ANALYSIS OF INVENTORY MODELS

A thesis submitted to Calicut University for the award of the degree of

Doctor of philosophy

in

Mathematics

By

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May 2023

CERTIFICATE

This is to certify that ,this is the revised version of the thesis **"Analysis of Inventory Models"** submitted by Sandhya E under my guidance , after incorporating the necessary corrections/suggestions made by the adjudicators. I also certify that the contents in the thesis and the soft copy are one and the same.

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This is to certify that the thesis titled **"Analysis of Inventory Models"** submitted by **Sandhya E** to the University of Calicut, Kerala in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy in **Mathematics** is a record of original research work carried out by her under my supervision. The content of this thesis, in full or in parts, has not been submitted by any other candidate to any other University for the award of any degree or diploma.

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DECLARATION

I, **Sandhya E**, hereby declare that the thesis titled **"Analysis of Inventory Models"** submitted to the University of Calicut, Kerala in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy in **Mathematics** is a record of original and independent research work done by me under the supervision of **Dr. C. Sreenivasan, Associate Professor, Government Victoria College, Palakkad, Kerala**. I also declare that this thesis or any part of it has not been submitted to any other University/Institute for the award of any degree.

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To

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തീസിസിന്റെ തലക്കെട്ട്:

"ഇൻവെന്ററി മോഡലുകളുടെ വിശകലനം"

തീസിസ് 6 അധ്യായങ്ങളായി തിരിച്ചിരിക്കുന്നു. ആദ്യ അധ്യായം ആമുഖമാണ്. ഈ അധ്യായത്തിൽ ക്യൂയിംഗ് മോഡലുകളുടെയും ഇൻവെന്ററി ക്യൂയിംഗ് മോഡലുകളുടെയും അടിസ്ഥാന ആശയങ്ങൾ വിശദീകരിക്കുന്നു. രണ്ടാമത്തെ അധ്യായത്തിൽ, പ്രദേശത്തെ അനുബന്ധ പ്രവർത്തനങ്ങളുടെ ഒരു ഹ്രസ്വ രൂപരേഖ ഞങ്ങൾ നൽകുന്നു.

"പോസിറ്റീവ് ലീഡ് സമയവും ബാക്ക്ലോഗുകളും ഉള്ള ഒരു ഇൻവെന്ററി മോഡലിനുള്ള വ്യക്തമായ പരിഹാരം" എന്നാണ് മൂന്നാമത്തെ അധ്യായത്തിന്റെ തലക്കെട്ട്. ഇൻവെന്ററി നൽകിയ ഒരു പോയിസൺ പ്രോസസ് അനുസരിച്ച് ഉപഭോക്താക്കൾ ഒരൊറ്റ സെർവർ കൗണ്ടറിലേക്ക് എത്തുന്ന ഒരു (s,S) ഇൻവെന്ററി മോഡൽ ഞങ്ങൾ ഇവിടെ പരിഗണിക്കുന്നു. (s,S) പോളിസി അനുസരിച്ച്

ഇൻവെന്ററി വീണ്ടും നിറയ്ക്കുന്നു, ഒരു എക്സ്പോണൻഷ്യൽ റാൻഡം വേരിയബിളാണ് നികത്തൽ സമയം. ഈ മോഡലിന് ഞങ്ങൾ നിസ്സാരമായ സേവന സമയം അനുമാനിക്കുന്നു. ഇൻവെന്ററി ലെവൽ പൂജ്യത്തിലേക്ക് താഴുമ്പോൾ ക്യൂവിൽ ചേരുന്ന ഉപഭോക്താക്കൾ ഒരു ക്യൂ ഉണ്ടാക്കുകയും ഇൻവെന്ററി നികത്തൽ യാഥാർത്ഥ്യമാകുന്നതുവരെ സിസ്റ്റത്തിൽ തുടരുകയും ചെയ്യുന്നു. സ്റ്റേഡി സ്റ്റേറ്റ് പ്രോബബിലിറ്റി വെക്ടറിന്റെ വ്യക്തമായ പദപ്രയോഗം ഉരുത്തിരിഞ്ഞു. ക്യൂവിൽ ഒരു ഉപഭോക്താവ് പ്രതീക്ഷിക്കുന്ന കാത്തിരിപ്പ് സമയത്തിന്റെ എക്സ്പ്രഷൻ ഉരുത്തിരിഞ്ഞതാണ്. പ്രകടന നടപടികളിൽ പാരാമീറ്ററുകളുടെ സ്വാധീനത്തെക്കുറിച്ചുള്ള ഒരു സംഖ്യാ പഠനം നടത്തി.

നാലാമത്തെ അധ്യായത്തിന്റെ തലക്കെട്ട് "പോസിറ്റീവ് ലീഡ് സമയവും സെർവർ തടസ്സങ്ങളുമുള്ള ഒരു ഇൻവെന്ററി മോഡലിനുള്ള വ്യക്തമായ പരിഹാരം". ഈ മാതൃകയിൽ ഞങ്ങൾ ഇൻവെന്ററിയുള്ള ഒരൊറ്റ സെർവർ ക്യൂയിംഗ് സിസ്റ്റം പരിഗണിക്കുന്നു.

ഉപഭോക്താക്കൾ ഒരു പോയിസൺ പ്രക്രിയയ്ക്ക് അനുസരിച്ചാണ് എത്തിച്ചേരുന്നത്, സേവന സമയങ്ങൾ എക്സ്പോണൻഷ്യൽ ഡിസ്ട്രിബ്യൂഷൻ പിന്തുടരുന്നു. എക്സ്പോണൻഷ്യൽ ഡിസ്ട്രിബ്യൂഷനെ പിന്തുടരുന്ന പോസിറ്റീവ് ലീഡ് ടൈം ഉപയോഗിച്ച് (കൾ, എസ്) പോളിസി അനുസരിച്ച് ഇൻവെന്ററി വീണ്ടും

നിറയ്ക്കുന്നു. സെർവർ ഒരു ഉപഭോക്താവിനെ സേവിക്കുമ്പോൾ, സേവനം തടസ്സപ്പെട്ടേക്കാം; ഒരു എക്സ്പോണൻഷ്യൽ വിതരണത്തെ തുടർന്നാണ് തടസ്സ സമയം. ഒരു തടസ്സത്തെത്തുടർന്ന്, എക്സ്പോണൻഷ്യൽ നിരക്കിൽ അറ്റകുറ്റപ്പണിക്ക് ശേഷം സേവനം പുനരാരംഭിക്കുന്നു. സെർവർ തടസ്സത്തിലായിരിക്കുമ്പോൾ, സേവനം ലഭിക്കുന്ന ഉപഭോക്താവ് തന്റെ സേവനം പൂർത്തിയാകുന്നതുവരെ അവിടെ കാത്തുനിൽക്കുമെന്നും തടസ്സം കാരണം ഇൻവെന്ററി നഷ്ടമാകില്ലെന്നും സെർവർ തടസ്സപ്പെടുമ്പോൾ വരവ് അനുവദിക്കില്ലെന്നും എന്തെങ്കിലും റദ്ദാക്കിയാൽ ഓർഡർ നൽകുമെന്നും ഞങ്ങൾ അനുമാനിക്കുന്നു. മുകളിലുള്ള സിസ്റ്റത്തിന്റെ സ്ഥിരത വിശകലനം ചെയ്യുകയും സ്റ്റേഡി സ്റ്റേറ്റ് പ്രോബബിലിറ്റി വെക്റ്റർ വ്യക്തമായി കണക്കാക്കുകയും ചെയ്യുന്നു. സിസ്റ്റത്തിൽ പ്രതീക്ഷിക്കുന്ന ഉപഭോക്താക്കളുടെ എണ്ണം, പ്രതീക്ഷിക്കുന്ന ഇൻവെന്ററി ലെവൽ, പ്രതീക്ഷിക്കുന്ന തടസ്സ നിരക്ക് തുടങ്ങിയ നിരവധി സിസ്റ്റം പ്രകടന അളവുകൾക്കായുള്ള എക്സ്പ്രഷനുകൾ ലഭിക്കും. വ്യക്തമായ പദപ്രയോഗങ്ങൾ ലഭിച്ചിട്ടുണ്ടെങ്കിലും, പ്രകടന നടപടികളിൽ പാരാമീറ്ററുകളുടെ സ്വാധീനത്തെക്കുറിച്ചുള്ള ഒരു സംഖ്യാ പഠനം നടത്തിയിട്ടുണ്ട്. മോഡലിന്റെ വില വിശകലനവും നടത്തിയിട്ടുണ്ട്.

അഞ്ചാമത്തെ അധ്യായത്തിന്റെ തലക്കെട്ട് "സെർവർ തടസ്സവും പുനഃപരിശോധനയും ഉള്ള ഒരു ഇൻവെന്ററി

മോഡലിനുള്ള വ്യക്തമായ പരിഹാരം". ഈ മാതൃകയിൽ ഉപഭോക്താക്കൾ ഇൻവെന്ററി നൽകുന്ന ഒരു പോയിസൺ പ്രക്രിയയ്ക്ക് അനുസൃതമായി ഒരൊറ്റ സെർവർ ക്യൂയിംഗ് മോഡലിലേക്ക് പ്രവേശിക്കുന്നു. എക്സ്പോണൻഷ്യൽ ഡിസ്ട്രിബ്യൂഷനെ തുടർന്നുള്ള സേവന സമയം. എത്തിച്ചേരുമ്പോൾ, സെർവർ തിരക്കിലാണെന്ന് കണ്ടെത്തി ഉപഭോക്താക്കൾ ഒരു പരിക്രമണപഥത്തിൽ പ്രവേശിക്കുന്നു, അവിടെ നിന്ന് സ്ഥിരമായ പുനരന്വേഷണ നിരക്കിൽ സേവനത്തിനായി വീണ്ടും ശ്രമിക്കുന്നു. സെർവർ ഒരു ഉപഭോക്താവിനെ സേവിക്കുമ്പോൾ, സേവനം തടസ്സപ്പെടാം, തടസ്സത്തിന്റെ ഇടയ്ക്ക് സംഭവിക്കുന്ന സമയം ഗണ്യമായി വിതരണം ചെയ്യുന്നു. ഒരു സേവന തടസ്സത്തെത്തുടർന്ന്, ക്രമാനുഗതമായി വിതരണം ചെയ്ത സമയത്തിന് ശേഷം സേവനം പുനരാരംഭിക്കുന്നു. ഇൻവെന്ററി (കൾ, എസ്) പോളിസി അനുസരിച്ച് വീണ്ടും നിറയ്ക്കുന്നു, നികത്തൽ തൽക്ഷണമാണ്. ചർച്ച ചെയ്യപ്പെടുന്ന മോഡലിന്, സെർവർ തടസ്സം കാരണം ഇൻവെന്ററികളൊന്നും നഷ്ടപ്പെടുന്നില്ലെന്ന് ഞങ്ങൾ അനുമാനിക്കുന്നു, തടസ്സം സംഭവിക്കുമ്പോൾ സേവനം നൽകുന്ന ഉപഭോക്താവ് തന്റെ സേവനം പൂർത്തിയാകുന്നതുവരെ അവിടെ കാത്തിരിക്കുന്നു, കൂടാതെ വരവുകളോ റീട്രയലുകളോ ഒന്നുമില്ല, സെർവർ ആയിരിക്കുമ്പോൾ എന്തെങ്കിലും റദ്ദാക്കിയാൽ ഓർഡർ നൽകും. തടസ്സം ന്. സ്റ്റേഡി സ്റ്റേറ്റ് പ്രോബബിലിറ്റികൾക്കായുള്ള വ്യക്തമായ പദപ്രയോഗം

കണക്കാക്കുകയും നിരവധി പ്രകടന നടപടികൾ വ്യക്തമായും സംഖ്യാപരമായും വിലയിരുത്തുകയും ചെയ്യുന്നു. പാരാമീറ്റർ മൂല്യങ്ങളുള്ള വിവിധ പ്രകടന അളവുകളുടെ വ്യത്യാസം കാണിക്കുന്ന ഗ്രാഫുകളും വരയ്ക്കുന്നു.

ആറാമത്തെ അധ്യായത്തിൽ, പ്രബന്ധത്തിൽ ചർച്ച ചെയ്ത മാതൃകകളെ അടിസ്ഥാനമാക്കി ഭാവി പ്രവർത്തനങ്ങളെക്കുറിച്ചും വിപുലീകരണങ്ങളെക്കുറിച്ചും ഞങ്ങൾ ചില ശുപാർശകൾ നൽകുന്നു.

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ABSTRACT

The thesis is divided into 6 chapters. The first chapter is Introduction. In this chapter the basic concepts of queuing models and inventory queuing models are explained. In the second chapter we give a brief outline of the related works in the area.

The third chapter is titled “An Explicit Solution for an Inventory Model with Positive Lead Time and Backlogs”. Here we consider an (s, S) inventory model in which customers arrive to a single server counter according to a Poisson process where inventory served. Inventory is replenished according to (s, S) policy, the replenishment time being an exponential random variable. We assume negligible service time for this model. The customers who join the queue when inventory level drops to zero form a queue and remain in the system until inventory replenishment is realized. The explicit expression for the steady state probability vector has been derived. The expression for expected waiting time of a customer in the queue has been derived. A numerical study of the effect of parameters on the performance measures has been done.

The fourth chapter is titled “An Explicit Solution for an Inventory Model with Positive Lead Time and Server Interruptions”. In this model we consider a single server queuing system with inventory. Customers arrive according to a Poisson process and service times follow exponential distribution. Inventory is replenished according to (s, S) policy with positive lead time which follows exponential distribution. While the server serves a customer, the service may be interrupted; the interruption time follows an exponential distribution.

Following an interruption the service restarts after repair at an exponential rate. We assume that while the server is on interruption, the customer being served waits there until his service is completed, no inventory is lost due to interruption, no arrivals are allowed when the server is on interruption and an order placed if any is cancelled. Stability of the above system is analyzed and the steady state probability vector is calculated explicitly. Expressions for several system performance measures such as expected number of customers in the system, expected inventory level, expected interruption rate etc. are obtained. Even though explicit expressions are obtained, a numerical study of the effect of parameters on the performance measures has been done. A cost analysis has also been done for the model.

The fifth chapter is titled "An Explicit Solution for an Inventory Model with Server Interruption and Retrials". In this model customers enter into a single server queuing model in accordance with a Poisson process where inventory is served. The inter service time follows exponential distribution. Upon arrival, finding the server busy the customers enter into an orbit from where they retry for service at a constant retrial rate. While the server serves a customer the service can be interrupted, the inter occurrence time of interruption being exponentially distributed. Following a service interruption the service restarts after an exponentially distributed time. Inventory is replenished according to (s, S) policy, replenishment being instantaneous. For the model under discussion we assume that no inventory is lost due to server interruption, the customer being served when interruption occurs waits there until his service is completed and no arrivals are entertained and an order placed if any is cancelled while the server is on interruption. Explicit expression for the steady state probabilities is calculated and several performance measures are evaluated explicitly and numerically. Graphs which show the variation of various performance measures with parameter values are also drawn.

In the sixth chapter we provide some recommendations about the future work and extensions which can be done based on the models discussed in the thesis.

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CHAPTER 1

Introduction

1.1 Description of the queuing problem

Here we study what happens when an individual or a group of people come and join queues. We are quite familiar with queues in our day to day life. Common examples are going to a doctor, going to a barber shop, customers waiting in a bank counter, patients waiting in a hospital, airplanes waiting to take off or for landing etc. A queuing system essentially happens when there are people or entities that arrive, called arrivals, who require some kind of service from another entity, called server. In most of the real life situations customers have to wait in a queue for getting service. This waiting time can be reduced by enhancing the service facilities. But this will result in additional cost and so a drop in the profit. On the other hand if the queue length is large it will result in lost sales. So the problem is to maximize the profit by achieving a balance between the cost associated with long queues and that associated with the reduction / prevention of waiting. Queuing theory is that branch of applied probability which studies service systems of the above type.

The basic characteristics of any queuing model are the following.

Arrival pattern of customers

The manner in which customers arrive and join a queuing system is described by the arrival pattern of customers. It is measured in terms of average number of arrivals per unit time which is called the mean arrival rate or the average time between successive arrivals which is called the mean inter arrival time. This is often expressed by means of a probability distribution.

Arrivals occur usually one by one, but there are also instances where customers arrive together. This is termed as Batch arrivals. If the queue is too long a customer may decide not to enter it upon arrival. This is known as balking. Sometimes a customer may enter the queue, but after some time lose patience and decide to leave. This behavior of the customer is known as reneging. Another case is, when there is more than one queue, customers may switch from one to another which is called jockeying.

Service pattern of servers

The manner in which service is provided to the customer is described by the service pattern of servers. It is measured in terms of average number of services per unit time which is called the mean service rate or the average time between successive services which is called the mean inter service time. This is also expressed by means of a probability distribution.

Service may also be in single or in batches. If there are no customers in the system, we say the server is idle. The servers that become idle may leave the system for a random period. This is known as server vacation. These vacations may be utilized to perform additional work assigned to the servers. However in retrial queues with no waiting space, each service is preceded and followed by an idle period.

Queue discipline

This specifies the manner in which the customers are selected for service when a queue is formed. The most common disciplines are FIFO (First in First out) and LIFO (Last in First out). SIRO (Service in Random order) is also a queue discipline. Sometimes a few customers are given priorities upon entering the system. The ones with higher priorities are selected for service ahead of those with lower priorities. Sometimes at the arrival of a high priority customer, the service of a low priority customer is interrupted and the high priority customer is taken for service. In other words high priority customers are never allowed to wait in favor of a low priority customer. This is known as a preemptive priority. On the other hand a non preemptive service never interrupts an ongoing service whether it is of a high priority or low priority customer.

Service channels

Service channels refer to the number of parallel service stations which can provide identical service to the customers.

Stages of service

A service may have several stages. A customer may have to pass through all these service stages before to leaving the system. Such queues are known as tandem queues.

1.2 Description of inventory systems

By inventory we mean stock of goods, commodities and other resources that are stored for the smooth conduct of business. In addition to usual features of queuing models, the availability of inventory is also considered in inventory models. The time required to replenish the inventory is called lead time. Instantaneous replenishment is considered in some inventory queuing models. In such cases we say lead time is zero. If the time required to serve the items to the customers and time required to replenish the items (lead time) are both negligible then no queue is formed except in the case when order for replenishment is placed only when a number of back orders accumulate. If either service time or lead time or both are taken to be positive then a queue is formed.

There are several policies for replenishing the inventory. The most commonly used is known as (s, S) inventory policy According to this policy an order is placed when inventory level drops to s. The order quantity is fixed as $Q = S - s$, where S is the maximum inventory level. Such an inventory model is called (s, S) inventory model. In (s, S) policy, s and S are control variables. Here s is called the reorder level and S is called the maximum inventory level. Here we use (s, S) policy in the sense defined in Stanfel and Sivazlian [1]: the on hand inventory, on reaching the level s, an order for the fixed quantity $S - s$ of the item is placed. There are several other ordering policies. In another policy, known as Order up to maximum S policy replenishment order is placed at levels $0 \leq i \leq s$, the replenishment quantity being S–i when inventory level is i $(0 \leq i \leq s)$ at replenishment epoch. In random order quantity policy, the order quantity can be anything between $s + 1$ and S–s. Yet another ordering policy is to place replenishment order when inventory level belongs to $\{0, 1, \ldots, s\}$.

1.3 Some basic concepts

Stochastic process

A stochastic process is defined as a family of random variables $\{X(t), t \in T\}$. That is for each $t \in T$, $X(t)$ is a random variable and is defined in some probability space. The parameter t is often referred to as time and $X(t)$ denotes the values taken by the random variable at time t. The set T is called the index set of the process. If T is a countable set then the stochastic process is said to be a discrete (time) stochastic process. If T is continuous, that is an interval of the real line then the stochastic process is said to be a continuous (time) process. For instance, $\{X_n, n = 0, 1, \ldots\}$ is a discrete time stochastic process indexed by the set of non negative integers, while $\{X(t), t \geq 0\}$ is a continuous time process indexed by non negative real numbers. The set of all values taken by $X(t)$ for all values of t is known as the state space of the process. The state space may also be discrete or continuous. Thus a stochastic process can have a discrete or continuous state space and may evolve at a discrete set of time points or continuously in time. If the state space is discrete the process is referred to as a chain.

Markov Process

A stochastic process whose conditional probability distribution function satisfies memory less property or Markov property is called a Markov process. Thus we can define discrete time Markov chains (DTMCs) and continuous time Markov chains (CTMCs) and also discrete time Markov process and continuous time Markov process.

Formally a DTMC is a stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$

which satisfies the Markov property, namely $Pr\{X_{n+1} = x_{n+1}/X_n = x_n, \ldots, X_0 = x_0\} = Pr\{X_{n+1} = x_{n+1}/X_n = x_n\}$. Similarly a CTMC is a stochastic process $\{X(t), t \in T\}$ which satisfies the condition $Pr\{X(t_n) = x_n/X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots, X(t_0) = x_0\} = Pr\{X(t_n) = x_n/X(t_{n-1}) = x_{n-1}\}$ for $t_0 < t_1 < \ldots < t_{n-1} < t_n$ and for every n; $x_0, x_1, \ldots, x_n$ are elements of the state space. This means that the distribution of any future occupancy depends only on the present state but not on the past.

Exponential distribution

A continuous random variable X is said to follow exponential distribution with parameter $\lambda > 0$ if its cumulative distribution function is given by $F(x) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$. The n^{th} moment of the distribution is given by $\dfrac{n!}{\lambda^n}$. This is the only continuous distribution which possess the memory less property; that is $Pr(X > t + s/X > t) = Pr(X > s)$ for all t, $s \geq 0$. Exponential distribution is relatively easy to work.

Renewal Process

Let $X_1, X_2, \ldots$ be a sequence of non negative independent random variables having a common probability distribution function $F(x) = Pr(X_i \leq x)$ $i = 1, 2, \ldots$. The random variable X_n denotes the interoccurrence time between the $(n-1)^{th}$ and n^{th} event in some specific probability problem.

Letting $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$; $S_0 = 0$, we have S_n is the epoch at which the n^{th} event occurs. Define $N(t) = max\{n : S_n \leq t\}$. The distribution of $N(t)$ is given by $P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$, where

F_n is the n-fold convolution of F with itself. The counting process $\{N(t) : t \geq 0\}$ is called the renewal process. The Poisson process is a renewal process where F is an exponential distribution.

Poisson Process

A renewal process $\{N(t), t \geq 0\}$ with interoccurrence times $X_1, X_2, \ldots$ is called a Poisson process if the interoccurrencs times have a common exponential probability density function $f(t) = \begin{cases} \lambda e^{-\lambda t} & , t \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

A Poisson process is also defined as follows.
A renewal process $\{N(t),\ t \geq 0\}$ is said to be a Poisson process having rate λ if

(i) $N(0) = 0$

(ii) The process has stationary and independent increments.

(iii) $P\{N(h) = 1\} = \lambda h + o(h)$.

(iv) $P\{N(h) \geq 2\} = o(h)$.

Using the above postulates it can be proved that for all $s, t \geq 0$, $P\{(N(t+s) - N(s)) = n\} = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$, $n = 0, 1, \ldots$. For a Poisson process having parameter λ the inter arrival time has an exponential distribution with mean $1/\lambda$.

Continuous-time Phase type (PH) distributions

Phase-type distributions were introduced by Neuts in 1975 as a generalization of the exponential distribution. A non negative random variable X is said to follow phase type distribution (PH distribution) if its distribution function is given by $F(t) = 1 - \alpha \exp(Tt)e$, where

(i) e is a column vector of ones

(ii) α is a row vector of order m with all elements non negative and $\alpha e \leq 1$.

(iii) T is a matrix of order $m \times m$ with all diagonal entries negative, all off diagonal entries non negative and row sums non positive. Also T is invertible.

Here T is called the generator and (α, T) is called the phase type representation of order m of the PH distribution. This may also be defined in the following manner.

Consider a Markov process Ω with $m+1$ states $\{1, 2, \ldots, m+1\}$ and infinitesimal generator matrix $Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$, where the matrix T satisfies $T_{ii} < 0$ for $1 \leq i \leq m$ and $T_{ij} \geq 0$ for $i \neq j$; T^0 is an $m \times 1$ coloumn matrix such that $Te + T^0 = 0$, where e is a column matrix of 1's of appropriate order. Let (α, α_{m+1}), where α is an $1 \times m$ dimensional row vector and α_{m+1} is a scalar such that $\alpha e + \alpha_{m+1} = 1$, be the initial probability vector of Ω. Since the sojourn time of state $m+1$ is zero it is an absorbing state for the Markov chain Ω. For eventual absorption into the absorbing state, starting from the initial state, it is necessary and sufficient that T is non singular. The probability distribution $F(.)$ of time until absorption in the state $m+1$ corresponding to the initial probability vector (α, α_{m+1}) can be calculated as $F(x) = 1 - \alpha e^{(Tx)} e$, $x \geq 0$. Thus a probability distribution $F(.)$ is a distribution of phase type if and only if it is the distribution of time until absorption of a finite Markov process described above. The moments about origin are given by $E(X^k) = \mu_k = (-1)^k k!(\alpha T^{-k} e)$ for $k \geq 0$. When $m = 1$ and $T = [-\lambda]$, the underlying PH-distribution is exponential.

PH-renewal process

A renewal process whose inter-renewal times have a PH distribution is called a PH-renewal process. To construct a PH-renewal process we consider a continuous time Markov chain with state space $\{1, 2, \ldots, m+1\}$ having infinitesimal generator $Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$. The matrix T is taken to be nonsingular so that absorption to the state $m+1$ occurs with probability 1 from any initial state. Let $(\alpha, 0)$ be the initial probability vector. When absorption occurs in the above chain we say a renewal has occurred. Then the process immediately starts anew in one of the states $\{1, 2, \ldots, m\}$ according to the probability vector α. Continuation of this process gives a non terminating stochastic process called PH-renewal process.

Level Independent Quasi-Birth-Death (LIQBD) process

A level independent quasi birth and death process is a Markov process on the state space $E = \{(i, j),\ i \geq 0,\ 1 \leq j \leq m\}$. Its infinitesimal generator matrix Q is given by

$$Q = \begin{bmatrix} B_0 & A_0 & & & & & \\ B_1 & A_1 & A_0 & & & & \\ & A_2 & A_1 & A_0 & & & \\ & & A_2 & A_1 & A_0 & & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix} \tag{1.3.1}$$

We partition the state space E into levels $\{\hat{0}, \hat{1}, \hat{2}, \ldots\}$, where $\hat{i} = \{(i, j),\ i \geq 0,\ 1 \leq j \leq m\}$. The states within the levels are called phases. The matrix B_0 denotes the transition rates within level $\hat{0}$, B_1 the transition rates from level $\hat{1}$ to level $\hat{0}$. A_2, A_1 and A_0 denote transition rates from level $\hat{i}$ to $(i \hat{-} 1)$, $\hat{i}$ to $\hat{i}$ and $\hat{i}$ to $(i \hat{+} 1)$ respectively.

Matrix Analytic Method

Matrix analytic approach to stochastic models was introduced by M.F Neuts to provide an algorithmic analysis for M/G/1 and G/M/1 type of queuing models. The following brief discussion gives an account of the method of solving an LIQBD using the matrix geometric method. For a detailed description, we refer to Neuts [2], Latouchi and Ramaswami [3].

Let $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ be the steady state vector of the QBD process whose generator matrix is Q. Each π_i can be partitioned as $\pi_i = (\pi(i,1), \pi(i,2), \ldots, \pi(i,m))$, m being the number of phases with in levels. Let $\pi_i = \pi_0 R^i$, $i \geq 1$. Then from $\pi Q = 0$ we get

$$\begin{aligned} \pi_0 A_0 + \pi_1 A_1 + \pi_2 A_2 &= 0 \\ \pi_0 A_0 + \pi_0 R A_1 + \pi_0 R^2 A_2 &= 0 \\ \pi_0 (A_0 + R A_1 + R^2 A_2) &= 0 \end{aligned}$$

Choose R such that $R^2 A_2 + R A_1 + A_0 = 0$. We also have

$$\begin{aligned} \pi_0 B_0 + \pi_1 B_1 &= 0 \\ \pi_0 B_0 + \pi_0 R B_1 &= 0 \\ i.e., \pi_0 (B_0 + R B_1) &= 0 \end{aligned}$$

First choose π_0 as the steady state vector of $B_0 + RB_1$. Then π_i, for $i \geq 1$ can be found using the formulae; $\pi_i = \pi_0 R^i$ for $i \geq 1$. Now the steady state probability distribution of the system is obtained by dividing each π_i, with the normalizing constant $[\pi_0 + \pi_1 + \ldots]e = (I - R)^{-1}e$.

The above discussion leads to the following theorem.

Theorem 1.3.1. *The QBD with infinitesimal generator Q of the form 1.3.1 is positive recurrent if and only if the minimal non negative solution R of the matrix quadratic equation $R^2A_2 + RA_1 + A_0 = 0$ has all its eigen values inside the unit disc and the finite system of equations $\pi_0(B_0 + RB_1) = 0$, $\pi_0(I - R)^{-1}e = 1$ has a unique solution π_0. If the matrix $A = A_0 + A_1 + A_2$ is irreducible, then $sp(R) < 1$ if and only if $xA_0e < xA_2e$, where x is the stationary probability vector of $A = A_0 + A_1 + A_2$. The stationary probability vector $\pi = (\pi_0, \pi_1, \ldots)$ of Q is given by $\pi_i = \pi_0 R^i$ for $i \geq 1$.*

Level Dependent Quasi Birth Death (LDQBD) Process

A level dependent Quasi-Birth–Death process is a Markov process on a state space $E = \{(i, j), i \geq 0, 1 \leq j \leq n_i\}$ with infinitesimal generator matrix Q given by

$$Q = \begin{bmatrix} A_{10} & A_{00} & & & & & \\ A_{21} & A_{11} & A_{01} & & & & \\ & A_{22} & A_{12} & A_{02} & & & \\ & & A_{23} & A_{13} & A_{03} & & \\ & & & - & - & - & - \\ & & & & - & - & - \\ & & & & & - & - \end{bmatrix} \quad (1.3.2)$$

The generator matrix Q is obtained in the above form by partitioning the state space E into levels $\{\hat{0}, \hat{1}, \hat{2}, \ldots\}$. Here the transitions take place only to the immediately preceding and succeeding levels for

$i \geq 1$. However the transition rate depends on the level i, unlike in the LIQBD, and therefore the spatial homogeneity of the associated process is lost.

A special class of LDQBD's is those which arise in retrial queueing models. We have described some of the basic tools for the analysis of queuing models. Now we provide a review of the work done in the theme of the present thesis.

CHAPTER 2

Literature Review

2.1 Review of related works

The pioneers in the study of queueing inventory models are Melikov and Molchano [4] and Sigman and Simchi- Levi [5]. In Sigman and Simchi- Levi customers are allowed to join even when there is no inventory in the system. They also discuss the case of non exponential lead time distribution. Later Berman and et al. [6] considered an inventory system where a processing time is required for serving the inventory. Here they considered deterministic service time and the model was discussed as a dynamic programming model. Berman and Kim [7] and Berman and Sapna [8] later discussed inventory queuing systems with exponential service time distribution and with arbitrary distribution.

There are several papers on inventory queuing models by Krishnamoorthy and his co-authors [9, 10, 11, 12, 13, 14, 15, 16, 17]. In [9] the authors consider an inventory system with two parallel service facilities and a certain number of customers are transferred from longer to shorter queue whenever their difference reaches a prescribed quantity. Along with this customer transfer, a certain quantity of inventory is also transferred, depending on availability. In [12] the authors numerically investigate a PH/PH/1 inventory model with reneging of customers and finite shortage of items. They assume both arrival process and service process to follow phase type renewal process. But the lead-time is taken as zero. Reneging of customers is also considered. In [10] the authors consider a single commodity inventory system operating under the (s, S) policy. There is a service time associated with each demand. Here items are serviced even in the absence of a demand and such processed items are stacked separately. The server keeps processing the items. The total of processed and unprocessed items cannot exceed S. Also when the total reaches s, an order for replenishment is placed and the order materialization takes place instantly. In [11] the authors consider an (s, S) retrial inventory model where inter arrival times follow a Batch Markovian arrival process. The demands enter into a buffer of finite capacity, equal to the number of inventory in hand. When the buffer is full the customers enter into an orbit from where they retry for service at an exponential rate. The service times and lead times both follow exponential distributions. In [12] the authors numerically investigate a PH/PH/1 inventory model with reneging of customers and finite shortage of items. Here arrivals occur according to a phase type renewal process and the service times are identically and independently distributed random variables having common phase type distribution. The lead-time is taken zero. Customers renege from the system at a constant rate and shortage is also permitted. In [13] the authors consider an (s, S) inventory system with a positive lead-time and positive service time. Those customers, encountering an idle server and positive inventory, are immediately taken into service and customers who at the time of arrival find an idle server with zero inventory are considered lost. A customer who finds the server busy, joins an orbit of infinite capacity and from there retries for service, with inter-retrial times exponentially distributed. At a service completion epoch, the server, with a certain probability, makes a search in the orbit and picks a customer, if any, randomly from the orbit, provided there is at least one item left in the inventory for the next service. The search time is assumed to be negligible. In [14] the authors introduce an additional control policy called the N-policy into an (s, S) inventory system with positive service time. Under specified interarrival and service time distributions, which are independent of each other, they obtain the necessary and sufficient condition for the system to be stable. They also obtain the optimal values of the control variables s, S, and N. Numerical illustrations are provided and several measures of performance of the system are evaluated. In [17] the authors consider an (s, S) inventory model with random positive service time. Arrivals are in accordance with a Markovian arrival process (MAP) and service times have phase type distribution. Lead time for replenishment of inventory follows a correlated process similar to the customer arrival process. At a service completion epoch if no customer is waiting or the inventory level is zero then the server goes on a vacation of random duration. The vacation time is also phase type distributed. On return from vacation if the server finds no customer waiting or inventory level is zero, the server goes on another vacation of random duration having the same phase type distribution as the earlier one.

The customer arrival process is subject to balking. While waiting for service, customers may become impatient and leave the system. In [18] the authors look at an opportunistic-type inventory replenishment in which there is an independent point process that is used to model events that are called opportunistic for replenishing inventory. When an opportunity (to replenish) occurs, a probabilistic rule that depends on the inventory level is used to determine whether to avail it or not. Assuming that the customers arrive according to a Markovian arrival process, the demands for inventory occur in batches of varying size, the demands require random service times that are modeled using a continuous-time phase-type distribution, and the point process for the opportunistic replenishment is a Poisson process. We refer to the papers [19, 20, 21] for a detailed description of several papers on inventory queuing models.

Several types of service interruption models are included in the literature. These include interruption due to server taking vacations, server breakdown, server interruptions, arrival of a priority customer etc. The first paper in this direction is due to White and Christie [22] where they consider an M/M/1 queuing model with exponentially distributed service interruption durations. Later Jaiswal [23, 24], Gaver [25], Keilson [26], Avi-Itzhak and Naor [27] and Thiruvengadom [28] analyzed queuing models with service interruptions, assuming general distribution for the service and interruption times. In all these papers it is assumed that the arrival of a high priority customer interrupts the service of a lower priority customer. Some other papers on service interruption models are due to Ibe and Trivedi [29], Federgruen and Green [30], Van Dijk [31] Takine and Sengupta [32], Masuyama and Takine [33]. Kulkarni and Choi [34] studied two models with server breakdowns in a single server retrial queue. In the first model, the customer whose service is interrupted, either leaves the system or re-joins the orbit; whereas in the second model the interrupted service is repeated after the repair is completed. Some other papers which study retrial queues with an unreliable server include Aissani and Artalejo [35], Artalejo and Gomez-Corral [36], Wang et al. [37], Sherman and Kharoufeh [38], Sherman et al. [39]. In [40] Krishnamoorthy and Ushakumari analyzed a queing model where disaster occur to the unit undergoing service. Another model by Wang et al. [41] discuss one with disaster and unreliable server.

Retrial queuing models are gaining more and more attention due to their use in communication and other fields. We refer to the books by Falin and Templeton [42] and Atralejo and Gomez Corral [43] for an extensive analysis of both theory and applications on retrial queues.

Inventory queing model with positive lead time and retrial of customers was first done by Artalejo et al. [44]. Ushakumari [40] arrived at an analytic solution for the problem discussed in the above paper. Following these, there were several papers in this direction. A few among Krishnamoorthy and Islam [45, 46], Krishnamoorthy et al. [47, 48] and Krishnamoorthy and Jose [49] are a few among them. In [45] the authors consider a production inventory model with retrial of customers. In [46] an analysis of a production inventory model with random shelf times of the items with retrials of the orbiting customers are considered. A study of inventory models with positive service time and retrial of customers from an orbit with an intermediate buffer of finite is done in [47], whereas a comparison of different (s, S) inventory models with an orbit of infinite capacity, having / not having a finite buffer is the content of [49]. Sivakumar B and his co authors

[50, 51, 52, 50, 53] have published several papers in queuing inventory models with retrials. In [50] the author considers a continuous review perishable (s, S) inventory system with a finite number of homogeneous sources of demands. The life time of each item and the lead times are assumed to be exponential. The author assumes that the demands occurring when inventory level drops to zero enter into the orbit. These orbiting demands retry for service at an exponential rate. In [52] the author analyzes a two-commodity inventory system under continuous review. It is assumed that primary demand for the i^{th} commodity is of unit size and primary demand time points form a Poisson process. The lead time is assumed to be exponential. Both the commodities are assumed to be substitutable in the sense that at the time of zero stock of any one commodity, the other one is used to meet the demand. When the inventory position of both commodities is zero, any arriving primary demand enters into an orbit of infinite size. The orbiting demands in the orbit send out signal to compete for their demand which is distributed as exponential. In [54] the authors consider a continuous review perishable (s, S) inventory system with a service facility consisting of a waiting line of finite capacity and a single server. Two types of customers, ordinary and negative, arrive according to a Markovian Arrival Process (MAP). An ordinary customer joins the queue and a negative customer removes some ordinary customers from the queue. A negative customer at an arrival epoch removes one or more ordinary waiting customers and the number of removals is a random variable depending on the number of waiting customers in the system. The life time of each item, the service time and the lead time of the reorders are all assumed to have independent exponential distributions.

CHAPTER 3

An Explicit Solution for an Inventory Model with Positive Lead Time and Backlogs

3.1 Introduction

The first study on queueing inventory systems is due to (Melikov and Molchano 1992) and (Sigman and Simchi- Levi 1992). Later inventory systems where a processing time is required for serving the inventory was done by Berman and et al. [6]. This was a deterministic model. Berman and Kim [7] and Berman and Sapna [8] were the first to discuss inventory with exponential distribution and with arbitrary distribution respectively. Krishnamoorthy and his co-authors used Matrix Analytic Methods to study a inventory models [9, 10, 11, 12, 13, 14, 15, 16, 17], where service time for providing the inventoried item is assumed. Padmavathi I et al. [55] studied a finite source (s, S) inventory system with postponed demands and server vacation. Krishnamoorthy and Islam [56] considered an (s, S) inventory system with postponed demands in which arrival follows a Poisson distribution and lead time exponential distribution. Sivakumar and Arivarignam [57] studied a perishable inventory system with postponed demands in which the demands that occur during the stock out period enter a pool with independent Bernoulli trial.

3.2 Model description

The system under consideration is described as follows. Customers arrive to a queueing system which has only one server according to a Poisson process of rate λ where inventory is served. Inventory is replenished according to (s, S) policy, the replenishment time being an exponential random variable with parameter δ. We assume negligible service time for this model. Customers who join the queue when the inventory level drops to zero form a queue and remains in the system until inventory replenishment is realized.

Let $N(t)$ be the number of customers in the system and $S(t)$ be the inventory level at time t. Then $\Omega = \{X(t) : t \geq 0\} = \{(N(t), S(t)) : t \geq 0\}$ is a Markov chain with state space $E = \{(0, k) : 0 \leq k \leq S\} \cup \{(i, 0) : i \geq 1\}$. The state space of the Markov chain can be partitioned into levels $\tilde{i}$ defined as $\tilde{0} = \{(0, 0), (0, 1), \ldots, (0, S)\}$ and $\tilde{i} = \{((i-1)Q + 1, 0), ((i-1)Q + 2, 0), \ldots, ((i-1)Q + Q, 0)\}; i \geq 1$. This makes the Markov chain under consideration, a level independent Quasi Birth Death(QBD) process. Here, $S - s = Q$, I_n denotes an identity matrix of order n and e denotes a column vector of 1's of appropriate order. The infinitesimal generator matrix of the process

Ω is

$$H = \begin{bmatrix} B_0 & B_1 & 0 & 0 & \_ & \_ & \_ \\ B_2 & A_1 & A_0 & 0 & 0 & \_ & \_ \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \_ \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix}$$

Here $B_0 = [b_{ij}]_{(s+1)\times(s+1)}$, where

$$b_{ij} = \begin{cases} -(\lambda+\delta) & : j = i; 1 \le i \le s+1 \\ -\lambda & : j = i; s+1 \le i \le S+1 \\ \lambda & : j = i-1; 2 \le i \le S+1 \\ \delta & : j = Q+i; 1 \le i \le s+1 \\ 0 & : \text{otherwise} \end{cases}$$

$$B_1 = [b_{ij}]_{(S+1)\times Q},\ \ b_{ij} = \begin{cases} \lambda & : i = j = 1 \\ 0 & : \text{otherwise} \end{cases}$$

$$B_2 = [b_{ij}]_{Q\times(S+1)},\quad b_{ij} = \begin{cases} \delta & : i+j = Q; 1 \le i \le Q \\ 0 & : \text{otherwise} \end{cases}$$

$$A_2 = \delta I_Q\,;\quad A_0 = [a_{ij}]_{Q\times Q};\quad a_{ij} = \begin{cases} \lambda & : i = Q, j = 1 \\ 0 & : \text{otherwise} \end{cases}$$

$$A_1 = [a_{ij}]_{Q\times Q};\quad a_{ij} = \begin{cases} -(\lambda+\delta) & : i = j \\ \lambda & : j = i+1; 1 \le i \le Q-1 \end{cases}$$

3.3 Analysis of the Model

Stability condition:
Define $A = A_0 + A_1 + A_2$ and let $\pi = (\pi_1, \pi_2, \ldots, \pi_Q)$ be the steady state vector of the generator matrix A. Then $\pi A = 0$ gives the following equations

$$-\lambda\pi_1 + \lambda\pi_Q = 0$$
$$-\lambda\pi_{i+1} + \lambda\pi_i = 0\,;\; 1 \leq i \leq Q-1$$

Hence $\pi_1 = \pi_2 = \ldots = \pi_Q$.
The QBD process with generator matrix H is stable if and only if the rate of drift to the left is larger than rate of drift to the right; that is $\pi A_0 e < \pi A_2 e$ (see Neuts) that is if and only if $\dfrac{\lambda}{Q\delta} < 1$.
Thus we have the following theorem for the stability of the system under study.

Theorem 3.3.1. *The Markov chain Ω is stable if and only if* $\dfrac{\lambda}{Q\delta} < 1$.

3.4 Computation of steady state vector

We find the steady state vector of Ω explicitly. Let $\pi = (\pi_0, \pi_1, \ldots)$ be the steady state vector where $\pi_0 = (\pi_0(0,0), \pi_0(0,1), \ldots, \pi_0(0,S))$ and

$$\pi_i = (\pi_i((i-1)Q+1, 0), \pi_i((i-1)Q+2, 0), \ldots, \pi_i((i-1)Q+Q, 0)); i \geq 1$$

.

$$\pi H = 0 \implies \pi_0 B_0 + \pi_1 B_2 = 0$$
$$\pi_0 B_1 + \pi_1 A_1 + \pi_2 A_2 = 0$$
$$\pi_i A_0 + \pi_{i+1} A_1 + \pi_{i+2} A_2 = 0;\; i \geq 1$$

Since Ω is a level independent QBD process its steady state vector is given by $\pi_{i+1} = \pi_1 R^i\,; i \geq 1$ (see Neuts), where R is the minimal non negative solution of the matrix quadratic equation $R^2A_2+RA_1+A_0 = 0$. Since all the rows of A_0 except the last are zeros so are that of R. Assume that

$$R = [r_{ij}]_{Q\times Q};\ r_{ij} = \begin{cases} r_j, & i = Q, 1 \leq j \leq Q \\ 0, & otherwise \end{cases}$$

Now $R^2A_2 + RA_1 + A_0 = 0$ gives us the following equations.

$$\begin{aligned} \lambda - (\lambda+\delta)r_1 + \delta r_1 r_Q &= 0; \\ \lambda r_1 - (\lambda+\delta)r_2 + \delta r_2 r_Q &= 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \lambda r_{Q-2} - (\lambda+\delta)r_{Q-1} + \delta r_{Q-1} r_Q &= 0; \\ \lambda r_{Q-1} - (\lambda+\delta)r_Q + \delta r_Q^2 &= 0 \end{aligned}$$

From the above equations we easily see that $r_j = r^j$ for every j; $r = r_1$. Then any of the above equations gives $\lambda - (\lambda+\delta)r + \delta r^{Q+1} = 0$. Dividing by $r-1$ we get $\delta r^Q + \delta r^{Q-1} + \ldots + \delta r - \lambda = 0$. Let $f(x) = \delta x^Q + \delta x^{Q-1} + \ldots + \delta x - \lambda$. Since $f(0)$ and $f(1)$ have opposite sign $f(x)$ has a root r between 0 and 1. Hence

$$R = [r_{ij}]_{Q\times Q};\ r_{ij} = \begin{cases} r^j, & i = Q, 1 \leq j \leq Q \\ 0, & otherwise \end{cases}$$

Now from $\pi_{i+1} = \pi_1 R^i$; $i \geq 1$ we get $\pi_2 = (r, r^2, \ldots, r^Q)\pi_1(Q,0)$; $\pi_3 = (r^{Q+1}, r^{Q+2}, \ldots, r^{2Q})\pi_1(Q,0)$ and so on. It remains to get π_0 and π_1

in terms of $\pi_1(Q,0)$.

$$\pi_0(0,0) = \left\{ \left(\frac{\lambda+\delta}{\lambda}\right)^Q - r\frac{\delta}{\lambda}\left(\frac{1-r^Q\left(\frac{\lambda+\delta}{\lambda}\right)^Q}{1-r\left(\frac{\lambda+\delta}{\lambda}\right)}\right)\right\}\pi_1(Q,0)$$

$$\pi_1(i,0) = \left\{ \left(\frac{\lambda+\delta}{\lambda}\right)^{Q-i} - r^{i+1}\frac{\delta}{\lambda}\left(\frac{1-r^{Q-i}\left(\frac{\lambda+\delta}{\lambda}\right)^{Q-i}}{1-r\left(\frac{\lambda+\delta}{\lambda}\right)}\right)\right\}\pi_1(Q,0);\ 1 \le i \le Q-1$$

$$\pi_0(0,1) = \left(\frac{\lambda+\delta}{\lambda}\right)\pi_0(0,0) - \frac{\delta}{\lambda}\pi_1(Q,0)$$

$$\pi_0(0,i+1) = \left(\frac{\lambda+\delta}{\lambda}\right)\pi_0(0,i) - \frac{\delta}{\lambda}\left\{ \left(\frac{\lambda+\delta}{\lambda}\right)^{i} - r^{Q-i+1}\frac{\delta}{\lambda}\left(\frac{1-r^{i}\left(\frac{\lambda+\delta}{\lambda}\right)^{i}}{1-r\left(\frac{\lambda+\delta}{\lambda}\right)}\right)\right\}\pi_1(Q,0);\ 1 \le i \le s$$

$$\pi_0(0,i+1) = \pi_0(0,i) - \frac{\delta}{\lambda}\left\{ \left(\frac{\lambda+\delta}{\lambda}\right)^{i} - r^{Q-i+1}\frac{\delta}{\lambda}\left(\frac{1-r^{i}\left(\frac{\lambda+\delta}{\lambda}\right)^{i}}{1-r\left(\frac{\lambda+\delta}{\lambda}\right)}\right)\right\}\pi_1(Q,0);\ s+1 \le i \le Q-1$$

$$\pi_0(0,Q+i+1) = \pi_0(0,Q+i) - \frac{\delta}{\lambda}\pi_0(0,i);\ 0 \le i \le s-1$$

Now $\pi_1(Q,0)$ is got from the condition $\pi_0 e_1 + \left(\sum_{i=1}^{\infty}\pi_i\right)e_2 = 1$ where e_1 and e_2 are column vector of one's of appropriate order.

3.5 System Performance Measures

The following system performance measures are calculated numerically.

3.5.1 Expected waiting Time in the System

For computing the expected waiting time in the queue of a tagged customer, who joins as the l^{th} customer in the queue, $(k-1)Q < l \le kQ$ we consider a Markov process $\psi = (\hat{N}(t))$, where $\hat{N}(t)$ denotes the rank, which is the position of the customer in the queue. The state space of the Markov chain is given by $\hat{E} = \{1,2,\ldots,k\} \cup \Delta$, where Δ is an absorbing state which corresponds to the tagged customer being taken for service. The infinitesimal generator matrix of the process ψ is given by $\hat{Q} = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$, where T^0 is a $k \times 1$ matrix such that

$$T^0(i,1) = \begin{cases} \delta; & i = k \\ 0; & otherwise \end{cases} \text{ and } T = \begin{bmatrix} -\delta & \delta & 0 & 0 & 0 & 0 \\ 0 & -\delta & \delta & 0 & 0 & 0 \\ 0 & 0 & -\delta & \delta & 0 & 0 \\ 0 & 0 & \_ & \_ & \_ & \_ \\ 0 & 0 & 0 & \_ & \_ & \_ \\ 0 & 0 & 0 & 0 & \_ & -\delta \end{bmatrix}. \text{ Now}$$

the waiting time W^k of the tagged customer is the time until absorption of the Markov process which is given by $W^k = -\alpha(T)^{-1}e$, where $\alpha = (1, 0, \ldots, 0, 0)$ and e is a column vector of ones of appropriate order. Since $T^{-1} = \begin{bmatrix} \frac{-1}{\delta} & \frac{-1}{\delta} & \frac{-1}{\delta} & \_ & \_ & \frac{-1}{\delta} \\ 0 & \frac{-1}{\delta} & \frac{-1}{\delta} & \_ & \_ & \frac{-1}{\delta} \\ 0 & 0 & \frac{-1}{\delta} & \frac{-1}{\delta} & \_ & \_ \\ 0 & 0 & 0 & \frac{-1}{\delta} & \frac{-1}{\delta} & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ \\ 0 & 0 & \_ & \_ & \_ & \frac{-1}{\delta} \end{bmatrix}$, we have $W^k = \dfrac{k}{\delta}$. Hence the expected waiting time of a general customer is given by

$$E(W_L) = \left(\sum_{k=1}^{\infty} \frac{k}{\delta} \pi_k \right) e = \frac{\lambda}{\delta^2} \left(\frac{2}{1 - r^Q} + \frac{r^Q}{(1 - r^Q)^2} \right) \pi_1(Q, 0) + \frac{\lambda}{\delta^2} \pi_0(0, 0).$$

In a similar manner, we can find the second moment of the waiting time of a customer as

$$E(W_L^2) = \sum_{k=1}^{\infty} W_2^k \pi_k = \frac{2\lambda}{\delta^3} \left[\frac{(1 - r^Q)^{-3} - 1}{r^Q} \right] \pi_1(Q, 0) + \frac{2\lambda}{\delta^3} \pi_0(0, 0),$$

where $W_2^k = 2\alpha(T^{-2})e = \frac{k(k+1)}{\delta^2}$.

3.5.2 Other Performance Measures

1. The expected number of customers in the system is given by

$$E(N) = \sum_{j=1}^{Q} \sum_{i=0}^{\infty} (iQ + j)\pi_{i+1}(iQ + j, 0).$$

2. The expected inventory level is given by $E(I) = \sum_{j=1}^{S} j\pi_0(0, j)$.

3. The expected replenishment rate $ERR = \delta \left(1 - \sum_{j=s+1}^{S} \pi_0(0, j)\right)$.

4. The probability that inventory level is zero
$P(I = 0) = \left(1 - \sum_{j=1}^{S} \pi_0(0, j)\right)$.

5. The probability that inventory level is greater than s
$P(I > s) = \sum_{j=s+1}^{S} \pi_0(0, j)$.

3.6 Numerical Illustration

In this section we provide numerical illustration of the system performance measures as underlying parameters vary.

3.6.1 Effect of reorder level s on various performance measures

In table 3.1 we see that as s increases the expected inventory level increases, expected number of customers in the system decreases and expected replenishment rate increases. The increase in expected inventory level $E(I)$ is as expected for orders are placed early. The decrease in expected number of customers in the system $E(N)$ is due to the fact that as expected inventory level increases more customers leave the system after getting served. The increase in expected replenishment rate ERR is obvious since as s increases there will be lesser number of states where order is not placed. This is clear from the formula $ERR = \delta\left(1 - \sum_{j=s+1}^{S} \pi_0(0,j)\right)$.

Table 3.2 shows a decrease in waiting time of a customer with an increase in s. As reorder level increases, with the maximum inventory level being the same, the time between two order placements decreases. Hence it becomes less probable for a customer to encounter shortage of inventory. This leads to a decrease in waiting time of the customer. The decrease in waiting time variance with increase in s is also in favor of the system performance.

Table 3.1: *Effect of s on the various performance measures*

$\lambda = 1 \quad \delta = 2 \quad S = 25$

s	$E(I)$	$E(N)$	$P(I=0)$	ERR	$P(I>s)$
5	13.52575	0.02635	0.01318	0.1	0.9
6	14.0176	0.0185	0.00925	0.10526	0.89474
7	14.51167	0.01303	0.00651	0.11111	0.88889
8	15.00717	0.00921	0.0046	0.11765	0.88235
9	15.50347	0.00653	0.00327	0.125	0.875
10	16.00004	0.00466	0.00233	0.13333	0.86667
11	16.49637	0.00335	0.00167	0.14286	0.85714

Table 3.2: *Variation of waiting time with re order level s*

$\lambda = 1 \quad \delta = 2 \quad S = 25$

s	$E(W_L)$	$E(W_L^2)$	$V(W_L)$
5	0.00879	0.01758	0.017503
6	0.00617	0.01235	0.012312
7	0.00435	0.0087	0.008681
8	0.00307	0.00615	0.006141
9	0.00218	0.00437	0.004365
10	0.00156	0.00312	0.003118
11	0.00112	0.00225	0.002249

3.6.2 Effect of maximum reorder level S on various performance measures

In table 3.3 we see that as S increases the expected inventory level increases, expected number of customers in the system decreases and expected replenishment rate also decreases. The increase in expected inventory level $E(I)$ is as expected for the order quantity $S - s$ increases as S increases. The decrease in expected number of customers in the system $E(N)$ is due to the fact that as expected inventory level increases more customers leave the system after getting served. The decrease in expected replenishment rate ERR is obvious since as S increases there will be more number of states where order is not placed. This is just the reverse to that with increase in s. This is clear from the formula $ERR = \delta \left(1 - \sum_{j=s+1}^{S} \pi_0(0, j)\right)$.

Table 3.4 shows a decrease in waiting time of a customer with an increase in S. As maximum inventory level increases, with the re order level being the same, even though the time between two order placements increases, the order quantity $S - s$ increases. Hence more customers will be served with each replenishment. This leads to a decrease in waiting time of the customer. It is also seen that the variance of waiting time also decreases with increase in S.

Table 3.3: *Effect of S on the various performance measures*

$\lambda = 1 \quad \delta = 2 \quad s = 5$

S	$E(I)$	$E(N)$	$P(I=0)$	ERR	$P(I>s)$
11	6.37214	0.10702	0.05289	0.3333	0.66667
12	6.94287	0.08514	0.04226	0.28571	0.71429
13	7.4823	0.07122	0.03544	0.25	0.75
14	8.00485	0.06156	0.03069	0.22222	0.77778
15	8.51776	0.05444	0.02716	0.2	0.8
16	9.02499	0.04893	0.02443	0.18182	0.81818
17	9.5288	0.04452	0.02224	0.16667	0.83333

Table 3.4: *Variation of waiting time with maximum inventory level S*

$\lambda = 1 \quad \delta = 2 \quad s = 5$

S	$E(W_L)$	$E(W_L^2)$	$V(W_L)$
11	0.03984	0.09237	0.090783
12	0.03056	0.06651	0.065576
13	0.02494	0.05246	0.051838
14	0.0212	0.04375	0.043301
15	0.01854	0.03782	0.037476
16	0.01655	0.03351	0.033236
17	0.01498	0.03021	0.029986

3.6.3 Effect of replenishment rate δ on various performance measures

Table 3.5 shows that the expected inventory level in the system $E(I)$ increases, expected number of customers in the system $E(N)$ decreases and the expected replenishment rate ERR remains constant as replenishment rate increases. The increase in $E(I)$ is obvious and decrease in expected number of customers in the system is due to fact that as δ increases, $E(I)$ increases as stated earlier and so more customers leave the system getting served. The expected replenishment rate is independent of replenishment rate for $ERR = \dfrac{\lambda}{(S-s)}$.

Table 3.6 shows that a decrease in expected waiting time which is expected. The variance of waiting time is also found to decrease as replenishment rate δ increases.

Table 3.5: *Effect of δ on the various performance measures*

$\lambda = 1 \quad s = 5 \quad S = 11$

δ	$E(I)$	$E(N)$	$P(I=0)$	ERR	$P(I>s)$
1	6.37214	0.10702	0.05289	0.3333	0.6667
1.2	6.76228	0.05091	0.03025	0.3333	0.72222
1.4	7.02939	0.02659	0.01847	0.3333	0.7619
1.6	7.22392	0.0149	0.01185	0.3333	0.79167
1.8	7.37208	0.00882	0.00789	0.3333	0.81481
2	7.48883	0.00545	0.00543	0.3333	0.83333
2.2	7.58329	0.00349	0.00383	0.3333	0.84848

Table 3.6: *Variation of waiting time with replenishment rate δ*

$\lambda = 1 \quad s = 5 \quad S = 11$

δ	$E(W_L)$	$E(W_L^2)$	$V(W_L)$
1	0.03984	0.09237	0.090783
1.2	0.01717	0.03118	0.030885
1.4	0.00825	0.01243	0.012362
1.6	0.0043	0.00557	0.005552
1.8	0.00238	0.00271	0.002704
2	0.00139	0.00141	0.001408
2.2	0.00084	0.00078	0.000779

Figure 3.1: *Reorder level verses Expected Inventory level*

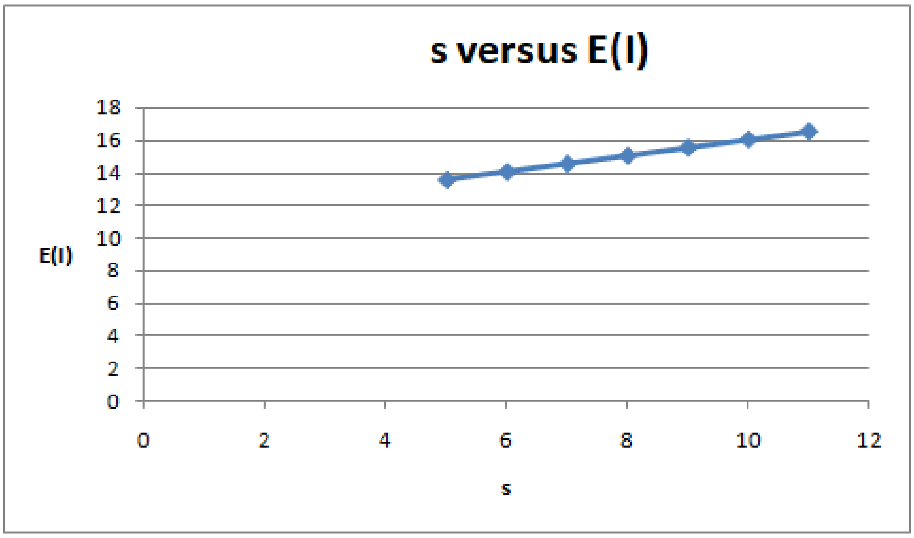


Figure 3.2: *Reorder level verses Expected Number of Customers in the System*

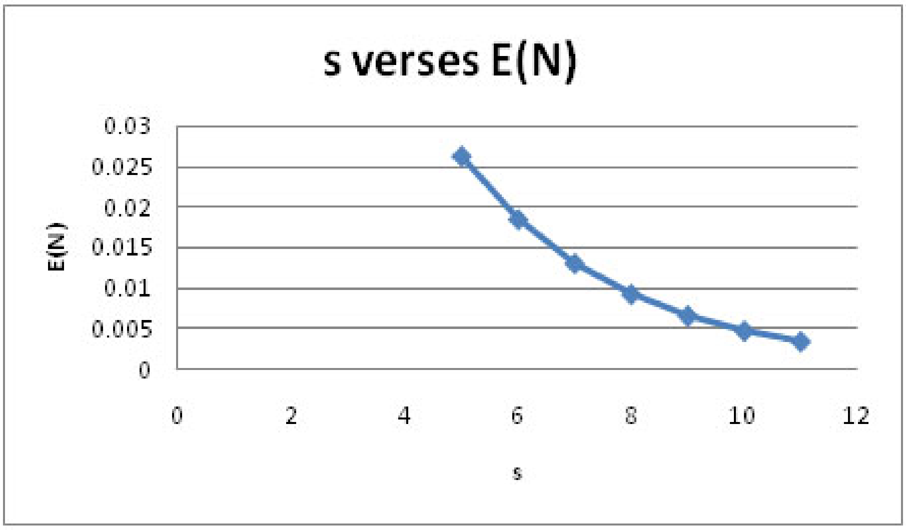


Figure 3.3: *Reorder level verses Expected Waiting Time*

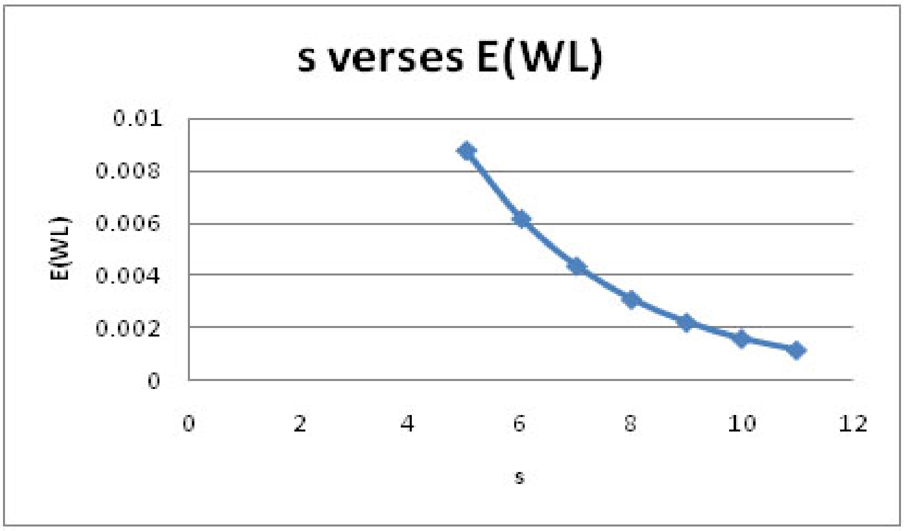


Figure 3.4: *Maximum Inventory level verses Expected Inventory Level*

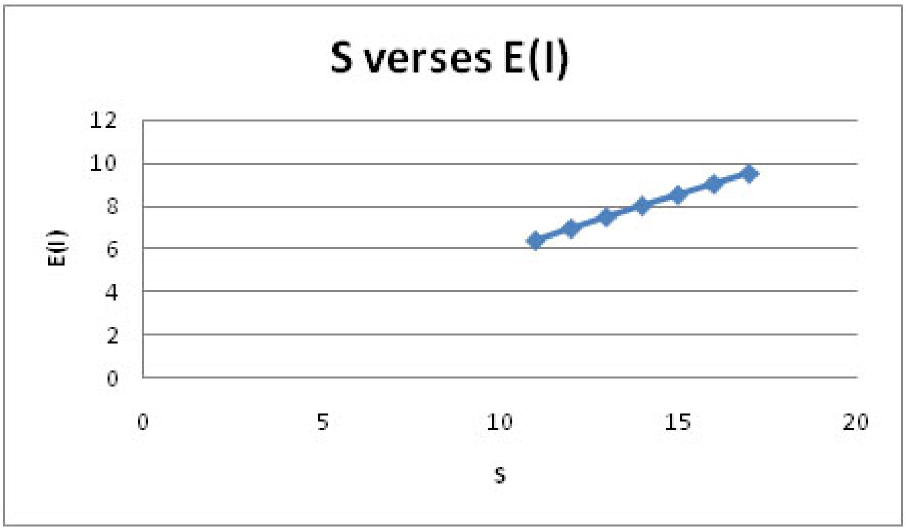


Figure 3.5: *Maximum Inventory level verses Expected Number of customers in the System*

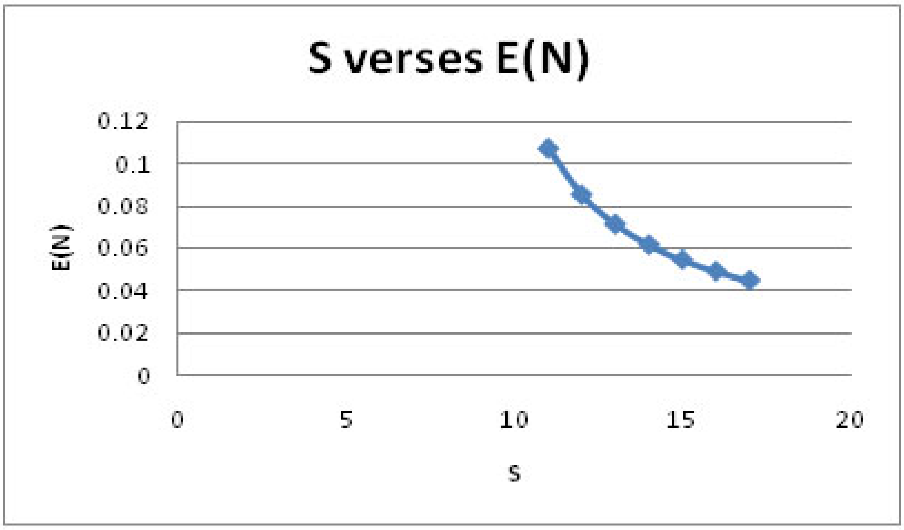


Figure 3.6: *Maximum Inventory level verses Expected waiting time*

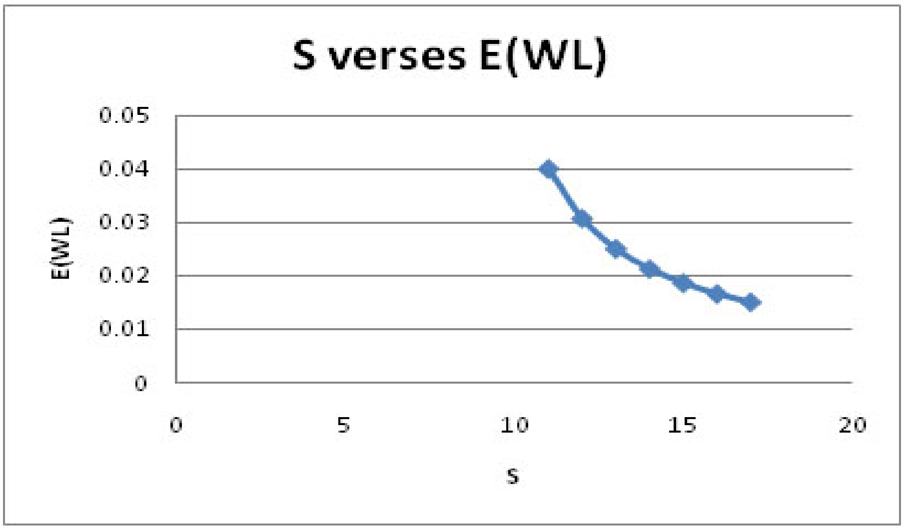


Figure 3.7: *Replenishment Rate verses Expected Inventory Level*

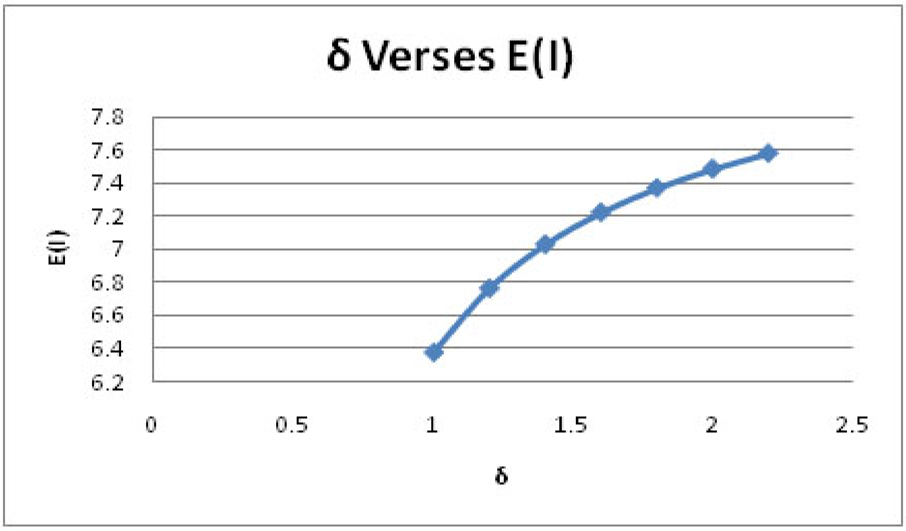


Figure 3.8: *Replenishment Rate verses Expected Number of Customer in the System*

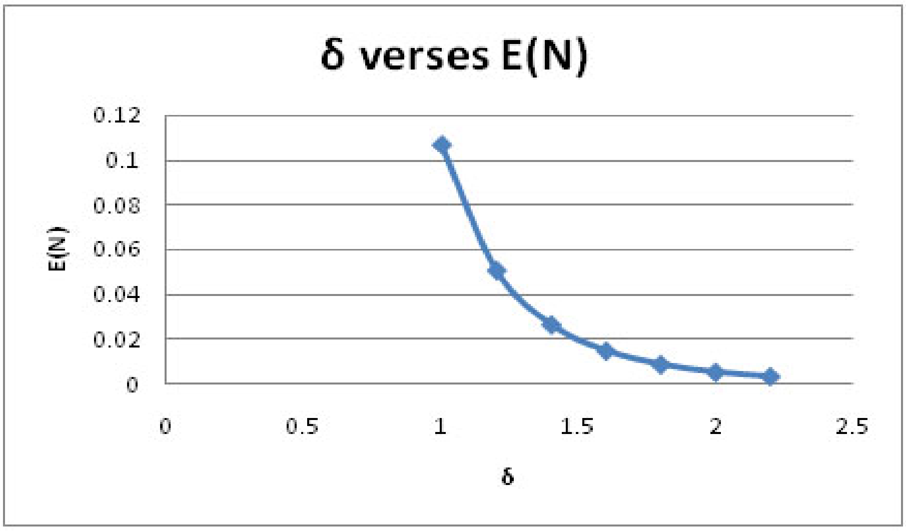
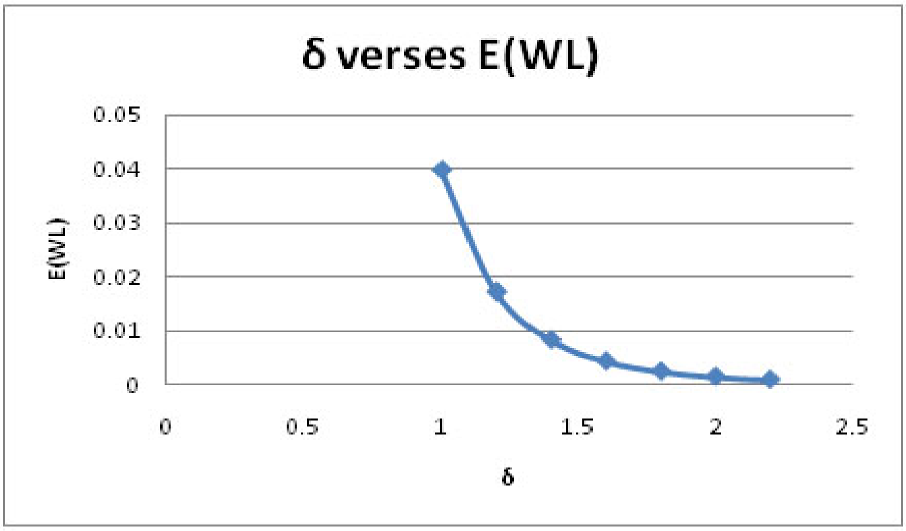


Figure 3.9: *Replenishment rate verses Waiting time*



Conclusion

We studied a single server queueing model with negligible service time and backlogs. We wish to extend this model to one with positive service time which may have many practical applications.

CHAPTER 4

An Explicit Solution for an Inventory Model with Positive Lead Time and Server Interruptions

4.1 Introduction

The pioneers in the study of queueing inventory models are Melikov and Molchanov [4] and Sigman and Simchi- Levi [5]. In Sigman and Simchi- Levi customers are allowed to join even when there is no inventory in the system. They also discuss the case of non exponential lead time distribution. Later Berman and et al. [6] considered an inventory system where a processing time is required for serving the inventory. Here they considered deterministic service time and the model was discussed as a dynamic programming model. Berman and Kim [7] and Berman and Sapna [8] later discussed inventory queueing systems with exponential service time distribution and with arbitrary distribution.

There are several papers on inventory queueing models by Krishnamoorthy and his co-authors [9, 10, 11, 12, 13, 14, 15, 16, 17]. They mainly used Matrix Analytic Methods to study these models. In most of the models service time for providing the inventoried item is assumed. Schwarz et al. [58] considered a queueing inventory model with Poisson arrivals and exponentially distributed service and lead times. They could obtain a product form solution for the system steady state. But they assumed that no customers join the system when the inventory level is zero.

4.2 Mathematical Model

The system under consideration is described as below. There is a single server counter where inventory is served to which customers arrive for service. The number of arrivals by time t follows a Poisson process with parameter λ. The service times are independently and identically distributed exponential random variables with parameter μ. Inventory is replenished according to (s, S) policy, in the sense that whenever inventory level drops to s an order is placed, order quantity being fixed as $Q = S - s$. The replenishment times follow exponential distribution with parameter η. While a customer is being served by the server, the service may be interrupted, the interruption rate being exponential with rate δ_1. Following a service interruption the service restarts at an exponential rate δ_2.

We make the following assumptions for the model under consideration.

i) There is no loss of inventory due to a service interruption.

ii) The customer being served when interruption occurs waits there until his service is completed.

iii) No arrival is entertained when the inventory level is zero.

iv) An order placed if any is cancelled while the server is on interruption.

v) We also assume that there are no arrivals while the server is on interruption.

We denote by $N(t)$ the number of the customers in the system including the one being served (if any), $L(t)$ the inventory level and $S(t)$ the server status at time t.

$$\text{Let } S(t) = \begin{cases} 0 & \text{if the server is idle} \\ 1 & \text{if the server is busy} \\ 2 & \text{if the server is on interruption} \end{cases}$$

Then $\Omega = X(t) = ((N(t), S(t), L(t))$ will be a Markov chain. The state space of this Markov chain can be described as
$E = \{(0,0,k) : 0 \leq k \leq S\} \cup \{(i,0,0) : i \geq 1\} \cup \{(i,j,k) : i \geq 1, j = 1,2; 1 \leq k \leq S\}$. The above state space can be partitioned into levels $L(i)$ where $L(0) = ((0,0,0),(0,0,1),\ldots,(0,0,S))$ and
$L(i) = ((i,0,0),(i,1,1),(i,1,2),\ldots,(i,1,S),(i,2,1),(i,2,2),\ldots,(i,2,S))$; $i \geq 1$. The Markov chain Ω described above is a level independent quasi birth death process whose infinitesimal generator matrix is given

by

$$T = \begin{bmatrix} B_0 & B_1 & 0 & 0 & . & . & . \\ B_2 & A_1 & A_0 & 0 & 0 & . & . \\ 0 & A_2 & A_1 & A_0 & 0 & . & 0 \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & . \\ & & & . & . & . & \\ & & & & . & . & . \end{bmatrix}$$

Here B_0, B_1, B_2 are matrices of orders $(S+1)\times(S+1)$, $(S+1)\times(2S+1)$ and $(2S+1)\times(S+1)$ respectively. All other matrices are square matrices of order $2S+1$. The different transitions in the Markov chain $\Omega = X(t) = ((N(t), S(t), L(t))$ are given below.

i) Transitions due to arrival of customers
$(i, j, k) \xrightarrow{\lambda} (i+1, j, k);\ i \geq 0,\ 0 < k \leq S,\ j = 0, 1$

ii) Transitions due to service completion of customers
$(i, j, k) \xrightarrow{\mu} (i-1, j, k-1);\ i > 0,\ 0 < k \leq S,\ j = 1$

iii) Transitions due to replenishment of inventory
$(i, j, k) \xrightarrow{\eta} (i, j, k+Q);\ i \geq 0,\ 0 \leq k \leq S,\ j = 0, 1$

iv) Transitions due to server interruption
$(i, 1, k) \xrightarrow{\delta_1} (i, 2, k);\ i \geq 1,\ 0 < k \leq S$

v) Transitions due to restart of service after a service interruption
$(i, 2, k) \xrightarrow{\delta_2} (i, 1, k);\ i \geq 1,\ 0 < k \leq S$

The matrix B_0 contains the transition rates within level $L(0)$, B_1 records the transition rates from $L(0)$ level to $L(1)$ and B_2 that from $L(1)$ to $L(0)$. Similarly the matrices $A_0.A_1, A_2$ contains the transitions from levels $L(i)$ to $L(i+1)$, $L(i)$ to itself and $L(i+1)$ to $L(i)$ for $i \geq 1$.

4.3 Analysis of the Model

Stability condition

Define $A = A_0 + A_1 + A_2$ and

$\pi = (\pi(0,0), \pi(1,1), \pi(1,2), \ldots, \pi(1,S), \pi(2,1), \pi(2,2), \ldots, \pi(2,S))$ be the steady state vector of A. We know the QBD process with generator matrix T is stable if and only if $\pi A_0 e < \pi A_2 e$ [2]. That is if and only if

$\lambda\left[\pi(1,1) + \pi(1,2) + \ldots + \pi(1,S)\right] < \mu\left[\pi(1,1) + \pi(1,2) + \ldots + \pi(1,S)\right]$, that is if and only if $\lambda < \mu$.

Thus we have the following theorem for the stability of the system under study.

Theorem 4.3.1. *The Markov chain is stable if and only if $\lambda < \mu$.*

4.4 Computation of steady state vector

We first consider a system identical to the above system except for service time is negligible. For this system $\tilde{\Omega} = \tilde{X}(t) = (S(t), L(t))$ will be a Markov chain where $S(t)$ and $L(t)$ are as defined for the original system. The state space of this Markov chain can be described as $\tilde{E} = \{(0,0), (1,1), (1,2)\ldots, (1,S), (2,1), (2,2), \ldots, (2,S)\}$. The infinitesimal generator matrix of the process is given by $\tilde{T} = \begin{bmatrix} \tilde{B}_0 & \tilde{B}_1 \\ \tilde{B}_2 & \tilde{B}_3 \end{bmatrix}$,

where $\tilde{B}_1 = \begin{bmatrix} 0 \\ \delta_1 I_s \end{bmatrix}_{(S+1)\times S}$, $\tilde{B}_2 = \begin{bmatrix} 0 & \delta_2 I_s \end{bmatrix}_{S\times(S+1)}$, $\tilde{B}_3 = -\delta_2 I_s$,

$\tilde{B}_0 = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$. Here

$$C_1 = \begin{bmatrix} -\eta & 0 \\ 0 & -(\lambda+\eta+\delta_1) I_{s-1} \end{bmatrix}_{(s+1)\times(s+1)} + \begin{bmatrix} 0 & 0 \\ \lambda I_{s-1} & 0 \end{bmatrix}_{(s+1)\times(s+1)}$$

$C_4 = -(\lambda + \delta_1) I_Q + \begin{bmatrix} 0 & 0 \\ \lambda I_{Q-1} & 0 \end{bmatrix}_{Q \times Q}, \ C_3 = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}_{Q \times (s+1)},$
$C_2 = \begin{bmatrix} 0 & \eta I_{s+1} \end{bmatrix}_{(s+1) \times Q}.$
Let $x = (x(0,0), x(1,1), \ldots, x(1,S), x(2,1), \ldots, x(2,S))$ be the steady state probability vector of the process $\tilde{\Omega}$. Then $x\tilde{T} = 0$ and $xe = 1$ gives

$$\begin{aligned} x(1,i) &= \frac{\eta}{\lambda}\left(\frac{\eta+\lambda}{\lambda}\right)^{i-1} x(0,0)\,;\ 1 \le i \le s+1 \\ x(1,s+1) &= x(1,s+2) = \ldots = x(1,Q) \\ x(1,Q+i) &= x(1,Q) - x(1,i)\,;\ 1 \le i \le s \\ x(2,i) &= \frac{\delta_1}{\delta_2} x(1,i)\,;\ 1 \le i \le S \end{aligned}$$

where $x(0,0) = \left[1 + Q\frac{\eta}{\lambda}\left(\frac{\eta+\lambda}{\lambda}\right)^{s}\left(\frac{\delta_1+\delta_2}{\delta_2}\right)\right]^{-1}$.
Let $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ be the steady state probability vector of the process Ω, where $\pi_0 = (\pi(0,0,0), \pi(0,0,1), \ldots, \pi(0,0,S)$ and
$\pi_i = (\pi(i,0,0), \pi(i,1,1), \pi(i,1,2), \ldots, \pi(i,1,S), \pi(i,2,1), \pi(i,2,2), \ldots,$
$\pi(i,2,S))$; $i \ge 1$. Then π satisfies $\pi T = 0$ and $\pi e = 1$. We have the equations

$$\begin{aligned} \pi_0 B_0 + \pi_1 B_2 &= 0 \\ \pi_0 B_1 + \pi_1 A_1 + \pi_2 A_2 &= 0 \\ \pi_i A_0 + \pi_{i+1} A_1 + \pi_{i+2} A_2 &= 0;\ i \ge 1 \end{aligned}$$

All the above equations are satisfied by taking

$$\pi_0 = \zeta(x(0,0), x(1,1), x(1,2), \ldots, x(1,S))$$

$$\pi_i = \zeta \left(\frac{\lambda}{\mu}\right)^i \Big(x(0,0), x(1,1), x(1,2), \ldots,$$
$$x(1,S), \frac{\delta_1}{\delta_2}x(1,1), \frac{\delta_1}{\delta_2}x(1,2), \ldots, \frac{\delta_1}{\delta_2}x(1,S)\Big);\ i \geq 1$$

The value of ζ is obtained from $\pi e = 1$ as $\zeta = \dfrac{(\mu-\lambda)\delta_2}{\delta_2\mu + \delta_1\lambda[1 - x(0,0)]}$

4.5 System Performance Measures

4.5.1 Expected waiting time of a customer in the queue

First we compute the expected waiting time of a customer who joins the queue as the r^{th} person. For that consider a Markov process $\psi = (\hat{N}(t), S(t), L(t))$, where $\hat{N}(t)$ represent the rank of the customer in the queue, $S(t)$ the server status and $L(t)$ the inventory level. The state space of the above Markov chain is $\hat{E} = \{(i,0,0), 1 \leq i \leq r-1\} \cup \{(i,j,k), 1 \leq i \leq r-1; j = 1,2;\ 1 \leq k \leq S\} \cup \{(r,1,k),\ 1 \leq k \leq S\} \cup \Delta$, where Δ correspond to the state, the r^{th} customer is taken for service. The generator matrix of the Markov chain is given by $\hat{Q} = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$, where T^0 is an $(r(S+1) - (S+1)) \times 1$ matrix where $T^0(i,1) = \mu$; $2 \leq i \leq S+1$ and

$$T = \begin{bmatrix} B & 0 & 0 & - & - & - & 0 \\ A_2 & B & 0 & - & - & - & 0 \\ 0 & A_2 & B & 0 & - & - & 0 \\ 0 & 0 & A_2 & B & & & \\ & & & - & - & - & - \\ & & & & - & - & - \\ & & & & & - & - \\ 0 & 0 & - & - & - & \hat{A}_2 & \hat{B} \end{bmatrix}$$

The different transitions in T are as follows.

i) $(i,0,k) \xrightarrow{\eta} (i,1,k+Q); 1 \le i \le r;\ 0 \le k \le s$

ii) $(i,j,k) \xrightarrow{\eta} (i,j,k+Q); 1 \le i \le r;\ j=1;\ 0 \le k \le s$

iii) $(i,1,k) \xrightarrow{\delta_1} (i,2,k); 1 \le i \le r;\ 1 \le k \le S$

iv) $(i,2,k) \xrightarrow{\delta_2} (i,1,k); 1 \le i \le r;\ 1 \le k \le S$

v) $\hat{B}(i,j) = B(i+1,j);\ \hat{A}(i,j) = A_2(i+1,j);\ 1 \le i \le S$

Now the waiting time of the customer who joins as the r^{th} customer is given by $W^r = \hat{I}_S(-T^{-1}e)$, where $\hat{I}_S = \begin{bmatrix} 0 & I_S \end{bmatrix}_{S\times(r(2S+1)-(S+1))}$.

So the expected waiting time of a general customer is given by $E(W_L) = \sum_{r=1}^{\infty} \hat{\pi}_r W^r$, where $\hat{\pi}_r(i) = \pi_r(i+1)$. Similarly the variance of waiting time of a general customer is also calculated numerically.

4.5.2 Expected number of interruptions encountered by a customer

For computing expected number of interruptions encountered by a customer we consider a Markov process

$$\{Y(t),\, t \ge 0\} = \{(N_1(t), S_1(t)),\, t \ge 0\},$$

where $N_1(t)$ denotes the number of interruptions that has occurred up to time t; $S_1(t) = 0$ or 1 according as the service is under interruption or not at time t. The state space of the process is $\{0,1,2,\ldots\} \times \{0,1\} \cup \{\Delta\}$, where Δ is an absorbing state which corresponds to service completion. The infinitesimal generator of the process is

$$\hat{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & - & - & - \\ \hat{C}_{00} & \hat{B}_{00} & \hat{B}_{01} & 0 & 0 & - & - & - \\ \hat{B}_2 & 0 & \hat{B}_1 & \hat{B}_0 & 0 & - & - & - \\ \hat{B}_2 & 0 & 0 & \hat{B}_1 & \hat{B}_0 & - & - & - \\ \hat{B}_2 & 0 & 0 & 0 & \hat{B}_1 & \hat{B}_0 & - & - \\ & & - & - & - & - & - & - \\ & & & - & - & - & - & - \\ & & & & & - & - & - \end{bmatrix},$$

where $\hat{C}_{00} = \begin{bmatrix} \mu \end{bmatrix}$, $\hat{B}_{00} = \begin{bmatrix} -(\mu+\delta_1) \end{bmatrix}$, $\hat{B}_{01} = \begin{bmatrix} \delta_1 & 0 \end{bmatrix}$, $\hat{B}_2 = \begin{bmatrix} 0 \\ \mu \end{bmatrix}$,
$\hat{B}_1 = \begin{bmatrix} -\delta_2 & \delta_2 \\ 0 & -(\mu+\delta 1) \end{bmatrix}$ and $\hat{B}_0 = \begin{bmatrix} 0 & 0 \\ \delta_1 & 0 \end{bmatrix}$. If y_k is the probability that absorption occurs with exactly k interruptions, then
$y_0 = -\hat{B}_{00}^{-1}\hat{C}_{00} = \dfrac{\mu}{\mu+\delta_1}$
$y_k = (-\hat{B}_{00}^{-1}\hat{B}_{01})(-\hat{B}_1^{-1}\hat{B}_0)^{k-1}(-\hat{B}_1^{-1}\hat{B}_2) = \dfrac{\mu}{\mu+\delta_1}\left(\dfrac{\delta_1}{\mu+\delta_1}\right)^k$,
$k = 1, 2, 3, \ldots$.
The expected number of interruptions before absorption is given by
$E_1 = \sum\limits_{k=0}^{\infty} k y_k = (-\hat{B}_{00}^{-1}\hat{B}_{01})\left[I_2 - (-\hat{B}_1^{-1}\hat{B}_0)\right]^{-1} e = \dfrac{\delta_1}{\mu}$.

4.5.3 Other performance Measures

1. The expected number of customers in the system,

$$\begin{aligned} L_s &= \sum_{i=1}^{\infty}\sum_{j=1}^{S} i\left\{\pi(i,1,j) + \pi(i,2,j)\right\} + \sum_{i=1}^{\infty} i\pi(i,0,0) \\ &= \zeta\frac{\lambda}{\mu}\left(\frac{\mu}{\mu-\lambda}\right)^2\left[1 + \frac{\delta_1}{\delta_2}(1 - x(0,0))\right]. \end{aligned}$$

2. The expected inventory level in the system,

$$
\begin{aligned}
INV_{mean} &= \sum_{i=1}^{\infty}\sum_{j=1}^{S} j\{\pi(i,1,j)+\pi(i,2,j)\} + \sum_{j=1}^{S} j\pi(0,0,j) \\
&= \zeta\frac{\lambda}{\mu}Q\left\{1+\frac{\delta_1}{\delta_2}\frac{\lambda}{\mu}\right\}\left(\frac{(S+s+1)}{2}\frac{\eta}{\lambda}\left(\frac{\eta+\lambda}{\lambda}\right)^{s}+\right. \\
&\qquad \left.\left[1-\left(\frac{\eta+\lambda}{\lambda}\right)^{s}\right]\right)x(0,0).
\end{aligned}
$$

3. The expected rate of ordering, $E_{or} = \sum\limits_{i=1}^{\infty} \mu\pi(i,1,s+1)$.

4. The expected replenishment rate,

$$
REP_{mean} = \sum_{i=0}^{\infty}\sum_{j=0}^{s} \eta\left\{\pi(i,0,j)+\pi(i,1,j)\right\}.
$$

5. The expected interruption rate, $INT_{mean} = \sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{S} \delta_1\pi(i,1,j) = \delta_1 P(busy)$.

6. The loss rate of customers, $LOSS_{mean} = \sum\limits_{i=0}^{\infty} \lambda\pi(i,0,0) + \sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{S} \lambda\pi(i,2,j) = \lambda\xi\frac{\mu}{\mu-\lambda}x(0,0) + \lambda P(int)$.

7. The probability that the server is busy, $P_{busy} = \sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{S} \pi(i,1,j) = \frac{\delta_2}{\delta_2\mu+\delta_1(1-x(0,0))}\frac{\lambda}{\mu-\lambda}Q\frac{\eta}{\lambda}\left(\frac{\eta+\lambda}{\lambda}\right)^{s}$.

8. The probability that the server is on interruption, $P_{int} = \sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{S} \pi(i,2,j) = \frac{\delta_1}{\delta_2}P(busy)$.

4.5.4 Cost analysis

We considered the following Cost function

$$Cost = CI \times INV_{mean} + CN \times L_s + CR \times E_{INTR} + (K + (S-s)K_1) \times E_{OR} + CL \times Loss_{mean},$$

where

CI : Cost of holding Inventory

CN : Cost of holding customers

CR : Cost incurred due to interruption of service

K : Fixed cost of ordering

K_1 : Cost of a single inventory

CL : Cost incurred due to loss of customers when inventory level drops to zero.

The effect of various parameters on the cost were studied.

4.6 Numerical Illustration

Eventhough we have explicit expressions for most of the system performance measures we provide numerical illustration of the effect of different parameters on the system performance measures in this section.

4.6.1 Effect of arrival rate λ

In table 4.1 we see that as arrival rate increases, there is an increase in both $P(busy)$, $P(int)$ and L_s. The increase in server busy probability is as expected since when arrival rate increases the mean number of customers in the system obviously increases and so the probability that server is busy increases. $P(int)$ is also seen to increase which may be due to the fact that an interruption to service occurs only when the server is busy. Also the decrease in INV_{mean} is due to the fact that the more customers get service when $P(busy)$ increases. Also notice the increase in mean waiting of a customer in the system due to increase in mean number of customers in the system.

4.6.2 Effect of service rate μ

In table 4.2 we see that as service rate increases, $P(busy)$, $P(int)$, L_s and $WAIT_{mean}$ all decrease. As the service rate increases, customers leave the system after getting service at a faster rate. Hence the mean waiting time in the system clearly decreases. Also the probability that the server is idle increases with increase in service rate and so $P(busy)$, $P(int)$ and L_s all decrease. It is seen from the tables that μ has no effect on INV_{mean}.

4.6.3 Effect of interruption rate δ_1

In table 4.3 we see that as interruption rate increases, $P(busy)$ increases whereas $P(int)$, $WAIT_{mean}$ and L_s decrease. The reason for decrease in the mean number of customers in the system is due to our assumption that when the server is on interruption no arrivals are entertained. The decrease in mean waiting time of a customer in the system is due to the increase in $P(busy)$. Also as mean number of customers in the system decreases, probability that server is idle increases and so $P(int)$ decreases. The interruption rate seems to have no effect on average inventory level in the system.

4.6.4 Effect of reorder level s

In table 4.4 we see that s has no considerable effect on the system performance measures $P(busy)$, $P(int)$ and L_S. The expected inventory level in the system increases with increase in re order level is as expected since orders are placed early with increase in s.

Table 4.1: *Effect of arrival rate on various performance measures*

$\mu = 10 \quad \eta = 2 \quad \delta_1 = 6 \quad \delta_2 = 7 \quad s = 5 \quad S = 12$

λ	$P(busy)$	$P(int)$	INV_{mean}	REP_{mean}	L_s	E_{OR}	$WAIT_{mean}$	$WAIT_{var}$
2	0.18749	0.06249	8.7142	0.2678	0.3125	0.2678	0.1597	0.0675
2.2	0.20496	0.06832	8.6856	0.2928	0.3504	0.2928	0.1643	0.0715
2.4	0.2222	0.074	8.6571	0.3174	0.3899	0.3174	0.1693	0.0761
2.6	0.2392	0.0797	8.6284	0.3417	0.4311	0.3417	0.1747	0.0812
2.8	0.2561	0.0853	8.5999	0.3658	0.4742	0.3658	0.1806	0.0868
3	0.2726	0.0909	8.5713	0.3895	0.5194	0.3895	0.187	0.0931

Table 4.2: *Effect of service rate on various performance measures*

$\lambda = 3 \quad \eta = 2 \quad \delta_1 = 6 \quad \delta_2 = 7 \quad s = 5 \quad S = 12$

μ	$P(busy)$	$P(int)$	INV_{mean}	REP_{mean}	L_s	E_{OR}	$WAIT_{mean}$	$WAIT_{var}$
9	0.2999	0.0999	8.5713	0.3856	0.5999	0.3856	0.2191	0.1256
9.2	0.2941	0.098	8.5713	0.3865	0.5819	0.3865	0.2118	0.1178
9.4	0.2884	0.0961	8.5713	0.3873	0.5649	0.3873	0.205	0.1107
9.6	0.2829	0.0943	8.5713	0.388	0.5489	0.3881	0.1986	0.1043
9.8	0.2777	0.0926	8.5713	0.3888	0.5337	0.3888	0.1927	0.0984
10	0.2727	0.0909	8.5713	0.3896	0.5195	0.3895	0.187	0.0931

Table 4.3: *Effect of interruption rate on various performance measures*

$\lambda = 3 \quad \mu = 9 \quad \eta = 2 \quad \delta_2 = 7 \quad s = 5 \quad S = 12$

δ_1	$P(busy)$	$P(int)$	INV_{mean}	REP_{mean}	L_s	E_{OR}	$WAIT_{mean}$	$WAIT_{var}$
6	0.2726	0.1818	8.5714	0.3506	0.6818	0.3506	0.2918	0.2219
6.2	0.2743	0.1769	8.5714	0.3526	0.6769	0.3526	0.2862	0.2128
6.4	0.2758	0.1723	8.5714	0.3546	0.6723	0.3546	0.281	0.2044
6.6	0.2772	0.168	8.5714	0.3564	0.668	0.3564	0.2761	0.1968
6.8	0.2786	0.1639	8.5714	0.3582	0.6639	0.3582	0.2716	0.1898
7	0.2799	0.1599	8.5714	0.3599	0.6599	0.3599	0.2673	0.1833

Table 4.4: *Effect of re orer level on various performance measures*

$\lambda = 3 \quad \mu = 9 \quad \eta = 2 \quad \delta_1 = 6 \quad \delta_2 = 7 \quad S = 21$

s	$P(busy)$	$P(int)$	INV_{mean}	REP_{mean}	L_s	E_{OR}	$WAIT_{mean}$	$WAIT_{var}$
5	0.2999	0.0999	13.071	0.1687	0.5999	0.1687	0.2197	0.1265
6	0.2999	0.0999	13.571	0.1799	0.5999	0.1799	0.2197	0.1265
7	0.2999	0.0999	14.071	0.1928	0.5999	0.1928	0.2196	0.1264
8	0.2999	0.0999	14.571	0.2076	0.5999	0.2076	0.2196	0.1263
9	0.2999	0.0999	15.071	0.2249	0.5999	0.2250	0.2195	0.1262
10	0.2999	0.0999	15.571	0.2454	0.5999	0.2454	0.2195	0.1261

Table 4.5: *Effect of arrival rate on Cost*

$CI = 40$	$CN = 30$	$CR = 75$	$K = 500$	$K_1 = 35$	$CZ = 750$
$\mu = 10$	$\eta = 2$	$\delta_1 = 6$	$\delta_2 = 7$	$s = 5$	$S = 12$

λ	2	2.2	2.4	2.6	2.8	3
Cost	566	586	605	624	643	662

Table 4.6: *Effect of service rate on Cost*

$CI = 40$	$CN = 30$	$CR = 75$	$K = 500$	$K_1 = 35$	$CZ = 750$
$\lambda = 3$	$\eta = 2$	$\delta_1 = 6$	$\delta_2 = 7$	$s = 5$	$S = 12$

μ	9	9.2	9.4	9.6	9.8	10
Cost	663.47	663.26	663.06	662.89	662.73	662.59

Table 4.7: *Effect of interruption rate on Cost*

$CI = 40$	$CN = 30$	$CR = 75$	$K = 500$	$K_1 = 35$	$CZ = 750$
$\lambda = 3$	$\eta = 2$	$\mu = 9$	$\delta_2 = 7$	$s = 5$	$S = 12$

δ_1	6	6.2	6.4	6.6	6.8	7
Cost	663.47	663.87	664.26	664.62	664.96	665.29

Table 4.8: *Effect of repair rate on Cost*

$CI = 40$	$CN = 30$	$CR = 75$	$K = 500$	$K_1 = 35$	$CZ = 750$
$\lambda = 3$	$\eta = 2$	$\mu = 9$	$\delta_1 = 6$	$s = 5$	$S = 12$

δ_2	6	6.2	6.4	6.6	6.8	7
Cost	663.47	663.91	664.34	664.74	665.12	665.49

Table 4.9: *Effect of reorder level on Cost*

$CI = 40$	$CN = 30$	$CR = 75$	$K = 500$	$K_1 = 35$	$CZ = 750$
$\lambda = 3$	$\eta = 2$	$\mu = 9$	$\delta_1 = 6$	$\delta_2 = 7$	$S = 21$

s	5	6	7	8	9	10
Cost	734	760	786	814	842	873

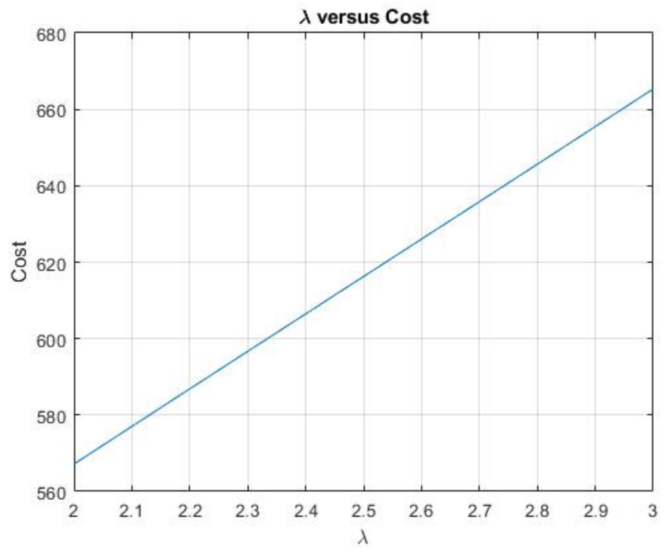


Figure 4.1: *Arrival rate versus Cost*

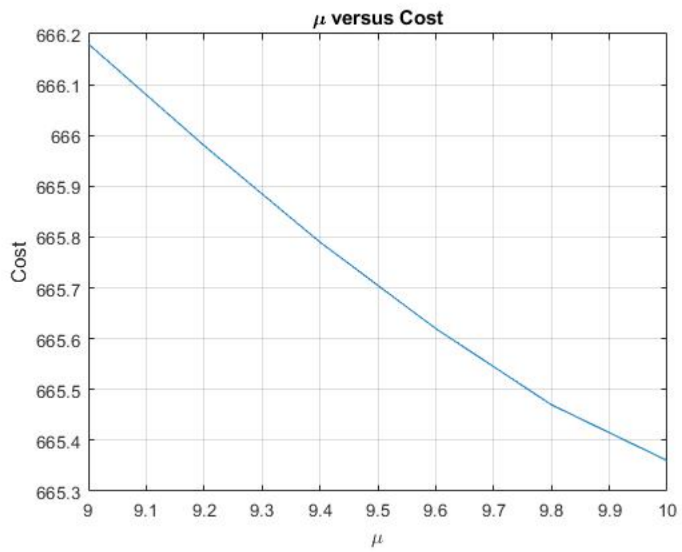


Figure 4.2: *Service rate versus Cost*

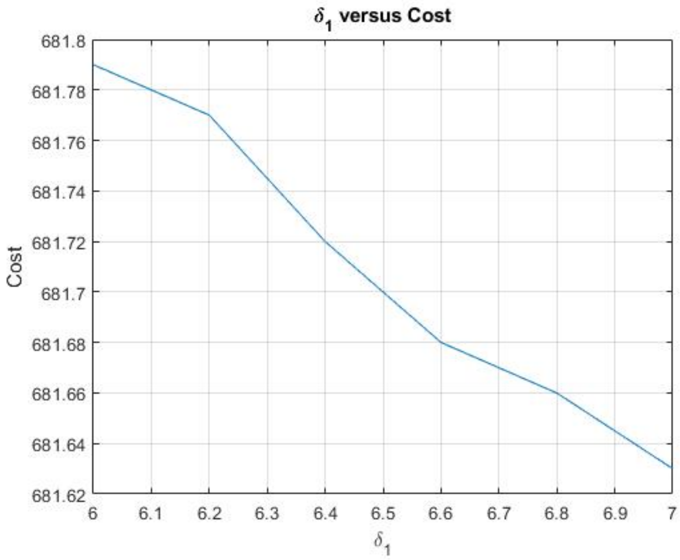


Figure 4.3: *Interruption rate versus Cost*

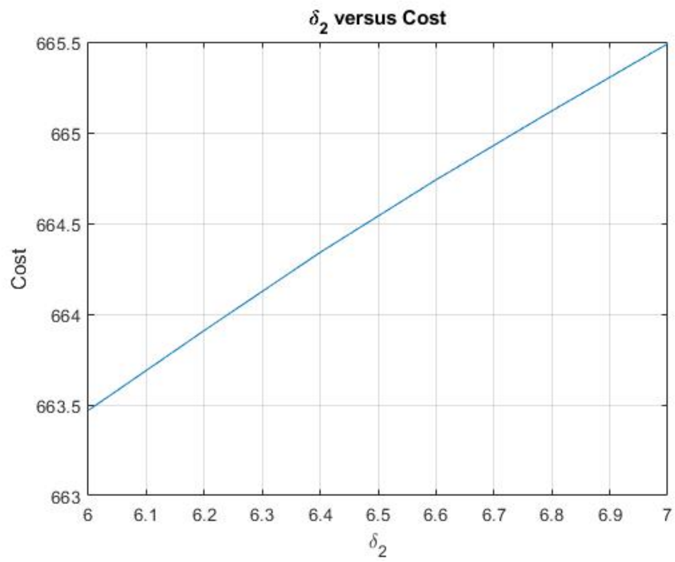


Figure 4.4: *Repair rate versus Cost*

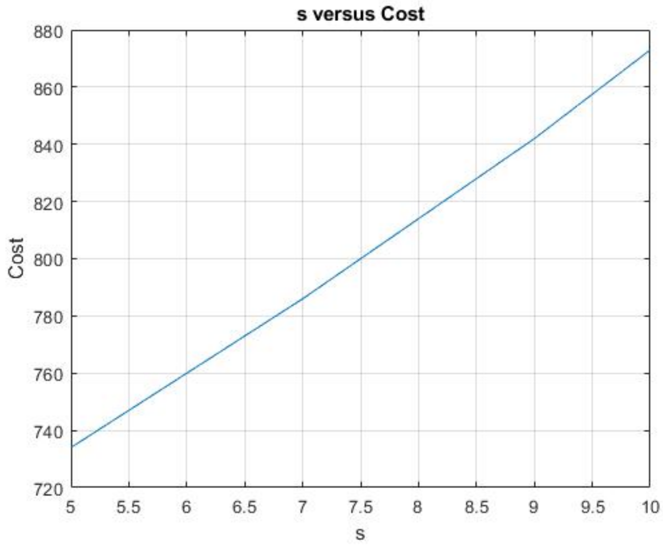


Figure 4.5: *Reorder level versus Cost*

Conclusion

We studied a single server queueing model with positive service time, positive lead time and service interruptions. We could arrive at an explicit expression for the steady state probability vector. We wish to extend this model by considering retrials as well.

CHAPTER 5

An Explicit Solution for an Inventory Model with Server Interruption and Retrials

Introduction

Melikov and Molchano [4] and Sigman and Simchi-Levi [5] were pioneers in the study of queueing inventory models. Customers could join the system in the Sigman and Simchi-Levi systems even when there is no product available. They also talk about the non-exponential lead time distribution scenario. Later, Berman and et al. [6] examined an inventory system where a processing time is needed for serving the inventory. Here, they took into account deterministic service time, and the paradigm was treated as a dynamic programming model. Later, inventory queueing systems with arbitrary distribution and exponential service time distribution were addressed by Berman and Kim [7] and Berman and Sapna [8].

Several studies on inventory queueing models have been published by Krishnamoorthy and his co-authors [9, 10, 11, 13, 14, 16, 15, 12]. To analyse these models, they largely used matrix analytic methods. The majority of models presume that the stocked item will be provided with service which require some time. An exponentially distributed service and lead time model with Poisson arrivals was taken into consideration by Schwarz et al. They got a product-form solution for the steady state of the system. But they assumed that no new users would sign up for the system once the inventory level was zero. For a detailed description of papers in inventory queuing models we refer to the papers [20, 21]. Melikov et a1. [59] studied a Queuing-Inventory System with Two Supply Sources and Destructive Customers. Melikov et a1. [60] also carried out a numerical analysis and long run total cost optimization of a perishable queuing inventory systems with delayed feedback. They also did a long run total cost optimization for the problem.

Retrial queuing models are widely used in communication and other fields. Hence they are gaining more and more attention. We refer to the books by Falin and Templeton [42] and Atralejo and Gomez Corral [43] for an extensive analysis of both theory and applications on retrial queues.

The first study of an inventory queing model with positive lead time and retrial of customers was made by Artalejo et al. [44]. Analytical solution to the problem discussed there could be found in Ushakumari [40]. Following these, several papers in this direction emerged. A few among them are the papers by Krishnamoorthy and Islam [45, 46], Krishnamoorthy et al. [47, 48] and Krishnamoorthy and Jose [49]. These papers are studies on a production inventory model with retrial of customers, analysis of a production inventory model with random shelf times of the items with retrials of the orbiting customers, study of inventory models with positive service time and retrial of customers from an orbit with an intermediate buffer of finite and comparison of different (s, S) inventory models with an orbit of infinite capacity, having/ not having a finite buffer.

5.1 Mathematical Model

The system under consideration is described as below. We consider a single server queuing model where inventory is served to which customers arrive for service. The number of arrivals of by time t follows a Poisson process with parameter λ. Inventory is replenished according to (s, S) policy, replenishment being instantaneous. Service times follow exponential distribution with parameter μ. Upon arrival, finding the server busy the customers enter into an orbit from where they retry for service at a constant retrial rate. The time between two successive retrials also follow exponential distribution with parameter θ. While the server serves a customer the service can be interrupted, the inter occurrence time of interruption being exponentially distributed with parameter δ_1. Following a service interruption the service restarts after an exponentially distributed time with parameter δ_2. For the model under study the following assumptions are made.

i) No inventory is lost due to server interruption.

ii) The customer being served when interruption occurs waits there until his service is completed.

iii) No arrivals are entertained when the server is on interruption.

iv) An order placed if any is cancelled while the server is on interruption.

We denote by $N(t)$ the number of the customers in the orbit, $I(t)$ the inventory level and $S(t)$ the server status at time t. Let

$$S(t) = \begin{cases} 0 & \text{if the server is idle} \\ 1 & \text{if the server is busy} \\ 2 & \text{if the server is on interruption} \end{cases}$$

Then $\Omega = X(t) = (N(t), S(t), I(t))$ will be a Markov chain. The state space of this Markov chain can be described as $E = \{(i, j, k) : i \geq 0;\ j = 0, 1, 2;\ s + 1 \leq k \leq S\}$. The above state space can be partitioned into levels $L(i)$ where
$L(i) = ((i, j, k);\ j = 0, 1, 2;\ k = s + 1, s + 2, \ldots, S)\,; i \geq 0$ in the lexicographic ordering. The Markov chain Ω described above is a level independent quasi birth death process whose infinitesimal generator matrix is given by $T = \begin{bmatrix} B_0 & A_0 & 0 & 0 & \_ & \_ & \_ \\ A_2 & A_1 & A_0 & 0 & 0 & \_ & \_ \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix}$. Here all the matrices are of order $3Q \times 3Q$ where $Q = S - s$. The different transitions in the Markov chain $\Omega = X(t) = (N(t), S(t), I(t))$ are given below.

i) Transitions due to arrivals
$(i, 0, k) \xrightarrow{\lambda} (i, 1, k),\ (i, 1, k) \xrightarrow{\lambda} (i + 1, 1, k);\ i \geq 0,\ s + 1 \leq k \leq S$

ii) Transitions due to service completion of customers
$(i, 1, k) \xrightarrow{\mu} (i - 1, 0, k - 1);\ i \geq 0,\ s + 2 \leq k \leq S$
$(i, 1, s + 1) \xrightarrow{\mu} (i - 1, 0, S);\ i \geq 0$

iii) Transitions due to retrials $(i, 0, k) \xrightarrow{\theta} (i-1, 1, k);\ i \geq 1$

iv) Transitions due to interruptions $(i, 1, k) \xrightarrow{\delta_1} (i, 2, k);\ i \geq 0$

v) Transitions due to repairs $(1, 2, k) \xrightarrow{\delta_2} (i, 1, k);\ i \geq 0$

The diagonal entries of B_0 and A_1 are such that each row sum of T is zero. The matrix B_0 contains the transition rates within level. Similarly the matrices A_0, A_1, A_2 contains the transitions from levels $L(i)$ to $L(i+1)$, $L(i)$ to $L(i)$ and $L(i)$ to $L(i-1)$ respectively.

5.2 Analysis of the Model

5.2.1 Stability condition

Define $A = A_0 + A_1 + A_2$ and $\pi = (\pi(0, s+1), \ldots, \pi(0, S), \pi(1, s+1), \ldots, \pi(1, S), \pi(2, s+1), \ldots, \pi(2, S))$ be the steady state vector of A. We know the QBD process with generator matrix T is stable if and only if $\pi A_0 e < \pi A_2 e$ (see Neuts). Since $A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda I_Q & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and

$A_2 = \begin{bmatrix} 0 & \theta I_Q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the stability condition reduces to

$\lambda \left[\pi(1, s+1) + \ldots + \pi(1, S)\right] < \theta \left[\pi(0, s+1) + \ldots + \pi(0, S)\right]$,

that is $\lambda \left(\frac{\lambda + \theta}{\mu}\right) < \theta$. Thus we have the following theorem for the stability of the system under study.

Theorem 5.2.1. *The Markov chain is stable if and only if*

$$\frac{\lambda}{\theta}\left(\frac{\lambda + \theta}{\mu}\right) < 1$$

5.2.2 Computation of steady state vector

We compute the steady state vector of the model explicitly. Let $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ be the steady state probability vector of the process Ω, where $\pi_i = (\pi(i,0,s+1), \ldots, \pi(i,0,S), \pi(i,1,s+1), \ldots, \pi(i,1,S),$ $\pi(i,2,s+1), \ldots, \pi(i,2,S))$; $i \geq 0$. Then π satisfies $\pi T = 0$ and $\pi e = 1$. We have the equations

$$\pi_0 B_0 + \pi_1 A_2 = 0$$

$$\pi_i A_0 + \pi_{i+1} A_1 + \pi_{i+2} A_2 = 0;\ i \geq 0$$

We first consider a system identical to the above system expect for no inventory is served. This system $\tilde{\Omega} = \tilde{X}(t) = (N(t), S(t))$ will be a Markov chain where $N(t)$ and $S(t)$ is as defined for the original system. The state space of this Markov chain can be described as $\tilde{E} = \{(i,0), (i,1), (i,2)\};\ i \geq 0$. The infinitesimal generator matrix of the process is given by $\tilde{T} = \begin{bmatrix} \tilde{B}_0 & \tilde{A}_0 & 0 & 0 & \_ & \_ & \_ \\ \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & 0 & 0 & \_ & \_ \\ 0 & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & 0 & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ & \_ \end{bmatrix}$, where

$\tilde{B}_0 = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu+\delta_1) & \delta_1 \\ 0 & \delta_2 & -\delta_2 \end{bmatrix}$, $\tilde{A}_1 = \begin{bmatrix} (\lambda+\theta) & \lambda & 0 \\ \mu & -(\lambda+\mu+\delta_1) & \delta_1 \\ 0 & \delta_2 & -\delta_2 \end{bmatrix}$,

$\tilde{A}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\tilde{A}_2 = \begin{bmatrix} 0 & \theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let $x = (x_0, x_1, \ldots)$, where $x_i = (x(i,0), x(i,1), x(i,2))$ be the steady state probability vector of the process $\tilde{\Omega}$. The steady state equations are given by $x\tilde{T} = 0$.

$$-\lambda x(0,0) + \mu x(0,1) = 0 \tag{5.2.1}$$

$$\lambda x(0,0) - (\lambda + \mu + \delta_1)x(0,1) + \delta_2 x(0,2) + \theta x(1,0) = 0 \tag{5.2.2}$$

$$\delta_1 x(0,1) - \delta_2 x(0,2) = 0 \tag{5.2.3}$$

$$-(\theta + \lambda)x(i,0) + \mu x(i,1) = 0;\ i \geq 1 \tag{5.2.4}$$

$$\lambda x(i-1,1) + \lambda x(i,0) - (\lambda + \mu + \delta_1)x(i,1) + \delta_2 x(i,2) + \theta x(i+1,0) = 0;\ i \geq 1 \tag{5.2.5}$$

$$\delta_1 x(i,1) - \delta_2 x(i,2) = 0;\ i \geq 1 \tag{5.2.6}$$

We know that $x_i = x_{i-1}R;\ i \geq 1$, where the matrix R satisfies $R^2A_2 + RA_1 + A_0 = 0$. Since the first and third rows of A_0 are zeros, so are that of R. From $R^2A_2 + RA_1 + A_0 = 0$, we obtain

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & r_1r_2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -(\lambda+\theta)r_1 + \mu r_2 & \lambda r_1 - (\lambda+\mu+\delta_1)r_2 + \delta_2 r_3 & r_2\delta_1 - r_3\delta_2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

We have the following equations.

$$\mu r_2 = (\lambda + \theta)r_1$$

$$\delta_1 r_2 = \delta_2 r_3$$

$$\lambda r_1 - (\lambda + \mu)r_2 + r_1r_2\theta + \lambda = 0$$

From the above 3 equations we obtain a quadratic in r_2 as $\mu\theta r_2^2 - (\lambda^2 + \mu\theta + \lambda\theta)r_2 + \lambda(\lambda + \theta) = 0$. Clearly the roots of this equation are 1 and $\dfrac{\lambda(\lambda+\theta)}{\theta\mu}$. It may be noted that the stability condition was

$\frac{\lambda(\lambda+\theta)}{\theta\mu} < 1$. Hence $R = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\lambda}{\theta} & \frac{\lambda(\lambda+\theta)}{\theta\mu} & \frac{\lambda(\lambda+\theta)\delta_1}{\theta\mu\delta_2} \\ 0 & 0 & 0 \end{bmatrix}$. Now from $x_0\tilde{B}_0 + x_1\tilde{A}_2 = 0$, we have the equations

$$-\lambda x(0,0) + \mu x(0,1) = 0$$
$$\delta_1 x(0,1) - \delta_2 x(0,2) = 0$$

Hence $x_0 = (x(0,0), x(0,1), x(0,2)) = \left(1, \frac{\lambda}{\mu}, \frac{\lambda\delta_1}{\mu\delta_2}\right) x(0,0)$ and from the normalizing condition $x_0(I-R)^{-1}e = 1$, where I is the identity matrix of order 3 and e is column vector of ones we get

$$x(0,0) = \frac{1 - \frac{\lambda(\lambda+\theta)}{\theta\mu}}{1 + \frac{\delta_1\lambda}{\delta_2\mu}}.$$

Now the equations $\pi T = 0$ are given by

$$\begin{aligned} &-\lambda\pi(0,0,i) + \mu\pi(0,1,i+1) = 0;\ s+1 \le i \le S-1 \\ &-\lambda\pi(0,0,S) + \mu\pi(0,1,s+1) = 0 \end{aligned} \tag{5.2.7}$$

$$\lambda\pi(0,0,i) - (\lambda+\mu+\delta_1)\pi(0,1,i) + \delta_2\pi(0,2,i) + \theta\pi(1,0,i) = 0;\ s+1 \le i \le S \tag{5.2.8}$$

$$\delta_1\pi(0,1,i) - \delta_2\pi(0,2,i) = 0;\ s+1 \le i \le S \tag{5.2.9}$$

$$\begin{aligned} &-(\lambda+\theta)\pi(i,0,j) + \mu\pi(i,1,j+1) = 0;\ i \ge 1,\ s+1 \le i \le S-1 \\ &-(\lambda+\theta)\pi(i,0,S) + \mu\pi(i,1,s+1) = 0;\ i \ge 1 \end{aligned} \tag{5.2.10}$$

$$\lambda\pi(i-1,0,i)+\lambda\pi(i,0,i)-(\lambda+\mu+\delta_1)\pi(i,1,i)+ \delta_2\pi(i,2,i)+\theta\pi(i+1,0,i)=0;\ i\geq 1,\ s+1\leq i\leq S \tag{5.2.11}$$

$$\delta_1\pi(i,1,j)-\delta_2\pi(i,2,j)=0;\ i\geq 1,\ s+1\leq j\leq S \tag{5.2.12}$$

We assume $\pi(i,j,k)=\dfrac{1}{Q}x(i,j);\ \ s+1\leq k\leq S$. Then the first Q equations of $\pi_0B_0+\pi_1A_2=0$ reduces to the first equation of $x_0\tilde{B}_0+x_1\tilde{A}_2=0$; the next Q equations of $\pi_0B_0+\pi_1A_2=0$ reduces to the second equation of $x_0\tilde{B}_0+x_1\tilde{A}_2=0$ and the last Q equations of $\pi_0B_0+\pi_1A_2=0$ reduces to the last equation of $x_0\tilde{B}_0+x_1\tilde{A}_2=0$. Similarly the first Q equations of $\pi_0A_0+\pi_1A_1+\pi_2A_2=0$ reduces to the first equation of $x_0\tilde{A}_0+x_1\tilde{A}_1+x_2\tilde{A}_2=0$; the next Q equations of $\pi_0A_0+\pi_1A_1+\pi_2A_2=0$ reduces to the second equation of $x_0\tilde{A}_0+x_1\tilde{A}_1+x_2\tilde{A}_2=0$ and the last Q equations of $\pi_0A_0+\pi_1A_1+\pi_2A_2=0$ reduces to the last equation of $x_0\tilde{A}_0+x_1\tilde{A}_1+x_2\tilde{A}_2=0$. The intuition behind this is that since the replenishment is instantaneous there is an equal probability for each inventory level to be visited. It is verified that the above values satisfy $\pi T=0$.

5.3 System Performance Measures

5.3.1 Expected number of interruptions encountered by a customer

For computing expected number of interruptions encountered by a customer we consider a Markov process $\{X_1(t),t\geq 0\}=\{(N_1(t),S_1(t))\,,\ t\geq 0\}$, where $N_1(t)$ denotes the number of interruptions that has occurred up to time t ; $S_1(t)=0$ or 1 according as the service is under interruption or not at time t . The Markov process $\{X_1(t),t\geq 0\}$ has state space $\{0,1,2,\ldots\}\times\{0,1\}\cup\{\Delta\}$, where Δ is an absorbing state which denotes service completion. The infinitesimal generator of the process is given by $\hat{U} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . \\ \hat{B}_{00} & \hat{A}_{00} & \hat{A}_{01} & 0 & 0 & . & . & . \\ \hat{A}_2 & 0 & \hat{A}_1 & \hat{A}_0 & 0 & . & . & . \\ \hat{A}_2 & 0 & 0 & \hat{A}_1 & \hat{A}_0 & . & . & . \\ \hat{A}_2 & 0 & 0 & 0 & \hat{A}_1 & \hat{A}_0 & . & . \\ & & . & . & . & . & . & . \\ & & & . & . & . & . & . \\ & & & & . & . & . & . \end{bmatrix}$,

$\hat{B}_{00} = [\mu]$, $\hat{A}_{00} = [-(\mu+\delta_1)]$, $\hat{A}_{01} = [\delta_1\ 0]$, $\hat{A}_2 = \begin{bmatrix} 0 \\ \mu \end{bmatrix}$,

$\hat{A}_1 = \begin{bmatrix} -\delta_2 & \delta_2 \\ 0 & -(\mu+\delta_1) \end{bmatrix}$ and $\hat{A}_0 = \begin{bmatrix} 0 & 0 \\ \delta_1 & 0 \end{bmatrix}$. If y_k is the probability that absorption occurs with exactly k interruptions, then

$y_0 = -\hat{A}_{00}^{-1}\hat{B}_{00} = \dfrac{\mu}{\mu+\delta_1}$

$y_k = (-\hat{A}_{00}^{-1}\hat{A}_{01})(-\hat{A}_1^{-1}\hat{A}_0)^{k-1}(-\hat{A}_1^{-1}\hat{A}_2) = \dfrac{\mu}{\mu+\delta_1}\left(\dfrac{\delta_1}{\mu+\delta_1}\right)^k$, $k = 1, 2, 3, \ldots$. The expected number of interruptions before absorption is given by

$E_1 = \sum_{k=0}^{\infty} k y_k = \left(-\hat{A}_{00}^{-1}\hat{A}_{01}\right)\left[I_2 - \left(-\hat{A}_1^{-1}\hat{A}_0\right)\right]^{-1} e = \dfrac{\delta_1}{\mu}$

5.3.2 Expected duration of an interrupted service

Here we calculate the average duration of an interrupted service. The service process with interruption can be viewed as a Markov process with two transient states 0 and 1, which denote whether the server is interrupted or is busy respectively, and a single absorption state Δ. Let $\hat{X}(t) = \{0, 1, \Delta\}$ be the corresponding process. The infinitesimal generator matrix of the process is given by $\hat{H} = \begin{bmatrix} \hat{B} & \hat{B}_0 \end{bmatrix}$, where $\hat{B} = \begin{bmatrix} -\delta_2 & \delta_2 \\ \delta_1 & -(\mu+\delta_1) \end{bmatrix}$ and $\hat{B}_0 = \begin{bmatrix} 0 \\ \mu \end{bmatrix}$. The probability distribution of

T, the time until absorption is given by $F(x) = 1 - \zeta \exp(\hat{B}x)e$, $x \geq 0$, where $\zeta = (0, 1)$. Its density function $F'(x)$ given by $F'(x) = \zeta \exp(\hat{B}x)\hat{B}_0$. The Laplace-Stieltjes transform $f(s)$ is $f(s) = \zeta(sI - \hat{B})^{-1}\hat{B}_0$. The expected time E_s for service completion is given by $E_s = \zeta(-\hat{B})^{-1}e = \dfrac{\delta_1 + \delta_2}{\mu\delta_2}$

5.3.3 Other Performance Measures

1. The probability that the server is busy
$PSB = \sum_{i=0}^{\infty} \sum_{k=s+1}^{S} \pi(i, 1, k) = \left[\dfrac{\lambda\theta}{\theta\mu - \lambda^2 - \lambda\theta}\right] x(0, 0)$

2. The probability that the server is on interruption
$PSI = \sum_{i=1}^{\infty} \sum_{k=s+1}^{S} \pi(i, 2, k) = \left(\dfrac{\delta_1}{\delta_2}\right) PSB$

3. The probability that the server is idle
$PSID = \sum_{i=0}^{\infty} \sum_{k=s+1}^{S} \pi(i, 0, k) = \left[1 + \dfrac{\lambda^2}{\theta\mu - \lambda^2 - \lambda\theta}\right] x(0, 0)$

4. The expected inventory level in the system
$EIL = \sum_{i=1}^{\infty} \sum_{j=0}^{2} \sum_{k=s+1}^{S} k\pi(i, j, k) = \dfrac{S + s + 1}{2}$

5. The expected number of customers in the orbit
$ENCO = \sum_{i=0}^{\infty} \sum_{j=0}^{2} \sum_{k=s+1}^{S} i\pi(i, j, k)$

6. The expected rate of ordering, $ERO = \sum_{i=1}^{\infty} \mu\pi(i, s + 1)$

5.4 Numerical Illustration

In this section we provide numerical illustration of the system performance measures as underlying parameters vary.

5.4.1 Effect of regular arrival rate on the various performance measures

In table 5.1 we see that as regular arrival rate increases, probability that server is busy increases and the probability that server is idle decreases which is obvious. The probability that server is on interruption also increases. This is due to the fact that the server gets interrupted while serving a customer and so probability of server interruption increases with server busy probability. The expected inventory level is independent of arrival rate. The increase in expected number of customers in the system is as expected and the increase in expected rate of ordering is due to the fact that, more the server is busy, inventory level drops to s at a faster rate.

Table 5.1: *Effect of regular arrival rate on the various performance measures*

$\mu = 9$ $\theta = 5$ $\delta_1 = 2$ $\delta_2 = 3$ $s = 5$ $S = 11$						
λ	PSB	PINT	PIDL	EIL	ENCO	ERO
3	0.4545	0.1818	0.3637	8.5	0.8701	0.1928
3.2	0.479	0.1916	0.3294	8.5	1.1112	0.2143
3.4	0.5029	0.2011	0.296	8.5	1.4354	0.2364
3.6	0.5274	0.2109	0.2617	8.5	1.9167	0.2603
3.8	0.5562	0.2201	0.2237	8.5	2.6055	0.2837
4	0.5603	0.2241	0.2156	8.5	3.0369	0.2943
4.2	0.5932	0.2372	0.1696	8.5	5.7195	0.3307
4.4	0.6145	0.2458	0.1397	8.5	10.9939	0.3555
4.6	0.6353	0.2541	0.1106	8.5	52.0843	0.3808

5.4.2 Effect of retrial rate on the various performance measures

In table 5.2 we see that the retrial rate has no effect on server busy probability, probability that server is on interruption and probability that server is idle. This is due to the fact that in our model no arrivals or retrials are entertained when the server is on interruption. It is seen that the expected number of customers in the system decreases and so the expected reorder rate decreases with increase in retrial rate. The expected inventory level depends only on s and S.

Table 5.2: *Effect of retrial rate on the various performance measures*

$\lambda = 3 \;\; \mu = 9 \;\; \delta_1 = 2 \;\; \delta_2 = 3 \;\; s = 5 \;\; S = 11$						
θ	PSB	PINT	PIDL	EIL	ENCO	ERO
4	0.4545	0.1818	0.3637	8.5	1.1272	0.241
4.2	0.4545	0.1818	0.3637	8.5	1.0575	0.229
4.4	0.4545	0.1818	0.3637	8.5	1.0031	0.2191
4.6	0.4545	0.1818	0.3637	8.5	0.953	0.2096
4.8	0.4545	0.1818	0.3637	8.5	0.909	0.2008
5	0.4545	0.1818	0.3637	8.5	0.8701	0.1928
5.2	0.4545	0.1818	0.3637	8.5	0.8353	0.1854
5.4	0.4545	0.1818	0.3637	8.5	0.8041	0.1785
5.6	0.4545	0.1818	0.3637	8.5	0.7759	0.1721

5.4.3 Effect of interruption rate on the various performance measures

In table 5.3 we see that as interruption rate increases probability that server is busy and probability that server is on interruption both increases. As interruption rate increases, probability that server is on interruption obviously increases and since $PSB = \frac{\delta_2}{\delta_1} PSI$, probability that server is busy also increases. The expected number of customers in the system is seen to increase. This is due to the fact that since probability that server is busy increases more arriving customers has to go into the orbit. The table also shows that the expected rate of ordering decreases with increase in interruption rate.

Table 5.3: *Effect of interruption rate on the various performance measures*

$\lambda = 3$ $\mu = 9$ $\theta = 5$ $\delta_2 = 3$ $s = 5$ $S = 11$						
δ_1	PSB	PINT	PIDL	EIL	ENCO	ERO
1	0.3999	0.0999	0.5002	8.5	0.8429	0.2076
1.2	0.4117	0.1176	0.4707	8.5	0.8487	0.2045
1.4	0.423	0.1346	0.4424	8.5	0.8543	0.2014
1.6	0.4339	0.1509	0.4152	8.5	0.8598	0.1985
1.8	0.4444	0.1666	0.389	8.5	0.865	0.1956
2	0.4545	0.1818	0.3637	8.5	0.8701	0.1928
2.2	0.4642	0.1964	0.3394	8.5	0.8749	0.1901
2.4	0.4736	0.2105	0.3159	8.5	0.8796	0.1875
2.6	0.4827	0.2241	0.2932	8.5	0.8842	0.1849

5.4.4 Effect of repair rate on the various performance measures

In table 5.4 we see that as interruption rate increases probability that server is busy and probability that server is on interruption both decreases. As repair rate increases, probability that server is on interruption obviously decreases and since $PSI = \frac{\delta_1}{\delta_2} PSB$, probability that server is busy also decreases. The expected number of customers in the system is seen to decrease. This is due to the fact that since probability that server is busy decreases less arriving customers has to go into the orbit. The table also shows that the expected rate of ordering increases with increase in repair rate.

Table 5.4: *Effect of repair rate on the various performance measures*

$\lambda = 3$ $\mu = 9$ $\theta = 5$ $\delta_1 = 2$ $s = 5$ $S = 11$						
δ_2	PSB	PINT	PIDL	EIL	ENCO	ERO
3	0.4545	0.1818	0.3637	8.5	0.8701	0.1928
3.2	0.4482	0.1724	0.3794	8.5	0.8669	0.1945
3.4	0.4426	0.1639	0.3935	8.5	0.8641	0.1961
3.6	0.4374	0.1562	0.4064	8.5	0.8616	0.1975
3.8	0.4328	0.1492	0.418	8.5	0.8592	0.1988
4	0.4285	0.1428	0.4287	8.5	0.8571	0.1999
4.2	0.4246	0.1369	0.4385	8.5	0.8551	0.201
4.4	0.421	0.1315	0.4475	8.5	0.8533	0.202
4.6	0.4177	0.1266	0.4557	8.5	0.8517	0.2029

5.4.5 Effect of service rate on the various performance measures

In table 5.5 we see that as service rate increases probability that server is busy decreases and since $PSI = \frac{\delta_1}{\delta_2} PSB$ probability that server is on interruption also decreases. This is due to the fact that as service rate increases, probability that server is idle obviously increases since the server completes its job at a faster rate. , probability that server is busy also decreases. The expected number of customers in the system is seen to decrease. This is due to the fact that since probability that server is busy decreases less arriving customers has to go into the orbit. The table also shows that the expected rate of ordering increases with increase in service rate.

Table 5.5: *Effect of service rate on the various performance measures*

$\lambda = 3$ $\theta = 5$ $\delta_1 = 2$ $\delta_2 = 3$ $s = 5$ $S = 11$						
μ	PSB	PINT	PIDL	EIL	ENCO	ERO
8	0.4999	0.1999	0.3002	8.5	1.1999	0.1846
8.2	0.4901	0.196	0.3139	8.5	1.1176	0.1863
8.4	0.4807	0.1923	0.327	8.5	1.0448	0.188
8.6	0.4716	0.1886	0.3398	8.5	0.9801	0.1897
8.8	0.4629	0.1851	0.352	8.5	0.9222	0.1913
9	0.4545	0.1818	0.3637	8.5	0.8701	0.1928
9.2	0.4464	0.1785	0.3751	8.5	0.823	0.1943
9.4	0.4385	0.1754	0.3861	8.5	0.7803	0.1958
9.6	0.431	0.1724	0.3966	8.5	0.7413	0.1972

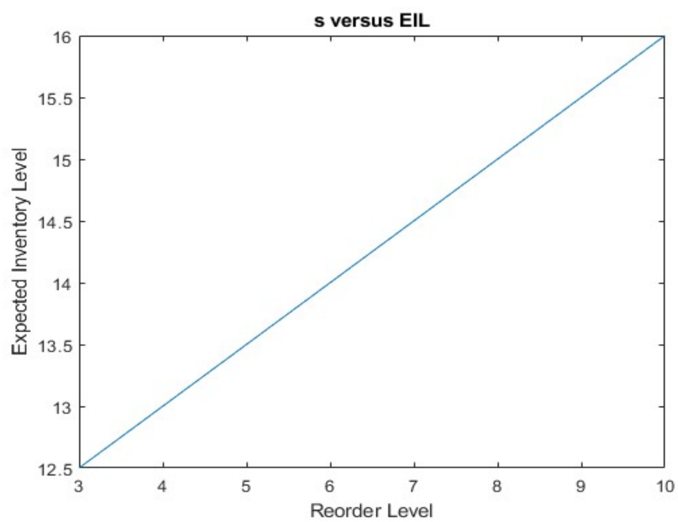
5.4.6 Effect of reorder level on the various performance measures

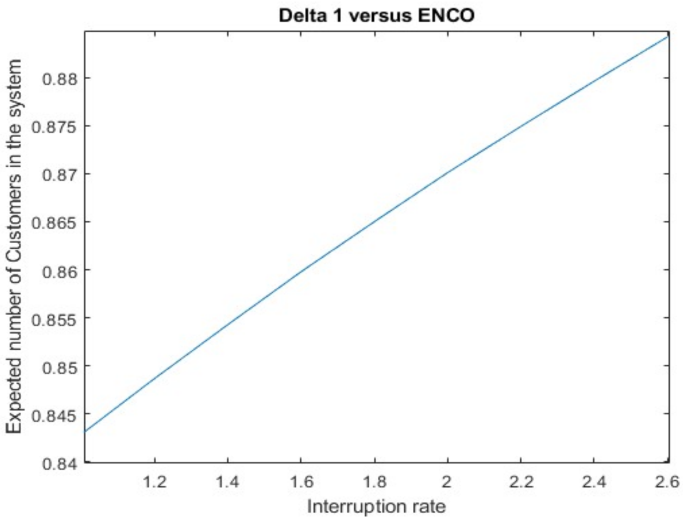
In table 5.6 we see that as reorder level the expected inventory level in the system increases since its value is $\frac{S+s+1}{2}$. The expected reorder rate also increases with increase in reorder level which due to the fact that orders are placed early. The table shows that the reorder level doesn't have much effect on the other performance measures.

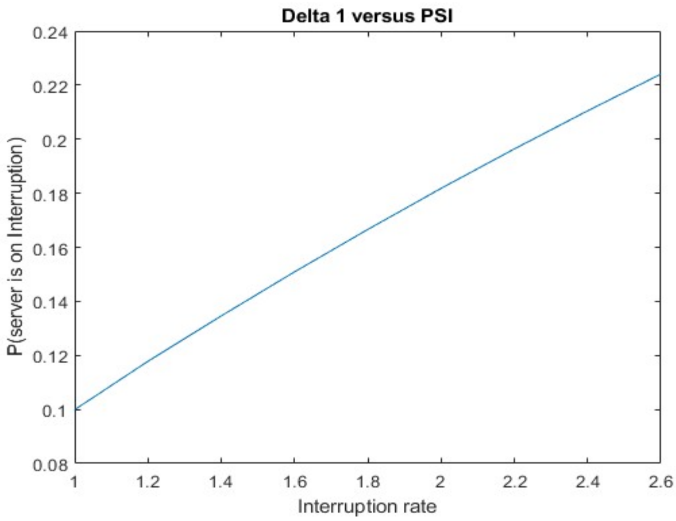
Table 5.6: *Effect of reorder level on the various performance measures*

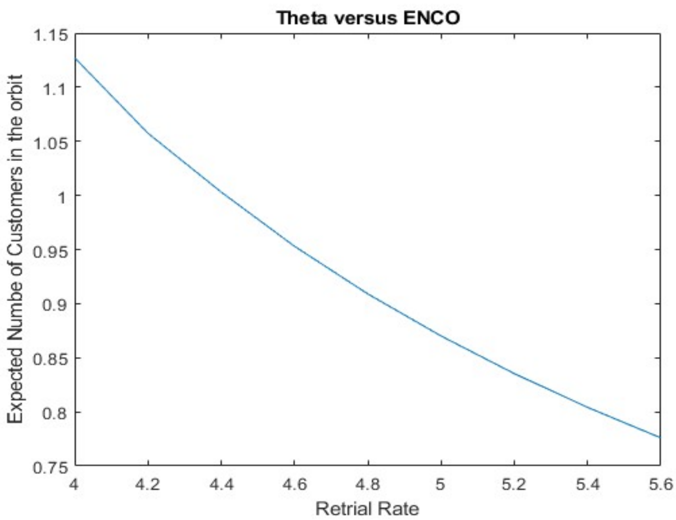
$\lambda = 3$ $\mu = 9$ $\theta = 5$ $\delta_1 = 2$ $\delta_2 = 3$ $S = 21$						
s	PSB	PINT	PIDL	EIL	ENCO	ERO
3	0.3333	0.1818	0.4849	12.5	0.8701	0.0642
4	0.3582	0.1818	0.46	13	0.8701	0.068
5	0.3863	0.1818	0.4319	13.5	0.8701	0.0723
6	0.4181	0.1818	0.4001	14	0.8701	0.0771
7	0.4545	0.1818	0.3637	14.5	0.8701	0.0826
8	0.4545	0.1818	0.3637	15	0.8701	0.089
9	0.4545	0.1818	0.3637	15.5	0.8701	0.0964
10	0.4545	0.1818	0.3637	16	0.8701	0.1056

5.5 Graphical Illustration




Delta 1 versus ENCO
Expected number of Customers in the system
Interruption rate
0.84
0.845
0.85
0.855
0.86
0.865
0.87
0.875
0.88
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6


Delta 1 versus PSI
P(server is on Interruption)
Interruption rate
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0.22
0.24
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6


Theta versus ENCO
Expected Numbe of Customers in the orbit
Retrial Rate
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
4
4.2
4.4
4.6
4.8
5
5.2
5.4
5.6

5.6 Conclusion

In this paper we could derive an explicit expression for the steady state probability vector of an inventory queuing model with retrial and server interruptions. Several other performance measures such as expected waiting time of a customer in the orbit, average duration of an interrupted service and so on can be calculated explicitly. An optimisation of a cost function may be done. We wish to extend this paper by considering positive lead time as well which may have several applications in real life situations. We also intend to do the transient analysis of this model and its extensions.

CHAPTER 6

Recommendations

In this thesis we have derived explicit expressions for the steady state probability vector of three inventory queuing models.

In the first model we considered a single server queuing model with negligible service time and backlogs. Several extensions of the model are possible. A simple extension of this model is the one identical to this model but with positive service time. We can also consider the same model or its extension with more general distributions for arrival, service and replenishment of inventory.

In the second model we considered a single server queuing model with positive service time, positive lead time and server interruptions. In this model we have assumed that no arrivals take place when the server is on interruption and the orders for inventory placed if any are cancelled. We can extend this model by relaxing one or both of the constraints and look for if any explicit solution can be obtained for the steady state probability vector. More generally we can think of considering the arrival process to be follow a Phase type distribution or MAP. Similarly for the service process also more general distributions can be considered.

In the third model we could arrive at an explicit expression for the steady state probability vector for an inventory model with retrials and server interruptions. Here we considered a constant retrial rate in the sense that a queue is formed in the orbit. Instead one can consider the same model with the classical retrial policy and look for explicit expressions for the steady state probability. We can also think of relaxing the assumption of no retrials are entertained when the server is on interruption and still look for explicit expressions for the steady state probability. Considering distributions other than exponential for arrival, service and retrials is another option.

Publications in Journals and Presentations

Publications:

1. E.Sandhya, C.Sreenivasan, Sajeev S Nair and M.P Rajan, "An Explicit solution for an Inventory model with Positive lead time and Backlogs", Stochastic Modelling and Applications, ISSN 0972-3641, Vol 25 No:1 January- June 2021, Chapter 11.
   Link to article/paper/abstract of the article:
   https://www.mukpublications.com/resources/SMAv25-1-11_new.pdf

2. E.Sandhya, C.Sreenivasan, Sajeev S Nair and M.P Rajan, "An explicit Solution for an Inventory Model with Positive Lead Time and Server Interruptions", Information Technologies and Mathematical Modelling , Queueing Theory and Applications, CCIS Volume 1605 2022, Pages 208-220.
   Link to article/paper/abstract of the article: https://link.springer.com/chapter/10.1007/978-3-031-09331-9_17

3. E.Sandhya, C. Sreenivasan, Smija Skaria and Sajeev S Nair, "An Explicit Solution for an Inventory Model with Retrial and Server Interruptions", Published in Information Technologies and Mathematical Modelling, Queueing Theory and Applications, CCIS Volume 1803: 2023, Pages 149-162.

Presentations:

1. Presented a paper titled "An Explicit Solution for an Inventory Model with Positive Lead Time and Backlogs" in the international Conference ITMM 2020 (Online) held at TOMSK Russia from 02-12-2020 to 05-12-2020.

2. Presented a paper titled "An Explicit Solution for an Inventory Model with Positive Lead Time and Server Interruptions" in the international Conference ITMM 2021 (Online) held at TOMSK Russia from 01-12-2021 to 05-12-2021.

3. Presented a paper titled "An Explicit Solution for an Inventory Model with Server Interruption and Retrials" in the international Conference ITMM 2022 (Online) held at KARSHI, UZBEKISTAN, Russia from 25-10-2022 to 29-10-2022.

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