of cycle tracking function to the set {0,1} then we have:

Theorem 9.1.11. *Let G be a graph. Then*

(i) the cycle tracking number $\tau_c(G)$ of a graph G is equal to $min\{w(f) : f : V(G) \to \{0,1\}$ is a minimal cycle tracking function on $G\}$.

(ii) the upper cycle tracking number $T_c(G)$ of a graph G is given by, $T(G) = max\{w(f) : f : V(G) \to \{0,1\}$ is a minimal dominating function on $G\}$.

Proof. (i) Let $m = min\{w(f) : f : V(G) \to \{0,1\}$ is a minimal tracking function on G$\}$. Let G be a graph and let S be a τ_c- set of G. Then by Theorem 9.1.8 the characteristic function χ_S is a minimal cycle tracking function from $V(G)$ to $\{0,1\}$ and $w(\chi_S) = \tau_c(G)$. Therefore $\tau_c(G) \geq m$.

If possible let $g : V(G) \to \{0,1\}$ be a cycle tracking function such that $\tau_c(G) > w(g)$. Let $S = \{v \in V : g(v) = 1\}$. Then $|S| = w(g) < \tau_c(G)$.

Since g is a cycle tracking function for every vertex $u \in V$, $g(T_G(u)) \geq 1$. That is $\sum\limits_{w \in T_G(u)} g(w) \geq 1$. That is $g(w) = 1$ for at least one vertex $w \in T_G(u)$. That is S is a cycle tracking set of G of cardinality less than $\tau_c(G)$, a contradiction.

(ii) Let $M = max\{w(f) : f : V(G) \to \{0,1\}$ is a minimal tracking function on G$\}$. Let G be a graph and let S be a minimal cycle tracking set of G with cardinality $T_c(G)$. Then the characteristic function is a minimal cycle tracking function from $V(G)$ to $\{0,1\}$ by Theorem 9.1.8 and $w(\chi_S) = T_c(G)$. Therefore $T_c(G) \leq M$. To prove the reverse inequality if possible let $h : V(G) \to \{0,1\}$ be a cycle tracking function such that $T_c(G) < w(h)$. Let $S = \{v \in V : h(v) = 1\}$. Since h is a cycle tracking function for every vertex $u \in V$, $h(T_G(u)) \geq 1$. That is $\sum\limits_{w \in T_G(u)} g(w) \geq 1$. That is $g(w) = 1$ for at least one vertex $w \in T_G(u)$. That is S is a cycle tracking set of G. Since $T_c(G) < w(h)$, S is not a minimal cycle

tracking set. So there exist a cycle tracking set $S^* \subset S$. Then the characteristic function χ_{S^*} is a cycle tracking function from $V(G)$ to $\{0,1\}$ and $\chi_{S^*}(v) \leq h(v)$ for every $v \in V$, a contradiction to the minimality of h.
Hence the theorem. $\square$

Theorem 9.1.12. *Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$ and let $f : V(G) \to [0,1]$. Let $\mathsf{X} = [f(v_1), f(v_2), ..., f(v_n)]^T$, the transpose of $[f(v_1), f(v_2), ..., f(v_n)]$. Then f is a cycle tracking function if and only if $TM(G).\mathsf{X} \geq 1$ for the tracking matrix $TM(G)$ of G.*

Proof. Let G be a graph with $V(G) = (v_1, v_2, ..., v_n)$ and let $f : V(G) \to [0,1]$ be a function. Let $\mathsf{X} = [f(v_1), f(v_2), ..., f(v_n)]^T$.
Suppose $TM(G).\mathsf{X} \geq 1$. Then the i^{th} entry of $TM(G).\mathsf{X} \geq 1$, for every i. That is $TM(G)_{i1}f(v_1) + TM(G)_{i2}f(v_2) + ... + TM(G)_{in}f(v_n) \geq 1$ for for every i. That is $\sum_{y \in T_G(v_i)} f(T_G(v_i)) \geq 1$ for every i. That is f is a cycle tracking function.

Conversely, suppose that f is a cycle tracking function of G. Then $\sum_{y \in T_G(v_i)} f(T_G(v_i)) \geq 1$ for every i. Therefore $1 \leq \sum_{y \in T_G(v_i)} f(T_G(v_i)) \leq TM(G)_{i1}f(v_1) + TM(G)_{i2}f(v_2) + ... + TM(G)_{in}f(v_n) = TM(G).\mathsf{X}$ for every i. That is i^{th} entry of $TM(G).\mathsf{X} \geq 1$ for every i. Hence $TM(G).\mathsf{X} \geq 1$. $\square$

Corollary 9.1.13. *Let G be a graph . Then $\tau_c(G) = min\{w(f) : f : V(G) \to \{0,1\}$ and $TM(G).[f(v_1), f(v_2), ..., f(v_n)]^T \geq 1\}$.*

Definition 9.1.14. *The fractional cycle tracking number of a graph G, $\tau_f(G)$ is defined by $\tau_f(G) = min\{w(g) : g$ is a minimal cycle tracking function of $G\}$.*

Definition 9.1.15. *The upper fractional cycle tracking number of a graph G, $T_f(G)$ is defined by $T_f(G) = max\{w(g) : g$ is a minimal cycle tracking function* *of G}.*

Definition 9.1.16. *A cycle tracking function $f : V(G) \to [0,1]$ is called a τ_f function if $w(f) = \tau_f(G)$.*

Remark 9.1.17. *As every minimal cycle tracking set induces a minimal cycle tracking function, $\tau_f(G) \leq \tau_c(G) \leq T_c(G) \leq T_f(G)$.*

Proposition 9.1.18. *For any graph G, $1 \leq \tau_f(G) \leq n$.*

The left inequality is sharp for track connected graphs and track connected floral graphs, and the right inequality is sharp for forests.

Theorem 9.1.19. *For a graph G, $\tau_f(G) = 1$ if and only if G is a track connected graph or a track connected floral graph.*

Proof. Let v be any vertex in G. Let $\tau_f(G) = 1$ and g be a τ_f function. Let $Q = \{v \in V(G) : g(v) > 0\}$. Then $w(g) = \sum\limits_{u \in Q} g(u) = \sum\limits_{u \in V(G)} g(u) = 1$. Since $g(T_G(v)) \geq 1$ for all $v \in V(G)$, $Q \subset T_G(v)$ for all $v \in V(G)$. Hence, if $u \in Q$, then $u \in T_G(v)$ for all $v \in V(G)$. That is $T_G(u) = n$. Therefore G is a track connected graph or a track connected floral graph.
Converse follow from Theorem 2.1.23 and from Remark 9.1.17. □

Corollary 9.1.20. *Let G be a track connected graph or a track connected floral graph and $v \in V(G)$. Then $g(v) = 0$ for any τ_f function g for which $|T_G(v)| < n$.*

Theorem 9.1.21. *For a graph G of order n, $\tau_f(G) \leq \frac{n}{t}$.*

Proof. Let G be a graph. Define a function $g' : V(G) \to [0,1]$ by $g'(v) = \frac{1}{t}$ for every $v \in V(G)$, where t is the minimum tracing number of G. For $v \in V$. Then

$g'(T_G(v)) = \sum\limits_{u \in T_G(v)} g(u) = \sum\limits_{u \in T_G(v)} \frac{1}{t} \geq t\frac{1}{t} = 1$. So g' is a cycle tracking function and $w(g) = \frac{n}{t}$. Therefore $\tau_f(G) \leq \frac{n}{t}$. □

Theorem 9.1.22. *For a positive integer k, let $f_i : V(G) \to [0,1]$ be a cycle tracking function of G, where $1 \leq i \leq k$. Then the function $f : V(G) \to [0,1]$ defined by $f(v) = \dfrac{1}{k}\sum\limits_{i=1}^{k} f_i(v)$ for any $v \in V$ is also a cycle tracking function of G.*

Proof. For $v \in V(G)$, $f(T_G(v)) = \sum\limits_{u \in T_G(v)} f(u) = \sum\limits_{u \in T_G(v)} \frac{1}{k}\sum\limits_{i=1}^{k} f_i(u)$
$= \frac{1}{k}\sum\limits_{i=1}^{k} \sum\limits_{u \in T_G(v)} f_i(u) \geq \frac{1}{k}\sum\limits_{i=1}^{k} 1 = 1$. Therefore f is a cycle tracking function of G. □

Theorem 9.1.23. *For a transitively tracked graph G, $\tau_c(G) = \tau_f(G)$.*

Proof. Let G be a transitively tracked graph. Then its vertex set can be partitioned into $V_1, V_2, ..., V_k$ of cardinality $m_1, m_2, ..., m_k$ respectively such that the graph $\langle V_i \rangle$ induced by each V_i is maximal track connected subgraph of G and $\tau_c(G) = k$. Moreover for every vertex $v \in V_i$, $T_G(v) = V_i$. So for every cycle tracking function f, $f(V_i) \geq 1$ for every $i = 1, 2, ..., k$ and $w(f) = \sum\limits_{i=1}^{k} f(V_i) \geq k$. Hence $\tau_f(G) \geq k = \tau_c(G)$. By Remark 9.1.17 $\tau_c(G) \geq \tau_f(G)$. Hence $\tau_c(G) = \tau_f(G)$. □

9.2 Trace Sigma Algebra

Let G be a graph with vertex set V and edge set E. The vertex set V(G) can be made into a measure space by taking power set as sigma algebra and counting measure as the measure. As the sigma algebra is the power set of V(G), every function $f : V(G) \to [0,1]$ is measurable and as μ is the counting measure, $\int\limits_{T_G(v)} f \; d\mu = \sum\limits_{u \in T_G(v)} f(u) = f(T_G(v))$. So cycle tracking function can be redefined as the function $f : V(G) \to [0,1]$ such that $\int\limits_{T_G(v)} f \; d\mu \geq 1$ for all $v \in V(G)$.

But this case is a least interesting one because in this case each subset of same cardinality has same weight. This will not be the case in general. In the general case different subsets may have different weight though they are of the same cardinality and some subsets may be neglected. Which sets are included, which are excluded, etc will depend on the choice of the sigma algebra under consideration. As the integrals over $T_G(v)$, for each $v \in V$ are to be evaluated, all $T_G(v)$ should be in that sigma algebra. So the most appropriate sigma algebra is the sigma algebra generated by $T_G(v)$s.

Definition 9.2.1. *Let $G = (V(G), E(G))$ be a graph. The sigma algebra generated by $\mathcal{G} = \{T_G(v) : v \in V(G)\}$ on $V(G)$ is called the trace sigma algebra of G and it is denoted by $\mathcal{T}_G$ (or simply $\mathcal{T}$ if there is no confusion) and $\mathcal{G}$ is called the generating set of $\mathcal{T}$.*

Throughout this section, by a graph G, we mean the graph with its trace sigma algebra $\mathcal{T}$ on the vertex set $V(G)$ and a subset of $V(G)$ is measurable means it is measurable with respect to the trace sigma algebra.

Definition 9.2.2. *Let G be a graph. For $v \in V(G)$, we define M_v^G (or simply M_v if there is no confusion) to be the intersection of all measurable sets containing v. Hence it is the smallest measurable set containing v.*

Example 9.2.3. *For the graph G, in Figure 9.1 the trace sigma algebra $\mathcal{T}$ is*

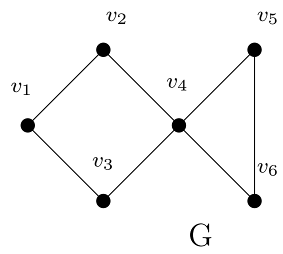


Figure 9.1: Graph G.

given by $\{\emptyset, \{v_1, v_2, v_3, v_4\}, \{v_4, v_5, v_6\}, \{v_5, v_6\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_5, v_6\}, \{v_4\}, V\}$. $M_{v_1} = M_{v_2} = M_{v_3} = \{v_1, v_2, v_3\}, M_{v_4} = \{v_4\}$ *and* $M_{v_5} = M_{v_6} = \{v_5, v_6\}$.

Since M_v is the smallest measurable set containing v, we have the following results.

Proposition 9.2.4. *Let G be a graph and $u, v \in V(G)$. Then*

1. *$u \in M_v$ if and only if $v \in M_u$.*
2. *$u \in M_v$ implies $M_u = M_v$.*
3. *$M_u \bigcap M_v \neq \emptyset$ implies $M_u = M_v$.*
4. *$\{M_u : u \in V(G)\}$ forms a partition of $V(G)$ and each measurable set can be written as a disjoint union of $M_v's$.*

Proposition 9.2.5. *Let G be a graph with vertex set V(G) and let $v \in V(G)$ be such that it is either a trace free vertex or the graph induced by $T_G(v)$ is a track connected floral graph. Then $M_v = \{v\}$.*

Proof. If v be a trace free vertex of G, then $T_G(v) = \{v\}$. Hence $M_u = \{u\}$. If the graph induced by $T_G(v)$ is a track connected floral graph. Then $T_G(v)$ contains at least two vertices u and w such that $T_G(u) \neq T_G(w)$ and $T_G(u) \cap T_G(w) = \{v\}$. Hence $M_v = \{v\}$. □

The converse of Proposition is not true. That is $M_v = \{v\}$ does not imply, v is a trace free vertex or the graph induced by $T_G(v)$ is a track connected floral graph.

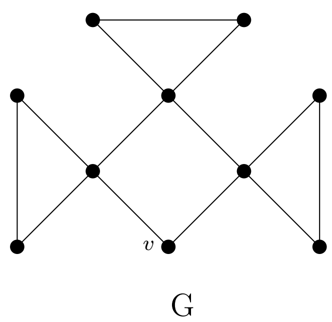


Figure 9.2: Graph G.

Consider the graph G in Figure 9.2. For the vertex v, $M_v = \{v\}$. But v is neither a trace free vertex nor the graph induced by $T_G(v)$ is a track connected floral graph.

For a track connected graph G, $T_G(v) = V(G)$ for every vertex $v \in V(G)$ and vice versa. Therefore;

Proposition 9.2.6. *A graph G is track connected if and only if $M_v = V(G)$, for all $v \in V(G)$.*

Proposition 9.2.7. *Let G be a graph with trace sigma algebra $\mathcal{T}$. Then every member of $\mathcal{T}$ can be expressed as the union of sets, each of which can be expressed* *as the intersection of members of $\mathcal{H}$, where $\mathcal{H} = \{T_G(v) : v \in V(G)\} \bigcup \{T_G(v)^c : v \in V(G)\}$.*

Proof. Let $\mathcal{J}$ consists of all subsets of $V(G)$ which can be expressed as unions of members of $\mathcal{G}$, where $\mathcal{G}$ is the family of all intersections of members of $\mathcal{H}$. Then $\mathcal{J}$ contains $\{T_G(v) : v \in V(G)\}$ and it is contained in $\mathcal{T}$. Also $\mathcal{J}$ itself is a sigma algebra. As $\mathcal{T}$ is generated by $\{T_G(v) : v \in V(G)\}$, $\mathcal{J} = \mathcal{T}$. Hence the proposition. □

Theorem 9.2.8. *Let G be a graph. Then for $v_1, v_2 \in V(G)$, $M_{v_1} = M_{v_2}$ if and only if $T_G(v_1) = T_G(v_2)$.*

Proof. Assume that $M_{v_1} = M_{v_2}$ for some $v_1, v_2 \in V(G)$. Suppose $T_G(v_1) \neq T_G(v_2)$. Without loss of generality, assume that there exists $u \in V(G)$ such that $u \in T_G(v_1)$ but $u \notin T_G(v_2)$. Therefore, $T_G(u) \bigcap T_G(v_1)$ is a measurable set containing v_1 but not v_2. This implies that $v_2 \notin M_{v_1}$.

Conversely, assume that $T_G(v_1) = T_G(v_2)$. Then for any $v \in V(G)$ either $v_1, v_2 \in T_G(v)$ or $v_1, v_2 \in T_G(v)^c$. Therefore, by Proposition 9.2.7, for every measurable set B, either $v_1, v_2 \in B$ or $v_1, v_2 \in B^c$. Therefore $M_{v_1} = M_{v_2}$. □

If u and v are two vertices of a graph G then $u \in M_v$ if and only if $M_u = M_v$. That is if and only if $T_G(u) = T_G(v)$. Thus we have:

Corollary 9.2.9. *Let G be a graph and $v \in V(G)$. Then $M_v = \{u \in V(G) : T_G(u) = T_G(v)\}$.*

Corollary 9.2.10. *Let G be a graph of order n and $v \in V(G)$ be such that $|T_G(v)| = n$. Then $M_v = \{v\}$ or $M_v = V(G)$.*

Proof. Let G be a graph of order n and $v \in V(G)$ be such that $|T_G(v)| = n$. That is if and only if G is a track connected graph or a track connected floral graph. If G is a track connected graph, then $T_G(u) = V(G)$ for every $u \in V$. So $M_v = \{u \in V(G) : T_G(u) = T_G(v)\} = V(G)$.
If G is a track connected floral graph there exists one and only one vertex v with tracing number n. So $M_v = \{u \in V(G) : T_G(u) = T_G(v)\} = \{v\}$. □

Theorem 9.2.11. *Let G be a graph with vertex set $V(G)$. For each $k \in \mathbb{N}$ with $1 \leq k \leq T$, the collection $S_k := \{v \in V(G) : |T_G(v)| = k\}$ is a measurable set.*

Proof. Let G be a graph with vertex set $V(G)$. Let $k \in \mathbb{N}$ be such that $1 \leq k \leq T$. If $S_k = \emptyset$, then it is measurable. So suppose that $S_k \neq \emptyset$. Let $v \in S_k$. Since $M_v = \{u \in V(G) : T_G(u) = T_G(v)\}$, $|T_G(u)| = |T_G(v)|$ for all $u \in M_v$. This implies that $M_v \subseteq S_k$ for all $v \in S_k$. Hence $S_k = \bigcup_{v \in S_k} M_v$. Therefore S_k is measurable. □

Corollary 9.2.12 follows from Theorem 9.2.11 and from the fact that the complement of a measurable set is measurable.

Corollary 9.2.12. *Let G be a graph with vertex set $V(G)$. For each $k \in \mathbb{N}$ with $1 \leq k \leq T$, the collection $\{v \in V(G) : |T_G(v)| \neq k\}$ is measurable.*

Here after a function defined on the vertex set of the given graph is measurable means which is measurable with respect to the trace sigma algebra of that graph.

Theorem 9.2.13. *Let G be a graph and $f : V(G) \longrightarrow [0, 1]$ be a function. Then f is measurable if and only if f is constant on M_v for all $v \in V(G)$.*

Proof. Let $v \in V(G)$ and $f(v) = c$. Suppose $f(u) = d$ for some $u \in M_v$. Without loss of generality assume that $c < d$. Then $f^{-1}(-\infty, d)$ is measurable and $v \in f^{-1}(-\infty, d)$. Therefore v belongs to the measurable set $f^{-1}(-\infty, d) \bigcap M_v$, which is a proper subset of M_v. This contradicts the fact that M_v is the smallest measurable set containing v.

Conversely assume that f is constant on M_v for all $v \in V(G)$. Let U be an open subset of [0,1]. Suppose that $f(V(G)) \bigcap U = \{k_1, k_2, ..., k_m\}$. Then $f^{-1}(U) = f^{-1}(\{k_1\}) \bigcup f^{-1}(\{k_2\}) \bigcup ... \bigcup f^{-1}(\{k_m\})$. Let $1 \leq i \leq m$. As f is constant on each M_v, $f^{-1}(\{k_i\}) = \bigcup\limits_{f(v_j)=k_i} M_{v_j}$. Hence $f^{-1}(k_i)$ is measurable for all $1 \leq i \leq m$. Therefore $f^{-1}(U)$ is measurable. Hence f is measurable. $\square$

As a consequence of Corollary 9.2.9 and Theorem 9.2.13, we have:

Corollary 9.2.14. *Let G be a graph with $u_1, u_2 \in V(G)$. If $f : V(G) \longrightarrow [0,1]$ is measurable and $T_G(u_1) = T_G(u_2)$ then $f(u_1) = f(u_2)$.*

9.3 Measurable Cycle Tracking Function of a Graph

Once we have a sigma algebra, next best thing that we can think of is that of measurable functions related to this sigma algebra. We call measurable functions related to the trace sigma algebra satisfying a particular condition as measurable cycle tracking function.

Definition 9.3.1. *Let G be a graph with vertex set $V(G)$ and let μ be a measure on G. A function $f : V(G) \rightarrow [0,1]$ is called a measurable cycle tracking function* *of G if the following conditions hold:*

(i) f is measurable

(ii) $\int\limits_{T_G(v)} f \; d\mu \geq 1$ for all $v \in V(G)$.

Remark 9.3.2. *Let f be a measurable cycle tracking function of a graph G. Then for all $v \in V(G)$, $f(T_G(v)) > 0$ and $\mu(T_G(v)) > 0$, where $f(T_G(v)) = \sum\limits_{u \in T_G(v)} f(u)$.*

Theorem 9.3.3. *Let f and g be two measurable cycle tracking functions of a graph G. Then all convex linear combinations of f and g are measurable cycle tracking functions of G.*

Proof. Let $\alpha \in \mathbb{R}$ be such that $0 \leq \alpha \leq 1$ and let $h = \alpha f + (1-\alpha)g$. Since f and g are measurable functions, h is also measurable. Then for $v \in V(G)$,

$$\begin{aligned}
\int\limits_{T_G(v)} h \; d\mu &= \int\limits_{T_G(v)} [\alpha f + (1-\alpha)g] \; d\mu \\
&= \int\limits_{T_G(v)} \alpha f \; d\mu + \int\limits_{T_G(v)} (1-\alpha)g \; d\mu \\
&= \alpha \int\limits_{T_G(v)} f \; d\mu + (1-\alpha) \int\limits_{T_G(v)} g \; d\mu \\
&\geq \alpha + (1-\alpha) \\
&= 1
\end{aligned}$$

Therefore, h is a measurable cycle tracking function of G. Hence the theorem. $\square$

Definition 9.3.4. *Let G be a graph with vertex set $V(G)$. A measurable cycle tracking function f of G is said to be minimal if there does not exist a measurable cycle tracking function g of G such that $g \leq f$ a.e and $g < f$ on some set of positive measure.*

Theorem 9.3.5 establishes a necessary and sufficient condition for a measurable cycle tracking function to be minimal.

Theorem 9.3.5. *Let G be a graph with vertex set $V(G)$. A measurable cycle tracking function f of G is minimal if and only if for every vertex $v \in V(G)$ with $\mu(M_v) > 0$ and $f > 0$ on M_v there exists a vertex $u \in T_G(v)$ with* $\int\limits_{T_G(u)} f \, d\mu = 1$.

Proof. Let f be a minimal measurable cycle tracking function of G. Suppose there exists a vertex $v \in V(G)$ with $\mu(M_v) > 0$ and $f > 0$ on M_v such that $\int\limits_{T_G(u)} f \, d\mu > 1$ for all $u \in T_G(v)$.

Let $m = \min \{ \int\limits_{T_G(u)\setminus M_v} f \, d\mu : u \in T_G(v)\}$.

We consider the cases $m \geq 1$ and $m < 1$ separately.

Case 1. $m \geq 1$.

Let $g = f - f\chi_{M_v}$, where χ_{M_v} denotes the characteristic function of M_v. That is for $w \in V(G)$, $g(w) = \begin{cases} 0 & \text{if } w \in M_v \\ f(w) & \text{if } w \notin M_v \end{cases}$. Since the product and difference of measurable functions are measurable, the function g is measurable. Also $g(w) \leq f(w)$ for every $w \in V(G)$ and $g < f$ on M_v.

For $u \in V(G)$ with $u \in T_G(v)$,

$$\begin{aligned}
\int\limits_{T_G(u)} g \, d\mu &= \int\limits_{M_v} g \, d\mu + \int\limits_{T_G(u)\setminus M_v} g \, d\mu \\
&= \int\limits_{T_G(u)\setminus M_v} g \, d\mu \\
&= \int\limits_{T_G(u)\setminus M_v} f \, d\mu \\
&\geq m \\
&\geq 1.
\end{aligned}$$

Also, for $u \in V(G)$ with $u \notin T_G(v)$,

$$\begin{aligned} \int\limits_{T_G(u)} g \; d\mu &= \int\limits_{T_G(u)} f \; d\mu \\ &\geq 1. \end{aligned}$$

Therefore, g is also a measurable cycle tracking function, a contradiction.

Case 2. $m < 1$.

For $u \in T_G(v)$, $\int\limits_{T_G(u)} f \; d\mu > 1$ by the assumption.

Suppose $f = c$ on M_v. Then,

$$\begin{aligned} \int\limits_{T_G(u)} f \; d\mu &= \int\limits_{M_v} f \; d\mu + \int\limits_{T_G(u) \setminus M_v} f \; d\mu \\ &= c\mu(M_v) + \int\limits_{T_G(u) \setminus M_v} f \; d\mu. \end{aligned}$$

Since $m < 1$, for at least one vertex $u \in T_G(v)$, $\int\limits_{T_G(u) \setminus M_v} f \; d\mu < 1$.

For such a u,

$$\begin{aligned} c\mu(M_v) &> 1 - \int\limits_{T_G(u) \setminus M_v} f \; d\mu \\ &> 0. \end{aligned}$$

This implies,

$$c > \frac{1 - \int\limits_{T_G(u) \setminus M_v} f \; d\mu}{\mu(M_v)} = R_u, \text{ say.}$$

Let $U = \{u \in T_G(v) : \int\limits_{T_G(u) \setminus M_v} f \; d\mu \; < 1\}$. Since $V(G)$ is finite, U is also finite.

Now choose d so that $c > d > R_u$ for all $u \in U$.

Let $h = f - (f - d)\chi_{M_v}$.

That is for $w \in V(G)$,

$$h(w) = \begin{cases} d & \text{if } w \in M_v \\ f(w) & \text{if } w \notin M_v. \end{cases}$$

The function h is measurable, since it is the difference of the measurable functions f and $(f - d)\chi_{M_v}$. Also $h(w) \leq f(w)$ for every $w \in V(G)$ and $h < f$ on M_v.

Let $u \in U$,

$$\begin{aligned} \int\limits_{T_G(u)} h \, d\mu &= \int\limits_{T_G(u)\setminus M_v} h \, d\mu + \int\limits_{M_v} h \, d\mu \\ &= \int\limits_{T_G(u)\setminus M_v} f \, d\mu + d\mu(M_v) \\ &> \int\limits_{T_G(u)\setminus M_v} f d\mu + R_u \mu(M_v) \\ &= \int\limits_{T_G(u)\setminus M_v} f \, d\mu + \left(1 - \int\limits_{T_G(u)\setminus M_v} f \, d\mu \right) \\ &= 1. \end{aligned}$$

Let $u \notin U$.

If $u \notin T_G(v)$,

$$\begin{aligned} \int\limits_{T_G(u)} h \, d\mu &= \int\limits_{T_G(u)} f \, d\mu \\ &\geq 1. \end{aligned}$$

If $u \in T_G(v)$, $\int\limits_{T_G(u)\setminus M_v} f \, d\mu \geq 1$.

Therefore,

$$\begin{aligned}
\int\limits_{T_G(u)} h\ d\mu &= \int\limits_{T_G(u)\setminus M_v} h\ d\mu + \int\limits_{M_v} h\ d\mu \\
&= \int\limits_{T_G(u)\setminus M_v} f\ d\mu + d\mu(M_v) \\
&> 1.
\end{aligned}$$

Therefore, h is a measurable cycle tracking function with $h(w) \leq f(w)$ for every $w \in V(G)$ and $h < f$ on M_v, a contradiction.

Conversely, let f be a measurable cycle tracking function of G such that for every vertex v with $\mu(M_v) > 0$ and $f > 0$ on M_v, there exists a vertex $u \in T_G(v)$ such that $\int\limits_{T_G(u)} f\ d\mu = 1$. Suppose f is not minimal. Then there exists a measurable cycle tracking function l such that $l \leq f$ a.e and $l < f$ on a set of positive measure. So there exists a $v \in V(G)$ with $\mu(M_v) > 0$ and $l < f$ on M_v. This implies $f(v) > 0$. Now by assumption, there exists a $u \in V(G)$ with $u \in T_G(v)$ and $\int\limits_{T_G(u)} f\ d\mu = 1$.

Therefore,

$$\begin{aligned}
1 &\leq \int\limits_{T_G(u)} l\ d\mu \\
&= \int\limits_{T_G(u)\setminus M_v} l\ d\mu + \int\limits_{M_v} l\ d\mu \\
&< \int\limits_{T_G(u)\setminus M_v} f\ d\mu + \int\limits_{M_v} f\ d\mu \\
&= 1, \text{ a contradiction.}
\end{aligned}$$

Therefore, f is a minimal measurable cycle tracking function. Hence the theorem.

□

Chapter 10

Track Closure Space Generated by a Graph

Topological structures are generalized methods for measuring similarity and dissimilarity between objects in the universe. Given a graph G with vertex set V, an easy way to associate a topology with due consideration to adjacency of vertices in G is to generate a topology with the doubletons formed by adjacent pairs of vertices as subbasis. Another way to generate a topology is by taking open neighborhoods of the vertices as subbasis. We can also generate a topology by taking closed neighborhoods of vertices as subbasis. Yet another way is to consider subsets A of V(G) satisfying the property that $x, y \in A$ if and only if $x \in T_G(y)$ and generate a topology with these subsets as subbasis.

Recall the relation '$\sim$' on the vertex set V of a graph G defined in section 2.2 : For every pair of vertices $u, v \in V$, we say that u is related to v ($u \sim v$) if $u \in T_G(v)$. Then '$\sim$' is a reflexive and symmetric relation. We define and investigate a new closure operator with respect to the relation '$\sim$' on the vertex set V of the graph G. In doing so, the idempotent condition, is achieved. The topology associated with this closure operator is studied in this chapter. Minimal neighborhood and accumulation points are also defined. We also investigate some properties of this topology.

10.1 Track Closure Space

Definition 10.1.1. *Let G be a graph and let $v \in V$. A subset $\langle v \rangle$ is the intersection of all $T_G(x)$ containing v.*

$$i.e; \langle v \rangle = \bigcap_{x \in T_G(v)} T_G(x).$$

So we have,

$$\begin{aligned} x \in \langle u \rangle \quad &\Longleftrightarrow \quad x \in \bigcap_{y \in T_G(u)} T_G(y) \\ &\Longleftrightarrow \quad x \in T_G(y) \ for \ every \ y \in T_G(u). \end{aligned}$$

Since $v \in T_G(x)$ for every $x \in T_G(v)$, $v \in \langle v \rangle$ for every vertex v in G.

Definition 10.1.2. *Let G be a graph and let A be a subset of V and the mapping $tc : P(V) \longrightarrow P(V)$ defined by $tc(A) = \{v \in V : \langle v \rangle \cap A \neq \emptyset\}$.*

For the graph G in figure 10.1 $\langle v_1 \rangle = \langle v_2 \rangle = \langle v_4 \rangle = \{v_1, v_2, v_3, v_4\}$,

$\langle v_5 \rangle = \langle v_6 \rangle = \{v_3, v_5, v_6\}$,

$\langle v_3 \rangle = \{v_3\}$,

$\langle v_7 \rangle = \{v_7\}$,

$tc(\{v_1\}) = tc(\{v_2\}) = tc(\{v_4\}) = \{v_1, v_2, v_4\}$,

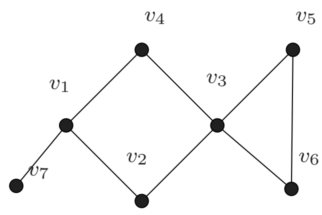


Figure 10.1: Graph G.

$tc(\{v_5\}) = tc(\{v_6\}) = \{v_5, v_6\},$

$tc(\{v_3\}) = \{v_1, v_2, v_3, v_4, v_5, v_6\},$

$tc(\{v_7\}) = \{v_7\}.$

Theorem 10.1.3. *Let G be a graph. Then (V, tc) is a closure space.*

Proof. Let G be a graph and tc is the mapping as defined in definition 10.1.2. Then,

1. $tc(\emptyset) = \emptyset$.

2. Let $v \in V$ and $A \subseteq V$. Then,

$$\begin{aligned} v \in A &\implies v \in \langle v \rangle \cap A, \ \textit{since } v \in \langle v \rangle \\ &\implies v \in tc(A). \end{aligned}$$

ie; $A \subset tc(A)$.

3. Let $v \in V$ and $A, B \subseteq V$. Then,

$$\begin{aligned} v \in A \cup B &\iff (\langle v \rangle \cap A) \cup (\langle v \rangle \cap B) \neq \emptyset \\ &\iff (\langle v \rangle \cap A) \neq \emptyset \ \textit{or} \ (\langle v \rangle \cap B) \neq \emptyset \\ &\iff v \in tc(A) \ \textit{or} \ v \in tc(B) \\ &\iff v \in tc(A) \cup tc(B). \end{aligned}$$

ie; $tc(A \cup B) = tc(A) \cup tc(B)$. $\square$

We call the operator, $tc : P(V) \longrightarrow P(V)$ the *track closure operator* associated with G. That is, track closure operator of a graph G is a mapping from $P(V)$ to $P(V)$ which associates with each subset A of V a subset $tc(A) = \{v \in V : \langle v \rangle \cap A \neq \emptyset\}$ of V.

Definition 10.1.4. *The closure space $(V(G), tc)$ is called the track closure space associated with the graph G.*

The closed sets [3] of the graph G in figure 10.1 are $\emptyset, \{v_1, v_2, v_4\}, \{v_5, v_6\}$,
$\{v_1, v_2, v_3, v_4, v_5, v_6\}, \{v_1, v_2, v_4, v_7\}, \{v_1, v_2, v_4, v_5, v_6\}, \{v_5, v_6, v_7\}$,
$\{v_7\}, \{v_1, v_2, v_4, v_5, v_6, v_7\},$ *and* $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

Lemma 10.1.5. *Let G be a graph and let $u, v \in V(G)$. If $u \in \langle v \rangle$ then $\langle u \rangle \subset \langle v \rangle$.*

Proof. Let G be a graph and let $u, v \in V(G)$. Suppose $u \in \langle v \rangle$. Then $u \in T_G(y)$ for every $y \in T_G(v)$. Let $x \in \langle u \rangle$. Then $x \in T_G(z)$ for every $z \in T_G(u)$. Hence $x \in T_G(y)$ for every $y \in T_G(v)$. Therefore $x \in \langle v \rangle$. $\square$

Proposition 10.1.6. *For $A \subset B$, $tc(A) \subseteq tc(B)$.*

Proof.

$$\begin{aligned} v \in tc(A) &\Longrightarrow \langle v \rangle \cap A \neq \emptyset \\ &\Longrightarrow \langle v \rangle \cap B \neq \emptyset \\ &\Longrightarrow v \in tc(B). \end{aligned}$$

ie; $tc(A) \subseteq tc(B)$. $\square$

Theorem 10.1.7. *For any graph G, $(V(G), tc)$ is idempotent, ie; $tc(tc(A)) = tc(A)$ for all $A \subset V$.*

Proof. It is enough to show that $tc(tc(A) \subseteq tc(A)$ for all $A \subset V$.
Let $u \in tc(tc(A))$. Then since $tc(tc(A) = \{v \in V : \langle v \rangle \cap tc(A) \neq \emptyset\}$, $\langle u \rangle \cap tc(A) \neq \emptyset$. Then there exists a vertex x such that $x \in \langle u \rangle \cap tc(A)$. That is $x \in \langle u \rangle and\ x \in tc(A)$. That is $x \in \langle u \rangle$ *and* $\langle x \rangle \cap A \neq \emptyset$.
Since $x \in \langle u \rangle$, $\langle x \rangle \subseteq \langle u \rangle$(by Lemma 10.1.5) and hence $\langle u \rangle \cap A \neq \emptyset$. So $u \in tc(A)$. Therefore $tc(tc(A) \subseteq tc(A)$. □

Theorem 10.1.8 follows from the fact that a closure space (X, cl) is a topological space iff $cl(cl(A)) = cl(A)$ for all $A \subseteq X$[3].

Theorem 10.1.8. *Every track closure space $(V(G), tc)$ is topological space.*

If $\mathcal{T}$ is the topology on X and the class $\mathcal{T}^c = \{A^c : A \in \mathcal{T}\}$. is also the topology on X, then $\mathcal{T}^c$ is the dual of $\mathcal{T}$[3].

Definition 10.1.9. *The dual $\mathcal{T}_t^G$ of the topology of the topological space $(V(G), tc)$ is called track topology of G.*

The track topology of the graph G in figure 10.1 is $\mathcal{T}_t^G = \{\emptyset, \{v_1, v_2, v_3, v_4\}$, $\{v_3, v_5, v_6\}, \{v_1, v_2, v_3, v_4, v_5, v_6\}, \{v_7\}, \{v_1, v_2, v_3, v_4, v_7\}$, $\{v_3, v_5, v_6, v_7\}, \{v_3\}\}, \{v_3, v_7\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}\}$.

As track topology of G is a topology on V, we can talk about interiors and closures of subsets of V with respect to this topology. We call them respectively by track interior and track closure.

Proposition 10.1.10. *Let G be a graph and $\mathcal{T}_t^G$ its track topology. If $A \subset V$ then $int(A) = \{v \in V : \langle v \rangle \subseteq A\}$.*

Proof. Since in any topological space X and for any subset A of X, $int(A) = (cl(A^c))^c$,

$$\begin{aligned} int(A) &= (tc(A^c))^c \\ &= (\{v \in V : \langle v \rangle \cap A^c \neq \emptyset\})^c \\ &= \{v \in V : \langle v \rangle \subseteq A\}. \end{aligned}$$

□

Definition 10.1.11. *If G and A are as in the Proposition 10.1.10 then, a point $v \in A$ is an interior point of A if $\langle v \rangle \subseteq A$.*

Remark 10.1.12. *From definition 10.1.11 it follows that $\langle v \rangle$ is the minimal neighborhood of v for all $v \in V$.*

So for a graph G, $\{\langle v \rangle : v \in V\}$ forms a basis for $\mathcal{T}_t^G$. Since $|V(G)|$ is finite, $\mathcal{T}_t^G$ is Alexandroff [3].

Proposition 10.1.13. *For each subset A of a track closure space $(V(G), tc)$, $tc(A) = \bigcup\limits_{x \in A} tc(x)$.*

Proposition 10.1.14. *If $\langle v \rangle = \{v\}$, then $\{v\}$ is open.*

Theorem 10.1.15 and 10.1.16 follows from Theorem 2.2.5 and 2.2.8

Theorem 10.1.15. *Let G be a transitively tracked graph. Then $\langle v \rangle = T_G(v)$, the maximal track connected component of G containing v.*

Theorem 10.1.16. *Let G be a transitively tracked graph with the track topology $\mathcal{T}_t^G$. Then the maximal track connected components of G forms a basis for $\mathcal{T}_t^G$.*

Corollary 10.1.17. *Let G be a transitively tracked graph. Then every cycle tracking set is dense in track topological space $\mathcal{T}_t^G$.*

Theorem 10.1.18. *The track topology $\mathcal{T}_t^G$ of the graph G is $T_\circ$ [3] if and only if G is a tree.*

Proof. If G is a tree then $\mathcal{T}_t^G$ is the discrete topology on V(G), which is clearly $T_\circ$.

If G is not a tree then there exists two vertices $u, v \in V$ such that $T_G(v) = T_G(u)$. Hence $\langle v \rangle = \langle u \rangle$. Since $\langle v \rangle$ is a minimal neighborhood of v for all $v \in V$, the topological space $\mathcal{T}_t^G$ is not $T_\circ$. □

Remark 10.1.19. *For any tree T, $(V(T), \mathcal{T}_t^T)$ is a metrizable space[3].*

A topological space is R_0 [3] if, for every two distinct points x and y of the space, either $cl(x) = cl(y)$ or $cl(x) \cap cl(y) = \emptyset$.

Theorem 10.1.20. *The topological space $(V, \mathcal{T}_t^G)$ of a graph G is R_0 if and only if G is transitively tracked.*

Proof. Let G be a transitively tracked graph. Then $tc(v) = T_G(v)$, the maximal track connected component of G containing v. Since $T_G(v) = T_G(u)$ or $T_G(v) \cap T_G(u) = \emptyset$ for every pair of vertices $u, v \in V$, $\mathcal{T}_t^G$ is R_0.

If G is not transitively tracked, then there exists a vertex $v \in V$ such that the graph induced by $T_G(v)$ is a track connected floral graph. Hence there exist vertices $u, w \in T_G(v)$ such that $T_G(v) \cap T_G(w) = \{v\}$. Hence $\mathcal{T}_t^G$ is not R_0. □

Definition 10.1.21. *A point $v \in V$ is called an accumulation point of a subset A of the vertex set V of the graph G iff $(\langle v \rangle - \{v\}) \cap A \neq \emptyset$. The set of all* *accumulation points of A is denoted by A',*

$$i.e;\ A' = \{v \in V : (\langle v\rangle - \{v\}) \cap A \neq \emptyset\}.$$

Proposition 10.1.22. $tc(A) = A \cup A'$.

Proof. Let $v \in tc(A)$. Then $\langle v\rangle \cap A \neq \emptyset$.

If $v \in A$ then $v \in A \cup A'$.

If $v \notin A$ then $\langle v\rangle - \{v\} \cap A \neq \emptyset$. That is $v \in A'$. Hence $tc(A) \subseteq A \cup A'$.

Conversely, assume that $v \in A \cup A'$. We have either $v \in A$ or $v \in A' \setminus A$. In the first case $v \in tc(A)$ and in the latter case $(\langle v\rangle - \{v\}) \cap A \neq \emptyset$, thus $\langle v\rangle \cap A \neq \emptyset$, hence $v \in tc(A)$. Hence $A \cup A' \subseteq tc(A)$. $\square$

Theorem 10.1.23. *Let G be a transitively tracked graph. Then $T_G(v) \in \mathcal{T}_t^G$.*

Proof. It is enough to show that $tc(T_G(v)^c) = T_G(v)^c$. And

$$\begin{aligned} u \in T_G(v)^c &\implies u \notin T_G(v) \\ &\implies T_G(u) \cap T_G(v) = \emptyset \\ &\implies T_G(u) \subset T_G(v)^c \\ &\implies \langle u\rangle \subset T_G(v)^c \\ &\implies \langle u\rangle \cap T_G(v)^c \neq \emptyset \\ &\implies u \in tc(T_G(v)^c). \end{aligned}$$

The reverse inclusion is trivial. $\square$

Corollary 10.1.24. *Let G be a transitively tracked graph and $A \subset V$. Then $A \in \mathcal{T}_t^G$ if and only if $A = \bigcup\limits_{v \in A} T_G(v)$.*

Corollary 10.1.25. *If G is transitively tracked, then $\mathcal{T}_t^G$ is the dual of itself.*

Epilogue

Some of the open problems that were thought about and where further research may be possible to enrich the theory of cycle tracking sets are discussed below;

1. Designing of an algorithm for finding $\tau_c - set$ of a given graph.
2. Determination of the family of all $\tau_c - set$ of a given graph G.
3. Determination of the family of all minimal cycle tracking set of a graph G.
4. Determination of the family of all $\tau_c - set$ of a graph G containing a vertex v.
5. Characterization of the graphs having unique τ_c- set.
6. Characterization of the vertices of a graph G which belongs to some $\tau_c - set$ of G.
7. Characterization of the vertices of a graph G which belongs to no $\tau_c - set$ of G.
8. Characterization of the cycle tracking sets S of a graph G, which has the property $|T_G(v) \cap S| = 1$ for every $v \in V$.

9. Given a psitive integer $k \geq 3$, characterize graphs G for which $|T_G(v)| \geq k$, for every $v \in V(G)$.
10. Determination of the energy of cycle tracking matrix of a graph.
11. Determination of the the maximum integer k such that vertex set of a graph can be partitioned into k pairwise disjoint cycle tracking sets.
12. Determination of the the maximum number of edges that can be removed from the given graph G without changing cycle tracking number.

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List of Publications

Papers Published/Accepted

1. Jalsiya M.P. and Raji Pilakkat, "Mixed Circuit Domination Number" , International Journal of Research in Advent Technology, Vol.6, No.10, October 2018.

2. Jalsiya M.P. and Raji Pilakkat, "Independent and Irredundant Cycle Tracking Sets of a Graph : An efficient approach to electrical circuit analysis" , Far East Journal of Mathematical Sciences (FJMS),Volume 111, Number 2, 2019, Pages 225-238.

3. Jalsiya M.P., Raji Pilakkat, "Track closure space generated by a graph" , International Journal of Research and Analytical Reviews (IJRAR), E-ISSN 2348-1269, P- ISSN 2349-5138, Volume.6, Issue 1, Page No pp.522-527, March 2019, Available at : http://www.ijrar.org/IJRAR19J2973.pdf.

4. Jalsiya M.P. and Raji Pilakkat, "Transitively tracked graphs" , Malaya Journal of Mathematik, Vol. S, No. 1, Pages 457-461, 2019.

5. Jalsiya M.P. and Raji Pilakkat, "Independent tracking polynomial of a graph" , Malaya Journal of Matematik, Vol. S, No. 1, Pages 462-465, 2019.

6. Jalsiya M.P. and Raji Pilakkat, "Cycle tracking polynomial of a graph" , Journal of Applied Science and Computations, June 2019, Volume IV, Issue IV, Pages 774-783.

7. Jalsiya M.P. and Raji Pilakkat, "Total Cycle Tracking Sets " , to appear, Far East Journal of Mathematical Sciences (FJMS).

Papers Presented

1. Presented a paper on "An efficient approach to circuit analysis through introduction of cycle tracking sets in a graph" in the International Conference on Discrete Mathematics and its Applications to Network Science organized by department of mathematics, Birla Institute of Technology and Science, Pilani, Goa on 7,8,9 and 10 July 2018.

2. Presented a paper on "Bounds on Total Cycle Tracking Set of a Graph" in the International Conference on Graph Theory and its Applications - ICGTA19 organized by department of mathematics, Amrita Vishwa Vidyapeetham, Coimbatore on 4,5 and 6 January 2019.

3. Presented a paper on "Independent cycle tracking polynomial of a graph" in the MESMAC international conference organized by MES Mambad college on 15,16 and 17 January 2019.

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